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Certaines classes d'opérateurs générés par une procédure d'interpolation

"Some classes of operators generated by an interpolation procedure"

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إهداء

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إلى كل أعضاء فرق البحث، بمخبر التحليل الدالي وهندسة الفضاءات تحت إدارة الأستاذ دحمان عشور.

Contents

ملخص: عديد من المثاليات الخطية للمؤثرات تم تعميمها حديثا من طرف الكثير من الباحثين إلى حالة المؤثرات الليبشيتزية. في هذه الرسالة قمنا بإعطاء تعريف للمثاليات الليبشتزية وبعض طرق إنشاءها. وتأكدنا من أن الأصناف التي درست حديثا تحقق شروط تعريف المثاليات الليبشيتزية. كما قمنا بتعريف ودراسة نوعين جديدين من المثاليات الليبشيتزية: المؤثرات الليبشيتزية p ـ جمعية بقوة و المؤثرات (p,σ) ـ ليبشيتزية مطلقا. و أعطينا مفهوم خاصية التقريبات المتعلق بالمثاليات الليبشيتزية وقمنا بدراسة أولية حول تفكيك المؤثرات الليبشيتزية وفق فضاءات الدوال البناخية.

Résumé: *La version lipschitzienne de plusieurs idéaux d'opérateurs linéaires, par exemple p-sommants, p-nucléaires, p-intégrals, compacts et faiblement compacts ont été développés et étudiés par plusieurs auteurs. Dans cette thèse, on a introduit la notion d'idéal d'opérateur lipchitzien et quelques méthodes de constructions et nous avons montré quelques exemples des classes d'opérateurs lipschitziens, qui sont parues dans la littérature, sont adaptées avec cette notion. Deux nouvelles classes d'opérateurs d'idéaux lipschitziens sont introduits et ont été étudiés dans notre thèse, à savoir les opérateurs fortement lipschitziens p-sommant et les opérateurs* (p,σ)*-absolument lipschitziens. De même nous avons étudié la propriété d'approximation pour l'idéal des opérateurs lipschitziens. On termine cette thèse, par l'étude de la factorisation des opérateurs lipschitziens entre des espaces de fonctions.*

Abstract: *Many classes of linears operator ideals for example the p-summing, p-nuclear, p-integral, compact and weakly compact operators have been developed and studied in the Lipschitz setting by several authors. In this thesis we introduce the notion of Lipschitz operator ideal and some methods of constructions. We show that a number of examples of classes of Lipschitz maps that have appeared in the literature fall within this notion of Lipschitz operator ideal. New classes of Lipschitz operators ideals are introduced and study in our thesis, namely the strongly Lipschitz p-summing operator and* (p,σ)*-absolutely Lipschitz operator, as an application we studied the approximation property for Lipschitz operator ideals. We finish this thesis by studying the factorization of Lipschitz operators on Banach function spaces.*

Keywords *: Arens–Eells space, Lipschitz mapping, Lipschitz operator ideal, Lipschitz absolutely p-summing operators, strongly Lipschitz p-summing, -absolutely Lipschitz mappings, Pietsch factorization theorem , Approximation property, Banach function spaces, factorization of operators, multiplication operators.*

Notations

$\mathbb{K}$	The field of real or complex numbers.
$\mathbb{R}_+$	The field of non negative real numbers.
p'	The conjugate of the number p $(1 \leq p \leq \infty)$, that is $\frac{1}{p} + \frac{1}{p'} = 1$
E^*	The topological dual of E.
B_E	The closed unit ball of E
$L(E, F)$	The set of all linear operators.
$\mathcal{L}(E, F)$	The set of all continuous linear operators.
w	The weak topology.
w^*	The weak * topology.
$\mathcal{I}$	The ideal of all linear operator.
$\mathcal{I}_{Lip}$	The ideal of all Lipschitz operators.
$\overline{\mathcal{I}}^{inj}$	The closed injective of the linear operators ideal $\mathcal{I}$.
Lip_0	The set of all Lipschitz operators that vanish at 0.
$X^\#$	The Lipschitz dual of the pointed metric space X.
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X.
$m_{xx'}$	The molecule defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$ for $x, x' \in X$, where χ_A is the characteristic function of the set A.
Æ(X)	The Arens-Eells space of X.
$\mathcal{F}(X)$	The Lipschitz free space of X.
T^*	The adjoint operator of T.
$T^\#$	The Lipschitz adjoint operator of T.
T_L	The linearization of the operator T.
T^t	The Lipschitz transpose map of T.
J_p	The canonical inclusion map defined between $C(K)$ and $L_p(\mu)$ $(1 \leq p < \infty)$.

$I_{\infty,p}$	The formal inclusion map defined between $L_\infty(\mu)$ and $L_p(\mu)$ $(1 \leq p < \infty)$.
i_E	The isometric embedding, $i_E : E \longrightarrow C(E^*)$ given by $i_E(x) := \langle x, . \rangle$.
$\mathcal{K}$	The set of all compact linear operators.
Lip_{0K}	The set of all Lipschitz compact operators.
$\mathcal{W}$	The set of all weakly compact linear operators.
Lip_{0W}	The set of all Lipschitz weakly compact operators.
$\mathcal{L}_f$	The set of all finite rank linear operators.
Lip_{0F}	The set of Lipschitz finite rank operators.
Π_p	The set of all linear p-summing operators $(1 \leq p < \infty)$.
Π_p^L	The set of all Lipschitz p-summing operator $(1 \leq p < \infty)$.
$\Pi_{p,r,s}^L$	The set of all Lipschitz (p, r, s)-summing operators $(1 \leq p, r, s \leq \infty)$.
$\mathcal{D}_p$	The set of all strongly linear p-summing operators $(1 < p \leq \infty)$.
$\mathcal{D}_{st,p}^L$	The set of all strongly Lipschitz p-summing operators $(1 < p \leq \infty)$.
$\Pi_{p,\sigma}$	The set of all (p, σ)-absolutely continuous linear operators $(1 \leq p < \infty,\ 0 \leq \sigma < 1)$.
$\Pi_{p,\sigma}^L$	The set of all (p, σ)-absolutely Lipschitz operators $(1 \leq p < \infty,\ 0 \leq \sigma < 1)$.
$\mathcal{I}_p$	The set of all p-integral linear operators $(1 \leq p < \infty)$.
$\mathcal{I}_p^L$	The set of all Lipschitz p-integral operators $(1 \leq p < \infty)$.
$\mathcal{N}_p$	The set of all p-nuclear linear operators $(1 \leq p < \infty)$.
$\mathcal{N}_p^L$	The set of all Lipschitz p-nuclear operators $(1 \leq p < \infty)$.

List of publications

1. Y. Rachid, D. Achour and P. Rueda, Absolutely summing Lipschitz conjugates. Mediterr. J. Math. 13 (2016), 1949-1961

2. D. Achour, P. Rueda, E.A. Sánchez-Pérez and R.Yahi, Lipschitz operator ideals and the approximation property. J. Math. Anal. Appl. 436 (2016) 217–236.

3. D. Achour, P. Rueda and R. Yahi, (p, σ)-Absolutely Lipschitz operators (2016). To appear in Annals of Functional Analysis.

Introduction

The theory of linear operator ideals between normed (or Banach) spaces have been developed by Albert Pietsch [59], and it is nowadays well established. A linear operator ideal $\mathcal{I}$ is a subclass of the class of all continuous linear operators, such that for every Banach spaces E and F, the set $\mathcal{I}(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$ that is invariant by the composition of linear operators on the right or and on the left and which contains the linear operators of finite rank.

Linear p-summing operators ($1 \leq p < \infty$) are an important classe of linear operator ideal. A linear operator T between two Banach spaces E and F is said to be *p-summing* if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1\leq i\leq n}$ in E

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}.$$

The fruitful development of the theory of absolute summability for linear operators produced several generalizations to the non linear context. The Lipschitz p-summing operators ($1 \leq p < \infty$) between metric spaces, a nonlinear generalization of p-summing operators, was introduced by Farmer and Johnson in [33]. Recall that a mapping $T : X \longrightarrow Y$ between two metric spaces is *Lipschitz p-summing* if there exists a constant $C \geq 0$ such that for all $(x_i)_{i\leq n}, (x'_i)_{i\leq n}$ in X and all $(a_i)_{i\leq n} \subset \mathbb{R}^+$

$$\left(\sum_{i=1}^{n} a_i d(T(x_i), T(x'_i))^p\right)^{\frac{1}{p}} \leq C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} a_i |f(x_i) - f(x'_i)|^p\right)^{\frac{1}{p}}.$$

Farmer and Johnson in [33] proved basic fundamental properties and leave to interested readers a list of open problems (*what results about p-summing operators have analogues for Lipschitz p-summing operators?*). Since then, many works have appeared related to this class of Lipschitz mappings and some of the problems have been solved. Their objective is to derive the full benefit that accrues from the successful theory of absolutely summing operators to the Lipschitz context.

Recently, many classes of linear operator ideals are introduced in Lipschitz context, just trying to establish Lipschitz versions of the corresponding linear operators. In [40] Lipschitz compact operators, Lipschitz weakly compact, Lipschitz finite-rank operators and Lipschitz approximable operators have been introduced and studied. Chen and Zheng [20] introduced and studied strongly Lipschitz p-integral, strongly Lipschitz p-nuclear operators and Lipschitz p-nuclear operators. Chávez-Domínguez defined and investigated Lipschitz (p, r, s)-summing operators and Lipschitz (q, p)-mixing operators in [16] and [17].

Like the above works on Lipschitz operators, the article of Farmer and Johnson [33] motivated us to study the Lipschitz version of some linear operator ideals. In [65] we studied the Lipschitz version of strongly p-summing operator (we mention that the concept of strongly p-summing linear operators $(1 < p \leq \infty)$ was introduced by J. S. Cohen in [21] as a characterization of linear operators having absolutely p-summing adjoint). The isomorphism between $Lip_0(X, E)$ and $\mathcal{L}(\text{Æ}(X), E)$ (also between $Lip_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^{\#}, w^*))$) provides a shortcut for the study of such summability property in the Lipschitz context of strongly summing operators.

In the mid way between continuous linear operators and absolutely summing operators, a scale of linear operators, namely (p, σ)-absolutely continuous operators $(1 \leq p < \infty$, $0 \leq \sigma < 1)$, were defined by Matter [51, 52] by applying an interpolative ideal procedure. The interpolated operator ideal $\Pi_{p,\sigma}$ of all (p, σ)-absolutely continuous operators was defined as an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators, and shares similar properties with absolutely p-summing operators. Although it was first thought as a tool for the study of super-reflexive Banach spaces, several works have been done concerning this class of operators: factorization properties and their representation as dual spaces of suitable tensor products can be found in [49, 61]. The nonlinear version (multilinear, m-homogeneous polynomial) of this concept has been recently introduced and studied (see [1] and the references therein) and applications to the theory of Pettis or Bochner integrable functions have been found in [57]. Connecting both the linear and the Lipschitz theories, we study the class of (p, σ)-absolutely Lipschitz mappings. These have to be considered as an attempt to interpolate Lipschitz mappings with Lipschitz absolutely p-summing mappings.

Following recent advances in the theory of operator ideals, we establish the basics of the theory of Lipschitz operators ideals with the aim of recovering several classes of Lipschitz maps related to absolute summability. Although in recent years there is an increasing interest for finding results on approximation properties related to Lipschitz functions and the free spaces Æ(X) of metric spaces X (see [23, 24, 35, 36, 44]), there are no explicit definitions of approximation properties on metric spaces. For example, [8] reproduces the approximation property for Banach spaces considering approximating Lipschitz mappings instead of linear operators. Recently, new extensions of the approximation property for Banach spaces related to operator ideals have been introduced (see [7, 45]). In [45], the definition is based on a notion of compactness depending on a given operator ideal, that has its roots in the paper by Carl and Stephani [14] and has been developed in the context of polynomials and holomorphy in [5, 6, 37]. The general notion of Lipschitz operator ideal provides a natural context for giving suitable definitions of such notions. Our purpose is to develop a basic theory of the Lipschitz operator ideals that unifies all the recently introduced new classes of Lipschitz mappings, and apply it to the establishment of a suitable context for the study of the approximation property intrinsic to metric spaces and Lipschitz mappings.

Our Thesis is organized as follows:

In chapter 1, we recall briefly some basic notions and terminologies needed in our thesis (we assume that the reader is familiar with basic functional analysis: normed spaces, the $L_p(\mu)$ spaces $(1 \leq p \leq \infty)$, the $\ell_p(E)$ $(1 \leq p \leq \infty)$ spaces, the $C(K)$ space, tensor product, the weak and weak* topologies, reflexive spaces, separable spaces, Hahn-Banach Theorem and its consequences...etc). The rest of this chapter is devoted to show some basic examples of linear operator ideals, the Lipschitz p-summing operators and the Lipschitz compact (weakly compact) operators .

In chapter 2 we contribute to the theory of absolutely p-summing Lipschitz mappings by studying the class of Lipschitz summing mappings whose linear analogue has found its natural place in the linear operator theory: the class of Lipschitz strongly p-summing mappings. The main properties of these mappings are established. We prove that, if the range space is a Hilbert space, every strongly Lipschitz p-summing is Lipschitz q-summing $(1 \leq q < \infty)$. As a consequence, we obtain a generalization to a result given by Chenz-Zheng [19, Theorem 3.1].

Chapter 3 is devoted to study (p, σ)-absolutely Lipschitz mappings as an interpolating class between Lipschitz mappings and Lipschitz absolutely p-summing mappings. We pay attention to the domination/factorization theorem, whose proof uses the abstract version of the Pietsch domination theorem given in [9] and [58]. When applying our factorization theorem to the particular class of absolutely p-summing Lipschitz mappings, we get an equivalent factorization to the one given by Farmer and Johnson in [33, Theorem 1]. In the final of this chapter, the duality for (p, σ)-absolutely Lipschitz operators is studied.

In chapter 4 we establish the basics of the theory of Lipschitz operator ideals with the aim of recovering several classes of Lipschitz maps related to absolute summability that have been introduced in the literature in the last years. As an application we extend the notion and main results on the approximation property for Banach spaces to the case of metric spaces.

The last chapter can be considered as a future research project. Let (X, d) be a pointed metric space and $T\colon X \to Y_1(\mu)$, $S\colon X \to Y_2(\mu)$ be two Lipschitz operators on two Banach function spaces over the same finite measure μ. We show which are the vector norm inequalities that an operator T must fulfill when T can be factored as $T = M_g \circ S$, where $M_g\colon Y_2 \to Y_1$ is a multiplication operator. We also show some Maurey-Rosenthal type theorems on factorization through L^p-spaces.

Chapter 1

Preliminaries

1.1 Basic notions and terminologies

In this chapter, we present some concepts which will use in the sequel of this thesis, such that : the Lipschitz space, Arens-Eells space and the linear operator ideal.
We will write $\mathbb{K}$ for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For $1 \leq p \leq \infty$, by p' we denote the conjugate of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$. Along this thesis the letters E and F denote Banach spaces. Given a Banach space Z, Z^* denotes its topological dual, and B_Z is its closed unit ball. By $\mathcal{L}(E, F)$ we denote the Banach space of all continuous linear operators between E and F with the norm

$$\|T\| = \sup_{x \in B_E} \|T(x)\|.$$

If $T \in \mathcal{L}(E, F)$, $T^* : F^* \longrightarrow E^*$ is the adjoint operator of T with $\|T\| = \|T^*\|$.
Two norms $\|.\|_1$ and $\|.\|_2$ on a linear space Z are equivalent if there exist positive numbers C, K such that

$$C \|x\|_1 \leq \|x\|_2 \leq K \|x\|_1, \forall x \in Z.$$

For a linear operator $T \in \mathcal{L}(E, F)$, the linear operator $S : F \longrightarrow E$ that satisfies $T \circ S = id_F$ and $S \circ T = id_E$ is called the inverse of T and is denoted here by T^{-1}.
A linear operator $T : Z_1 \longrightarrow Z_2$ between two normed spaces Z_1 and Z_2 is an *isomorphism* if T is a continuous bijection whose inverse T^{-1} is also continuous. In such case the spaces Z_1 and Z_2 are said to be *isomorphic*. T is an *isometric isomorphism* when $\|T(x)\| = \|x\|$ for all $x \in Z_1$. In particular, any linear operator between normed spaces of the same finite dimension is an isomorphism.

A linear operator T is an *embedding of* E *into* F if T is an isomorphism onto its image $T(E)$. In this case we say that E *embeds in* F. If $T : E \longrightarrow F$ is an embedding such that $\|T(x)\| = \|x\|$ for all $x \in E$, then T is said to be an *isometric embedding.*

1.2 Arens-Eells space

The notion of metric space was formalized by Maurice Fréchet in his thesis "Doctorat d'Etat" in 1906 "Sur quelques points du calcul fonctionnel" (see [34]). He was among the first who used the word space. Recall that a metric or distance on a non empty set X is a function

$$d : X \times X \longrightarrow \mathbb{R}_+$$

with the following properties:

1. (i) (Positivity) For all $x, y \in X$, $d(x, y) \geq 0$ with equality if and only if $x = y$.
2. (ii) (Symmetry) For all $x, y \in X$, $d(x, y) = d(y, x)$.
3. (iii) (Triangle inequality) For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

The set X equipped with the distance d is called a metric space.
The natural morphisms between metric spaces are the Lipschitz functions. A map $T : X \longrightarrow Y$ between two metric spaces is called Lipschitz if there is a positive constant C such that

$$\forall x, y \in X, \quad d(T(x), T(y)) \leq Cd(x, y). \tag{2.1}$$

For a Lipschitz map $T : X \longrightarrow Y$ its Lipschitz constant is given by

$$Lip\,(T) = \sup \left\{ \frac{d(T(x), T(y))}{d(x, y)}, x \neq y \right\}$$

A pointed metric space X is a metric space with a base point in X , that is, a designated special point, which we will always denote by 0. Along this thesis, X and Y denote pointed metric spaces. We will consider a normed space E over $\mathbb{K}$ as a pointed metric space with the distance defined by its norm and the zero vector as the base point. We denote by $Lip_0(X, Y)$ the set of all base-point preserving Lipschitz maps from X to Y. If E is a Banach space, $Lip_0(X, E)$ is a Banach space under the Lipschitz norm given by

$$Lip\,(T) = \sup \left\{ \frac{\|T(x) - T(y)\|}{d(x, y)}, x \neq y \right\}$$

For $E = \mathbb{K}$, we designate $Lip_0(X, \mathbb{K}) = Lip_0(X) = X^{\#}$. The Banach space $X^{\#}$ of Lipschitz functions is called also Lipschitz dual it has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case.

Now we are going to present some concepts about the space of molecules, the reader can see [63] or [4] for more details.

Definition 1.1. *Let X be a metric space. A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x \in X} m(x) = 0$. We denote by $\mathcal{M}(X)$ the linear space of all molecules on X.*

For $x, x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$ we can write

$$m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'}$$

for some suitable scalars λ_j, and we write

$$\|m\|_{\mathcal{M}(X)} = \inf \left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j, x_j'), \ m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'} \right\},$$

where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [4] in 1956. The terminology *Arens-Eells space* Æ(X) is due to Weaver [63]. A different notation was used in [35] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X)$.

Let X be a pointed metric space

1. The map $\delta_X : X \longrightarrow$ Æ(X) defined by

$$\delta_X(x) = m_{x0}$$

   is an isometric embedding of X into Æ(X) (see [63, Theorem 2.2.4]).

2. The map $Q_X : X^{\#} \longrightarrow$ Æ$(X)^*$ defined by

$$Q_X(f) = f_L, \text{ where } f_L(m) = \sum_{x \in X} f(x) m(x),$$

establishes an isometric isomorphic between $X^{\#}$ and $\text{Æ}(X)^{*}$. On bounded subsets of $X^{\#}$ its weak * topology agrees with the topology of pointwise convergence (see [63, Theorem 2.2.2]).

The Banach space $\text{Æ}(X)$ has some remarkable properties. We mention the following ones.

Theorem 1.2. *([63, Theorem 2.2.4])*
Let $T : X \longrightarrow E$ be a Lipschitz map which preserves the base point; that is $T(0) = 0$. Then there is a unique bounded linear map $T_L : \text{Æ}(X) \longrightarrow E$ such that $T = T_L \circ \delta_X$ that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & E, \\ & {\scriptstyle \delta_X} \searrow \quad \nearrow {\scriptstyle T_L} & \\ & \text{Æ}(X), & \end{array}$$

commutes. Furthermore $\|T_L\| = Lip(T)$

The operator T_L is referred to as the linearization of T. The correspondence

$$T \longleftrightarrow T_L$$

establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(\text{Æ}(X), E)$.

Theorem 1.3. *([42, Lemma 3.1])*
Let $T : X \longrightarrow Y$ be a Lipschitz map which preserves the base point. Then there is a unique bounded linear map $\hat{T} : \text{Æ}(X) \longrightarrow \text{Æ}(Y)$ such that $\widehat{T}\delta_X = \delta_Y T$ that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & Y \\ \downarrow {\scriptstyle \delta_X} & & \downarrow {\scriptstyle \delta_Y} \\ \text{Æ}(X) & \xrightarrow{\;\hat{T}\;} & \text{Æ}(Y) \end{array}$$

commutes. Furthermore $\left\|\hat{T}\right\| = Lip(T)$.

Sawashima [62] defined the Lipschitz adjoint (or dual) of $T \in Lip_0(X, Y)$ as the continuous linear operator

$$\begin{array}{rccc} T^{\#} : & Y^{\#} & \longrightarrow & X^{\#} \\ & g & \longmapsto & T^{\#}(g) = g \circ T \end{array}$$

If $Y = E$ is a Banach space, the restriction of $T^{\#}$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t. The correspondence

$$T \longleftrightarrow T^t$$

establishes an isomorphism between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^{\#}, w^*))$, where w^* denote the weak* topology (see [40, Theorem 3.1]).

1.3 Linear operator ideals

Recall that a linear operator $T \in \mathcal{L}(E, F)$ is said to have finite rank if $T(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle \, y$$

i.e. if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^{n} x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$ (see [59, Page 25]).

Definition 1.4. *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E, F) := \mathcal{L}(E, F) \cap \mathcal{I}$ satisfy:*

(i) $\mathcal{I}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the finite rank operators.

(ii) The ideal property: if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(G, H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ satisfies:

(i') $(\mathcal{I}(E, F), \|.\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces E and F,

(ii') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$,

(iii') if $v \in \mathcal{L}(G, E)$, $u \in \mathcal{I}(E, F)$ and $w \in \mathcal{L}(F, H)$,

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \, \|v\|_{\mathcal{I}} \, \|u\| \, ,$$

then $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal.

The operator ideal $\mathcal{I}$ is said to be *closed* if each $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ for the sup norm.

Definition 1.5. *(injective operator ideal)*
A normed operator ideal $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is said to be injective if for every metric injection $i : F \hookrightarrow G$ and every $u \in \mathcal{L}(E,F)$ it follows from $i \circ u \in \mathcal{I}(E,G)$ that $u \in \mathcal{I}(E,F)$. Moreover

$$\|i \circ u\|_{\mathcal{I}} = \|u\|_{\mathcal{I}},$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ the largest one [59, Theorem 1.2.2].

1.3.1 Ideal of p-summing linear operators.

The theory of p-summing operators is based on a crucial criterion due to Pietsch [59]. We mention that the linear p-summing operators are the starting point in the study of Lipschitz p-summing mappings.
Let $1 \leq p < \infty$. A linear operator $T : E \longrightarrow F$ is said to be p-summing if there exists a constant $C \geq 0$ such that for all finite sequence $(x_i)_{1\leq i\leq n}$ in E

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{E^*}\leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p(T)$. The collection of all p-summing operators between E and F is denoted by $\Pi_p(E,F)$. We mention that (Π_p, π_p) is an injective Banach operator ideal.

Theorem 1.6. *[32, Page 39] If $1 \leq p \leq q < \infty$, then $\Pi_p(E,F) \subset \Pi_q(E,F)$. Moreover, for $T \in \Pi_p(E,F)$ we have $\pi_q(T) \leq \pi_p(T)$.*

The following basic result about p-summing operators is due to A. Pietsch, and it characterizes the p-summability by means of a domination theorem.

Theorem 1.7. *(Pietsch Domination Theorem) [32, page 44]*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E,F)$. Then T is p-summing if and only if there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak star topology) so that

$$\|T(x)\| \leq C \left(\int_{B_{E^*}} |\langle x, x^*\rangle|^p \, d\mu(x^*)\right)^{\frac{1}{p}}, \qquad x \in E. \tag{1.2}$$

In this case, $\pi_p(T)$ is the least of all the constants C such that (1.2) holds.

In order to adapt the previous result into a factorization theorem, we present basic examples of p-summing linear operators.

Example 1.8. *see [32, Example 2.9 (b),(d)]*
(1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K, and let $1 \leq p < \infty$. The canonical inclusion

$$J_p : C(K) \longrightarrow L_p(\mu),$$

is p-summing with $\pi_p(J_p) = \|J_p\| = \mu(K)^{\frac{1}{p}}$.
(2) Let (Ω, Σ, μ) be a finite measure space and let $1 \leq p < \infty$. The formal inclusion map

$$I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu),$$

is p-summing, with $\pi_p(I_{\infty,p}) = \mu(\Omega)^{\frac{1}{p}}$.

We denote by i_E the isometric embedding $E \longrightarrow C(B_{E^*})$ given by $i_E(x) = \langle x, . \rangle$.

Corollary 1.9. *[32, page 45] (Pietsch Factorization Theorem)*
Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. The following are equivalent

(i) T is p-summing.

(ii) There exist a regular Borel probability measure μ on B_{E^*} (with the weak star topology), a closed subspace E_p of $L_p(\mu)$ and a linear continuous operator $\widetilde{u} : E_p \longrightarrow F$ such that $J_p \circ i_E(E) \subset E_p$ and $\widetilde{u} \circ J_p \circ i_E(x) = T(x)$ for all $x \in E$.
In other words, if $\overline{J_p}$ is the map $i_E(E) \longrightarrow E_p$ induced by J_p, then the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ i_E \downarrow & & \uparrow \widetilde{T} \\ i_E(E) & \xrightarrow{\overline{J_p}} & E_p \\ \cap & & \cap \\ C\left(B_{E^*}\right) & \xrightarrow{J_p} & L_p(\mu). \end{array}$$

In addition, we may choose μ and $\widetilde{T}$ so that $\left\|\widetilde{T}\right\| = \pi_p(T)$.

1.3.2 Ideal of compact and weakly compact linear operators.

We say that a bounded linear operator $T : E \rightarrow F$ is compact (weakly compact) if $T(B_E)$ is relatively compact (respectively, relatively weakly compact)in F. By $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$

we denote the sets of compact linear operators and weakly compact linear operators from E to F, respectively.
Schauder theorem about the compactness of the adjoint of a compact linear operator between two Banach spaces is well known. We will remind it as we need in the sequel of this thesis.

Theorem 1.10. *Let $T \in \mathcal{L}(E,F)$. Then T is compact if and only if T^* is compact.*

The weakly version is due to Gantmacher:

Theorem 1.11. *Let $T \in \mathcal{L}(E,F)$. Then T is weakly compact if and only if T^* is weakly compact.*

Proposition 1.12. *([59]) The classes $\mathcal{K}, \mathcal{W}$ consitute closed injective Banach operator ideals, where the ideal norm is the operator norm.*

Theorem 1.13. *[32, Page 50] Let $1 \leq p < \infty$.*
(1) Every p-summing operator between Banach spaces is weakly compact.
(2) If E is a reflexive space, then every p-summing operator between E and F is compact.

1.3.3 Ideal of strongly p-summing linear operators.

Pietsch has shown in [59, page 338] that the identity $Id_{1,2} : \ell_1 \to \ell_2$ is 1-absolutely summing but the adjoint operator is not 1-summing. For this Cohen [21] introduces the class of strongly p-summing linear operators to characterize those operators whose adjoins are absolutely p'-summing linear operators. Recall that a linear operator $T : E \longrightarrow F$ is strongly p-summing $(1 < p \leq \infty)$ if there exists $C \geq 0$ such that for all $n \in \mathbb{N}$, $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{\frac{1}{p}} \sup_{\varphi \in B_{F^{**}}} \Big(\sum_{i=1}^{n} |\varphi(y_i^*)|^{p'} \Big)^{\frac{1}{p'}}.$$

This class is denoted by $\mathcal{D}_p(E,F)$ and the infimum of all C by $d_p(T)$. The collection of all strongly p-summing linear operators, denoted by $\mathcal{D}_p$ which is a Banach ideal with the ideal norm. The following result is due to Cohen .

Theorem 1.14. *[21, Theorem 2.2.2]*
(1) Let $1 \leq p < \infty$. A linear operator T belongs to $\Pi_p(E,F)$ if and only if the adjoint operator T^* belongs to $\mathcal{D}_{p^*}(F^*, E^*)$. In this case $\pi_p(T) = d_{p^*}(T^*)$.

(2) Let $1 < p \leq \infty$. A linear operator T belongs to $\mathcal{D}_p(E,F)$ if and only if the adjoint operator T^* belongs to $\Pi_{p^*}(F^*,E^*)$. In this case $d_p(T) = \pi_{p^*}(T^*)$.

Also, there is a Domination Theorem for the strongly p-summing linear operators.

Theorem 1.15. *A linear operator $u \in \mathcal{L}(E,F)$ is strongly p-summing if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{E^{**}}$,(with the weak star topology) so that for all $x \in E$ and for all $y^* \in F^*$, the inequality*

$$|\langle u(x), y^*\rangle| \leq C \|x\| \left(\int_{B_{F^{**}}} |\langle y^*, \psi\rangle|^{p'} d\mu(\psi) \right)^{\frac{1}{p'}}, \tag{1.3}$$

holds.

As a consequence of Theorem 1.6 and 1.14 we have the inclusion theorem.

Theorem 1.16. *If $p \leq q$. Then $\mathcal{D}_q(E,F) \subset \mathcal{D}_p(E,F)$.*

1.4 Lipschitz p-summing operators

The Lipschitz version of p-summing operators ($1 \leq p < \infty$) was introduced by J.D. Farmer and W.B. Johnson in [33]. A mapping $T \in Lip_0(X,Y)$ is *Lipschitz p-summing* if there exists a constant $C \geq 0$ such that for all $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$ in X and all $(a_i)_{i \leq n} \subset \mathbb{R}^+$

$$\sum_{i=1}^{n} a_i d(T(x_i), T(x'_i))^p \leq C^p \sup_{f \in B_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(x'_i)|^p.$$

The infimum of all such constants $C \geq 0$ is denoted by $\pi_p^L(T)$. This class of mappings is denoted by $\Pi_p^L(X,Y)$.

The theorem characterizing the Lipschitz p-summing operators is the following.

Theorem 1.17. *[33, Theorem 1] Let $1 \leq p < \infty$. The following properties are equivalent for a mapping $T \in Lip_0(X,Y)$ and a positive constant C.*

1. *$\pi_p^L(T) \leq C$.*

2. *There is a probability μ on $B_{X^\#}$ such that*

$$d(T(x), T(y)) \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p d\mu(f) \right)^{\frac{1}{p}}.$$

(Pietsch domination theorem).

3. *For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization*

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \\ \uparrow{\scriptstyle A} & & & & \downarrow{\scriptstyle B} \\ X & \xrightarrow{T} & Y & \xrightarrow{J} & Z \end{array}$$

with μ a probability and $Lip(A)\cdot Lip(B) \leq C$.

(Pietsch factorization theorem).

The domination theorem immediately implies the monotonicity of the Lipschitz $p-$summing norm. Then we have the following theorem.

Theorem 1.18. *If $1 \leq p \leq q < \infty$, then $\Pi_p^L(X,Y) \subset \Pi_q^L(X,Y)$. Moreover, $\pi_p^L(T) \leq \pi_q^L(T)$.*

For a linear operator $T \in \mathcal{L}(E,F)$ it is clear that $\pi_p^L(T) \leq \pi_p(T)$. J.D. Farmer and W.B. Johnson proved that the reverse inequality is true. This justifies that the notion of Lipschitz p-summing operator is really a generalization of the concept of linear p-summing operator.

Theorem 1.19. *[33, Theorem 2] Let T be a bounded linear operator from E into F and $1 \leq p < \infty$. Then $\pi_p^L(T) = \pi_p(T)$.*

In [16] vector valued molecules were naturally considered. An E-valued molecule on X is a finitely supported function $m : X \longrightarrow E$ such that $\sum_{x\in X} m(x) = 0$. The vector space of all E-valued molecules on X is denoted by $\mathcal{M}(X,E)$. An *E-valued atom* is a function of the form $vm_{xx'}$ with $v \in E$, x,$x^{'} \in X$. Note that any molecule may be expressed (in a nonunique way) as a finite sum of atoms. Chávez-Domínguez in [16] defined the p-Chevet-Saphar norm of a molecule $m \in \mathcal{M}(X,E)$ by

$$cs_p(m) = \inf\left\{\left(\sum_j \lambda_j^p \|v_j\|^p\right)^{\frac{1}{p}}\left(\sum_j \lambda_j^{-p'}\left|f(x_j)-f(x_j^{'})\right|^{p'}\right)^{\frac{1}{p'}} : m = \sum_j v_j m_{x_j x_j'},\ \lambda_j > 0\right\}.$$

The space $\mathcal{M}(X,E)$ with the norm $cs_p(.)$ is denoted by $CS_p(X,E)$.

Theorem 1.20. *[16, Theorem 4.3] The spaces $CS_p(X,E)^*$ and $\Pi_{p'}^L(X,E^*)$ are isometrically isomorphic via the canonical pairing, $\langle m,T\rangle = \sum_{j=1}^{n}\left\langle v_j, T(x_j) - T(x_j')\right\rangle$.*

1.5 Lipschitz compact and weakly compact operators

Following [40] a Lipschitz map $T \in Lip_0(X,E)$ is *Lipschitz compact* (*Lipschitz weakly compact*) if the set

$$\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$$

is relatively compact (respectively, relatively weakly compact) in E. Denote by $Lip_{0\mathcal{K}}(X,E)$ and $Lip_{0\mathcal{W}}(X,E)$ the sets of Lipschitz compact operators and Lipschitz weakly compact operators from X to E, respectively. In [40], the relationship between the compactness (weakly compactness) of a Lipschitz operator $T \in Lip_0(X,E)$ and the compactness (respectively, weakly compactness) of its linearization $T_L \in \mathcal{L}(\text{Æ}(X), E)$ has been established:

Proposition 1.21. *[40, Proposition 2.4] Let $T \in Lip_0(X,E)$. The following statements are equivalent:*

(1) T is compact Lipschitz.

(2) T_L is compact from Æ(X) to E.

The Lipschitz versions of Schauder's theorem (respectively, Gantmacher's theorem) on the compactness (respectively, weak compactness) of the adjoint of a compact (respectively, weakly compact) linear operators are also studied in terms of the Lipschitz transpose map of a Lipschitz operator [40].

Proposition 1.22. *[40, Proposition 3.4](Gantmacher's theorem) Let $T \in Lip_0(X,E)$. The following statements are equivalent:*

(1) T is weakly compact Lipschitz.

(2) T^t is weakly compact from E^* to $X^\#$.

Proposition 1.23. *[40, Proposition 3.5](Schauder's theorem) Let $T \in Lip_0(X,E)$. The following statements are equivalent:*

(1) T is Lipschitz compact.

(2) T^t is compact from E^* to $X^\#$.

Chapter 2

Absolutely summing Lipschitz conjugates

The results obtained in this chapter have been published in Mediterranean Journal of Mathematics [65]. In this chapter we contribute to the theory of absolutely p-summing Lipschitz mappings by studying the class of Lipschitz summing mappings whose linear analogue has found its natural place in the linear operator theory: the class of Lipschitz strongly p-summing mappings. The aim is to characterize those mappings whose Lipschitz conjugates are absolutely p-summing. We emphasize the relation of these mappings with the known linear theory using two different ways: via their linearization and via their Lipschitz conjugates. After giving equivalences for these operators in terms of a summability inequality and its correspondent domination formula, we prove that a map T is Lipschitz strongly p-summing if, and only if, its Lipschitz conjugate $T^t : E^* \to X^\#$ is absolutely p'-summing. Note that the Lipschitz conjugate is a linear operator, and so relates the new class of Lipschitz strongly p-summing mappings with the classical theory of absolutely p-summing linear operators. Identifying the Lipschitz strongly p-summing mappings, with values in a dual space, with a subclass of Lipschitz (p, r, s)-summing mappings in the sense of Chávez-Domínguez, we obtain a predual space for our class. On the other hand, Chen-Zheng [19, Theorem 3.1] recently proved that if a Lipschitz mapping T from a pointed metric space X into a Hilbert space H, with $T(0) = 0$, is such that T^t is p'-summing $(1 < p \leq \infty)$, then T is 1-summing. We provide another proof of this result by using the concept of strongly Lipschitz p-summing. We end this chapter with a factorization theorem for Lipschitz strongly p-summing mappings.

2.1 Strongly Lipschitz p-summing mappings

The concept of strongly p-summing linear operators ($1 \leq p \leq \infty$) was introduced by J.S. Cohen in [21] as a characterization of linear operators having absolutely p-summing adjoint. In [21, Theorem 2.2.2] it is shown that $T \in \mathcal{D}_p(E,F)$ if and only if $T^* \in \Pi_{p'}(F^*, E^*)$ whenever $1 < p \leq \infty$. A similar condition on the Lipschitz transpose (namely, being p-summing) has been considered in [19, Theorem 3.1]. Let us introduce the Lipschitz version of strongly p-summing operators.

Definition 2.1. *Let $1 < p \leq \infty$. A mapping $T \in Lip_0(X,E)$ is called strongly Lipschitz p-summing, if there is a Banach space F and an linear operator $S \in \mathcal{D}_p(F,E)$ (i.e., $S^* \in \Pi_{p'}(E^*, F^*)$) such that*

$$|\langle y^*, T(x) - T(x') \rangle| \leq d(x,x') \, \|S^*(y^*)\| \tag{2.1}$$

for all $x, x' \in X, y^* \in E^*$. We write $\mathcal{D}^L_{st,p}(X,E)$ for the set of all strongly Lipschitz p-summing mappings between X and E. For each $T \in \mathcal{D}^L_{st,p}(X,E)$, let $d^L_{st,p}(T)$ denote the infimum of $d_p(S)$, for S satisfying the inequality (2.1).

For $p = 1$ we define $\mathcal{D}^L_{st,1}(X,E) = Lip_0(X,E)$. Note that if T is a linear operator between two Banach spaces E and F then $T \in \mathcal{D}^L_{st,p}(E,F)$ if and only if $T \in \mathcal{D}_p(E,F)$. Indeed if $T \in \mathcal{D}^L_{st,p}(E,F)$ be a linear operator then there is a Banach space Z and a linear operator $S \in \mathcal{D}_p(Z,F)$ such that

$$|\langle y^*, T(x) \rangle| \leq \|x\| \, \|S^*(y^*)\| \tag{2.2}$$

for all $x \in E, y^* \in F^*$. Since S^* is p'-summing so, using the Pietsch domination theorem, there exists a probability measure μ on $B_{F^{**}}$ such that for all $x \in X, y^* \in E^*$ we have

$$|\langle y^*, T(x) \rangle| \leq \pi_{p'}(S^*) \, \|x\| \left(\int_{B_{F^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi) \right)^{\frac{1}{p'}}. \tag{2.3}$$

Thus by Theorem 1.15 T is strongly p-summing, and

$$d_p(T) \leq d_p(S).$$

Taking the infimum of $d_p(S)$, for S satisfying the inequality (2.3), we obtain

$$d_p(T) \leq d^L_{st,p}(T).$$

For the converse let $T \in \mathcal{D}_p(E,F)$ so, using the Pietsch domination theorem, there exists a probability measure μ on $B_{F^{**}}$ such that for all $x, x' \in E, y^* \in F^*$ we have

$$|\langle y^*, T(x) - T(x')\rangle| \leq d_p(T) \|x - x'\| \left(\int_{B_{F^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi) \right)^{\frac{1}{p'}}. \tag{2.4}$$

Let $i_{F^*} : F^* \longrightarrow C(B_{F^{**}})$ be the isometric injection given by $i_{F^*}(y^*)(\varphi) = \varphi(y^*)$, $y^* \in F^*, \varphi \in B_{F^{**}}$ and let $j_{p'} : C(B_{F^{**}}) \longrightarrow L_{p'}(\mu)$ denote the canonical map.

Then,

$$\begin{aligned} |\langle y^*, T(x) - T(x')\rangle| &\leq d_p(T) \|x - x'\| \left(\int_{B_{F^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi) \right)^{\frac{1}{p'}} \\ &= \|x - x'\| \left(\int_{B_{F^{**}}} |d_p(T)\, j_{p'} i_{F^*}(y^*)(\varphi)|^{p'} d\mu(\varphi) \right)^{\frac{1}{p'}} \\ &= \|x - x'\| \, \|d_p(T)\, j_{p'} i_{F^*}(y^*)\|_{L_{p'}(\mu)}. \end{aligned}$$

Consequently there is a Banach space $G = L_{p'}(\mu)$ and a linear p'-summing operator $g = d_p(T)\, j_{p'} i_{F^*}$ such that

$$|\langle y^*, T(x) - T(x')\rangle| \leq \|x - x'\| \, \|g(y^*)\|.$$

Thus by Definition 2.1 T is strongly Lipschitz p-summing, and

$$d^L_{st,p}(T) \leq d_p(g) \leq d_p(T).$$

Let $1 < p \leq q \leq \infty$. It is well known that every absolutely q'-summing operator is absolutely p'-summing. Then, it is clear from the Definition 2.1 that

$$\mathcal{D}^L_{st,q}(X,E) \subseteq \mathcal{D}^L_{st,p}(X,E). \tag{2.5}$$

Our first result concerning strongly Lipschitz p-summing mappings relates them with strongly p-summing linear operators. This relation is given via the linearization of the Lipschitz mapping, and provides a shortcut for the study of such summability property in the Lipschitz context.

Proposition 2.2. *Let $1 < p \leq \infty$ and $T \in Lip_0(X,E)$. Then $T \in \mathcal{D}^L_{st,p}(X,E)$ if, and only if, $T_L \in \mathcal{D}_p(Æ(X), E)$. In this case, $d^L_{st,p}(T) = d_p(T_L)$.*

Proof. If we assume that T_L is strongly p-summing, from

$$\begin{aligned} |\langle y^*, T(x) - T(x')\rangle| &= |\langle y^*, T_L(m_{xx'})\rangle| \\ &= |(\langle T_L)^*(y^*), m_{xx'}\rangle| \\ &\leq \|m_{xx'}\|_{\mathcal{M}(X)}\|(T_L)^*(y^*)\| \\ &= \|x - x'\|\|(T_L)^*(y^*)\| \end{aligned}$$

we get that $T \in \mathcal{D}^L_{st,p}(X, E)$. Conversely, take $m \in \mathcal{M}(X)$ and $y^* \in F^*$. Given $\epsilon > 0$ choose a representation $m = \sum_{i=1}^n \alpha_i m_{x_i x_i'}$ such that

$$\sum_{i=1}^n |\alpha_i| d(x_i, x_i') \leq \|m\|_{\mathcal{M}(X)} + \epsilon.$$

Then,

$$\begin{aligned} |\langle y^*, T_L(m)\rangle| &\leq \sum_{i=1}^n |\alpha_i \langle y^*, T_L(m_{x_i x_i'})\rangle| \\ &= \sum_{i=1}^n |\alpha_i \langle y^*, T(x_i) - T(x_i')\rangle| \\ &\leq \sum_{i=1}^n |\alpha_i| d(x_i, x_i')\|S^*(y^*)\| \\ &\leq (\|m\|_{\mathcal{M}(X)} + \epsilon)\|S^*(y^*)\|, \end{aligned}$$

for some $S \in \mathcal{D}_p(F, E)$. The result follows easily. $\square$

Theorem 2.3. *Let $1 < p \leq \infty$ and $T \in Lip_0(X, E)$. The following statements are equivalent*

1. *$T \in \mathcal{D}^L_{st,p}(X, E)$;*

2. *there exist a constant $C \geq 0$ and a probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X, y^* \in E^*$ we have*

$$|\langle y^*, T(x) - T(x')\rangle| \leq C d(x, x')\Big(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi)\Big)^{\frac{1}{p'}};$$

3. *there exist a constant $C \geq 0$ such that for all $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ in X, all $(y_i^*)_{i=1}^n \subset E^*$ and $(b_i)_{i=1}^n$ in $\mathbb{R}$ we have*

$$\sum_{i=1}^n |b_i| \, |\langle y_i^*, T(x_i) - T(x_i')\rangle| \leq C\Big(\sum_{i=1}^n |b_i|^p \, d(x_i, x_i')^p\Big)^{\frac{1}{p}} \sup_{\varphi \in B_{E^{**}}} \Big(\sum_{i=1}^n |\varphi(y_i^*)|^{p'}\Big)^{\frac{1}{p'}}.$$

Furthermore, the infimum of the constants $C \geq 0$ in (2) and (3) is $d^L_{st,p}(T)$.

Proof. (1) $\Rightarrow$ (2) Let $T \in \mathcal{D}^L_{st,p}(X,E)$. By Proposition 2.2 $T_L \in \mathcal{D}_p(\text{Æ}(X),E)$ and so, using the Pietsch domination theorem, there exists a probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X, y^* \in E^*$ we have

$$|\langle y^*, T(x) - T(x')\rangle| \leq \pi_{p'}\left((T_L)^*\right) d(x,x') \Big(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu\left(\varphi\right) \Big).$$

Thus $\inf\{C : C \text{ as in (2)}\} \leq \pi_{p'}\left((T_L)^*\right) = d_p\left(T_L\right)$. That is, $\inf\{C : C \text{ as in (2)}\} \leq d^L_{st,p}(T)$.

(2) $\Rightarrow$ (1) By Proposition 2.2 it suffices to show that $T_L \in \mathcal{D}_p(\text{Æ}(X),E)$. Take $m \in \mathcal{M}(X)$ and $y^* \in F^*$. Given $\epsilon > 0$ choose a representation $m = \sum_{i=1}^n \alpha_i m_{x_i x_i'}$ such that $\sum_{i=1}^n |\alpha_i| d(x_i, x_i') \leq \|m\|_{\mathcal{M}(X)} + \epsilon$. Then,

$$\begin{aligned}
|\langle y^*, T_L(m)\rangle| &\leq \sum_{i=1}^n |\alpha_i \langle y^*, T_L(m_{x_i x_i'})\rangle \\
&\leq C \sum_{i=1}^n d(x_i, x_i')|\alpha_i| \left(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi) \right)^{1/p'} \\
&\leq C(\|m\|_{\mathcal{M}(X)} + \epsilon) \left(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi) \right)^{1/p'}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary the result follows easily by [21, Theorem 2.3.1]. Moreover, $d^L_{st,p}(T) \leq \inf\{C : C \text{ as in (2)}\}$.

(2) $\Rightarrow$ (3) From (2) it follows that

$$\begin{aligned}
&\sum_{i=1}^n |b_i|\, |\langle y_i^*, T(x_i) - T(x_i')\rangle| \\
&\quad \leq C \sum_{i=1}^n \Big(|b_i|\, d(x_i, x_i') \Big(\int_{B_{E^{**}}} |\varphi(y_i^*)|^{p'} d\mu\left(\varphi\right) \Big)^{\frac{1}{p'}} \Big) \\
&\quad \leq C \Big(\sum_{i=1}^n |b_i|^p\, d(x_i, x_i')^p \Big)^{\frac{1}{p}} \Big(\sum_{i=1}^n \Big(\int_{B_{E^{**}}} |\varphi(y_i^*)|^{p'} d\mu\left(\varphi\right) \Big) \Big)^{\frac{1}{p'}} \\
&\quad \leq C \Big(\sum_{i=1}^n |b_i|^p\, d(x_i, x_i')^p \Big)^{\frac{1}{p}} \sup_{\varphi \in B_{E^{**}}} \Big(\sum_{i=1}^n |\varphi(y_i^*)|^{p'} \Big)^{\frac{1}{p'}}.
\end{aligned}$$

Thus $\inf\{C : C \text{ as in (3)}\} \leq \inf\{C : C \text{ as in (2)}\}$.

(3) $\Rightarrow$ (2) It follows from [60, Theorem 2.3]. However, it is unnecessary the proof in [60, Theorem 2.3] (a repetition of the classical argument) as this is just a particular case of the unified Pietsch domination theorem given in [56] (see also [9, 58]). Considering

$$R_1 : B_{E^{**}} \times (X \times X) \times \mathbb{R} \to [0; \infty[$$

given by $R_1(\varphi,(x,x'),b)=|b|\,d(x,x')$,

$$R_2 : B_{E^{**}} \times (X \times X) \times E^* \to [0;\infty[$$

given by $R_2(\varphi,(x,x'),y^*)=|\varphi(y^*)|$ and

$$S : Lip_0(X,E) \times (X \times X) \times \mathbb{R} \times E^* \to [0;\infty[$$

given by $S(T,(x,x'),b,y^*) = |b|\,|\langle y^*, T(x)-T(x')\rangle|$, it follows that T is R_1,R_2S-abstract (p,p')-summing. Indeed,

$$\begin{aligned}
\sum_{i=1}^{n} S(T,(x_i,x_i'),b_i,y_i^*) &= \sum_{i=1}^{n} |b_i|\,|\langle y_i^*, T(x_i)-T(x_i')\rangle| \\
&\le C\Big(\sum_{i=1}^{n} |b_i|^p\, d(x_i,x_i')^p\Big)^{\frac{1}{p}} \sup_{\varphi\in B_{E^{**}}} \Big(\sum_{i=1}^{n} |\varphi(y_i^*)|^{p'}\Big)^{\frac{1}{p'}} \\
&= C\Big(\sum_{i=1}^{n} R_1(\varphi,(x_i,x_i'),b_i)^p\Big)^{\frac{1}{p}} \sup_{\varphi\in B_{E^{**}}} \Big(\sum_{i=1}^{n} R_2(\varphi,(x_i,x_i'),y_i^*)^{p'}\Big)^{\frac{1}{p'}} \\
&\le \sup_{\varphi\in B_{E^{**}}} \Big(\sum_{i=1}^{n} R_1(\varphi,(x_i,x_i'),b_i)^p\Big)^{\frac{1}{p}} \sup_{\varphi\in B_{E^{**}}} \Big(\sum_{i=1}^{n} R_2(\varphi,(x_i,x_i'),y_i^*)^{p'}\Big)^{\frac{1}{p'}}.
\end{aligned}$$

By [56, Theorem 4.6], there is a constant $C \ge 0$ and Borel probability measures μ_1,μ_2 on $B_{E^{**}}$ such that

$$\begin{aligned}
&S(T,(x,x'),b,y^*) \\
&\quad \le \Big(\int_{B_{E^{**}}} R_1(\varphi,(x,x'),b)^p d\mu_1(\varphi)\Big)^{\frac{1}{p}} \Big(\int_{B_{E^{**}}} R_2(\varphi,(x,x'),y^*)^{p'} d\mu_2(\varphi)\Big)^{\frac{1}{p'}}
\end{aligned}$$

and, consequently,

$$|\langle y^*, T(x)-T(x')\rangle| \le C d(x,x')\Big(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi)\Big)^{\frac{1}{p'}},$$

where $\mu := \mu_2$.

To end with the proof, $\inf\{C : C \text{ as in } (2)\} \le \inf\{C : C \text{ as in } (3)\}$. □

Part (3) of the above theorem asserts that Lipschitz Cohen strongly p-summing operators introduced recently in [60] coincide with strongly Lipschitz p-summing operators.

The space $\Pi^L_{p,r,s}(X,F)$ of Lipschitz (p,r,s)-summing mappings is introduced in [16] as the space of all $T \in Lip_0(X,F)$ for which there is a constant $C \ge 0$ such that for all positive integer n and all $x_j, x_j^{'} \in X, v_j \in F^*$, and $\lambda_j, k_j > 0$, $j = 1,\ldots,n$ we have

$$\begin{aligned}
&\Big\|\big(\lambda_j \langle v_j, T(x_j)-T(x_j')\rangle\big)_{j=1}^n\Big\|_p \\
&\quad \le C \sup_{f\in B_{X^\#}} \Big\|\Big(\lambda_j k_j^{-1}\big[f(x_j)-f(x_j^{'})\big]\Big)_{j=1}^n\Big\|_r \sup_{\varphi\in B_{F^{**}}} \Big(\sum_{j=1}^{n} |k_j\varphi(v_j)|^s\Big)^{\frac{1}{s}}.
\end{aligned}$$

This space is endowed with the norm given by the least constant C fulfilling the above inequality. We need the following theorem which states the Domination/Factorization theorem related to the space of (p, r, s)-summing operators.

Theorem 2.4. *([16, Theorem 5.2]) Suppose* $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$ *and let* $T \in Lip_0(X, E^*)$, $C > 0$. *The following are equivalent:*

(a) $|\langle T, m\rangle| \leq C\mu_{p,r,s}(m)$ *for all* $m \in \mathcal{M}(X, E)$.

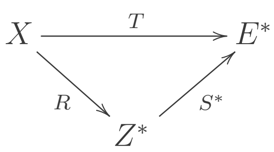
(b) There exist regular Borel probability measures μ *and* ν *on the weak***-compact unit balls* $B_{X^\#}, B_{E^*}$ *such that for all* $x, x' \in X$ *and* $v \in E$,

$$|\langle T(x) - T(x'), v\rangle| \leq C \left(\int_{B_{X^\#}} (|f(x) - f(x')|)^r d\mu(f) \right)^{\frac{1}{r}} \left(\int_{B_{E^*}} |(v^*(v)|^s \, d\nu(v^*) \right)^{\frac{1}{s}}.$$

(c) There exist a Banach space Z, *a Lipschitz* r*-summing operator* $R : X \longrightarrow Z^*$ *and a linear* s*-summing operator* $S : E \longrightarrow Z$ *such that* $\pi_r^L(R) . \pi_s(S) \leq C$ *and*

$$\langle T(x), v\rangle = \langle R(x), S(v)\rangle \text{ for all } x \in X, v \in E;$$

that is the following diagram commutes



The statement (a) of the above theorem means that the space $\Pi^L_{p',r,s}(X, F^*)$ has as a predual the space $\mathcal{M}_{p,r,s}(X, F)$, which is defined as the space $\mathcal{M}(X, F)$ of all F-valued molecules on X endowed with the norm

$$\begin{aligned}
&\mu_{p,r,s}(m) \\
&= \inf \left\{ \left\| (\lambda_j)_{j=1}^n \right\|_p \sup_{f \in B_{X^\#}} \left\| \left(k_j^{-1} \lambda_j^{-1} \left[f(x_j) - f(x_j') \right] \right)_{j=1}^n \right\|_r \left\| (k_j v_j)_{j=1}^n \right\|_{s,w} : \right. \\
&\qquad \left. \lambda_j, k_j > 0, \; m = \sum_{j=1}^n v_j m_{x_j x_j'} \right\}.
\end{aligned}$$

Let $\mathcal{M}_{p,p'}(X, F)$ denote $\mathcal{M}_{p,\infty,p'}(X, F)$, and consider

$$\begin{aligned}
\mu_{p,\infty,p'}(m) = \inf \Big\{ &\left\| (\lambda_j)_{j=1}^n \right\|_p \max_{1 \leq j \leq n} k_j^{-1} \lambda_j^{-1} d(x_j, x_j') \left\| (k_j v_j)_{j=1}^n \right\|_{p',w} : \\
&\lambda_j, k_j > 0, \; m = \sum_{j=1}^n v_j m_{x_j x_j'} \Big\}
\end{aligned}$$

Remark 2.5. *Saadi [60] introduces the norm*

$$\mu_p(m) = \inf\Big\{\Big(\sum_{j=1}^{n} \lambda_j^p d(x_j, x_j')^p\Big)^{1/p} \sup_{v' \in B_{F^*}} \Big(\sum_{j=1}^{n} |v'(v_j)|^{p'}\Big)^{1/p'}\Big\},$$

where the infimum varies over all the representations $m = \sum_{j=1}^{n} \lambda_j v_j m_{x_j x_j'}$ *of the molecule* $m \in \mathcal{M}(X,F)$. *Equivalently,*

$$\mu_p(m) = \inf\Big\{\Big(\sum_{j=1}^{n} \lambda_j^p d(x_j, x_j')^p\Big)^{1/p} \sup_{v' \in B_{F^*}} \Big(\sum_{j=1}^{n} \lambda_j^{-p'} |v'(v_j)|^{p'}\Big)^{1/p'}\Big\},$$

where the infimum varies over all the positive real numbers $\lambda_1, \ldots, \lambda_n$ *and all the representations* $m = \sum_{j=1}^{n} v_j m_{x_j x_j'}$ *of the molecule* $m \in \mathcal{M}(X,F)$. *An easy calculation shows that this norm coincides with the norm* $\mu_{p,\infty,p'}(m)$.

Immediate consequences of Theorem 2.3 and 2.4 are the following results.

Corollary 2.6. *For any* $1 < p \leq \infty$, $\mathcal{D}_{st,p}^L(X,E) = \Pi_{p',\infty,p'}^L(X,E)$.

Corollary 2.7. *[16, Theorem 5.4] Let* $1 < p \leq \infty$ *and* $T \in Lip_0(X,E)$. *Then,* $T \in \mathcal{D}_{st,p}^L(X,E)$ *if and only if there exist a Banach space* G, $u \in \mathcal{D}_p(G,E)$ *and* $S \in Lip_0(X,G)$ *such that* $T = u \circ S$.

Corollary 2.8. *The spaces* $\mathcal{D}_{st,p}^L(X,E^*)$ *and* $\mathcal{M}_{p,p'}(X,E)^*$ *are isometrically isomorphic.*

2.2 Lipschitz transpose of a strongly Lipschitz p-summing mapping

The next result establishes the relation of strongly Lipschitz p-summing mapping $T : X \to E$ and its Lipschitz transpose $T^t : E^* \to X^\#$.

Theorem 2.9. *Let* $1 < p \leq \infty$ *and* $T \in Lip_0(X,E)$. *Then* $T \in \mathcal{D}_{st,p}^L(X,E)$ *if, and only if,* $T^t \in \Pi_{p'}(E^*, X^\#)$. *In this case,* $d_{st,p}^L(T) = \pi_{p'}(T^t)$.

Proof. Assume that $T \in \mathcal{D}_{st,p}^L(X,E)$. By Proposition 2.2, $T_L \in \mathcal{D}_p(Æ(X),E)$, that is, $(T_L)^* \in \Pi_{p'}(E^*, Æ(X)^*)$. Then, by the ideal property $T^t = \delta_X^t \circ (T_L)^* \in \Pi_{p'}(E^*, X^\#)$.

Assume now that $T^t \in \Pi_{p'}(E^*, X^\#)$. Let μ be Pietsch measure on $B_{E^{**}}$ for T^t. Then, for all $y^* \in E^*$ we have

$$Lip(T^t(y^*)) \leq \pi_{p'}(T^t)\left(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi)\right)^{\frac{1}{p'}}.$$

Hence, for all $x, x' \in X$,

$$\left|T^t(y^*)(x) - T^t(y^*)(x')\right| \leq \pi_{p'}(T^t)d(x,x')\left(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi)\right)^{\frac{1}{p'}}.$$

Thus,

$$|\langle y^*, T(x) - T(x')\rangle| \leq \pi_{p'}(T^t)d(x,x')\left(\int_{B_{E^{**}}} |\varphi(y^*)|^{p'} d\mu(\varphi)\right)^{\frac{1}{p'}},$$

and Theorem 2.3 gives that $T \in \mathcal{D}^L_{st,p}(X,E)$. □

Corollary 2.10. *Let $1 < p \leq \infty$.*

1. *Every strongly Lipschitz p-summing operator is Lipschitz weakly compact.*
2. *If E is a reflexive Banach space, then every strongly Lipschitz p-summing operator is Lipschitz compact.*

Using Proposition 2.2 twice again we can show the ideal property for this class of Lipschitz operators in a short way. Therefore, all strongly Lipschitz p-summing mappings form a Lipschitz operator ideal (see Chapter 4).

Proposition 2.11. *Let Y and X be pointed metric spaces and let E and F be Banach spaces. Let $v \in Lip_0(Y,X)$ and $u \in \mathcal{L}(E,F)$. If $T \in \mathcal{D}^L_{st,p}(X,E)$, then $u \circ T \circ v \in \mathcal{D}^L_{st,p}(Y,F)$.*

Proof. We can see that the linerization of $u \circ T \circ v$ is $u \circ T_L \circ \hat{v}$, where $\hat{v} \in \mathcal{L}(Æ(Y), Æ(X))$ is the unique linear operator such that $\hat{v} \circ \delta_X = \delta_Y \circ v$. (see [42, Lemma 3.1]). Assume that $T \in \mathcal{D}^L_{st,p}(X,E)$, then $T_L \in \mathcal{D}_p(X,E)$ by Proposition 2.2. Since $\mathcal{D}_p(X,E)$ is Banach operator ideal, then $(u \circ T \circ v)_L \in \mathcal{D}_p(X,E)$. Hence Proposition 2.2 gives that $u \circ T \circ v \in \mathcal{D}^L_{st,p}(Y,F)$. □

It is easy to show the next proposition.

Proposition 2.12. *Let $1 \leq p < \infty$ and $T \in Lip_0(X,E)$. If T^t is strongly p'-summing then, T is Lipschitz p-summing and $\pi^L_p(T) \leq d_p(T^t)$.*

Remark 2.13. *Let $1 \leq p < \infty$ and $T \in Lip_0(X,E)$. From the definition it follows that T is Lipschitz p-summing and $\pi^L_p(T) \leq \pi_p(T_L)$ whenever T_L is p-summing (see also [60, Proposition 3.2] for an alternative proof). The converse is not true in general (see [60, Remark 3.11]). This fact gives that the converse of Proposition 2.12 does not hold either in general. Indeed, if we assume that $T^t \in \mathcal{D}_{p'}(E^*, X^{\#})$ whenever $T \in \Pi^L_p(X,E)$, then as $Q_X \circ (T_L)^* = T^t \in \mathcal{D}_{p'}(E^*, X^{\#})$, by injectivity $(T_L)^* \in \mathcal{D}_{p'}(E^*, Æ(X)^*)$. Hence [21, Theorem 2.2.2] yields that $T_L \in \Pi_p(Æ(X), E)$.*

Theorem 2.14. *Let $2 < p \leq \infty$. If E^* has cotype 2, then*

$$\mathcal{D}^L_{st,p}(X,E) = \mathcal{D}^L_{st,2}(X,E)$$

Proof. By (2.5), we have that $\mathcal{D}^L_{st,p}(X,E) \subseteq \mathcal{D}^L_{st,2}(X,E)$. Let $T \in \mathcal{D}^L_{st,2}(X,E)$. By Proposition 2.2, $T_L \in \mathcal{D}_2(\text{Æ}(X), E)$ and so, its adjoint operator $(T_L)^* : E^* \longrightarrow \text{Æ}(X)^*$ is 2−summing. Then, by [32, Corollary 11.16 (a)], $(T_L)^* \in \Pi_1(E^*, \text{Æ}(X)^*)$ and hence $(T_L)^* \in \Pi_{p'}(E^*, \text{Æ}(X)^*)$. Then, by [21, Theorem 2.2.2], $T_L \in \mathcal{D}_p(\text{Æ}(X), E)$. Thus $T \in \mathcal{D}^L_{st,p}(X,E)$. □

From [21, Theorem 2.2.2] and [11, Main Theorem], if the range space is a Hilbert space, then all Cohen strongly p−summing linear operators are absolutely q-summing ($1 < p \leq \infty$ and $1 \leq q < \infty$). Let us show that this is also true in the Lipschitz case.

Theorem 2.15. *Let H be a Hilbert space, $1 < p \leq \infty$ and $1 \leq q < \infty$. Then*

$$\mathcal{D}^L_{st,p}(X,H) \subseteq \Pi^L_q(X,H).$$

Proof. Let $T \in \mathcal{D}^L_{st,p}(X,H)$. Then $T_L \in \mathcal{D}_p(\text{Æ}(X), H)$. Hence, by [21, Theorem 2.2.2] $(T_L)^* \in \Pi_{p'}(H^*, \text{Æ}(X)^*)$. By [11, Main Theorem], $(T_L)^* \in \mathcal{D}_{q'}(H^*, \text{Æ}(X)^*)$, and by [21, Theorem 2.2.2] $T_L \in \Pi_q(\text{Æ}(X), H)$. Therefore, $T \in \Pi^L_q(X,H)$. □

As a consequence, for $q = 1$, we have the following corollary.

Corollary 2.16. *[19, Theorem 3.1] Let H be a Hilbert space and $1 < p \leq \infty$. If $T \in Lip_0(X;H)$ is such that T^t is p'-summing, then T is Lipschitz 1-summing.*

Proof. By Theorem 2.9, $T^t \in \Pi_{p'}(H^*, X^{\#})$ if and only if $T \in \mathcal{D}^L_{st,p}(X,H)$. Then, by Theorem 2.15, we have $T \in \Pi^L_1(X,H)$. □

2.3 Factorization Theorem

Let us show how the Pietsch factorization theorem is modeled on the class of Lipschitz strongly p-summing operators. Recall the map $K_X : X \longrightarrow X^{\#*}$ is the evaluation map $K_X(x)(f) = f(x)$, $x \in X, f \in X^{\#}$. Given a Banach space E, let $J_E : E \longrightarrow E^{**}$ be the canonical injection and let $i_E : E \longrightarrow C(B_{E^*})$ be the isometric injection given by $i_E(x)(x^*) = x^*(x)$. If B_{E^*} is endowed with the weak* topology and μ is a regular Borel measure on B_{E^*}, let $j_p : C(B_{E^*}) \longrightarrow L_p(\mu)$ denote the canonical map.

Theorem 2.17. *Let $1 < p \leq \infty$ and $T \in Lip_0(X, E)$. The following statements are equivalent:*

1. $T \in \mathcal{D}^L_{st,p}(X, E)$.

2. *There exist a regular Borel probability measure μ on $B_{E^{**}}$ (with the weak* topology), a closed subspace $S_{p'}$ of $L_{p'}(\mu)$ and a Lipschitz mapping $v : X \to (S_{p'})^*$ such that the following diagram commutes,*

$$\begin{array}{ccccc} X & \xrightarrow{\;T\;} & E & \xrightarrow{\;J_E\;} & E^{**} \\ \big\downarrow{\scriptstyle v} & & & & \big\uparrow{\scriptstyle i^*_{E^*}} \\ (S_{p'})^* & & \xrightarrow{\;j^*_{p'}\;} & & (i_{E^*}(E^*))^* \end{array} .$$

Proof. (1)⇒(2) Let $T \in \mathcal{D}^L_{st,p}(X, E)$. By Theorem 2.9 we have that $T^t : E^* \to X^\#$ is p'-summing. By Pietsch factorization theorem there exists a regular Borel probability measure μ on $B_{E^{**}}$, a subspace $S_{p'}$, defined as the closure of $j_{p'}(i_{E^*}(E^*))$ in $L_{p'}(\mu)$, and a continuous linear operator $u : S_{p'} \to X^\#$ such that the following diagram commutes,

$$\begin{array}{ccc} E^* & \xrightarrow{\;T^t\;} & X^\# \\ \big\downarrow{\scriptstyle i_{E^*}} & & \big\uparrow{\scriptstyle u} \\ i_{E^*}(E^*) & \xrightarrow{\;j_{p'}\;} & S_{p'} \end{array} .$$

Dualizing we get

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & E \\ \big\downarrow{\scriptstyle K_X} & & \big\downarrow{\scriptstyle J_E} \\ (X^\#)^* & \xrightarrow{\;(T^t)^*\;} & E^{**} \\ \big\downarrow{\scriptstyle u^*} & & \big\uparrow{\scriptstyle i^*_{E^*}} \\ (S_{p'})^* & \xrightarrow{\;j^*_{p'}\;} & (i_{E^*}(E^*))^* \end{array}$$

Define the Lipschitz mapping v by $v := u^* \circ K_X$. For any x, x' we have

$$\begin{aligned} \|v(x) - v(x')\| &\leq \|u^*\| \, \|K_X(x) - K_X(x')\| \\ &= \|u\| \, d(x, x') \end{aligned}$$

and so $Lip(v) \leq \|u\|$.

(2)⇒(1) Note that $J^*_E \circ J_{E^*} = id_{E^*}$. Then, since $J_E \circ T = i^*_{E^*} \circ j^*_{p'} \circ v$ it follows that $T^t \circ J^*_E = (i^*_{E^*} \circ j^*_{p'} \circ v)^t$ and so $T^t = (i^*_{E^*} \circ j^*_{p'} \circ v)^t \circ J_{E^*}$. Since the mapping $j_{p'}$ belongs to $\Pi_{p'}(i_{E^*}(E^*), S_{p'})$

its adjoint $j_{p'}^{*}$ belongs to $\mathcal{D}_p((S_{p'})^{*}, (i_{E^*}(E^*))^{*})$ and so to $\mathcal{D}^L_{st,p}((S_{p'})^{*}, (i_{E^*}(E^*))^{*})$. By the ideal property (Proposition 4.18) $i_{E^*}^{*} \circ j_{p'}^{*} \circ v \in \mathcal{D}^L_{st,p}(X, E^{**})$. Applying Theorem 2.9 and the ideal property we conclude that $T^t = (i_{E^*}^{*} \circ j_{p'}^{*} \circ v)^t \circ J_{E^*} \in \Pi_{p'}(E^*, X^{\#})$. Thus, once more Theorem 2.9 gives $T \in \mathcal{D}^L_{st,p}(X, E)$. □

Using Theorem 2.17, Theorem 2.9, the ideal property and [21, Theorem 2.2.2] we obtain the following corollary.

Corollary 2.18. *Let* $1 < p \leq \infty$ *and* $T \in Lip_0(X, E)$. *The following statements are equivalent:*

1. $T \in \mathcal{D}^L_{st,p}(X, E)$.
2. $J_E \circ T \in \mathcal{D}^L_{st,p}(X, E^{**})$.
3. $(T^t)^* \in \mathcal{D}_p((X^{\#})^*, E^{**})$.

Chapter 3

(p, σ)-absolutely Lipschitz operators

The results obtained in this chapter are going to appear in Annals of Functional Analysis [3]. Our aim in this chapter is to apply the interpolating method developed by Matter to the spaces of absolutely summing Lipschitz operators. So chapter 3 is organized as follows: in Section 3.1 we fix notations and basic concepts related to the classes of (p, σ)-absolutely continuous operators. In Section 3.2 we extend to Lipschitz mappings the concept of (p, σ)-absolutely continuous operator and give the first results. Section 2.3 is devoted mainly to analyze factorization theorems for (p, σ)-absolutely Lipschitz mappings. In the final section of this chapter, the duality for (p, σ)-absolutely Lipschitz operators is studied.

3.1 Ideal of (p, σ)-absolutely continuous linear operators.

Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. The interpolated operator ideal $\Pi_{p,\sigma}$ of the (p, σ)-absolutely continuous operators was introduced by Matter [51]. Essentially, it is defined to be an intermediate operator ideal between the ideal Π_p of the absolutely p-summing linear operators and the ideal of all continuous operators (see [41]). In the nineties, several papers describing and analysing the properties and applications of this interpolated class appeared. They were mainly devoted to the study of the factorization properties and the trace duality for these operators, finding in particular the class of tensor norms that represent these operator ideals (see [61],[48],[49]). It must be said that, due to the interpolative procedure that was used for defining them, (p, σ)-absolutely continuous operators preserve some of the characteristic properties of the absolutely p-summing linear operators. However, the emergence of new classes whenever $0 < \sigma < 1$, different from absolutely p-summing linear operators, yields that the theory of p-summing linear operators cannot be applied. Therefore, these new classes are a useful tool to deal with summability properties of operators weaker than absolutely p-summability (see for instance [61],[49]).

Definition 3.1. *[51, Definition 3.1] Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Recall that a linear operator $T \in \mathcal{L}(E, F)$ between two Banach spaces E and F is called (p, σ)-absolutely continuous if there exist a Banach space G and a linear operator $S \in \Pi_p(E, G)$ such that*

$$\|T(x)\| \leq \|x\|^{\sigma} \|S(x)\|^{1-\sigma}, \quad x \in E. \tag{3.1}$$

Let $\pi_{p,\sigma}(T) = \inf \pi_p(S)^{1-\sigma}$, where the infimum is taken over all Banach spaces G and $S \in \Pi_p(E, G)$ such that (3.1) holds. By $\Pi_{p,\sigma}(E, F)$ we denote the Banach space of all (p, σ)-absolutely continuous operators between E and F.

Note that if $T \in \Pi_{p,\sigma}(E, F)$ then $\|T\| \leq \pi_{p,\sigma}(T)$. The class $(\Pi_{p,\sigma}, \pi_{p,\sigma}(.))$ is an injective Banach operator ideal (see [51, Theorem 3.2]).

Proposition 3.2. *[22, Proposition 1] Let $1 \leq p \leq q < \infty$ and $0 \leq \sigma < 1$. Then the following holds*

$$\Pi_{p,\sigma}(E, F) \subset \Pi_{q,\sigma}(E, F)$$

for all Banach spaces E and F.

The Pietsch Domination Theorem concerning the (p, σ)-absolutely continuous linear operators is proved by Matter in [51]. As usual, the unit ball B_{E^*}, of E^* is considered as a compact space with respect to the weak star topology.

Theorem 3.3. *[51, Theorem 4.1]*
For a linear operator $T \in \mathcal{L}(E, F)$ the following statements are equivalent.
(i) $T \in \Pi_{p,\sigma}(E, F)$
(ii) There are a constant $C > 0$ and a regular Borel probability measure μ on B_{E^*}, such that

$$\|T(x)\| \leq C \left(\int_{B_{E^*}} \left(|\langle x, x^* \rangle|^{1-\sigma} \|x\|^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu(x^*) \right)^{\frac{1-\sigma}{p}}, \quad x \in E. \tag{3.2}$$

(iii) There is a constant $C \geq 0$ such that for all $(x_i)_{i=1}^n$ in E we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} \left(|\langle x^*, x_i\rangle|^{1-\sigma} \|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

In addition, $\pi_{p,\sigma}(T)$ is the smallest number C for which (ii) and (iii) hold.

Lipschitz p-dominated operators between Banach spaces are introduced in [20]. A map $T \in \mathrm{Lip}_0(E, F)$ is *Lipschitz p-dominated* if there exist a Banach space G and a linear operator $S \in \Pi_p(E, G)$ such that $\|T(x) - T(x')\| \leq \|S(x) - S(x')\|$ for all $x, x' \in E$. Let $d_p^L(T)$ denote the infimum of all $\pi_p(S)$, when S varies over all linear p-summing operators defined on E that fulfill the above condition. This is a norm for the space $D_p^L(E, F)$ of all Lipschitz p-dominated mappings between E and F. Any mapping in $D_p^L(E, F)$ is Lipschitz and $\mathrm{Lip} \leq d_p^L$. Note that

$$D_p^L(E, F) \subset \Pi_p^L(E, F). \tag{3.3}$$

3.2 (p, σ)-absolutely Lipschitz mappings

Let us introduce the Lipschitz version of (p, σ)-absolutely continuous operators.

Definition 3.4. *Let $1 \leq p < \infty$, and $0 \leq \sigma < 1$. Let X be a pointed metric space and E be a Banach space. A base point preserving mapping $T \in \mathrm{Lip}_0(X, E)$ is called (p, σ)-absolutely Lipschitz if there exist a Banach space F and a Lipschitz operator $S \in \Pi_p^L(X, F)$ such that*

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x, x')^{\sigma}$$

for all $x, x' \in X$. Let $\pi_{p,\sigma}^L(T)$ denote the infimum of all $\pi_p^L(S)^{1-\sigma}$, when S varies over all Lipschitz p-summing operators defined on X that fulfill the above condition. The space of all (p, σ)-absolutely Lipschitz mappings between X and E is denoted by $\Pi_{p,\sigma}^L(X, E)$.

An easy calculation shows that

$$\Pi_p^L \subset \Pi_{p,\sigma}^L \subset \mathrm{Lip}_0 \tag{3.4}$$

and $\mathrm{Lip} \leq \pi_{p,\sigma}^L \leq \pi_p^L$ for every $0 < \sigma < 1$.

Remark 3.5. *When $\sigma = 0$, $(p, 0)$-absolutely Lipschitz mappings extend the notion of Lipschitz p-dominated operators, that was introduced in [20]. A Lipschitz p-dominated operator*

T is defined between Banach spaces and satisfies a similar condition to the one given in Definition 3.4, but there the dominating mapping S is a linear absolutely p-summing operator. We don't know if the Lipschitz mapping S in our definition can be replaced with a linear one, even if T is linear. In particular, we don't know if being (p,σ)-absolutely Lipschitz implies (p,σ)-absolute continuity whenever the mapping T is linear. The converse is, of course, clearly true. Our aim is to work in the setting of Lipschitz mappings.

Let $p > 1, 1/p + 1/p' = 1$ and $0 < \sigma < 1$. López Molina and Sánchez-Pérez in [49, Example 1.9] proved that the operator $u : \ell_{p'} \to \ell_{\frac{p}{1-\sigma}}$ defined by $u(e_i) = \left(\frac{1}{i}\right)^{\frac{1}{p}} e_i$, where $(e_i)_{i=1}^{\infty}$ is the unit vector basis of $\ell_{p'}$, is (p,σ)-absolutely continuous and $u \notin \Pi_p(\ell_{p'}, \ell_{\frac{p}{1-\sigma}})$. Then u is trivially (p,σ)-absolutely Lipschitz, but by Theorem 2 in [33] $u \notin \Pi_p^L(\ell_{p'}, \ell_{\frac{p}{1-\sigma}})$, then the inclusion $\Pi_p^L \subset \Pi_{p,\sigma}^L$ is strict.

Proposition 3.6. *Let X be a pointed metric space and E be a Banach space, then for $1 \le p < \infty$ and $0 \le \sigma < 1$ the space $\Pi_{p,\sigma}^L(X,E)$ is a Banach space.*

Proof. We prove the triangle inequality. Consider $T_1, T_2 \in \Pi_{p,\sigma}^L(X,E)$, F_1, F_2 Banach spaces and $S_i \in \Pi_p^L(X, F_i)$, $i = 1, 2$, such that

$$\|T_i(x) - T_i(x')\| \le \|S_i(x) - S_i(x')\|^{1-\sigma} d(x,x')^{\sigma} \text{ for all } x, x' \in X.$$

Let F be the ℓ_1-sum of F_1 and F_2 and let $I_i : F_i \to F$ be the canonical injections. The map

$$S := \pi_p^L(S_1)^{-\sigma} I_1 \circ S_1 + \pi_p^L(S_2)^{-\sigma} I_2 \circ S_2$$

belongs to $\Pi_p^L(X,F)$ and

$$\pi_p^L(S) \le \pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}.$$

Using Hölder's inequality we get

$$\begin{aligned}
&\|(T_1+T_2)(x) - (T_1+T_2)(x')\| \\
&\qquad \le \|T_1(x) - T_1(x')\| + \|T_2(x) - T_2(x')\| \\
&\qquad \le \sum_{i=1}^{2} \left\|\pi_p^L(S_i)^{-\sigma}(S_i(x) - S_i(x'))\right\|_{F_i}^{1-\sigma} \left(\pi_p^L(S_i)^{1-\sigma}\right)^{\sigma} d(x,x')^{\sigma} \\
&\qquad \le \left(\sum_{i=1}^{2} \left\|\pi_p^L(S_i)^{-\sigma}(S_i(x) - S_i(x'))\right\|_{F_i}\right)^{1-\sigma} \left(\sum_{i=1}^{2} \pi_p^L(S_i)^{1-\sigma}\right)^{\sigma} d(x,x')^{\sigma} \\
&\qquad = \left(\pi_p^L(S_1)^{1-\sigma} + \pi_p^L(S_2)^{1-\sigma}\right)^{\sigma} \|S(x) - S(x')\|_F^{1-\sigma} d(x,x')^{\sigma}
\end{aligned}$$

for all $x, x' \in X$. Thus $T_1 + T_2 \in \Pi^L_{p,\sigma}(X, E)$ and

$$\begin{aligned}\pi^L_{p,\sigma}(T_1 + T_2) &\leq \left(\pi^L_p(S_1)^{1-\sigma} + \pi^L_p(S_2)^{1-\sigma}\right)^\sigma \pi^L_p(S)^{1-\sigma} \\ &\leq \pi^L_p(S_1)^{1-\sigma} + \pi^L_p(S_2)^{1-\sigma}.\end{aligned}$$

Taking the infimum we finally get that $\pi^L_{p,\sigma}(T_1 + T_2) \leq \pi^L_{p,\sigma}(T_1) + \pi^L_{p,\sigma}(T_2)$.

To prove the completeness of the space, take a sequence $(T_n)_n$ in $\Pi^L_{p,\sigma}(X, E)$ such that $\sum_{n=1}^{\infty} \pi^L_{p,\sigma}(T_n) < \infty$. Since $\mathrm{Lip} \leq \pi^L_{p,\sigma}$ and $(\mathrm{Lip}_0(X, E), \mathrm{Lip})$ is a Banach space, there exists $T := \sum_{n=1}^{\infty} T_n \in \mathrm{Lip}_0(X, E)$. We prove that $\sum_{n=1}^{\infty} T_n = T$ for $\pi^L_{p,\sigma}$. Let $\epsilon > 0$ and, for each $n \in \mathbb{N}$, let $S_n \in \Pi^L_p(X, F_n)$ be such that

$$\|T_n(x) - T_n(x')\| \leq \|S_n(x) - S_n(x')\|^{1-\sigma} d(x, x')^\sigma$$

for all $x, x' \in X$ and $\pi^L_p(S_n)^{1-\sigma} \leq \pi^L_{p,\sigma}(T_n) + \epsilon/2^n$. Then

$$\left(\sum_{n=1}^{\infty} \pi^L_p(S_n)\right)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi^L_p(S_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi^L_{p,\sigma}(T_n) + \epsilon < \infty.$$

Let $S = \sum_{n=1}^{\infty} \pi^L_p(S_n)^{-\sigma}(I_n \circ S_n) \in \Pi^L_p(X, F)$, where F is the ℓ_1-sum of all F_n and $I_n : F_n \to F$ is the natural inclusion. Hence,

$$\begin{aligned}\|T(x) - T(x')\| &\leq \sum_{n=1}^{\infty} \|T_n(x) - T_n(x')\| \\ &\leq \sum_{n=1}^{\infty} \|S_n(x) - S_n(x')\|^{1-\sigma}_{F_n} d(x, x')^\sigma \\ &\leq \|S(x) - S(x')\|^{1-\sigma}_F \left(\sum_{n=1}^{\infty} \pi^L_p(S_n)^{1-\sigma}\right)^\sigma d(x, x')^\sigma.\end{aligned}$$

This implies $T \in \Pi^L_{p,\sigma}(X, E)$ and

$$\pi^L_{p,\sigma}(T) \leq \sum_{n=1}^{\infty} \pi^L_p(S_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi^L_{p,\sigma}(T_n) + \epsilon.$$

We have

$$\pi^L_{p,\sigma}\Big(T - \sum_{k=1}^{n} T_k\Big) = \pi^L_{p,\sigma}\Big(\sum_{k=n+1}^{\infty} T_k\Big) \leq \sum_{k=n+1}^{\infty} \pi^L_p(S_k)^{1-\sigma}.$$

Thus, $\sum_{n=1}^{\infty} T_n = T$ for $\pi^L_{p,\sigma}$. □

Note that $\Pi^L_{p,0} = \Pi^L_p$ and $\pi^L_{p,0} = \pi^L_p$. Therefore, the class $\Pi^L_{p,\sigma}$ for $0 \leq \sigma < 1$ can be considered as an interpolating class between Π^L_p and Lip_0.

The next result is an extension of [20, Theorem 3.2] for (p,σ)-absolutely Lipschitz mappings. To prove the domination part, we will use a alternative technique: the unified abstract version of the Pietsch domination theorem given in [9, Theorem 2.2] (see also [58]).

Theorem 3.7. *Let* $1 \leq p < \infty$, $0 \leq \sigma < 1$ *and* $T \in \mathrm{Lip}_0(X,E)$. *The following statements are equivalent.*

1. $T \in \Pi^L_{p,\sigma}(X,E)$.

2. *There is a constant* $C \geq 0$ *and a regular Borel probability measure* μ *on* $B_{X^\#}$ *such that*

$$\|T(x) - T(x')\| \leq C \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x,x')^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$

   for all $x, x' \in X$.

3. *There is a constant* $C \geq 0$ *such that for all* $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ *in* X *and all* $(a_i)_{i=1}^n \subset \mathbb{R}^+$ *we have*

$$\left(\sum_{i=1}^n a_i \|T(x_i) - T(x'_i)\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n a_i \left(|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i,x'_i)^\sigma \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

Furthermore, the infimum of the constants $C \geq 0$ *in* (2) *and* (3) *is* $\pi^L_{p,\sigma}(T)$.

Proof. (1)⇒(2) If $T \in \Pi^L_{p,\sigma}(X,E)$, then there exist a Banach space F and $S \in \Pi^L_p(X,F)$ such that

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x,x')^\sigma$$

for all $x, x' \in X$. By [33, Theorem 1], since S is Lipschitz p-summing then there exists a regular Borel probability measure μ on $B_{X^\#}$ such that

$$\begin{aligned} \|T(x) - T(x')\| &\leq \|S(x) - S(x')\|^{1-\sigma} d(x,x')^\sigma \\ &\leq \pi^L_p(S)^{1-\sigma} \left(\int_{B_{X^\#}} |f(x) - f(x')|^p d\mu(f) \right)^{\frac{1-\sigma}{p}} d(x,x')^\sigma \\ &= \pi^L_p(S)^{1-\sigma} \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x,x')^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}} \end{aligned}$$

for all $x, x' \in X$.

(2)⇒(1) Let A be the natural isometric embedding from X into $C(B_{X^\#})$ composed with the formal identity from $C(B_{X^\#})$ into $L_\infty(\mu)$ given by $A(x)(f) = f(x)$, $x \in X, f \in B_{X^\#}$. Let $I_{\infty,p} : L_\infty(\mu) \longrightarrow L_p(\mu)$ be the canonical mapping $I_{\infty,p}(g) = g$. Note that $\pi_p^L(I_{\infty,p}) = 1$. Therefore by (2)

$$\|T(x) - T(x')\| \leq C \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x,x')^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}}$$
$$= \left(\int_{B_{X^\#}} |C^{\frac{1}{1-\sigma}} I_{\infty,p} A(x)(f) - C^{\frac{1}{1-\sigma}} I_{\infty,p} A(x')(f)|^p d\mu(f) \right)^{\frac{1-\sigma}{p}} d(x,x')^\sigma.$$

Consequently there is a Banach space $F = L_p(\mu)$ and a Lipschitz p-summing operator $S = C^{\frac{1}{1-\sigma}} I_{\infty,p} A$ such that

$$\|T(x) - T(x')\| \leq \|S(x) - S(x')\|^{1-\sigma} d(x,x')^\sigma,$$

as required.

(2)⇒(3) If

$$\|T(x) - T(x')\| \leq C \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x,x')^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \right)^{\frac{1-\sigma}{p}},$$

for all $x, x' \in X$ then, for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}^+$ and $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^n a_i \|T(x_i) - T(x'_i)\|^{\frac{p}{1-\sigma}} \\
&\quad \leq C^{\frac{p}{1-\sigma}} \sum_{i=1}^n \int_{B_{X^\#}} a_i \left(|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i,x'_i)^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \\
&\quad = C^{\frac{p}{1-\sigma}} \int_{B_{X^\#}} \sum_{i=1}^n a_i \left(|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i,x'_i)^\sigma \right)^{\frac{p}{1-\sigma}} d\mu(f) \\
&\quad \leq C^{\frac{p}{1-\sigma}} \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i \left(|f(x_i) - f(x'_i)|^{1-\sigma} d(x_i,x'_i)^\sigma \right)^{\frac{p}{1-\sigma}}.
\end{aligned}$$

(3)⇒(2) We will use the unified abstract version of the Pietsch domination theorem given in [9, Theorem 2.2].

Let $R : B_{X^\#} \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0, \infty[$ be given by

$$R(f, (x,x',a), \lambda) = |a|^{\frac{1-\sigma}{p}} |f(x) - f(x')|^{1-\sigma} d(x,x')^\sigma |\lambda|$$

and let $S : \mathrm{Lip}_0(X,E) \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0, \infty[$ be given by

$$S(T, (x,x',a), \lambda) = |a|^{\frac{1-\sigma}{p}} \|T(x) - T(x')\| |\lambda|.$$

Then T is R-S-abstract $p/(1-\sigma)$-summing (see [9, Definition 2.1]):

$$\begin{aligned}
&\left(\sum_{i=1}^{n} S(T,(x_i,x_i',a_i),\lambda_i)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \left(\sum_{i=1}^{n} |a_i|\,|\lambda_i|^{\frac{p}{1-\sigma}}\,\|T(x_i)-T(x_i')\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} |a_i|\,|\lambda_i|^{\frac{p}{1-\sigma}} \left(|f(x_i)-f(x_i')|^{1-\sigma} d(x_i,x_i')^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= C \sup_{f\in B_{X^\#}} \left(\sum_{i=1}^{n} R\left(f,(x_i,x_i',a_i),\lambda_i\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Then, by [9, Theorem 2.2] there is a constant $C \geq 0$ and a regular Borel probability measure μ on $B_{X^\#}$ such that

$$S(T,(x,x',a),\lambda) \leq C\left(\int_{B_{X^\#}} R(f,(x,x',a),\lambda)^{\frac{p}{1-\sigma}} d\mu\,(f)\right)^{\frac{1-\sigma}{p}}$$

for all $(x,x',a)\in X\times X\times\mathbb{R}$ and $\lambda\in\mathbb{R}$. In particular, we have

$$\|T(x)-T(x')\| \leq C\left(\int_{B_{X^\#}} \left(|f(x)-f(x')|^{1-\sigma} d(x,x')^{\sigma}\right)^{\frac{p}{1-\sigma}} d\mu\,(f)\right)^{\frac{1-\sigma}{p}}$$

for all $x,x'\in X$. □

Remark 3.8. *The notion of (p,σ)-absolutely Lipschitz mapping can be defined for Lipschitz mappings between pointed metric spaces. Given pointed metric spaces X and Y, a map $T\in \mathrm{Lip}_0(X,Y)$ is called (p,σ)-absolutely Lipschitz, if there exist a constant $k\geq 0$, a pointed metric space G and a Lipschitz operator $S\in \Pi_p^L(X,G)$ such that*

$$d(T(x),T(x')) \leq kd(S(x),S(x'))^{1-\sigma} d(x,x')^{\sigma}$$

for all $x,x'\in X$. In this case $\pi_{p,\sigma}^L(T)$ denotes the infimum of all $k\pi_p^L(S)^{1-\sigma}$. Theorem 3.7 can be easily adapted whenever $T\in \mathrm{Lip}_0(X,Y)$ and X and Y are pointed metric spaces.

Proposition 3.9 (Ideal property). *Let X,Y,X_0,Y_0 be pointed metric spaces. If $v: X_0 \longrightarrow X$, $w: Y\longrightarrow Y_0$ are Lipschitz mappings and $T: X\longrightarrow Y$ is (p,σ)-absolutely Lipschitz then, wTv is (p,σ)- absolutely Lipschitz and*

$$\pi_{p,\sigma}^L(wTv) \leq \mathrm{Lip}(w)\mathrm{Lip}(v)\pi_{p,\sigma}^L(T).$$

Proof. Since T is (p,σ)-absolutely Lipschitz then there exist a constant $k \geq 0$, a pointed metric space G and a Lipschitz operator $S \in \Pi_p^L(X,G)$ such that

$$d(T(x),T(x')) \leq kd(S(x),S(x'))^{1-\sigma} d(x,x')^{\sigma} \text{ for all } x,x' \in X.$$

Let $x_0, x_0' \in X_0$, then

$$\begin{aligned} d(wTv(x_0), wTv(x_0')) &\leq \operatorname{Lip}(w) d(Tv(x_0), Tv(x_0')) \\ &\leq k\operatorname{Lip}(w) d(S \circ v(x_0), S \circ v(x_0'))^{1-\sigma} d(v(x_0), v(x_0'))^{\sigma} \\ &\leq k\operatorname{Lip}(w)\operatorname{Lip}(v)^{\sigma} d(S \circ v(x_0), S \circ v(x_0'))^{1-\sigma} d(x_0, x_0')^{\sigma}. \end{aligned}$$

Since $S \circ v \in \Pi_p^L(X_0, G)$ it follows that $w \circ T \circ v \in \Pi_{p,\sigma}^L(X_0, Y_0)$ and

$$\begin{aligned} \pi_{p,\sigma}^L(wTv) &\leq k\operatorname{Lip}(w)\operatorname{Lip}(v)^{\sigma} \pi_p^L(S \circ v)^{1-\sigma} \\ &\leq k\operatorname{Lip}(w)\operatorname{Lip}(v) \pi_p^L(S)^{1-\sigma}. \end{aligned}$$

Taking infimum we get

$$\pi_{p,\sigma}^L(wTv) \leq \operatorname{Lip}(w)\operatorname{Lip}(v)\pi_{p,\sigma}^L(T).$$

□

Remark 3.10. *Using (3.4), Proposition 3.6 and Proposition 3.9 it can be shown that all (p,σ)-absolutely Lipschitz mappings form a Lipschitz operator ideal (see Chapter 4).*

3.3 Factorization Theorem

Let μ be a Borel probability measure on $B_{X^\#}$. Consider the canonical inclusion $i : Æ(X) \longrightarrow C(B_{X^\#})$, given by $i(\sum_{j=1}^{n} \lambda_j m_{x_j x_j'}) := \sum_{j=1}^{n} \lambda_j \langle m_{x_j x_j'}, .\rangle$. On $i(Æ(X))$, we define the semi-norm

$$\|i(m)\|_{p,\sigma} := \inf \left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j, x_j')^{\sigma} \Big(\int_{B_{X^\#}} \left| f(x_j) - f(x_j') \right|^p d\mu(f) \Big)^{\frac{1-\sigma}{p}} \right\}$$

where the infimum is taken over all representations of m of the form $m = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'}$. Consider on $i \circ \delta_X(X)$ the pseudo-metric induced by $\|.\|_{p,\sigma}$:

$$d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) := \|i \circ \delta_X(x) - i \circ \delta_X(x')\|_{p,\sigma}$$

and the relation of equivalence $\mathcal{R}$ given by

$$i \circ \delta_X(x)\mathcal{R}i \circ \delta_X(x') \Leftrightarrow d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = 0.$$

We set $X^{\mu}_{p,\sigma} := \frac{i \circ \delta_X(X)}{\mathcal{R}}$ and let $q : i \circ \delta_X(X) \longrightarrow X^{\mu}_{p,\sigma}$ be the projection.

Note that, if we consider the canonical map $j_p : C(B_{X^\#}) \longrightarrow L_p(\mu)$, then $i \circ \delta_X(x)\mathcal{R}i \circ \delta_X(x')$ if and only if $j_p(i \circ \delta_X(x)) = j_p(i \circ \delta_X(x'))$. Hence $X^{\mu}_{p,\sigma}$ can be seen as a subset of $L_p(\mu)$ via the formal identity I.

Theorem 3.11. *Let $1 \le p < \infty$ and $0 \le \sigma < 1$. Let X and Y be pointed metric spaces and $T \in \mathrm{Lip}_0(X, Y)$. The following statements are equivalent.*

1. *$T \in \Pi^L_{p,\sigma}(X, Y)$.*

2. *There exist a regular Borel probability measure μ on $B_{X^\#}$ and a Lipschitz operator $v : X^{\mu}_{p,\sigma} \to Y$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & Y \\ \downarrow{\scriptstyle \delta_X} & & \uparrow{\scriptstyle v} \\ \delta_X(X) & \xrightarrow{\ q\circ i\ } & X^{\mu}_{p,\sigma} \end{array}$$

Proof. (1)⇒(2) Assume first that $T \in \Pi^L_{p,\sigma}(X, Y)$. By Theorem 3.7 and Remark 3.8 there is a regular Borel probability measure μ on $B_{X^\#}$ such that

$$d(T(x), T(x')) \le \pi^L_{p,\sigma}(T) \left(\int_{B_{X^\#}} \left(|f(x) - f(x')|^{1-\sigma} d(x, x')^{\sigma} \right)^{\frac{p}{1-\sigma}} d\mu\,(f) \right)^{\frac{1-\sigma}{p}}$$

for all $x, x' \in X$. Define $v(q \circ i \circ \delta_X(x)) := T(x)$, $x \in X$. If $x, x' \in X$ are so that $q(i \circ \delta_X(x)) = q(i \circ \delta_X(x'))$ then $0 = d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = \|\langle m_{xx'}, .\rangle\|_{p,\sigma}$. Therefore, given $\epsilon > 0$, there exists a representation of $m_{xx'}$, $m_{xx'} = \sum\limits_{j=1}^{n} \lambda_j m_{x_j x'_j}$ such that

$$\sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j)^{\sigma} \Big(\int_{B_{X^\#}} \left| f(x_j) - f(x'_j) \right|^p d\mu \Big)^{\frac{1-\sigma}{p}} < \epsilon.$$

Let $g \in B_{Y^\#}$. Then,

$$\begin{aligned} |g\,(T(x)) - g\,(T(x'))| &= |\langle m_{xx'}, g \circ T\rangle| \\ &\le \sum_{j=1}^{n} |\lambda_j| \left| \langle m_{x_j x'_j}, g \circ T\rangle \right| \le \sum_{j=1}^{n} |\lambda_j|\, d(T(x_j), T(x'_j)) \\ &\le \pi^L_{p,\sigma}(T) \sum_{j=1}^{n} |\lambda_j|\, d(x_j, x'_j)^{\sigma} \Big(\int_{B_{X^\#}} \left| f(x_j) - f(x'_j) \right|^p d\mu \Big)^{\frac{1-\sigma}{p}} < \epsilon \pi^L_{p,\sigma}(T). \end{aligned}$$

Letting $\epsilon \longrightarrow 0$ it follows that $g\left(T(x)\right) - g\left(T(x')\right) = 0$ for all $g \in B_{Y^\#}$. Hence $T(x) = T(x')$. This proves that v is well defined.

We now show that v is Lipschitz. Take $g \in B_{Y^\#}$ and let $m_{xx'} = \sum_{j=1}^{n} \lambda_j m_{x_j x_j'}$. Then, by Proposition 3.9

$$\begin{aligned}
&\left| g \circ v\left(q \circ i \circ \delta_X(x)\right) - g \circ v\left(q \circ i \circ \delta_X(x')\right) \right| \\
&\quad = \left| g \circ T(x) - g \circ T(x') \right| = \left| \langle m_{xx'}, g \circ T \rangle \right| \\
&\quad \leq \pi^L_{p,\sigma}(T) \sum_{j=1}^{n} |\lambda_j|\, d(x_j, x_j')^{\sigma} \Big(\int_{B_{X^\#}} \left| f(x_j) - f(x_j') \right|^p d\mu \Big)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Taking the infimum over all representations of $m_{xx'}$, we get

$$\left| g \circ v\left(q \circ i \circ \delta_X(x)\right) - g \circ v\left(q \circ i \circ \delta_X(x')\right) \right| \leq \pi^L_{p,\sigma}(T) d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x'))$$

for all $g \in B_{Y^\#}$. We conclude now that

$$d\left(v\left(q \circ i \circ \delta_X(x)\right), v\left(q \circ i \circ \delta_X(x')\right)\right) \leq \pi^L_{p,\sigma}(T) d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')).$$

(2)⇒(1) Assume that T factors as in (2). By Proposition 3.9 it suffices to prove that $q \circ i : \delta_X(X) \longrightarrow X^\mu_{p,\sigma}$ is (p,σ)-absolutely Lipschitz, but this is clear as

$$d_{p,\sigma}(i \circ \delta_X(x), i \circ \delta_X(x')) = \|i(m_{xx'})\|_{p,\sigma} \leq \|m_{xx'}\|^{\sigma} \Big(\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\mu \Big)^{\frac{1-\sigma}{p}}.$$

□

Farmer and Johnson [33, Theorem 1] proved that $\pi^L_p(T) \leq C$ if and only if for some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization

$$\begin{array}{ccccc}
L_\infty(\mu) & & \xrightarrow{\quad I_{\infty,p} \quad} & & L_p(\mu) \\
{\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle B} \\
X & \xrightarrow{\ T\ } & Y & \xrightarrow{\ J\ } & Z
\end{array}$$

with μ a probability and $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \leq C$.

Letting $\sigma = 0$ in Theorem 3.11, we obtain a factorization theorem for Lipschitz absolutely p-summing operators which is equivalent to the above. In that case, $X^\mu_{p,0} = j_p \circ i \circ \delta_X(X)$, where $j_p : C(B_{X^\#}) \to L_p(\mu)$ is the canonical mapping, and the induced metric $d_{p,0}$ generates the L_p-norm on $X^\mu_{p,0}$. So, Theorem 3.11 is a generalization of the Farmer and Johnson factorization.

Theorem 3.12. *Let $1 \leq p < \infty$. Let X and Y be pointed metric spaces. The following statements are equivalent for a mapping $T \in \mathrm{Lip}_0(X,Y)$ and a positive constant C.*

1. *$T \in \Pi_p^L(X,Y)$ and $\pi_p^L(T) \leq C$.*

2. *There exist a regular Borel probability measure μ on $B_{X^\#}$ such that*
$$d(T(x),T(x')) \leq C \left(\int_{B_{X^\#}} |\langle \delta_X(x) - \delta_X(x'), f\rangle|^p \, d\mu(f) \right)^{1/p}$$
for all $x, x' \in X$.

3. *There exist a regular Borel probability measure μ on $B_{X^\#}$ and a Lipschitz operator $v : X^\mu_{p,0} \to Y$ such that the following diagram commutes*
$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ \Big\downarrow{\scriptstyle i\circ\delta_X} & & \Big\uparrow{\scriptstyle v} \\ i\circ\delta_X(X) & \xrightarrow{\quad j_p \quad} & X^\mu_{p,0} \end{array}$$

Furthermore, the infimum of the constants $C \geq 0$ in (1) and (2) is $\pi_p^L(T)$.

Let us end showing the duality for (p,σ)-absolutely Lipschitz operators.

Let $1 \leq p, r < \infty$ and $0 \leq \sigma < 1$ such that $r' = \frac{p'}{1-\sigma}$, where p' is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. For $x_1, ..., x_n, x_1', ..., x_n'$ in X and scalars $\lambda_1, \ldots, \lambda_n$, we define
$$\delta^{Lip}_{p,\sigma}\left((\lambda_j, x_j, x_j')_{j=1}^n\right) := \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^n \left(|\lambda_j| \, |f(x_j) - f(x_j')|^{1-\sigma} d(x_j, x_j')^\sigma \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$
If we denote
$$w^{Lip}_{\frac{p}{1-\sigma}}\left((\lambda_j, x_j, x_j')_{j=1}^n\right) := \sup_{f \in B_{X^\#}} \left(\sum_{j=1}^n \left(|\lambda_j| \, |f(x_j) - f(x_j')| \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$
then, we have
$$w^{Lip}_{\frac{p}{1-\sigma}}\left((\lambda_j, x_j, x_j')_{j=1}^n\right) \leq \delta^{Lip}_{p,\sigma}\left((\lambda_j, x_j, x_j')_{j=1}^n\right).$$

As a remark, the above inequality shows that $\Pi^L_{p/(1-\sigma)}(X,Y) \subset \Pi^L_{p,\sigma}(X,Y)$.

For a molecule $m \in \mathcal{M}(X,E)$ we define its (p,σ)-Chevet–Saphar norm by
$$cs_{p,\sigma}(m) = \inf \left\{ \left\| (\lambda_j \|v_j\|)_{j=1}^n \right\|_r \delta^{Lip}_{p',\sigma}\left((\lambda_j^{-1}, x_j, x_j')_{j=1}^n\right) : m = \sum_{j=1}^n v_j m_{x_j x_j'}, \lambda_j > 0 \right\}.$$
We denote by $CS_{p,\sigma}(X,E)$ the space $\mathcal{M}(X,E)$ endowed with the norm $cs_{p,\sigma}$.

The following theorem can be proved as in [16, Theorem 4.3].

Theorem 3.13. *The spaces* $CS_{p,\sigma}(X,E)^*$ *and* $\Pi^L_{p',\sigma}(X,E^*)$ *are isometrically isomorphic via the canonical pairing,* $\langle m, T\rangle = \sum_{j=1}^{n} \left\langle v_j, T(x_j) - T(x'_j)\right\rangle$.

Chapter 4

Lipschitz operator ideals and the approximation property

Our results in this chapter have been published in collaboration with D. Achour, P. Rueda and E. A. Sánchez Pérez in the Journal of Mathematical Analysis and Applications [2] and are presented as follows. Section 4.1.1 contains the basics of the theory of Lipschitz operator ideals, that are introduced in a natural way. Especial emphasis is made on examples: we recover the main classes of Lipschitz mappings such as approximable Lipschitz mappings, Lipschitz p-summing mappings, Lipschitz p-integral mappings, Lipschitz p-nuclear mappings, Lipschitz compact and Lipschitz weakly compact mappings. In Section 4.3 we extend to the Lipschitz mappings setting some linear procedures to construct ideals of operators from a given operator ideal $\mathcal{I}$: by composition with linear operators from $\mathcal{I}$ and, by considering those Lipschitz mappings whose Lipschitz transposes belong to $\mathcal{I}$. It is proved that the Lipschitz dual of a given operator ideal $\mathcal{I}$ coincides with the Lipschitz composition ideal with the dual ideal of $\mathcal{I}$. A direct consequence is the well-known Lipschitz variant of Schauder's (respectively Gantmacher's) theorem: a Lipschitz operator is compact (respectively weakly compact) if, and only if, its transpose is compact (respectively weakly compact). In Section 4.4 we initiate a study of approximation properties for metric spaces, that come from considering composition ideals of Lipschitz mappings. The main aim is to relate the linear and the Lipschitz theories. To do so, we strongly make use of linearization techniques

4.1 Lipschitz operator ideals

4.1.1 Definitions

The aim of this section is to introduce the concept of Lipschitz operator ideal. We will follow the spirit of the definition of linear operator ideal explained in the excellent monographs [26, 59].

Definition 4.1. *A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a subclass of Lip_0 such that for every pointed metric space X and every Banach space E the components*

$$\mathcal{I}_{Lip}(X,E) := Lip_0(X,E) \cap \mathcal{I}_{Lip}$$

satisfy:

(i) $\mathcal{I}_{Lip}(X,E)$ is a linear subspace of $Lip_0(X,E)$.

(ii) $vg \in \mathcal{I}_{Lip}(X,E)$ for $v \in E$ and $g \in X^{\#}$.

(iii) The ideal property: if $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then the composition wTS is in $\mathcal{I}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is $\|.\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,+\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathcal{I}_{Lip}(X,E), \|.\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}$ for all $T \in \mathcal{I}_{Lip}(X,E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1$.

(iii') If $S \in Lip_0(Y,X)$, $T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then

$$\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip(S) \|T\|_{\mathcal{I}_{Lip}} \|w\| .$$

From conditions (i'), (ii') and (iii') we get for any $v \in E$ and any $g \in X^{\#}$,

$$\begin{aligned} \|vg\|_{\mathcal{I}_{Lip}} &= \|(v\ Id_{\mathbb{K}}) \circ Id_{\mathbb{K}} \circ g\|_{\mathcal{I}_{Lip}} \leq \|v\ Id_{\mathbb{K}}\| \|Id_{\mathbb{K}}\|_{\mathcal{I}_{Lip}} Lip(g) \\ &= \|v\| Lip(g) = Lip(vg) \leq \|vg\|_{\mathcal{I}_{Lip}}, \end{aligned}$$

that is, $\|vg\|_{\mathcal{I}_{Lip}} = \|v\| Lip(g) = Lip(vg)$.

Following [40] a mapping $T \in Lip_0(X,E)$ has Lipschitz finite dimensional rank if the linear hull of the set $\left\{\frac{T(x)-T(x')}{d(x,x')}, x, x' \in X, x \neq x'\right\}$ is a finite dimensional subspace of E. We denote by $Lip_{0\mathcal{F}}(X,E)$ the set of all Lipschitz finite rank mappings from X to E. Clearly,

$Lip_{0\mathcal{F}}(X,E)$ is a linear subspace of $Lip_0(X,E)$. It is proved in [40, Proposition 2.4] that having finite dimensional rank is equivalent to having Lipschitz finite dimensional rank. This is also equivalent to saying that the linearization T_L of T has finite rank. Such a mapping admits a finite representation $T=\sum\limits_{k=1}^{n} u_k f_k$, with $(u_k)_{k\leq n}\in E, (f_k)_{k\leq n}\in X^{\#}$. Indeed, given $T\in Lip_{0\mathcal{F}}(X,E)$, according to [40, Proposition 2.4], $T_L\in\mathcal{L}(\text{Æ}(X),E)$ has finite rank. Thus

$$T_L=\sum_{i=1}^{n} m_i^{*}\otimes u_i, (u_i)_{i\leq n}\subset E, (m_i^{*})_{i\leq n}\subset \text{Æ}(X)^{*}.$$

Since $\text{Æ}(X)^{*}$ and $X^{\#}$ are isometrically isomorphic then, there is $(f_i)_{i\leq n}\subset X^{\#}$ such that $(f_i)_L=m_i^{*}$.

For $x\in X$ we obtain

$$\begin{aligned} T(x) &= T_L\circ\delta_X(x)=T_L(m_{x0})=\sum_{i=1}^{n}(f_i)_L\otimes u_i(m_{x0}) \\ &= \sum_{k=1}^{n}\langle (f_i)_L, m_{x0}\rangle\, u_i \\ &= \sum_{i=1}^{n} f_i(x)u_i. \end{aligned}$$

Thus $T=\sum\limits_{i=1}^{n} u_i f_i$. The class $Lip_{0\mathcal{F}}$ is the smallest Lipschitz operator ideal.

Definition 4.2. *(1) A Lipschitz operator ideal $\mathcal{I}_{Lip}$ is said to be injective if for every metric linear injection $I: E\hookrightarrow F$ (that is, I is linear and $\|I(x)\|=\|x\|$ for all $x\in E$) and every $T\in Lip_0(X,E)$, T is in $\mathcal{I}_{Lip}$ whenever $I\circ T\in\mathcal{I}_{Lip}(X,F)$. A normed Lipschitz operator ideal is called injective if moreover $\|T\|_{\mathcal{I}_{Lip}}=\|I\circ T\|_{\mathcal{I}_{Lip}}$. In a straightforward way, the injective hull $\mathcal{I}_{Lip}^{inj}$ can be defined as the smallest injective Lipschitz operator ideal that contains $\mathcal{I}_{Lip}$. Note that each Lipschitz operator ideal is contained in an injective Lipschitz operator ideal, and so the notion of injective hull makes sense. Indeed, as in the linear case, $\mathcal{I}_{Lip}^{inj}(X,E)$ is formed by all Lipschitz mappings $T: X\to E$ that satisfy that $J_F\circ T\in\mathcal{I}(X,\ell_\infty(B_{E^*}))$, where J_E is the canonical inclusion $J_E: E\to\ell_\infty(B_{E^*})$. The space $\mathcal{I}_{Lip}^{inj}(X,E)$ is endowed with the norm $\|T\|_{\mathcal{I}_{Lip}}^{inj}:=\|J_E\circ T\|_{\mathcal{I}_{Lip}}$, for all $T\in\mathcal{I}_{Lip}^{inj}(X,E)$ (see for example [26, 9.7]).*
(2) The closed hull $\overline{\mathcal{I}_{Lip}}$ of a Lipschitz operator ideal $\mathcal{I}_{Lip}$ consists of all Lipschitz mappings that can be approximated, with respect to the Lipschitz norm, by a sequence of Lipschitz mappings in $\mathcal{I}_{Lip}$.

4.1.2 Approximable Lipschitz operators

Following [40], a Lipschitz mapping $T \in Lip_0(X, E)$ is said to be *approximable* if it is the limit of a sequence of Lipschitz finite rank operators from X to E in the Lipschitz norm Lip.

The collection of all approximable Lipschitz operators $T \in Lip_0(X, E)$ is denoted by $\overline{Lip_{0\mathcal{F}}(X, E)}$. It is clear that $\overline{Lip_{0\mathcal{F}}(X, E)}$ is an example of Lipschitz ideal.

The following is inspired in [10].

Lemma 4.3. *Let $T \in Lip_0(X, E)$. The following statements are equivalent.*

1. $T \in Lip_{0\mathcal{F}}(X, E)$.
2. $T_L \in \mathcal{L}_f(Æ(X), E)$.
3. $T^t \in \mathcal{L}_f(E^*, X^\#)$.

Proof. The equivalence between (1) and (2) appears in [40, Proposition 2.4]. The equivalence with (3) follows from the fact that $(T_L)^* = Q_X \circ T^t$ and the ideal property. □

Corollary 4.4. *Let $T \in Lip_0(X, E)$. The following statements are equivalent.*

1. *T can be approximated by finite rank Lipschitz operators.*
2. *T_L can be approximated by finite rank linear operators.*
3. *T^t can be approximated by finite rank linear operators.*

Proof. It is a direct consequence of Lemma 4.3 the fact that the correspondence $T \in Lip_0(X, E) \longmapsto T_L \in \mathcal{L}(Æ(X), E)$ is an isomorphism, and Hutton's theorem [39, Theorem 2.1] that assures that T_L can be approximated by finite rank operators if, and only if, $(T_L)^*$ can be approximated by finite rank operators. □

4.2 Examples

4.2.1 Lipschitz p-summing operators

Let $1 \leq p < \infty$. In [33] Lipschitz $p-$summing operators defined between metric spaces are introduced (see section 1.4). It is well known that $\Pi_p^L(X, E)$ with the norm $\pi_p^L(.)$ is a Banach space that satisfies the ideal property (this is straightforward, see for instance [33]) and that $Lip(T) \leq \pi_p^L(T)$ for every $T \in \Pi_p^L(X, E)$. A direct calculation shows that $\pi_p^L(vg) \leq \|v\| Lip(g)$ for any $v \in E$ and $0 \neq g \in X^\#$. Theorem 2 in [33] assures that $\pi_p^L(Id_{\mathbb{K}}) = \pi_p(Id_{\mathbb{K}}) = 1$, where $Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda$. Moreover, consider a metric linear injection $I : E \hookrightarrow F$ and $T \in Lip_0(X, E)$ such that $I \circ T \in \Pi_p^L(X, E)$. Since I is a metric injection, given $(x_i)_{i \leq n}, (x'_i)_{i \leq n}$ in X we get

$$\begin{aligned} \left(\sum_{i=1}^n \|T(x_i) - T(x'_i)\|^p \right)^{\frac{1}{p}} &= \left(\sum_{i=1}^n \|IT(x_i) - IT(x'_i)\|^p \right)^{\frac{1}{p}} \\ &\leq \pi_p^L(IT) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n |f(x_i) - f(x'_i)|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Hence $T \in \Pi_p^L(X, E)$ and $\pi_p^L(T) \leq \pi_p^L(IT)$. The ideal property gives the reverse inequality and then $\pi_p^L(T) = \pi_p^L(IT)$. That proves that Π_p^L is injective. Summarizing, we have the following.

Proposition 4.5. (Π_p^L, π_p^L) *is an injective Banach Lipschitz operator ideal.*

4.2.2 Lipschitz p-integral operators

Let $1 \leq p \leq \infty$ Lipschitz $p-$integral operators from metric spaces to pointed metric spaces are introduced in [33] (see also [20]). A map $T \in Lip_0(X, E)$ is *Lipschitz p-integral* if there are a probability measure space (Ω, Σ, μ) and two Lipschitz mappings $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{T} & E & \xrightarrow{k_E} & E^{**} \\ \downarrow{\scriptstyle B} & & & & \uparrow{\scriptstyle A} \\ L_\infty(\mu) & & \xrightarrow{i_p^\mu} & & L_p(\mu). \end{array}$$

where $i_p^\mu : L_\infty(\mu) \longrightarrow L_p(\mu)$ is the canonical mapping and $k_E : E \longrightarrow E^{**}$ is the canonical isometric embedding. For short, we will say that (A, i_p^μ, B) is a factorization for T. Denote by $\iota_p^L(T)$ the infimum of $Lip(A)Lip(B)$ taken over all factorizations as above. This class of mappings is denoted by $\mathcal{I}_p^L(X, E)$.

Using some standard techniques from [12, Proposition 2.1] (see also [32, Theorem 5.2]) we can show that $\mathcal{I}_p^L(X, E)$ is a Banach Lipschitz operator ideal with the norm $\iota_p^L(.)$. For the convenience of the reader, we include the proof.

Proposition 4.6. *$(\mathcal{I}_p^L, \iota_p^L)$ is a Banach Lipschitz operator ideal.*

Proof. We check all conditions in Definition 4.1.

(i) We start checking that $Lip(T) \leq \iota_p^L(T)$ for all $T \in \mathcal{I}_p^L(X,E)$. Let (A, i_p^μ, B) be a factorization for a $T \in \mathcal{I}_p^L(X,E)$. Clearly $k_E T \in Lip_0(X, E^{**})$ and $Lip(T) = Lip(k_E T) \leq Lip(A)Lip(B)$. Taking the infimum we get $Lip(T) \leq \iota_p^L(T)$.

We check now that $\mathcal{I}_p^L(X,E)$ is a linear subspace of $Lip_0(X,E)$. Let $T \in \mathcal{I}_p^L(X,E)$ and let (A, i_p^μ, B) be a decomposition for T. For any $\alpha \in \mathbb{K}$, $k_E(\alpha T) = (\alpha A) i_p^\mu B$. Since $\alpha A \in Lip_0(L_p(\mu), E^{**})$ then αT is Lipschitz $p-$integral and

$$\iota_p^L(\alpha T) \leq Lip(\alpha A) Lip(B) = |\alpha| \, Lip(A) Lip(B).$$

It follows that $\iota_p^L(\alpha T) \leq |\alpha| \, \iota_p^L(T)$. Then, for $\alpha \neq 0$, we have $\iota_p^L(T) = \iota_p^L(\alpha^{-1}\alpha T) \leq |\alpha^{-1}| \, \iota_p^L(\alpha T)$, hence $|\alpha| \, \iota_p^L(T) \leq \iota_p^L(\alpha T)$ and so $\iota_p^L(\alpha T) = |\alpha| \, \iota_p^L(T)$.

Let $T_1, T_2 \in \mathcal{I}_p^L(X,E)$ and $\varepsilon > 0$. Consider factorizations $(A_i, i_p^{\mu_i}, B_i)$ for T_i such that $Lip(B_i) = \frac{1}{2}$ and $Lip(A_i) < \iota_p^L(T_i) + \frac{\varepsilon}{2}$, for $i = 1, 2$. We may assume also that $\Omega_1 \cap \Omega_2 = \emptyset$.

Take $\Omega := \Omega_1 \cup \Omega_2$, $\Sigma := \{S \subset \Omega : S \cap \Omega_i \in \Sigma_i, i = 1, 2\}$ and define the probability measure μ on Ω by

$$\mu(S) := \frac{Lip(A_1)\mu_1(S \cap \Omega_1) + Lip(A_2)\mu_2(S \cap \Omega_2)}{Lip(A_1) + Lip(A_2)}.$$

Define $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ by

$$\begin{aligned} A(f) &= A_1(f|_{\Omega_1}) + A_2(f|_{\Omega_2}), \\ B(x) &= B_1(x) \cdot \chi_{\Omega_1} + B_2(x) \cdot \chi_{\Omega_2}, \end{aligned}$$

where χ_{Ω_i} is the characteristic function of Ω_i for $i = 1, 2$. We have $A(0) = 0$. If $\frac{1}{p} + \frac{1}{p'} = 1$, using Hölder's inequality we get

$$\begin{aligned} \|A(f) - A(f')\| &\leq \sum_{i=1}^{2} \|A_i(f|_{\Omega_i}) - A_i(f'|_{\Omega_i})\| \\ &\leq \sum_{i=1}^{2} Lip(A_i)^{1/p} Lip(A_i)^{1/p'} \|(f - f')|_{\Omega_i}\|_{L_p(\mu_i)} \\ &\leq \left(\sum_{i=1}^{2} Lip(A_1)\right)^{1/p} \left(\sum_{i=1}^{2} Lip(A_i)^{1/p'} \|(f - f')|_{\Omega_i}\|_{L_p(\mu_i)}\right) \\ &\leq \left(\sum_{i=1}^{2} Lip(A_1)\right)^{1/p} \left(\sum_{i=1}^{2} Lip(A_i)\right)^{1/p'} \left(\sum_{i=1}^{2} \|(f - f')|_{\Omega_i}\|_{L_p(\mu_i)}^p\right)^{1/p} \\ &= (Lip(A_1) + Lip(A_2)) \, \|(f - f')\|_{L_p(\mu)} \end{aligned}$$

for all $f, f' \in L_p(\mu)$. Hence $A \in Lip_0(L_p(\mu), E^{**})$ and $Lip(A) \leq Lip(A_1)+Lip(A_2)$. Moreover $B \in Lip_0(X, L_\infty(\mu))$ with $Lip(B) \leq 1$ because $B(0) = 0$ and

$$\begin{aligned} \|B(x) - B(x')\|_{L_\infty(\mu)} &\leq \|B_1(x) - B_1(x')\|_{L_\infty(\mu_1)} + \|B_2(x) - B_2(x')\|_{L_\infty(\mu_2)} \\ &\leq (Lip(B_1) + Lip(B_2))\, d(x, x') \\ &= d(x, x') \end{aligned}$$

for all $x, x' \in X$. For each $x \in X$, we have

$$\begin{aligned} Ai_p^\mu B(x) &= Ai_p^\mu (B_1(x)\chi_{\Omega_1} + B_2(x)\chi_{\Omega_2}) \\ &= A_1 i_p^{\mu_1} B_1(x) + A_2 i_p^{\mu_2} B_2(x) \\ &= k_E T_1(x) + k_E T_2(x) \\ &= k_E(T_1 + T_2)(x), \end{aligned}$$

and thus $Ai_p^\mu B = k_E(T_1 + T_2)$. Hence $T_1 + T_2 \in \mathcal{I}_p^L(X, E)$ and

$$\iota_p^L(T_1 + T_2) \leq Lip(A)Lip(B) \leq Lip(A) \leq \iota_p^L(T_1) + \iota_p^L(T_2) + \varepsilon.$$

Since ε was arbitrary, it follows that $\iota_p^L(T_1 + T_2) \leq \iota_p^L(T_1) + \iota_p^L(T_2)$. To prove the completeness of the space $\mathcal{I}_p^L(X, E)$, take a sequence (T_n) in $\mathcal{I}_p^L(X, E)$ such that $\sum\limits_{n=1}^{\infty} \iota_p^L(T_n) < \infty$. Since $Lip(.) \leq \iota_p^L(.)$ and $(Lip_0(X, E), Lip(.))$ is a Banach space, there exists $T := \sum\limits_{n=1}^{\infty} T_n \in Lip_0(X, E)$. We prove that $\sum\limits_{n=1}^{\infty} T_n = T$ for $\iota_p^L(.)$. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ we can find a probability space $(\Omega_n, \Sigma_n, \mu_n)$, Lipschitz operators $B_n \in Lip_0(X, L_\infty(\mu_n)), A_n \in Lip_0(L_p(\mu_n), E^{**})$ with $Lip(B_n) = 1/2^n$ and $Lip(A_n) \leq \iota_p^L(T_n)+\varepsilon/2^n$ such that $k_E T_n$ factors as

$$k_E T_n = A_n i_p^{\mu_n} B_n : X \xrightarrow{B_n} L_\infty(\mu_n) \xrightarrow{i_p^{\mu_n}} L_p(\mu_n) \xrightarrow{A_n} E^{**}.$$

We can assume that the Ω_n's are pairwise disjoint and set $\Omega := \cup_{n\in\mathbb{N}}\Omega_n$ and

$$\Sigma := \{S \subset \Omega : S \cap \Omega_n \in \Sigma_n,\ \forall n \in \mathbb{N}\}.$$

Define the probability measure μ on Σ by

$$\mu(S) := \frac{\sum_{n=1}^{\infty} \mu_n(S \cap \Omega_n)Lip(A_n)}{\sum_{n=1}^{\infty} Lip(A_n)},$$

$S \in \Sigma$. Define $A : L_p(\mu) \longrightarrow E^{**}$ and $B : X \longrightarrow L_\infty(\mu)$ by

$$\begin{aligned} A(f) &= \sum_{n=1}^{\infty} A_n(f\,|_{\Omega_n}), \\ B(x) &= \sum_{n=1}^{\infty} B_n(x)\chi_{\Omega_n}. \end{aligned}$$

Clearly $A(0) = 0$ and $B(0) = 0$. When $p = \infty$ it is clear that $Lip(A) \leq \sum_{n=1}^{\infty} Lip(A_n) < \infty$. For $1 \leq p < \infty$, using a similar argument as above we get

$$\|A(f) - A(f')\| \leq \left(\sum_{n=1}^{\infty} Lip(A_n)\right) \|f - f'\|_{L_p(\mu)}$$

and

$$\|B(x) - B(x')\|_{L_\infty(\mu)} \leq d(x, x'),$$

for all $f, f' \in L_p(\mu)$ and all $x, x' \in X$. Hence, $A \in Lip_0(L_p(\mu), E^{**})$ with $Lip(A) \leq \sum\limits_{n=1}^{\infty} Lip(A_n) \leq \sum\limits_{n=1}^{\infty} \iota_p^L(T_n) + \epsilon$, and $B \in Lip_0(X, L_\infty(\mu))$ with $Lip(B) \leq 1$.

For each $x \in X$, we have

$$\begin{aligned} Ai_p^\mu B(x) &= A\left(\sum_{n=1}^{\infty} i_p^\mu(B_n(x)\chi_{\Omega_n})\right) \\ &= \sum_{m=1}^{\infty} A_m\left(\Big(\sum_{n=1}^{\infty} i_p^\mu(B_n(x)\chi_{\Omega_n})\Big)|_{\Omega_m}\right) \\ &= \sum_{m=1}^{\infty} A_m i_p^{\mu_m} B_m(x) = \sum_{m=1}^{\infty} k_E T_m(x) = k_E T(x) \end{aligned}$$

and thus $Ai_p^\mu B = k_E T$. Hence, $T \in \mathcal{I}_p^L(X, E)$ and

$$\iota_p^L(T) \leq Lip(A)Lip(B) \leq Lip(A) \leq \sum_{n=1}^{\infty} \iota_p^L(T_n) + \varepsilon.$$

Since ε was arbitrary, it follows that $\iota_p^L(T) \leq \sum_{n=1}^{\infty} \iota_p^L(T_n)$.

We now show that $T = \sum_{k=1}^{\infty} T_k$ for the ι_p^L norm. For each $n \in \mathbb{N}$, define $t_n : L_p(\mu) \longrightarrow E^{**}$ by $t_n(f) = \sum_{k=n+1}^{\infty} A_k(f\,|_{\Omega_k})$. By the same argument used above we obtain, $t_n \in Lip_0(L_p(\mu), E^{**})$ with $Lip(t_n) \leq \sum_{k=n+1}^{\infty} Lip(A_k)$ and so $\lim\limits_{n\to\infty} Lip(t_n) = 0$. It is easy to see that $T - \sum_{k=1}^{n} T_k = t_n i_p^\mu B$. Then, $\lim\limits_{n\to\infty} \iota_p^L(\ T - \sum_{k=1}^{n} T_k) = 0$.

(ii) Fix a point $x_0 \in X$ and take $\Omega = \{x_0\}$, $\Sigma = \{\Omega, \emptyset\}$ and $\mu : \Sigma \longrightarrow \mathbb{R}$ defined by $\mu(\Omega) = 1, \mu(\emptyset) = 0$. Then (Σ, Ω, μ) is a probability space. Clearly, $L_\infty(\mu)$ and $L_p(\mu)$ contain only constant functions.

Let $g \in X^\#$ and $v \in E$. Define $B \in Lip_0(X, L_\infty(\mu))$ and $A \in Lip_0(L_p(\mu_i), E^{**})$ by $A(t\mathbf{1}) = tk_E(v)$ for all $t \in \mathbb{K}$ and $B(x) = g(x)\mathbf{1}$ for all $x \in X$, where $\mathbf{1}$ is the constant function equal to 1 on Ω. It is clear that

$$(k_E(vg))(x) = g(x)k_E(v) = g(x)A(\mathbf{1}) = A(g(x)\mathbf{1}) = Ai_p^\mu(g(x)\mathbf{1}) = Ai_p^\mu B(x)$$

for all $x \in X$. Then $vg \in \mathcal{I}_p^L(X, E)$ and $\iota_p^L(vg) \leq Lip(A)Lip(B) = \|v\|\, Lip(g)$.

Let $Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda$. By [20, p. 5275] $\iota_p^L(Id_{\mathbb{K}}) = \iota_p(Id_{\mathbb{K}}) = 1$.

(iii) Let $v \in Lip_0(Y, X)$, $T \in \mathcal{I}_p^L(X, E)$ and $w \in \mathcal{L}(E, F)$. Since T is Lipschitz $p-$integral then we have the following factorization

$$\begin{array}{ccccccccc}
Y & \xrightarrow{v} & X & \xrightarrow{T} & E & \xrightarrow{w} & F & \xrightarrow{k_F} & F^{**} \\
 & & \downarrow B & & & \searrow k_E & & \nearrow w^{**} & \\
 & & & & & & E^{**} & & \\
 & & & & & \nearrow A & & & \\
 & & L_\infty(\mu) & \xrightarrow{i_p^\mu} & L_p(\mu) & & & &
\end{array}$$

Since the mappings $B \circ v$ and $w^{**} \circ A$ are Lipschitz then $w \circ T \circ v$ is Lipschitz p-integral and

$$\iota_p^L(w \circ T \circ v) = \iota_p^L(w^{**} \circ A \circ i_p^\mu \circ B \circ v) \leq \|w\|\, Lip(A)Lip(B)Lip(v).$$

Taking the infimum, we obtain

$$\iota_p^L(w \circ T \circ v) \leq \|w\|\, \iota_p^L(T)Lip(v).$$

□

4.2.3 Lipschitz p-nuclear operators

Let $1 \leq p \leq \infty$. Lipschitz p-nuclear operators from metric spaces to Banach spaces are introduced in [20]. A map $T \in Lip_0(X, E)$ is *Lipschitz p-nuclear* if there are two Lipschitz mappings $A : \ell_p \longrightarrow E$ and $B : X \longrightarrow \ell_\infty$ and $\lambda = (\lambda_n)_n \in \ell_p$ such that the following diagram commutes:

$$\begin{array}{ccc}
X & \xrightarrow{T} & E \\
B \downarrow & & \uparrow A \\
\ell_\infty & \xrightarrow{M_\lambda} & \ell_p
\end{array}$$

where $M_\lambda((x_n)_n) := (\lambda_n x_n)_n$, for all $(x_n)_n \in \ell_\infty$. Let $\nu_p^L(T)$ denote the infimum of $Lip(A)\,\|M_\lambda\|\,Lip(B)$ over all factorizations as above; note that $\|M_\lambda\| = \|\lambda\|_{\ell_p}$. The set of all such T is denoted by $\mathcal{N}_p^L(X,E)$.

Proposition 4.7. *$(\mathcal{N}_p^L, \nu_p^L)$ is a Banach Lipschitz operator ideal.*

Proof. We start proving that $Lip(T) \leq \nu_p^L(T)$ for any $T \in \mathcal{N}_p^L(X,E)$. Assume that $T \in \mathcal{N}_p^L(X,E)$. Then, there exist $A \in Lip_0(X,\ell_\infty)$, $B \in Lip_0(\ell_p, E)$ and a diagonal operator $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ with $\lambda \in \ell_p$ such that T factors as

$$T : X \xrightarrow{A} \ell_\infty \xrightarrow{M_\lambda} \ell_p \xrightarrow{B} E.$$

Clearly $T \in Lip_0(X,E)$ and

$$Lip(T) = Lip(BM_\lambda A) \leq Lip(B)\,\|M_\lambda\|\,Lip(A).$$

Taking the infimum we get $Lip(T) \leq \nu_p^L(T)$.

The spaces $L_p(\mu)$ and $L_\infty(\mu)$ are the standard sequence spaces ℓ_p and ℓ_∞ respectively, whenever μ is the counting measure on $\mathbb{N}$. Then, an easy adaptation of the proof of Proposition 4.6 proves that $\mathcal{N}_p^L$ is a Banach Lipschitz operator ideal. □

4.2.4 Lipschitz compact and Lipschitz weakly compact operators

Lipschitz compact and weakly compact operators are introduced in [40] (see section 1.5). Those classes of mappings are Banach Lipschitz operator ideals with the norm *Lip*. Indeed the ideal property has been proved in [40, Proposition 2.3]. Now we prove that $Lip_{0\mathcal{K}}(X,E)$ (respectively $Lip_{0\mathcal{W}}(X,E)$) contains the finite dimensional rank Lipschitz operators. Let T be a finite dimensional rank Lipschitz operator, then by [40, Proposition 2.4] the linearization T_L has finite dimensional rank and so is compact (respectively weakly compact) and by [40, Proposition 2.1 and Proposition 2.2] T is Lipschitz compact (respectively Lipschitz weakly compact). By [40, Corollary 2.6] the spaces $Lip_{0\mathcal{K}}(X,E)$ and $Lip_{0\mathcal{W}}(X,E)$ are closed subspaces of $Lip_0(X,E)$). Then $Lip_{0\mathcal{K}}$ and $Lip_{0\mathcal{W}}$ are Banach Lipschitz operator ideals with the norm *Lip*. We can get the next result regarding the classical approximation property; we will adapt this notion to the Lipschitz setting in a more general way in the last section of this chapter.

Proposition 4.8. *If E has the approximation property and $T \in Lip_0(X, E)$, then T is Lipschitz compact if, and only if, T can be approximated by Lipschitz finite rank operators.*

Proof. If T is Lipschitz compact then T_L is compact and by the approximation property of rE we conclude that T_L can be approximated by finite rank operators. Then, by Corollary 4.4 T can be approximated by Lipschitz finite rank operators. For the converse see [40, Propositions 2.1 and 2.4]. □

Remark 4.9. *If $\mathcal{I}_{Lip}$ is a Banach Lipschitz operator ideal and we define $\mathcal{I}_{Lip} \cap \mathcal{L}$ as the class of all linear operators between Banach spaces that belong to $\mathcal{I}_{Lip}$, then $\mathcal{I}_{Lip} \cap \mathcal{L}$ is a linear operator ideal. It is shown in [33, Theorem 2] that for a linear operator T, $\pi_p^L(T)$ coincides with the absolutely $p-$summing norm and then, $\Pi_p^L \cap \mathcal{L}$ is the ideal of absolutely $p-$summing operators. The same thing occurs for Lipschitz $p-$integral operators: for a linear operator T, $\iota_p^L(T)$ coincides with the $p-$integral norm and then, $\mathcal{I}_p^L \cap \mathcal{L}$ is the ideal of $p-$integral operators. Also, from [20, Theorem 2.1] it follows that $\mathcal{N}_p^L(X, E^*) \cap \mathcal{L}(X, E^*) = \mathcal{N}_p(X, E^*)$ whenever X is a separable Banach space and E^* is a dual Banach space.*

4.3 Methods to produce Lipschitz operator ideals

We undertake here some methods to produce Lipschitz operator ideals.

4.3.1 Composition ideal of Lipschitz mappings

Definition 4.10. *(Composition Ideals) Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X, E)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ Lip_0$, denoted $T \in \mathcal{I} \circ Lip_0(X, E)$, if there are a Banach space F, a Lipschitz operator $S \in Lip_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal we write $\|T\|_{\mathcal{I} \circ Lip_0} = \inf \|u\|_{\mathcal{I}} \, Lip(S)$, where the infimum is taken over all u, S as above.*

Proposition 4.11. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in Lip_0(X, E)$:*

(1) $T \in \mathcal{I} \circ Lip_0(X, E)$.

(2) $T_L \in \mathcal{I}(Æ(X), E)$.

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L\|_{\mathcal{I}} \, .$$

Proof. (1) $\Rightarrow$ (2) Assume that $T \in \mathcal{I}\circ Lip_0(X,E)$. Then there is a Banach space F, a Lipschitz operator $S \in Lip_0(X,F)$ and an operator $u \in \mathcal{I}(F,E)$ such that $T = u \circ S$. Since $T_L = u \circ S_L$, the ideal property ensures that $T_L \in \mathcal{I}(Æ(X),E)$. If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed then,

$$\|T_L\|_{\mathcal{I}} = \|u \circ S_L\|_{\mathcal{I}} \leq \|u\|_{\mathcal{I}} \|S_L\| = \|u\|_{\mathcal{I}} Lip(S).$$

Taking the infimum over all such factorizations we obtain $\|T_L\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}\circ Lip_0}$.

(2) $\Rightarrow$ (1) Consider the factorization of T given by $T = T_L \circ \delta_X$. Since δ_X is Lipschitz and $T_L \in \mathcal{I}(Æ(X),E)$ then $T \in \mathcal{I}\circ Lip_0(X,E)$ and, if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed we have

$$\|T\|_{\mathcal{I}\circ Lip_0} = \|T_L \circ \delta_X\|_{\mathcal{I}\circ Lip_0} \leq \|T_L\|_{\mathcal{I}} Lip(\delta_X) = \|T_L\|_{\mathcal{I}}.$$

□

Corollary 4.12. *If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I}\circ Lip_0$ is a (respectively normed, closed, Banach) Lipschitz operator ideal.*

Proof. Let us check that a composition Lipschitz operator ideal is a Lipschitz operator ideal. Using Proposition 4.11,

$$\begin{aligned} T, S \in \mathcal{I} \circ Lip_0(X,E) &\iff T_L, S_L \in \mathcal{I}(Æ(X),E) \\ &\iff (\alpha T + \beta S)_L = \alpha T_L + \beta S_L \in \mathcal{I}(Æ(X),E), \forall \alpha, \beta \in \mathbb{K} \\ &\iff \alpha T + \beta S \in \mathcal{I} \circ Lip_0(X,E), \forall \alpha, \beta \in \mathbb{K}. \end{aligned}$$

As a consequence of Proposition 4.11, $\mathcal{I}\circ Lip_0(X,E)$ contains the finite dimensional rank Lipschitz operators. To prove the ideal property consider $v \in Lip_0(Y,X)$, $T \in \mathcal{I}\circ Lip_0(X,E)$ and $w \in \mathcal{L}(E,F)$. There are a Banach space G, a Lipschitz operator $S \in Lip_0(X,G)$ and an operator $u \in \mathcal{I}(G,E)$ such that $T = u \circ S$. From the ideal property, $w \circ u \in \mathcal{I}(G,E)$. It is clear that $S \circ v \in Lip_0(Y,E)$ and we conclude that $w \circ T \circ v \in \mathcal{I}\circ Lip_0(Y,F)$. Therefore $\mathcal{I}\circ Lip_0$ is a Lipschitz operator ideal.

Now we show that $\mathcal{I}\circ Lip_0$ is closed whenever $\mathcal{I}$ is. Consider a sequence $(T_i)_i \in \mathcal{I}\circ Lip_0(X,E)$ converging to T in $Lip_0(X,E)$. From

$$\lim_{i\to\infty} \|(T_i)_L - T_L\| = \lim_{i\to\infty} \|(T_i - T)_L\| = \lim_{i\to\infty} Lip(T_i - T) = 0.$$

it follows that $T_L \in \mathcal{I}(Æ(X),E)$ and so, $T \in \mathcal{I}\circ Lip_0(X,E)$.

Clearly, $(\mathcal{I}\circ Lip_0, \|\cdot\|_{\mathcal{I}\circ Lip_0})$ is a normed Lipschitz ideal whenever $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is normed. Just for the sake of completeness note that if $v \in Lip_0(Y, X)$, $T \in \mathcal{I}\circ Lip_0(X, E)$ and $w \in \mathcal{L}(E, F)$ then, by [42, Lemma 3.1]) the linearization of $w \circ T \circ v$ is $w \circ T_L \circ \hat{v}$ for a suitable linear operator $\hat{v} \in \mathcal{L}(Æ(Y), Æ(X))$, with $\|\tilde{v}\| = Lip(v)$.

$$\begin{aligned} \|w \circ T \circ v\|_{\mathcal{I}\circ Lip_0} &= \|w \circ T_L \circ \hat{v}\|_{\mathcal{I}\circ Lip_0} \\ &\leq \|w\| \, \|T_L\|_{\mathcal{I}} \, \|\hat{v}\| \\ &= \|w\| \, \|T\|_{\mathcal{I}\circ Lip_0} \, Lip(v). \end{aligned}$$

To finish the proof, it easily follows from Proposition 4.11 that if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a Banach operator ideal then, $(\mathcal{I}\circ Lip_0, \|.\|_{\mathcal{I}\circ Lip_0})$ is a Banach Lipschitz operator ideal. □

Example 4.13. *In chapter 2 we introduced the class of strongly Lipschitz p-summing* $(1 < p \leq \infty)$ *and we proved that a Lipschitz operator* T *is strongly Lipschitz p-summing if, and only if,* T_L *is strongly p-summing. Then by Proposition 4.11 every strongly Lipschitz p-summing mapping* T *can be considered the composition of a Lipschitz mapping and a strongly p-summing mapping, that is,* $\mathcal{D}^L_{st,p} = D_p \circ Lip_0$.

Example 4.14. *The following variant of Lipschitz integral mapping is introduced in [12]. A mapping* $T \in Lip_0(X, E)$ *is Lipschitz Grothendieck-integral if there are a probability measure space* (Ω, Σ, μ), *a linear operator* $A \in \mathcal{L}(L_1(\mu), E^{**})$ *and a Lipschitz mapping* $B \in Lip_0(X, L_\infty(\mu))$ *such that* $k_E \circ T = A \circ i_1^\mu \circ B$. *The Lipschitz G-integral norm of* T *is defined by* $Lip_{GI}(T) = \inf \|A\| Lip(B)$, *where the infimum is taken over all factorizations as above. This class of mappings is denoted by* $Lip_{0GI}(X, E)$. *If we denote by* $(\mathcal{I}_1, \iota_1)$ *the Banach ideal of integral linear operators (We refer to [32, 26] for integral linear operators) then,* $\mathcal{I}_1(F, E) \subset Lip_{0GI}(F, E) \subset \mathcal{I}_1^L(F, E)$, *for all Banach spaces* E *and* F. *The maximality of* $\mathcal{I}_1$ *implies that* $\iota(T) = Lip_{GI}(T) = \iota_1(T)$, *for all* $T \in \mathcal{I}_1(F, E)$. *In [12, Proposition 2.1] it is proved that* Lip_{0GI} *is a Banach Lipschitz operator ideal (a similar notion of Lipschitz operator ideal is considered in [12], without going into a general study). Proposition 2.4 in [12] shows that* $T \in Lip_{0GI}(X, E)$ *if, and only if,* $T_L \in \mathcal{I}_1(Æ(X), E)$. *Hence, by Proposition 4.11,* $Lip_{0GI} = \mathcal{I}_1 \circ Lip_0$.

Proposition 4.15. *Let* X *be a pointed metric space and* E *be a Banach space. We have*

1. $Lip_{0\mathcal{K}}(X, E) = \mathcal{K} \circ Lip_0(X, E)$ *isometrically.*

2. $Lip_{0\mathcal{W}}(X,E) = \mathcal{W} \circ Lip_0(X,E)$ isometrically.

Proof. It is a consequence of Propositions 2.1 and 2.2 in [40] and Proposition 4.11. $\square$

Remark 4.16. *In general, a Lipschitz operator ideal $\mathcal{I}_{Lip}$ does not coincide with $\mathcal{I} \circ Lip_0$. For example, the ideal of Lipschitz p-summing operators Π_p^L and the corresponding composition $\Pi_p \circ Lip_0$ do not coincide. Indeed, $\delta_{\mathbb{K}}$ is Lipschitz p-summing, but its linearization is the identity map on the infinite dimensional space Æ($\mathbb{K}$) and so, it cannot be absolutely p-summing. Proposition 4.11 ensures now that $\delta_{\mathbb{K}}$ does not belong to $\Pi_p \circ Lip_0$. The same situation occurs for Lipschitz p-nuclear and Lipschitz p-integral operators.*

4.3.2 The dual Lipschitz ideal

The *dual* of an operator ideal $\mathcal{I}$ is defined as follows: for Banach spaces E and F,

$$\mathcal{I}^{dual}(E,F) = \{u \in \mathcal{L}(E,F) : u^* \in \mathcal{I}(F^*, E^*)\},$$

where $u^* : F^* \longrightarrow E^*$ is the adjoint of u. It is well known that $\mathcal{I}^{dual}$ is an operator ideal. We write $\|u\|_{\mathcal{I}^{dual}} = \|u^*\|_{\mathcal{I}}$. We mention that $\mathcal{K} = \mathcal{K}^{dual}$ and $\mathcal{W} = \mathcal{W}^{dual}$ (see [59])

Definition 4.17. *The Lipschitz dual of a given operator ideal $\mathcal{I}$ is defined by*

$$\mathcal{I}^{Lip_0-dual}(X,E) = \left\{T \in Lip_0(X,E) : T^t \in \mathcal{I}(E^*, X^{\#})\right\}$$

If $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is a quasinormed operator ideal, define

$$\|T\|_{\mathcal{I}^{Lip_0-dual}} = \left\|T^t\right\|_{\mathcal{I}}.$$

Theorem 4.18. *If $\mathcal{I}$ is an operator ideal then,*

$$\mathcal{I}^{Lip_0-dual} = \mathcal{I}^{dual} \circ Lip_0$$

Moreover, if $(\mathcal{I}, \|.\|_{\mathcal{I}})$ is normed then,

$$\|\cdot\|_{\mathcal{I}^{Lip_0-dual}} = \|\cdot\|_{\mathcal{I}^{dual} \circ Lip_0}.$$

Proof. Assume that $T \in \mathcal{I}^{Lip_0-dual}$, that is, $T^t \in \mathcal{I}(E^*, X^\#)$. Consider the operator $Q_X : X^\# \to \text{Æ}(X)^*$ defined in Section 1.2. As we said there, it is linear, continuous and $\|Q_X\| \leq 1$. Since $(T_L)^* = Q_X \circ T^t \in \mathcal{I}(E^*, \text{Æ}(X)^*)$ then, $T_L \in \mathcal{I}^{dual}(\text{Æ}(X), E)$. By Proposition 4.11 we conclude $T \in \mathcal{I}^{dual} \circ Lip_0$. Besides,

$$\begin{aligned} \|T\|_{\mathcal{I}^{dual} \circ Lip_0} &= \|T_L\|_{\mathcal{I}^{dual}} = \|(T_L)^*\|_{\mathcal{I}} = \left\|Q_X \circ T^t\right\|_{\mathcal{I}} \\ &\leq \|Q_X\| \left\|T^t\right\|_{\mathcal{I}} = \|T\|_{\mathcal{I}^{Lip_0-dual}}. \end{aligned}$$

Assume that $T \in \mathcal{I}^{dual} \circ Lip_0(X, E)$. Then $T = u \circ S$ with $u^* \in \mathcal{I}(E^*, F^*)$ and $S \in Lip_0(X, F)$ for some Banach space F. We have $T^t = S^t \circ u^*$. By the ideal property we conclude that $T^t \in \mathcal{I}(E^*, X^\#)$, that is $T \in \mathcal{I}^{Lip_0-dual}$. To finish the proof, we show the equality of the norms. Let $\epsilon > 0$, choosing F, S and u such that $\|u\|_{\mathcal{I}^{dual}} \cdot Lip(S) \leq (1 + \epsilon) \|T\|_{\mathcal{I}^{dual} \circ Lip_0}$ we get

$$\begin{aligned} \|T\|_{\mathcal{I}^{Lip_0-dual}} &= \left\|T^t\right\|_{\mathcal{I}} = \left\|S^t \circ u^*\right\|_{\mathcal{I}} \leq \|u^*\|_{\mathcal{I}} \left\|S^t\right\| \\ &= \|u\|_{\mathcal{I}^{dual}} Lip(S) \\ &\leq (1 + \epsilon) \|T\|_{\mathcal{I}^{dual} \circ Lip_0}. \end{aligned}$$

Letting $\epsilon \longrightarrow 0$ we obtain $\|T\|_{\mathcal{I}^{Lip_0-dual}} \leq \|T\|_{\mathcal{I}^{dual} \circ Lip_0}$. □

Propositions 3.4 and 3.5 in [40] can be proved directly from our previous results.

Corollary 4.19. *A Lipschitz operator is compact (weakly compact) if, and only if, its transpose is a compact (weakly compact) linear operator.*

Proof. We get the results directly from Proposition 4.15, Schauder's (Gantmacher's) theorem and Theorem 4.18; indeed, they give the equalities $Lip_{0\mathcal{K}} = \mathcal{K} \circ Lip_0 = \mathcal{K}^{dual} \circ Lip_0 = \mathcal{K}^{Lip_0-dual}$ and $Lip_{0\mathcal{W}} = \mathcal{W} \circ Lip_0 = \mathcal{W}^{dual} \circ Lip_0 = \mathcal{W}^{Lip_0-dual}$. □

4.4 Applications: the approximation property for Lipschitz operator ideals

Recall first the classical approximation property: a Banach space E has the *approximation property* if for every Banach spaces F (or just for E) every linear operator $T : E \to F$ can be approximated uniformly on compact sets by finite rank operators or, equivalently, for every Banach space F every compact linear operator $T : F \to E$ can be approximated uniformly on bounded sets by finite rank operators [38, Proposition 35]. The approximation property has been widely studied and the reader can find several books where it has been treated (for example in [26, Ch.I,§5] and [15]). Roughly speaking, three are the main ingredients that play an important role in the approximation property: the approximating operators (the finite rank operators), the approximated operators (the linear or compact operators) and the bornology where the convergence is considered (compact or bounded sets). In the last decades, some variants of the approximation property related to an operator ideal $\mathcal{I}$ have been considered. These variants concern to what extent the approximation property can spread whenever its ingredients are replaced by some others related to $\mathcal{I}$. The main situations that have been considered are:

• To replace finite rank operators by operators belonging to an operator ideal. The main purpose in [7] is the study of Banach spaces E for which every linear operator $T \in \mathcal{L}(E, E)$ can be approximated uniformly on compact subsets of E by operators in $\mathcal{I}(E, E)$.

• To replace bounded linear or compact operators by elements in a general ideal $\mathcal{I}$. This variant of approximation property on a Banach space E related to an operator ideal $\mathcal{I}$ studies when the space of finite rank operators $\mathcal{L}_f(F, E)$ is dense in $\mathcal{I}(F, E)$ for every Banach space F (see for example [54] and the references therein).

• To replace compact sets by another class of sets with some kind of compactness related to $\mathcal{I}$. The new class of sets is formed by $\mathcal{I}$-compact sets. This notion was introduced by Carl and Stephani [14] and the related approximation property has been studied in e.g. [29, 45, 46].

Let $\mathcal{I}$ be an operator ideal. A subset B of a Banach space E is *relatively $\mathcal{I}$-compact* if there is a Banach space G and an operator $S \in \mathcal{I}(G, E)$ such that $B \subseteq S(M)$, where M is a compact subset of G. The associated operator definition of this set-theoretic notion is the following. Let T be a linear operator between Banach spaces E and F and consider an operator ideal $\mathcal{I}$. It is said that T is *$\mathcal{I}$-compact* if $T(B_E)$ is a relatively $\mathcal{I}$-compact subset of F. The operator ideal formed by all linear $\mathcal{I}$-compact operators is denoted by $\mathcal{K}_{\mathcal{I}}$ (see [45, Sec.2]).

Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a Banach operator ideal. Following [54], a Banach space E has the *$\mathcal{I}$-approximation property* if for every Banach space F, the space of finite rank linear operators

$\mathcal{L}_f(F,E)$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}(F,E)$. In [45] it is also considered the situation of replacing the norm $\|\cdot\|_{\mathcal{I}}$ by the operator norm $\|\cdot\|$ in $\mathcal{L}$: a Banach space E has the *$\mathcal{I}$-uniform approximation property* if for every Banach space F, the space of finite rank linear operators $\mathcal{L}_f(F,E)$ is $\|\cdot\|$-dense in $\mathcal{I}(F,E)$. Note that, whenever the ideal of compact operators $\mathcal{K}$ is considered, the $\mathcal{K}$-(uniform) approximation property is just the approximation property.

In particular, a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property if for every Banach space F, $\mathcal{L}_f(F,E)$ is norm dense in $\mathcal{K}_{\mathcal{I}}(F,E)$, that is, every $\mathcal{I}$-compact operator from F into E can be uniformly approximated by finite rank operators. In [45, Proposition 3.2] it is proved that a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property if, and only if, the identity $Id_E : E \to E$ can be approximated uniformly on relatively $\mathcal{I}$-compact sets by finite rank operators.

This is the starting point of our analysis of the "Lipschitz version" of these notions, that is explained in what follows. We will consider Lipschitz ideals $\mathcal{I}_{Lip}$ that are composition ideals, that is, $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ for some linear operator ideal $\mathcal{I}$. Taking into account that each Lipschitz operator can be factored as

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & E, \\ & {\scriptstyle \delta_X} \searrow \quad \nearrow {\scriptstyle T_L} & \\ & \text{Æ}(X)\,, & \end{array}$$

$\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ satisfies that a Lipschitz map T belongs to $\mathcal{I}_{Lip}(X,E)$ if, and only if, $T_L \in \mathcal{I}(\text{Æ}(X),E)$ (see Proposition 4.11).

We consider the following natural definitions.

Definition 4.20. *Let X and Y be pointed metric spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal.*

A set $K \subseteq X$ is relatively $\mathcal{I}$-Lipschitz compact if there is a pointed metric space Y and a Lipschitz operator $S : Y \to \text{Æ}(X)$ in $\mathcal{I}_{Lip}$ such that $\delta_X(K) \subseteq S(M)$, where M is a compact subset of Y.

Let Y and X be pointed metric spaces. We say that a Lipschitz operator $\phi : Y \to X$ is $\mathcal{I}$-Lipschitz compact if $\phi(B_Y) = \phi(\{x \in Y : d(x,0) \leq 1\})$ is relatively $\mathcal{I}$-Lipschitz compact.

Note that, if E is a Banach space then, a Lipschitz operator $T : X \to E$ is $\mathcal{I}$-Lipschitz compact if $T(\{x \in X : d(x,0) \leq 1\})$ is a relatively $\mathcal{I}$-Lipschitz compact subset of E.

We will write $\mathcal{K}^L_{\mathcal{I}}(X,E)$ for the class of all $\mathcal{I}$-Lipschitz compact operators from X to E, considered as a topological subspace of the Lipschitz operators. This class satisfies some good composition properties. For example, taking into account that the Lipschitz image of a relatively $\mathcal{I}$-Lipschitz compact set is again relatively $\mathcal{I}$-Lipschitz compact, we obtain the following fact: if $T : X \to Y$ is $\mathcal{I}$-Lipschitz compact and $R \in Lip_0(Y,Z)$, then $R \circ T : X \to Z$ is also $\mathcal{I}$-Lipschitz compact.

Proposition 4.21. *Let X be a pointed metric space and let $A \subset X$. Then A is relatively $\mathcal{I}$-Lipschitz compact if, and only if, $\delta_X(A)$ is relatively $\mathcal{I}$-compact.*

Moreover, if E is a Banach space, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact.

Proof. Assume that A is relatively $\mathcal{I}$-Lipschitz compact. There is a pointed metric space Y, $M \subset Y$ compact and $S \in \mathcal{I}_{Lip}(Y, Æ(X))$ such that $\delta_X(A) \subseteq S(M)$. Since we can write $S = S_L \circ \delta_Y$ with $S_L \in \mathcal{I}(Æ(Y), Æ(X))$, and $\delta_Y(M)$ is compact, we conclude that $\delta_X(A)$ is a relatively $\mathcal{I}$-compact set.

Now assume that $\delta_X(A)$ is relatively $\mathcal{I}$-compact. Let Z be a Banach space, K a compact subset of Z and $T \in \mathcal{I}(Z, Æ(X))$ such that $\delta_X(A) \subseteq T(K)$. This implies that $T \in \mathcal{I}_{Lip}(Z, Æ(X))$, which now does give the result.

Finally, assume that X is a Banach space E. Let us see that every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Take a relatively $\mathcal{I}$-Lipschitz compact subset K of E. This means that there is a Lipschitz map $S : Y \to Æ(E)$ in $\mathcal{I}_{Lip}$ from a pointed metric space Y and a compact subset $M \subseteq Y$ such that $\delta_E(K) \subseteq S(M)$. We have the canonical factorization for S given by $S_L \circ \delta_Y$, and since $\mathcal{I}_{Lip}$ is a composition Lipschitz ideal, we have that S_L belongs to the linear ideal $\mathcal{I}$. Then the compact subset $M_1 := \delta_Y(M)$ of $Æ(Y)$ satisfies that there is a linear operator S_L in $\mathcal{I}$ such that $\delta_E(K) \subseteq S_L(M_1)$. Consider $(Id_E)_L : Æ(E) \to E$. Then we have that $K = (Id_E)_L(\delta_E(K)) \subseteq (Id_E)_L \circ S_L(M_1)$, where $(Id_E)_L \circ S_L$ is a linear map in $\mathcal{I}$, and so K is a relatively $\mathcal{I}$-compact subset of E.

□

From Proposition 4.21, we obtain the next

Corollary 4.22. *A linear map $T : F \to E$ between Banach spaces is $\mathcal{I}$-Lipschitz compact if, and only if, it is linear $\mathcal{I}$-compact.*

In what follows we will show more concrete information about the relation between the linear and the Lipschitz $\mathcal{I}$-compactness for operators. We start with a lemma that is already known. The linearization $\beta_E : \AE(E) \to E$ of the identity map in E is known as the barycentric map, and it is in fact a quotient map (see for example [43, p.25] and [35, Lemma 2.4], where this map is denoted by β). This gives a proof of the lemma; however, we write a direct proof for the aim of completeness.

Lemma 4.23. *Let E be a Banach space and let $\beta_E : \AE(E) \to E$ be the linearization of the identity map $Id^E : E \to E$. Then, $\beta_E(B_{\AE(E)}) = B_E$.*

Proof. Since $\beta_E \circ \delta_E(x) = Id^E(x) = x$ and $\delta_E(B_E) \subset B_{\AE(E)}$, the inclusion $B_E \subset \beta_E(B_{\AE(E)})$ is trivial. Let us prove the other inclusion. It is enough to prove that $\beta_E(B_{\mathcal{M}(E)}) = B_E$. Assume that there is $m \in B_{\mathcal{M}(E)}$ such that $\beta_E(m)$ does not belong to B_E. The Hahn-Banach theorem gives a linear functional $\phi \in E^*$ of norm $\|\phi\| = 1$ such that $|\phi(\beta_E(m))| > 1$. Write $f := \phi \circ \beta_E \in \mathcal{M}(E)^*$ and take a real number a so that $1 < a < |f(m)|$. Let $\epsilon > 0$. Consider a representation of m, $m = \sum_{i=1}^n \lambda_i m_{x_i, x_i'}$ so that $\sum_{i=1}^n |\lambda_i| \|x_i - x_i'\| < 1 + \epsilon$. Then,

$$a < |f(m)| = |\sum_{i=1}^n \lambda_i \phi(\beta_E(\delta_E(x_i) - \delta_E(x_i')))| = |\phi(\sum_{i=1}^n \lambda_i (x_i - x_i'))| < 1 + \epsilon.$$

Letting $\epsilon \to 0$, we get a contradiction. □

The map $(\delta_E \circ T)_L$ appearing in (iii) of the next result is already known (see [35, Lemma 2.2]): it is the unique linear map $\hat{T} : \AE(F) \to \AE(E)$ such that $\hat{T} \circ \delta_F = \delta_E \circ T$.

Proposition 4.24. *Let F and E be Banach spaces, and $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ a composition Lipschitz operator ideal. Consider a Lipschitz map $T : F \to E$.*

(i) If T is linear and $\mathcal{I}$-Lipschitz compact, then $T_L : \AE(F) \to E$ is (linear) $\mathcal{I}$-compact.

(ii) If $\delta_E \circ T_L : \AE(F) \to \AE(E)$ sends the unit ball to a relatively $\mathcal{I}$-compact set, then T is $\mathcal{I}$-Lipschitz compact.

(iii) If $(\delta_E \circ T)_L : \AE(F) \to \AE(E)$ is (linear) $\mathcal{I}$-compact, then T is $\mathcal{I}$-Lipschitz compact.

(iv) If T is linear, then T is $\mathcal{I}$-Lipschitz compact if, and only if, $T_L : \AE(F) \to E$ is (linear) $\mathcal{I}$-compact.

Proof. (i) Assume that T is linear and $\mathcal{I}$-Lipschitz compact. By definition, $T(B_F)$ is relatively Lipschitz $\mathcal{I}$-compact. By Proposition 4.21, $T(B_F)$ is relatively $\mathcal{I}$-compact. Note that since T is linear so is $T \circ \beta_F : Æ(F) \to E$ (where β_F is the barycentric map defined above), and

$$(T \circ \beta_F) \circ \delta_F = T \circ (\beta_F \circ \delta_F) = T.$$

But T_L is by definition the unique linear map $Æ(F) \to E$ such that $T = T_L \circ \delta_F$, so we obtain that $T_L = T \circ \beta_F$. Now, using Lemma 4.23,

$$T_L(B_{Æ(F)}) = T \circ \beta_F(B_{Æ(F)}) = T(B_F),$$

so $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-compact set and thus T_L is linear $\mathcal{I}$-compact.

(ii) This is an immediate consequence of Proposition 4.21. By hypothesis we have that $\delta_E \circ T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. By Proposition 4.21, we have that $T_L(B_{Æ(F)})$ is a relatively $\mathcal{I}$-Lipschitz compact set. Since $\delta_F(B_F) \subset B_{Æ(F)}$ and $T = T_L \circ \delta_F$, it follows that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact and thus T is $\mathcal{I}$-Lipschitz compact.

(iii) If $(\delta_E \circ T)_L$ is $\mathcal{I}$-compact, then $(\delta_E \circ T)_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact. Since $(\delta_E \circ T)(B_F) = ((\delta_E \circ T)_L) \circ \delta_F)(B_F) \subseteq (\delta_E \circ T)_L(B_{Æ(F)})$ it follows that $(\delta_E \circ T)(B_F)$ is relatively $\mathcal{I}$-compact and thus by Proposition 4.21 we conclude that $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact.

(iv) We only need to prove ($\Longleftarrow$), since (i) gives the other implication.

Suppose that T_L is $\mathcal{I}$-compact. Then $T_L(B_{Æ(F)})$ is relatively $\mathcal{I}$-compact and thus it is relatively $\mathcal{I}$-Lipschitz compact by Proposition 4.21. Since $\delta_F(B_F) \subset B_{Æ(F)}$,

$$T(B_F) = (T_L \circ \delta_F)(B_F) \subset T_L(B_{Æ(F)}),$$

so $T(B_F)$ is relatively $\mathcal{I}$-Lipschitz compact, and thus T is $\mathcal{I}$-Lipschitz compact.

□

Proposition 4.24 gives that for linear operators, being (Lipschitz) compact and being $\mathcal{L}$-Lipschitz compact coincide. The first three statements of the following result have been proved in [40, Proposition 2.1].

Corollary 4.25. *Let F and E be Banach spaces and consider a linear operator $T : F \to E$. The following assertions are equivalent.*

(i) T is compact.

(ii) $T_L : Æ(F) \to E$ *is compact.*

(iii) T *is Lipschitz compact as a Lipschitz map.*

(iv) T *is* $\mathcal{L}$*-Lipschitz compact as a Lipschitz map.*

4.4.1 The $\mathcal{I}$-approximation property for Lipschitz operators.

Consider a linear operator ideal $\mathcal{I}$ and let $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$ be the associated composition Lipschitz operator ideal. On $Lip_0(X, E)$, we define the topology *Lipschitz-*$\tau_{\mathcal{I}}$ of uniform convergence on $\mathcal{I}$-Lipschitz compact sets in the space of operators $Lip_0(X, E)$ as the one generated by the seminorms

$$q_K(T) := \sup_{x \in K} \|T(x)\| = \sup_{m \in \delta_K(K)} \|T_L(m)\|_E,$$

where K is a relatively $\mathcal{I}$-Lipschitz compact set of X.

Note that this topology induces on the space $\mathcal{L}(F, E)$, of linear operators between Banach spaces F and E, the topology $\tau_{\mathcal{I}}$ of uniform convergence on $\mathcal{I}$-compact sets.

Definition 4.26. *Let* X *be a pointed metric space. Consider a class of operators* $\mathcal{O}(X, Æ(X)) \subseteq Lip_0(X, Æ(X))$ *with the operations inherited from this linear space. We say that* X *has the* $\mathcal{I}$*-Lipschitz approximation property with respect to* $\mathcal{O}(X, Æ(X))$ *if* $\delta_X : X \to Æ(X)$ *belongs to the Lipschitz-*$\tau_{\mathcal{I}}$*-closure of* $\mathcal{O}(X, Æ(X))$.

Of course, when looking for a genuine version of the approximation property for metric spaces, the elements of $\mathcal{O}$ must have some sort of finite-range-type property. In fact, the case $\mathcal{O} = Lip_{0\mathcal{F}}$ provides the main classical characterization of an approximation type property, as we will show in what follows. There are also two more interesting cases of sets of operators $\mathcal{O}$ that will be analyzed in the next section.

The first result is a natural extension of Proposition 3.6 of [46] for the Lipschitz case that can be obtained as a consequence of the factorization of the Lipschitz operators through $Æ(X)$.

Proposition 4.27. *Let* X *be a pointed metric space. The following assertions are equivalent.*

(i) X *has the* $\mathcal{I}$*-Lipschitz approximation property with respect to* $Lip_{0\mathcal{F}}(X, Æ(X))$.

(ii) For every Banach space E, $Lip_{0\mathcal{F}}(X,E)$ is Lipschitz-$\tau_{\mathcal{I}}$ dense in $Lip_0(X,E)$.

Proof. (i) $\Rightarrow$ (ii). Consider a Banach space E and $\phi \in Lip_0(X,E)$. Take $\varepsilon > 0$ and an $\mathcal{I}$-Lipschitz compact subset K of X. Let $g_\varepsilon \in Lip_0(X, Æ(X))$ satisfying that $\sup_{x\in K} \|\delta_X(x) - g_\varepsilon(x)\|_{Æ(X)} < \varepsilon$. Since there is a factorization for ϕ given by $\phi = \phi_L \circ \delta_X$, for each $x \in K$ we get that

$$\|\phi(x) - \phi_L \circ g_\varepsilon(x)\| \leq \|\phi_L\| \|\delta_X(x) - g_\varepsilon(x)\|_{Æ(X)} \leq \varepsilon \, Lip(\phi).$$

This gives the proof. (ii) $\Rightarrow$ (i) is obvious.

□

The second main property related to the approximation property concerns the approximation of compact operators by finite rank ones. Let us show the Lipschitz version.

Proposition 4.28. *Let X be a pointed metric space and $\mathcal{I}$ be an operator ideal. If X has the Lipschitz $\mathcal{I}$-approximation property with respect to $Lip_{0\mathcal{F}}(X, Æ(X))$ then, for any pointed metric space Z and any $\mathcal{I}$-Lipschitz compact mapping $\phi : Z \to X$, the mapping $\delta_X \circ \phi$ can be approximated by finite rank operators of $Lip_{0\mathcal{F}}(Z, Æ(X))$ uniformly on B_Z.*

Proof. Assume that ϕ is Lipschitz $\mathcal{I}$-compact. There is a relatively $\mathcal{I}$-Lipschitz compact subset K of X such that $\phi(B_Z) \subseteq K$. Fix $n \in \mathbb{N}$. Then by the approximation property of X there is a finite rank Lipschitz map g_n such that $\sup_{x\in K} \|\delta_X(x) - g_n(x)\|_{Æ(X)} < 1/n$. Consequently,

$$\sup_{z\in B_Z} \|\delta_X \circ \phi(z) - g_n \circ \phi(z)\|_{Æ(X)} \leq \sup_{x\in K} \|\delta_X(x) - g_n(x)\|_{Æ(X)} < \frac{1}{n}.$$

This gives a sequence $(g_n \circ \phi)_n$ of finite rank Lipschitz maps converging to $\delta_X \circ \phi$ uniformly on B_Z, and the result follows.

□

4.4.2 The relation between Lipschitz and linear approximation properties.

The purpose of this section is to show that the new concepts and results we have stated for the Lipschitz setting fit with the definitions and properties of the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. To establish the connection, we consider the $\mathcal{I}$-Lipschitz approximation property with respect to the sets $\mathcal{O}_1(E, Æ(E)) = \delta_E \circ Lip_{0\mathcal{F}}(E, E)$ and $\mathcal{O}_2(X, Æ(X)) = Lip_{0\mathcal{F}}(Æ(X), Æ(X)) \circ \delta_X$. We will show that the choice of $\mathcal{O}_1$ or $\mathcal{O}_2$ depends on which version of the two canonical cases we want to get: either when X is a Banach space and has the approximation property or when $Æ(X)$ has the approximation property.

Our first aim now is to prove that the $\mathcal{I}$-Lipschitz approximation property is weaker than the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property when they can be compared, that is, if a Banach space E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then it has the $\mathcal{I}$-Lipschitz approximation property with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E, E)$ too. This clearly provides a lot of examples of our new approximation property for pointed metric spaces.

Proposition 4.29. *Let $\mathcal{I}$ be an operator ideal. Let E be a Banach space with the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then it has the $\mathcal{I}$-Lipschitz approximation property as a metric space with respect to the set $\delta_E \circ Lip_{0\mathcal{F}}(E, E)$.*

Proof. Suppose that E has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property. Then there is a sequence of finite rank operators $(T_n)_n$ that converges to Id_E in the $\tau_{\mathcal{I}}$ topology. Let us show that the sequence $(\hat{T}_n)_n$ of Lipschitz maps defined as $\hat{T}_n(x) = \delta_E(T_n(x)) \in Æ(E)$, $x \in E$, converges to δ_E in the Lipschitz-$\tau_{\mathcal{I}}$ topology. For a fixed relatively $\mathcal{I}$-Lipschitz compact subset K of E, consider the seminorm

$$q_K(R) := \sup_{x \in K} \|R(x)\|_{Æ(E)}, \quad R \in Lip_0(E, Æ(E)).$$

Recall that, by Proposition 4.21, every relatively $\mathcal{I}$-Lipschitz compact subset of E is relatively $\mathcal{I}$-compact. Thus, K is also a relatively $\mathcal{I}$-compact subset of E, and then this formula defines also a seminorm of the topology $\tau_{\mathcal{I}}$. We get

$$q_K(\delta_E - \hat{T}_n) = \sup_{x \in K} \|m_{x,0} - \delta_E(T_n(x))\|_{Æ(E)}$$

$$= \sup_{x \in K} \|m_{x,0} - m_{T_n(x),0}\|_{Æ(E)} \leq \sup_{x \in K} \|x - T_n(x)\|_E.$$

This proves the result. □

Some results are known about the standard approximation property for the free spaces $Æ(X)$ (see [23, 24, 36] and the references therein). In this direction, we can also show that under the hypothesis that $Æ(X)$ has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, we have that X has the $\mathcal{I}$-Lipschitz approximation property too with respect to a finite-range-type class of operators, showing that our definition can also be applied in these cases.

Proposition 4.30. *Let $\mathcal{I}$ be an operator ideal and consider the associated composition Lipschitz operator ideal $\mathcal{I}_{Lip} = \mathcal{I} \circ Lip_0$. Let X be a pointed metric space. If Æ(X) has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property, then X has the $\mathcal{I}$-Lipschitz approximation property with respect to the class $\mathcal{F}(Æ(X), Æ(X)) \circ \delta_X$.*

Proof. Suppose that Æ(X) has the $\mathcal{K}_{\mathcal{I}}$-uniform approximation property as a Banach space. Fix a relatively $\mathcal{I}$-Lipschitz compact subset K of X. By Proposition 4.21 $\delta_X(K)$ is a relatively $\mathcal{I}$-compact set.

By [45, Proposition 3.2], there is a sequence of linear finite rank operators $T_n : Æ(X) \to Æ(X)$ such that T_n converges to $Id_{Æ(X)}$ uniformly on $\delta_X(K)$. Consider the finite rank Lipschitz maps $\tilde{T_n} := T_n \circ \delta_X$, that define Lipschitz operators from X to Æ(X). It follows that

$$\begin{aligned}\sup_{x \in K} \|\delta_X(x) - \tilde{T}_n(x))\|_{Æ(X)} &= \sup_{x \in K} \|\delta_X(x) - T_n \circ \delta_X(x)\|_{Æ(X)} \\ &= \sup_{w \in \delta_X(K)} \|w - T_n(w)\|_{Æ(X)}\end{aligned}$$

and this finishes the proof. □

Chapter 5

Factorization of Lipschitz operators on Banach function spaces

5.1 Introduction

Let (X, d) be a metric space. Let (Ω, Σ, μ) be a finite measure space and consider two Lipschitz operators $T\colon X \to Y_1$ and $S\colon X \to Y_2$ on Banach function spaces Y_1 and Y_2 over μ. In this chapter we characterize when T factors through S as $T = M_g \circ S$ by means of vector norm inequalities, where $M_g\colon Y_2 \to Y_1$ is a multiplication operator defined by a measurable function g.

In the case of linear operators, this kind of factorization is called *strong factorization* for T through S. A recent study of this problem for the case of linear operators can be found in [31]. We show how these ideas can be adapted to the case of Lipschitz operators; thus, we characterize the factorization of a Lipschitz operator T through a given Lipschitz operator S by using inequalities among these maps.

There are many classical results relating inequalities for linear operators and factorizations. Probably, the ones that have found more applications are the nowadays called Maurey-Rosenthal theorems, that give concavity and convexity conditions on a linear operator to get a strong factorization via L^p spaces (see for instance [64, Proposition III.H.10] and [25, 27, 28]). The direct application of these factorization theorems is the characterization of subspaces and sublattices of Banach function spaces that can be identified isomorphically with an L^p-space for a certain $1 \leq p < \infty$. The last part of the present chapter is devoted to show that these arguments can be translated to the Lipschitz setting by extending the main factorization results of the Maurey-Rosenthal theory to this non-linear case. In particular, as a consequence of Theorem 5.5 and Corollary 5.6 we will provide an answer to the following problem. Let $1 \leq p < q < \infty$ and consider the standard metrics $d_{\|\cdot\|_{L^q(\mu)}}$ and $d_{L^0(\mu)}$ provided by the norm L^q in the first case and the metric of $L^0(\mu)$ in the second one. Let A be a subset of $L^q(\mu)$ such that topologies generated by $d_{\|\cdot\|_{L^q(\mu)}}$ and $d_{L^0(\mu)}$ coincide on A. When can we say that the topology on A coincides also with the one generated by an L^p norm?

5.2 Preliminaries

Let (Ω, Σ, μ) be a finite measure space. Let $L^0(\mu)$ be the space of classes of measurable real functions on Ω that are equal μ-a.e. A Banach function space over μ is a Banach space X_1 of elements of $L^0(\mu)$ with a norm $\|\cdot\|_{X_1}$ satisfying that if $f \in L^0(\mu)$, $g \in X_1$ and $|f| \leq |g|$ μ-a.e. then $f \in X_1$ and $\|f\|_{X_1} \leq \|g\|_{X_1}$. A Banach function space X_1 is a Banach lattice with the pointwise μ-a.e. order; X_1 is order continuous if for every $f, f_n \in X_1$ such that $0 \leq f_n \uparrow f$ μ-a.e., $f_n \to f$ in norm and has the Fatou property if for every sequence $(f_n) \subset X_1$ such that $0 \leq f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_{X_1} < \infty$, it follows that $f \in X_1$ and $\|f_n\|_{X_1} \uparrow \|f\|_{X_1}$. The reader can find more information about in [55, Ch.2], [47, p.28ff] and [66, Ch. 15].

If Y_1 and Y_2 are Banach function spaces over the same measure μ, the space of multiplication operators from Y_1 to Y_2 is

$$Y_1^{Y_2} = \big\{h \in L^0(\mu) : \, hf \in Y_2 \text{ for each } f \in Y_1\big\}.$$

The function $\|h\|Y_1^{Y_2} := \sup_{f \in B_{Y_1}} \|hf\|_{Y_2}$ for all $h \in Y_1^{Y_2}$, is clearly a seminorm on $Y_1^{Y_2}$. It is also a norm only if Y_1 is *saturated*, i.e. there is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ-a.e. for all $f \in Y_1$. In this case, $Y_1^{Y_2}$ is a Banach function space. If $h \in Y_1^{Y_2}$, we write $M_h \colon Y_1 \to Y_2$ for the corresponding multiplication operator $f \rightsquigarrow fh$. See [13, 50] for more information on these spaces.

If Y_1 and Y_2 are Banach function spaces, the product space $Y_1 \pi Y_2$ is the space of functions $f \in L^0(\mu)$ such that $|f| \leq \sum_{i\geq 1} |x_i y_i|$ μ-a.e. for some sequences $(x_i) \subset Y_1$ and $(y_i) \subset Y_2$ with $\sum_{i\geq 1} \|x_i\|_{Y_1} \|y_i\|_{Y_2} < \infty$. The natural norm for this space is

$$\|f\|_{Y_1\pi Y_2} = \inf\Big\{\sum_{i\geq 1} \|x_i\|_{Y_1}\|y_i\|_{Y_2}\Big\}, \quad f \in Y_1 \pi Y_2,$$

where the infimum is computed over all sequences $(x_i) \subset Y_1$ and $(y_i) \subset Y_2$ such that $|f| \leq \sum_{i\geq 1} |x_i y_i|$ μ-a.e. and $\sum_{i\geq 1} \|x_i\|_{Y_1} \|y_i\|_{Y_2} < \infty$. If Y_1, Y_2 and $Y_1^{Y_2'}$ are saturated Banach function spaces then $Y_1 \pi Y_2$ is a saturated B.f.s. with norm $\|\cdot\|_{Y_1\pi Y_2}$ (see for example [30]).

5.2.1 Strong factorizations between couples of Lipschitz operators

Let (X, d) be a metric space and Y_1 and Y_2 two Banach function spaces over a finite measure μ, such that Y_1 and $Y_2 \pi Y_1'$ are order continuous and that Y_1 has the Fatou property. Under these conditions, the product space $Y_2 \pi Y_1'$ is a saturated Banach function space, as a consequence of [30, Proposition 2.2]. For the sake of clarity, let us explain some topological aspects concerning these Banach function spaces. Since $Y_2 \pi Y_1'$ is order continuous, by [47, p. 29] the topological dual $(Y_2 \pi Y_1')^*$ and the Köthe dual $(Y_2 \pi Y_1')'$ coincide. The Fatou property of Y_1 provides the isometry $Y_1 = Y_1''$ ([47, p. 30]), and by [30, Proposition 2.2] we obtain the isometric equalities $(Y_2 \pi Y_1')' = Y_2^{Y_1''} = Y_2^{Y_1}$. Then for each $\xi \in Y_2 \pi Y_1'$ there exists $\xi' \in B_{Y_2^{Y_1}}$ such that $\|\xi\|_{Y_2 \pi Y_1'} = \langle \xi', \xi \rangle = \int \xi \, \xi' \, d\mu$. Conditions under which the product space of two Banach function spaces is order continuous —which is a requirement in the results above— can be found in Section 5 of [31].

Consider two Lipschitz operators $T : X \to Y_1$ and $S : X \to Y_2$. In this subsection we characterizewhen T factorizes strongly through S, that is, when there is a function $g \in Y_2^{Y_1}$ so that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y_1 \\ & \searrow^{S} & \uparrow{\scriptstyle M_g} \\ & & Y_2 \end{array} \quad . \tag{5.1}$$

This factorization is inspired in the following result regarding linear operators between Banach function spaces:

Theorem 5.1. ([31, Theorem 4.1]) *Let X_1, X_2, Y_1, Y_2 be Banach function spaces and let $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$ be continuous linear operators. Assume that Y_1 and $Y_2 \pi Y_1'$ are order continuous and that Y_1 has the Fatou property. The following statements are equivalent.*

(i) There exists a function $h \in X_1^{X_2}$ such that

$$\sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu \leq \Big\| \sum_{i=1}^{n} S(h x_i) y_i' \Big\|_{Y_2 \pi Y_1'}$$

for each $n \in \mathbb{N}$, *every* $x_1, \ldots, x_n \in X_1$ *and* $y_1', \ldots, y_n' \in Y_1'$.

(ii) There exist functions $f \in X_1^{X_2}$ *and* $g \in Y_2^{Y_1}$ *such that* $T(x) = g(S(fx))$ *for all* $x \in X_1$.

It worth mentioning that if we take $Z := X_1 = X_2$ in the above theorem, the Banach lattice structure on Z does not play any relevant role in its proof. The reader can easily check that the theorem remains valid for any normed linear space Z as domain of both linear operators T and S. Now, we are going to see that we can use such a variant of Theorem 5.1 for metric domains in order to prove a strong factorization theorem for Lipschitz operators.

Theorem 5.2. *Let* $T : X \to Y_1(\mu)$ *and* $S : X \to Y_2(\mu)$ *be Lipschitz operators such that* $T(0) = S(0) = 0$. *Suppose that* $Y_1(\mu)$ *and* $Y_2(\mu)\pi Y_1(\mu)'$ *are order continuous and that* Y_1 *has the Fatou property. The following statements are equivalent.*

(i) The inequality

$$\int \sum_{i=1}^{n} \big(T(x_i^1) - T(x_i^2)\big)\, y_i'\, d\mu \leq \Big\| \sum_{i=1}^{n} \big(S(x_i^1) - S(x_i^2)\big) y_i' \Big\|_{Y_2(\mu)\pi Y_1(\mu)'}$$

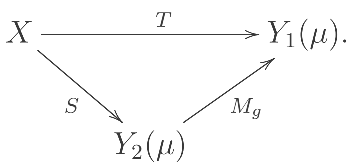
holds for each $n \in \mathbb{N}$ *and each finite set* $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in X$ *and* $y_1', ..., y_n' \in Y_1(\mu)'$.

(ii) There exists a function $g \in Y_2(\mu)^{Y_1(\mu)}$ *such that*

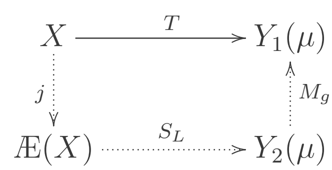
$$T(x^1) - T(x^2) = g(S(x^1) - S(x^2))$$

for all $x^1, x^2 \in X$.

(iii) There is a function $g \in Y_2(\mu)^{Y_1(\mu)}$ *such that* $T(x) = gS(x)$ *for all* $x \in X$, *i.e.* T *factors through* S *as*



(iv) T *factors through* S_L *as*



for a certain function $g \in Y_2(\mu)^{Y_1(\mu)}$.

(v) T_L factors through S_L as

$$\begin{array}{ccc} \text{Æ}(X) & \xrightarrow{\ T_L\ } & Y_1(\mu) \\ & \overset{S_L}{\searrow} & \uparrow M_g \\ & & Y_2(\mu) \end{array}$$

for a certain function $g \in Y_2(\mu)^{Y_1(\mu)}$.

In that case, the g's in (ii), (iii), (iv) *and* (v) *coincide.*

Proof. We will write Y_1 and Y_2 instead of $Y_1(\mu)$ $Y_2(\mu)$ for the aim of clarity in the proof. Consider the linearizations S_L and T_L of S and T respectively. If we assume (i), we can rewrite the inequality as

$$\sum_{i=1}^{n} \int T_L(m_{x_i^1,x_i^2})\, y_i'\, d\mu \leq \Big\| \sum_{i=1}^{n} \big(S_L(m_{x_i^1,x_i^2})\big) y_i' \Big\|_{Y_2 \pi Y_1'}$$

for every finite set $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in X$ and $y_1', ..., y_n' \in Y_1'$. It is clear then that the inequality

$$\sum_{i=1}^{n} \int T_L(m_i)\, y_i'\, d\mu \leq \Big\| \sum_{i=1}^{n} \big(S_L(m_i)\big) y_i' \Big\|_{Y_2 \pi Y_1'}$$

holds for every $m_1, \ldots, m_n \in \mathcal{M}(X)$. By Theorem 5.1, there exists a function $g \in Y_2^{Y_1}$ such that $T_L(m) = gS_L(m)$ for all $m \in \mathcal{M}(X)$. In particular, $T_L(m_{x^1,x^2}) = gS_L(m_{x^1,x^2})$ for all $x^1, x^2 \in X$. Therefore, $T(x^1) - T(x^2) = g(S(x^1) - S(x^2))$ for all $x^1, x^2 \in X$, and (ii) is proved. The implication (ii) $\Rightarrow$ (iii) is obvious. If we assume (iii), the linearization of S fulfills $T(x) = gS_L(m_{x,0})$ for all $x \in X$, which gives (iv). From (iv), $T_L(m) = gS_L(m)$ for all $m \in \mathcal{M}(X)$ and (v) follows. That (v) implies (i) follows easily from Theorem 5.1. □

5.2.2 Maurey-Rosenthal type Theorems on factorization of Lipschitz maps through L^p-spaces

Chávez-Domínguez [18] has found several results concerning factorization of Lipschitz operators through L^p-spaces, including some variants of the Maurey-Rosenthal Theorem. In this subsection we will show a different way of proving similar results on factorization. However, it must be noted that unlike what happens in the results of [18], the measure μ over which $Y(\mu)$ is defined as a Banach function space is the same that appears in the factorization space $L^p(\mu)$. Also the fact that the closing operator is a multiplication map $L^p(\mu) \to Y(\mu)$ provides a different meaning to our result when comparing with the ones of Chávez-Domínguez, that are given for abstract Banach lattices. Let $Y(\mu)$ be a Banach function space. Although the definition given in [18] for Lipschitz p-convexity allows to prove factorization results, for the aim of coherence we prefer to define Lipschitz p-convexity in a slightly different way. This will allow to obtain Maurey-Rosenthal type factorizations in the classical sense. We will say that $T \in \mathrm{Lip}_0(X, Y(\mu))$ is Lipschitz strongly p-convex ($1 \leq p < \infty$), if there exists a constant $K \geq 0$ such that for all $(x_i^j)_{i\leq n, j\leq m}$, $(y_i^j)_{i\leq n, j\leq m}$ in X and $(\lambda_i^j)_{i\leq n, j\leq m}$ in $\mathbb{R}$,

$$\Big\| \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} \lambda_i^j (T(x_i^j) - T(y_i^j)) \Big|^p \Big)^{1/p} \Big\| \leq K \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} \lambda_i^j d(x_i^j, y_i^j) \Big|^p \Big)^{1/p}. \tag{5.2}$$

The definition of Chávez-Domínguez is a bit weaker; we get it if we make $m = 1$. This is the reason we use the term “strongly” in our definition. We can prove a result that is similar to the one given in [18, Theorem 3.3].

Lemma 5.3. *Let $T : X \to Y(\mu))$ be a Lipschitz operator with $T(0) = 0$. If T is Lipschitz strongly p-convex, then $T_L : Æ(X) \to Y(\mu)$ is p-convex and so, well-defined and continuous.*

Proof. The proof is straightforward. Assume that T is Lipschitz strongly p-convex and use (5.2). Consider the molecules $m_i \in \mathcal{M}$, that can be written as $m_i = \sum_{j=1}^{r_i} \lambda_i^j m_{x_i^j, y_i^j}$ for $i = 1, ..., n$. Write $m = \max\{r_i : i = 1, ..., n\}$ and complete the sums to m terms by adding $\lambda_i^j = 0$ for $r_i < j \leq m$ for each i.

$$\begin{aligned}
\Big\| \Big(\sum_{i=1}^{n} |T_L(m_i)|^p \Big)^{1/p} \Big\| &= \Big\| \Big(\sum_{i=1}^{n} \Big| T_L \Big(\sum_{j=1}^{m} \lambda_i^j m_{x_i^j, y_i^j} \Big) \Big|^p \Big)^{1/p} \Big\| \\
&= \Big\| \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} \lambda_i^j (T(x_i^j) - T(y_i^j)) \Big|^p \Big)^{1/p} \Big\| \\
&\leq K \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} \lambda_i^j d(x_i^j, y_i^j) \Big|^p \Big)^{1/p} \\
&\leq K \Big(\sum_{i=1}^{n} \Big(\sum_{j=1}^{m} |\lambda_i^j| d(x_i^j, y_i^j) \Big)^p \Big)^{1/p}.
\end{aligned}$$

Since this computation can be done for each set of representations of the molecules m_i and the left hand side of the inequality does not depend on the representations, we get that

$$\Big\| \Big(\sum_{i=1}^{n} |T_L(m_i)|^p \Big)^{1/p} \Big\| \leq K \Big(\sum_{i=1}^{n} \Big\| m_i \Big\|_{Æ(X)}^p \Big)^{1/p}.$$

Thus, T_L is p-convex. $\square$

Theorem 5.4. *Let $Y(\mu)$ be a Banach function space and let $T : X \to Y(\mu)$ be a Lipschitz operator with $T(0) = 0$. If $Y(\mu)$ is p-concave and T is strongly Lipschitz p-convex ($1 \leq p < \infty$), then there is g and a Lipschitz map v ($v(0) = 0$) such that $f = M_g \circ v$.*

Proof. By Lemma 5.3, T_L : Æ$(X) \to Y(\mu)$ is p-convex. Since $Y(\mu)$ is concave, it is in particular order continuous, and by the Maurey-Rosenthal Theorem [25, Corollary 2], there exists a positive multiplication operator $M_g : L^p(\mu) \to Y(\mu)$ and a continuous operator u : Æ$(X) \to L^p(\mu)$ such that $T_L = M_g \circ u$. Defining $v := u \circ \delta_X$ we complete the proof. □

5.3 Applications: Lipschitz isomorphisms among subsets of measurable functions with different metrics

Factorization of operators among Banach function spaces are useful tools for the study of the geometric properties of subspaces of these spaces. For example, Maurey-Rosenthal type factorizations of operators are the key for proving the structure of reflexive subspaces of L^1-spaces. The Lipschitz version of these results can also be applied for analyzing the corresponding substructures of the metric spaces with some further properties, as we will show in this section. Therefore, we will center our attention in the applications of our results to the analysis of the metric structure of subsets of Banach function spaces.

Let (Ω, Σ, μ) be a finite measure space. Consider a subset of the (metric) space $L^0(\mu)$ of all the μ-a.e equal classes of functions endowed with its natural metric

$$d_0(f, g) = \int_\Omega \frac{|f(w) - g(w)|}{1 + |f(w) - g(w)|} \, d\mu(w).$$

In this section, we characterize when there is a Lipschitz isomorphism between subsets of two different (metric) spaces of (classes of) measurable functions. Concretely, among other results, we solve the following problem: *if $1 \leq p \leq q < \infty$ and A is a subset of a space $L^q(\mu)$ satisfying that the topologies generated by $d_{\|\cdot\|_{L^q(\mu)}}$ and d_0 coincide on it, when can we say that the topology on A coincides also with the one generated by an L^p-norm?*

We will develop our results in the following setting. Suppose that we have a Lipschitz copy —that is, a (bi)Lipschitz bijection from a metric space to a metric subspace of a fixed Banach function space $Y_1(\mu)$, and we want to know if this metric space can be found as a metric subspace of a weighted L^p space defined on the same measure space (Ω, Σ, μ). A direct application of Theorem 5.2 to the identity map gives the following result.

We will say that a set of functions A has measurable support if the union of all the supports of all the functions belonging to it is a measurable set.

Theorem 5.5. *Let $A \subset Y_1(\mu)$ such that $0 \in A$ and has measurable support. Suppose that $Y_1(\mu)$ and $Y_2(\mu)\pi Y_1(\mu)'$ are order continuous and that $Y_1(\mu)$ has the Fatou property. The following statements are equivalent:*

(i) There is a Lipschitz map $S : (A, d_{\|\cdot\|_{Y_1(\mu)}}) \to Y_2(\mu)$ with $S(0) = 0$ such that the inequality

$$\int \sum_{i=1} \left(x_i^1 - x_i^2\right) y_i' \, d\mu \leq \left\| \sum_{i=1}^n \left(S(x_i^1) - S(x_i^2)\right) y_i' \right\|_{Y_2(\mu)\pi Y_1(\mu)'}$$

holds for each $n \in \mathbb{N}$ and every pair of finite set $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in A$ and $y_1', ..., y_n' \in Y_1(\mu)'$.

(ii) There exists a function $g \in Y_2(\mu)^{Y_1(\mu)}$ such that the multiplication map $1/g : (A, d_{\|\cdot\|_{Y_1(\mu)}}) \to Y_2(\mu)$ defines a Lipschitz isomorphism.

Proof. For the aim of clarity, we will write Y_1 and Y_2 instead of $Y_1(\mu)$ and $Y_2(\mu)$ in all the proof.

(i) $\Rightarrow$ (ii). Suppose that there is such a Lipschitz map S. We consider the metric space $X = (A, d_{\|\cdot\|_{Y_1}})$ for applying Theorem 5.2. Using it we obtain that there is a function $g \in Y_2^{Y_1}$ such that $x = gS(x)$ for all $x \in A$. In particular, we can assume that g is non-zero in any subset of positive measure of the measurable support of all the functions in the subspace generated by A, that coincides with union of the supports of all the functions in A. We can assume that $g = 1$ outside this support. Since we are giving a factorization of the identity map, this implies that $S(x) = x/g$ for all $x \in A$. Moreover, since S is Lipschitz, we obtain that there is a constant $K > 0$ such that if $x_1, x_2 \in A$,

$$d_{\|\cdot\|_{Y_2}}(x_1/g, x_2/g) = d_{\|\cdot\|_{Y_2}}(S(x_1), S(x_2)) \leq K d_{\|\cdot\|_{Y_1}}(x_1, x_2),$$

and, for $f_1, f_2 \in \frac{A}{g} \subseteq Y_2$,

$$d_{\|\cdot\|_{Y_1}}(gf_1, gf_2) \leq \|g\|_{Y_2^{Y_1}} \cdot \|f_1 - f_2\|_{Y_2} = \|g\|_{Y_2^{Y_1}} \cdot d(f_1, f_2)_{\|\cdot\|_{Y_2}},$$

that is, if $x_1 = gf_1 \in A$ and $x_2 = gf_2 \in A$, we obtain that

$$d_{\|\cdot\|_{Y_1}}(x_1, x_2) \leq \|g\|_{Y_2^{Y_1}} \cdot d_{\|\cdot\|_{Y_2}}(x_1/g, x_2/g).$$

Consequently, the multiplication $1/g$ defines a bijective map that gives a Lipschitz isomorphism, as we wanted to prove.

(ii) $\Rightarrow$ (i). Note first that by the factorization we can assume w.l.o.g that g cannot be equal to 0 in any set of positive measure. Take $n \in \mathbb{N}$ and finite sets $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in Y_1$ and $y_1', ..., y_n' \in Y_1'$. Then, if we define the operator S as $S(x) := \|g\|_{Y_2^{Y_1}} \cdot x/g$, we obtain

$$\sum_{i=1}^{n} \int (x_i^1 - x_i^2) y_i' \, d\mu = \int g \Big(\sum_{i=1}^{n} (x_i^1/g - x_i^2/g) y_i' \Big) \, d\mu$$

$$\leq \|g\|_{Y_2^{Y_1}} \cdot \Big\| \sum_{i=1}^{n} (x_i^1/g - x_i^2/g) y_i' \Big\|_{Y_2 \pi Y_1'} = \Big\| \sum_{i=1}^{n} (S(x_i^1) - S(x_i^2)) y_i' \Big\|_{Y_2 \pi Y_1'}.$$

Since clearly $S(0) = 0$, these computations show (i) and finish the proof.

□

Theorem 5.5 provides the key tool of the present chapter. As we will show, it can be used to provide complete characterizations of subsets of classical Banach function spaces that are also Lipschitz isomorphic to subspaces of other spaces that can be associated to them by means of a product formula, or as spaces of multiplication operators. For example, it is well-known that for $1 < q < p < \infty$, the space of multiplication operators $L^p(\mu)^{L^q(\mu)}$ can be identified with $L^s(\mu)$ isometrically and in the order, where $1/q = 1/p + 1/s$. Next result characterizes when a subset of $L^q(\mu)$ —considered as a metric space— can be identified metrically with a subset of a space $L^p(hd\mu)$ for a certain weight h. We will consider $Y_1(\mu) = L^q(\mu)$ and $Y_2(\mu) = L^p(\mu)$ in Theorem 5.5 for obtaining the next result.

Corollary 5.6. *Let $1 < q < p < \infty$ and let $1/r = 1/p + 1/q'$. Let $A \subseteq L^q(\mu)$ such that $0 \in A$ and has measurable support. The following statements are equivalent.*

(i) There is a Lipschitz operator $S : (A, d_{\|\cdot\|_{L^q(\mu)}}) \to L^p(\mu)$ such that

$$\int \sum_{i=1}^{n} (x_i^1 - x_i^2) y_i' \, d\mu \leq \Big\| \sum_{i=1}^{n} (S(x_i^1) - S(x_i^2)) y_i' \Big\|_{L^r(\mu)}$$

holds for each $n \in \mathbb{N}$ and each pair of finite sets $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in A$ and $y_1', ..., y_n' \in L^{q'}(\mu)$.

(ii) There is a function $g \in L^{r'}(\mu)$ such that the identity between $(A, d_{\|\cdot\|_{L^q(\mu)}})$ and $(A, d_{\|\cdot\|_{L^p(\mu/g^p)}})$ is a Lipschitz isomorphism.

Proof. (i) $\Rightarrow$ (ii). Note that, by the conditions on p and q, we have that $1/q = 1/p + 1/r'$, and so $L^p(\mu)^{L^q(\mu)} = L^{r'}(\mu)$ and $L^p(\mu)\pi L^{q'}(\mu) = L^r(\mu)$. All these spaces are order continuous and have the Fatou property. Call B to the support of A, that is, the union of all the supports of the functions of A. By Theorem 5.5, we have that for a certain $g \in L^{r'}(\mu)$ –that can be assumed to be positive—, the operator $S(x) = x/g$ for $x \in A$ closes the diagram. The map $y \rightsquigarrow gy$ defines an isometry from $L^p(\mu|_B)$ on $L^p(d\mu|_B/g^p)$. Therefore, we have that the identity is a Lipschitz isomorphism between $(A, d_{\|\cdot\|_{L^q(\mu)}})$ and $(A, d_{\|\cdot\|_{L^p(d\mu/g^p)}})$.

Theorem 5.5 for the spaces $L^q(\mu)$ and $L^p(\mu)$ and the last computations give (ii) $\Rightarrow$ (i). $\square$

Let us finish this section with the general result that solves the question formulated at the beginning of the work. The proof follows the lines of the corollary above, as a direct application of Theorem 5.5.

Corollary 5.7. *Let $1 \leq q \leq p < \infty$ and let $1/r = 1/p + 1/q'$. Let $A \subseteq L^0(\mu)$ such that $0 \in A$ and has measurable support, and the metrics d_0 and $d_{\|\cdot\|_{L^q(\mu)}}$ are equivalent on it. The following statements are equivalent.*

(i) There is a Lipschitz operator $S : (A, d_0) \to L^p(\mu)$ with $S(0) = 0$ such that

$$\int \sum_{i=1}^{n} (x_i^1 - x_i^2) y_i' \, d\mu \leq \Big\| \sum_{i=1}^{n} (S(x_i^1) - S(x_i^2)) y_i' \Big\|_{L^r(\mu)}$$

holds for each $n \in \mathbb{N}$ and functions $x_1^1, ..., x_n^1, x_1^2, ..., x_n^2 \in A$ and $y_1', ..., y_n' \in L^{q'}(\mu)$.

(ii) There exists a function $g \in L^{r'}(\mu)$ such that the identity between (A, d_0) and $(A, d_{\|\cdot\|_{L^p(\mu/g^p)}})$ is a Lipschitz isomorphism.

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