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Compatibility of fuzzy relations

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Thesis submitted in fulfillment of the requirements for the degree of Science Doctor of Mathematics

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Introduction

Equivalence relations and orderings are fundamental concepts of mathematics. Various generalizations of these notions to fuzzy set theory have already been proposed and successfully used. Despite its simplicity, the notion of a fuzzy relation is one of the most important concepts in fuzzy set theory, and was introduced by Zadeh in the early years [85]. From the very beginning [44], it was clear that the lattice-theoretic setting adopted also in the present paper is the most appropriate one for developing the theory of fuzzy sets in general and the calculus of fuzzy relations in particular. On the engineering side, fuzzy relations can be used as condensed representations of Mamdani-type fuzzy rule-based models, which has considerably advanced their understanding [26, 65]. On the mathematical side, fuzzy relations appear as generalizations of various key types of mathematical constructs such as equivalence relations, order relations, and so on. In the symmetric setting, this has contributed to the understanding of existing similarity relations, whether transitive or not [28], and has led to the key concept of a fuzzy equivalence relation, also called indistinguishability relation in earlier years [67, 68]. In the non-symmetric setting, various approaches to the fuzzification of the concept of an order relation have been proposed [5, 14]. All of these notions, symmetric as well as non-symmetric, blend nicely in the fields of preference modelling and decision making, to which fuzzy set theory has largely contributed [39].

Fuzzy equalities, and the more general fuzzy equivalence relations, play a fundamental role, as they allow to shape the universe of discourse or set of alternatives, by stating in a gradual and transitive way how alike elements or alternatives are. Probably the first ones to realize the primal role of this notion and its far-reaching impact are Höhle and Blanchard [47]. They introduced the property of extensionality of a mapping between two universes endowed with fuzzy equalites. Loosely speaking, extensionality expresses here that elements that are similar to related elements are related as well. In the context of (additive) fuzzy preference structures, a similar primal role is played by the indifference relations [27, 37, 79]. Although first neglected for some years, the extensionality property, or compatibility property as it also often called, has become quite established since. It appears, among others, in the study of fuzzy functions by Perfilieva and colleagues [30, 63, 64, 66], in the study of fuzzy order relations by Bodenhofer and colleagues [12, 16, 15] and has been profoundly used in the lattice-theoretic approach to concept lattices by Bělohlávek [7].

Given the importance of the compatibility property, it is remarkable that no individual paper seems to have been dedicated to it. The aim of the present thesis is to strip the compatibility property of its originating context of fuzzy equivalence relations. After recalling the necessary basic definitions in Chapter1, we will generalize the property of compatibility to arbitrary fuzzy relations. In particular, left compatibility, right compatibility and compatibility will be introduced. In order to facilitate several interesting characterizations of these compatibility notions, we will first provide some additional properties of traces of fuzzy relations [40] in terms of the sub- and super-composition of fuzzy relations introduced by Bandler and Kohout [1, 2]. The main findings of the second chapter of this thesis are that the left and right compatibility can be expressed in terms of inclusions, which facilitates their understanding.

One such relatively new notion is that of compatibility of a fuzzy relation R with a fuzzy equivalence relation E, stating that

$$R(x_1, y_1) * E(x_1, x_2) * E(y_1, y_2) \leq R(x_2, y_2)\,,$$

for any x_1, x_2, y_1, y_2 in the universe of discourse X, introduced by Bělohlávek [5]. It expresses that elements that are similar to related elements are related as well. Actually, this notion is equivalent to the older notion of extensionality introduced by Höhle and Blanchard [47]. This compatibility notion is witnessing increased attention. It appears, among others, in the study of fuzzy lattices [5, 34, 58, 81, 88], in the study of fuzzy functions [31, 33, 30, 66], in the definition of strict fuzzy orderings by Bodenhofer and Demirci [16], and so on. Furthermore, it is key to the fuzzy approach to concept lattices of Bělohlávek [7].

In several of the mentioned works, one encounters the implicit standing assumption freely stated as: *"Consider a fuzzy order relation R that is compatible with a given fuzzy equivalence relation E"*. Having in mind the ever lurking danger of voidness results [7, 8, 30], in this thesis we would like to challenge the even more basic assumption *"Consider an order relation $\leqslant$ that is compatible with a given fuzzy equivalence relation E"*. We will show that this is indeed a void assumption, but can be circumvented by considering the strict part of the order relation only. This enquiry turns out to stumble upon some previously unexplored notions in the theory of partially ordered sets.

In this thesis, we recall the compatibility notion of Bělohlávek and show the negative result that the only reflexive fuzzy relation an order relation can be compatible with is the crisp equality. The notion of clone relation is introduced in full detail. We characterize the weak compatibility of a strict order with a fuzzy tolerance (resp. fuzzy equivalence) relation in terms of the clone tolerance relation. Whereas the clone relation is a tolerance relation in general, we show that it is intransitive for comparable clones, while it is transitive for incomparable clones. In order to provide deeper insights, we also study the clone relation of the union and linear sum of posets. Here, we characterize the fuzzy tolerance (resp. equivalence) relations a strict order relation is weakly compatible with, in terms of two fuzzy tolerance (resp. equivalence) relations, one restricted to comparable clones, the other to incomparable clones.

Another notion of compatibility has been introduced by Bodenhofer and colleagues [12, 11, 14, 15]. Indeed, Bodenhofer has defined the compatibility of a fuzzy equivalence relation with a crisp order as follows: the two outer elements of an ascending three element chain are at most as similar as any two elements of this chain. Here, we show that these two concepts of compatibility are not equivalent in general. To avoid ambiguity and overlap of these two notions of compatibility, we say only compatibility to mean the compatibility as coined Bělohlávek [7].

In this work, we mainly focus on the (weak) compatibility of fuzzy relations. This dissertation is organized as follows.

1. In Chapter 1, we provide generalities on ordered sets, residuated lattices and L-relations that we need throughout this thesis.

2. In Chapter 2, we focus on the notion of the compatibility of fuzzy relations. Indeed, after recalling the necessary basic definitions, we will generalize the property of compatibility to arbitrary fuzzy relations, in particular, left compatibility, right compatibility and compatibility will be introduced. In order to facilitate several interesting characterizations of these compatibility notions, we will first provide some additional properties of traces of fuzzy relations [40] in terms of the sub- and super-composition of fuzzy relations introduced by Bandler and Kohout [1, 2]. The main are that the left and right compatibility can be expressed in terms of inclusions, which facilitates their understanding.

   In Chapter 3, we focus on the compatibility of crisp orders (resp. crisp strict orders) with fuzzy tolerance and fuzzy equivalence relations. On one hand, we will show that the crisp equality is the only fuzzy tolerance and fuzzy equivalence relation where a crisp order is compatible with it. On the other hand, we provide characterizations of the weak compatibility of a crisp strict order with fuzzy tolerance and fuzzy equivalence relations, for this reason, we have introduced a new crisp tolerance relation called "clone relation". In this case, the name clone relation is carefully chosen. If $x \approx y$, we say that they are clones of one another, meaning that they hold the same relation ($<, >$ or $\|$) with any other element. The findings results obtained in this chapter, is that an L-tolerance(resp. equivalence) relation on a universe X, can be characterized in terms of the weak compatibility of the strict order with it, as union of two L-tolerance(resp. equivalence) relations defined on two specific parts of the universe X whose elements are clones of one another..

   In Chapter 4, we focus on the right and left (weak) compatibility of crisp orders (resp. crisp strict orders) with fuzzy tolerance and fuzzy equivalence relations. We will characterize these tolerance and equivalences relations in

which a strict order is weakly right compatible with, weakly left compatible with and both these characterizations leads to the same result obtained in Chapter 3.
We finish with a generale conclusion and some open future works. Note that Chapters 2, have been described in [52].

1 Generalities on ordered sets, residuated lattices and L-relations

In this first chapter, we recall some useful notions on Posets, lattices, residuated lattices, t-norms and fuzzy relations.

1.1. Ordered sets and Lattices

1.1.1. Ordered sets

A partial order (order, for short) is a binary relation $\leqslant$ over a set X which is reflexive ($a \leqslant a$ for any $a \in X$), antisymmetric ($a \leqslant b$ and $b \leqslant a$ implies $a = b$ for any $a, b \in X$) and transitive ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$ for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset). A strict order is a binary relation $<$ on a set X that is irreflexive ($a < a$ does not hold for any $a \in X$), asymmetric (if $a < b$, thus $b < a$ does not hold for any $a, b \in X$) and transitive. A given binary relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric ($a \sim b$ implies $b \sim a$ for any $a, b \in X$) and transitive. If $\leqslant$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \;\text{ if }\; a \leqslant b \;\text{ and }\; a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leqslant$ is the reflexive closure given by

$$a \leqslant b \;\text{ if }\; a < b \;\text{ or }\; a = b.$$

Two elements x and y of X are called comparable if $x \leqslant y$ or $y \leqslant x$; otherwise they are called incomparable, and we write $x \parallel y$. Using the strict order $<$, the relation (x is covered by y) denoted as $x \ll y$, if $x < y$ and there exists no $z \in X$ such that $x < z < y$. A poset can be conveniently represented by a Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y. For more details about order, strict order and equivalence relations we refer to [21, 74].

•A binary relation $\lesssim$ [15] on a given non-empty domain X is called a weak order if it has the following two properties for all $x, y, z \in X$:

- if $x \lesssim y$ and $y \lesssim z$ then $x \lesssim z$ (transitivity),
- $x \lesssim y$ or $y \lesssim x$ (completeness).

Note that completeness trivially implies reflexivity, i.e. $x \lesssim x$ holds for all $x \in X$. Obviously the only difference between weak orders and linear orders is that weak orders need not be antisymmetric, i.e. a weak order $\lesssim$ is a linear order if and only if the additional property if $x \lesssim y$ and $y \lesssim x$ then $x = y$ (antisymmetry) holds for all $x, y \in X$.
It is easy to see that the ranking of linearly ordered properties of objects constitutes a weak order, e.g. ranking cars by their maximum speed, ranking persons by their height or weight, ranking products by their price, and so forth. This basic fact is not only a fundamental construction principle, but also a fundamental representation of weak orders.

1.1.2. Sum of ordered sets

Definition 1.1. *(See [21]) Let P and Q be (disjoint) ordered sets. The linear sum $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \leqslant y$ if and only if:*
$(x, y \in P$ and $x \leqslant y$ in $P)$, $(x, y \in Q$ and $x \leqslant y$ in $Q)$ or $(x \in P$ and $y \in Q)$.
A diagram for $P \oplus Q$ (when P and Q are finite) is obtained by placing a diagram of P directly below a diagram of Q and then adding a line segment from each maximal element of P to each minimal element of Q. The horizontal sum of two bounded ordered sets P and Q , is obtained from their disjoint union $P \cup Q$ by identifying the bottoms of the two ordered sets and also identifying the tops. The vertical sum of P and Q is obtained from the linear sum $P \oplus Q$ by identifying the top of P with the bottom of Q.

Definition 1.2. *A poset $(P, \leqslant)$ is called a subposet of a poset $(Q, \leqslant')$ if*

(i) *$P \subseteq Q$,*

(ii) *For any $x, y \in P$: $x \leqslant y$ if and only if $x \leqslant' y$.*

Remark 1.1. *In general, the second condition in Definition 1.2 is just implication. The clone relation introduced in this work, leads to introduce a new definition of subposet of a poset, using the equivalence instead of implication.*

1.1.3. Lattices

We shall be particularly interested in ordered sets $(X, \leq)$ in which $\sup\{x, y\}$, $\inf\{x, y\}$ exist for all $x, y \in X$.

Notation 1.1. *Looking ahead, we shall adopt the following neater notation: we write $x \vee y$ in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ in place of $\inf\{x, y\}$ when it exists. Similarly we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when these exist.*

Definition 1.3. *([21]) Let* $(X, \leq)$ *be an ordered set.*

(i) *If* $x \vee y$, $x \wedge y$ *exist for all* $x, y \in X$, *then* $(X, \leq)$ *is called a lattice.*

(ii) *If* $\bigvee S$, $\bigwedge S$ *exist for all* $S \subseteq X$, *then* $(X, \leq)$ *is called a complete lattice.*

1.2. Residuated lattices

1.2.1. Basic Concept

Residuated lattices play an important role because they provide an algebraic frameworks to fuzzy logic and fuzzy reasoning. we recall in this chapter some important properties of residuated lattices which are related to compatibility of fuzzy relations and fuzzy algebraic lattice that are the aim of this work.

Definition 1.4. *[6] A residuated lattice is an algebra* $(L, \wedge, \vee, *, \rightarrow, 0, 1)$, *or simply,* $(L, *, \rightarrow)$ *where:*

(i) $(L, \wedge, \vee, 0, 1)$ *is a lattice (the corresponding order will be denoted by* $\leq$*) with the least element* 0 *and the greatest element* 1*;*

(ii) $(*, \rightarrow)$ *forms an adjoint couple on* L, *i.e. for any* $a, b, c \in L$*:*

$(R1)$ *If* $a \leq b$ *and* $c \leq d$ *then* $a * c \leq b * d$,

$(R2)$ *If* $b \leq c$ *then* $a \rightarrow b \leq a \rightarrow c$,

$(R3)$ *If* $a \leq b$ *then* $b \rightarrow c \leq a \rightarrow c$,

$(R4)$ $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$ *(adjointness condition).*

(iii) $(L, *, 1)$ *forms a commutative monoid, i.e. for any* $a, b, c \in L$,

$(R5)$ $(a * b) * c = a * (b * c)$;

$(R6)$ $a * b = b * a$;

$(R7)$ $1 * a = a$.

Moreover, if L is a chain, then L is called a residuated chain.

Example 1.1. *Consider* $L = (0, a, b, c, d, m, 1)$ *with* $0 < a < b < m < 1$ *and* $0 < c < d < m < 1$, *but elements* $\{a, c\}$ *and* $\{b, d\}$ *are pairwise incomparable (Fig 1). Then ([50], page 23)* $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ *becomes a residuated lattice relative to the following operations:*

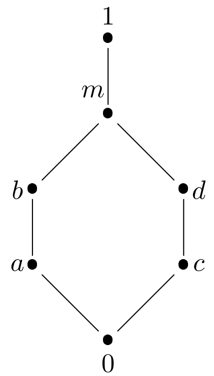

Figure 1.1

$\to$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$*$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

1.2.2. Main properties

For the sake of convenience, we use $\pounds$ to denote the class of all residuated lattices and we always suppose that$L$ is a bounded lattice with the smallest element 0 and greatest element 1, $*$ and $\to$ are two binary operation on L. In addition, we often use the following derived operations:

$$a^0 = 1, \quad a^n = a^{n-1} * a, \quad n \in N, a, b \in L$$

The following Proposition lists the fundamental properties of residuated lattices.

Proposition 1.1. *[62, 69] If $(L, *, \to) \in \pounds$ then*

$(R8)$ $a \leq b \to a * b$;

$(R9)$ $(a \to b) * a \leq b$;

$(R10)$ $f_a : L \to L, x \mapsto x * a$ *preserves all joins existing in* L, *i.e.*

$$(\bigvee_{i \in I} a_i) * a = \bigvee_{i \in I} (a_i * a)$$

(*R11*) $g_a : L \to L, x \mapsto a \to x$ *preserves all meets existing in* L, *i.e.*

$$a \to (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I}(a \to a_i)$$

(*R12*) $h_a : L \to L, x \mapsto a * x$ *preserves all joins existing in* L, *i.e.*

$$a * (\bigvee_{i\in I} a_i) = \bigvee_{i\in I}(a * a_i)$$

(*R13*) $k_a : L \to L, x \mapsto x \to a$ *changes all joins existing in* L *to meets, i.e.*

$$(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I}(a_i \to b)$$

(*R14*) $b \to c \leq (a \to b) \to (a \to c)$ *and* $b \to a \leq (a \to c) \to (b \to c)$

(*R15*) $a = 1 \to a$;

(*R16*) $a \leq b \Leftrightarrow a \to b = 1$;

(*R17*) $a \leq b \to c \Leftrightarrow b \leq a \to c$;

(*R18*) $a \to b \leq a * c \to b * c$ *and* $(a \to b) * (b \to c) \leq (a \to c)$

(*R19*) $a * b \to c = a \to (b \to c)$;

(*R20*) $a \to (b \to c) = b \to (a \to c)$;

(*R21*) $a * b \leq a \wedge b$;

(*R22*) $a^n \leq a^m, n, m \in N, m \leq n$;

Proposition 1.2. *[55] Let* $(L, *, \to)$ *a residuated lattice. The following two properties hold:*

1. $x * (y \wedge z) \leq (x * y) \wedge (x * z)$,
2. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

The following Proposition shows that under suitable conditions, the operations $*$ and $\to$ are not independent.

Proposition 1.3. *[62, 69] Suppose that* L *is a complete lattice.*

(i) *If* $*$ *is a binary operation on* L *satisfying conditions* $(R1)$ *and* $(R10)$, *then there exists a binary operation* $\to$ *satisfying conditions* $(R2), (R3)$ *and* $(R4)$. *Such operation is unique which is determined by the following formula:*

$$a \to b = \bigvee\{x \in L \mid x * a \leq b\},\ a, b \in L$$

(ii) *If* $\rightarrow$ *is a binary operation on* L *satisfying conditions* $(R2), (R3)$ *and* $(R11)$, *then there exists a binary operation* $*$ *satisfying conditions* $(R1)$ *and* $(R4)$, *and such operation is unique which is determined by the following formula:*

$$a * b = \bigwedge \{x \in L \mid a \leq b \rightarrow x\}, \ a, b \in L$$

Clearly, the conditions in Definition 1. are not independent. We now characterize the residuated lattices with the following propositions.

Proposition 1.4. *[62]* $(L, *, \rightarrow) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R4)$ $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$;

(ii) $(R7)$ $1 * a = a$;

(iii) $(R20)$ $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$.

Proposition 1.5. *[62]* $(L, *, \rightarrow) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R8)$ $(a \rightarrow b) * a \leq b$;

(ii) $(R9)$ $a \leq b \rightarrow a * b$;

(iii) $(R7)$ $1 * a = a$.

(iv) $(R20)$ $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$;

(v) $(R23)$ $(a \vee b) * c = (a * c) \vee (b * c)$;

(vi) $(R24)$ $a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c)$.

Proposition 1.6. $(L, *, \rightarrow) \in \pounds$ *if and only if the following conditions hold, for all* $a, b, c \in L$*:*

(i) $(R4)$ $a * b \leq c \Leftrightarrow a \leq b \rightarrow c$

(ii) $(R'7)$ $1 \rightarrow a = a$;

(iii) $(R20)$ $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$.

Proof. It is easy to see that (i),(ii) and (iii) hold in any residuated lattice. Conversely, it suffices to show that (i),(ii) and (iii) imply that $(L, *, 1)$ is a commutative monoid. We have $x * 1 \leq t$ iff $x \leq (1 \rightarrow t)$ iff (by $(R'7)$) $x \leq t$, which implies $x * 1 = x$. Furthermore, $x * y \leq t$ iff $x \leq (y \rightarrow t)$ iff $1 \rightarrow x \leq y \rightarrow t$ iff (by (R2))$x \rightarrow (1 \rightarrow x) \leq x \rightarrow (y \rightarrow t)$ iff (by (R20)) $(1 \rightarrow (x \rightarrow x) \leq y \rightarrow (x \rightarrow t)$ iff $1 \leq y \rightarrow (x \rightarrow t)$ iff (by (R4)) $1 * y \leq x \rightarrow t$ iff $y \leq 1 \rightarrow (x \rightarrow t)$ iff $y \leq x \rightarrow t$ iff $y * x \leq t$, i.e. $x * y = y * x$. Finally, $(x * y) * z \leq t$ iff ...iff $1 \rightarrow x \leq y \rightarrow (z \rightarrow t)$ iff $1 \leq x \rightarrow (y \rightarrow (z \rightarrow t))$ iff (by (R20)) $1 \leq z \rightarrow (y \rightarrow (x \rightarrow t))$ iff...iff $x * (y * z) \leq t$, i.e. $(x * y) * z = x * (y * z)$. Therefore $(L, *, 1)$ is a commutative monoid. □

Remark 1.2.

([62] Definition 3.1) If $(L, *, \rightarrow) \in \pounds$, *and satisfies the following condition:* $(R25)$ $a \rightarrow b \vee c = (a \rightarrow b) \vee (a \rightarrow c)$, *for any* $a, b, c \in L$, *then* $(L, *, \rightarrow)$ *is called a normal residuated lattice. Moreover, if* L *is a chain, then* L *is called a normal residuated chain.*

• A residuated lattice satisfies the prelinearity axiom [6] if and only if $(x \rightarrow y) \vee (y \rightarrow x) = 1$ holds. A residuated lattice is *divisible* [6] if and only if $x \wedge y = x * (x \rightarrow y)$. It can be shown [6] that divisibility is equivalent to the following condition: for each $x \leq y$ there is z such that $x = y*z$. A residuated lattice satisfies the law of double negation(and is called integral, commutative Girard-monoid [6]) if and only if $x = (x \rightarrow 0) \rightarrow 0$ holds. A Heyting algebras is a residuated lattice where $x * y = x \wedge y$. A BL-algebras [6] is a residuated lattice which is divisible and satisfies the prelinearity axiom. An MV-algebras [6, 19] is a residuated lattice in which $x \vee y = (x \rightarrow y) \rightarrow y$ holds. Equivalently [49], an MV-algebras is a residuated lattice which is divisible and satisfies the law of double negation. Thus, each BL-algebras satisfying the law of double negation is an MV-algebras(which is the way MV-algebras are defined in [46]). A Π-algebras(product algebras)[46] is a BL-algebras satisfying $(z \rightarrow 0) \rightarrow 0 \leq ((x * z) \rightarrow (y * z)) \rightarrow (x \rightarrow y)$ and $x \wedge (x \rightarrow 0) = 0$. A G-algebras (Gödel algebras) is a BL-algebras which satisfies $x * x = x$ (i.e. a Heyting algebras satisfying the prelinearity axiom). A Boolean algebras is a residuated lattice which is both a Heyting algebras and an MV-algebras(relation to the usual axiomatization is $x \rightarrow y = x' \vee y$).
More information about residuated and complete residuated lattices can be found in [5, 6, 9, 18, 21, 46, 74, 43, 69].

1.3. T-norms

The history of triangular-norms(*t-norms*) started with Menger [57]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar [72] provided the axioms of *t-norms*, as they are used today.

Definition 1.5. *A function* $T : [0,1]^2 \rightarrow [0,1]$ *is called a t-norm if for all* $x, y, z \in [0,1]$ *the following four axioms are satisfied:*

(T1) $T(x, y) = T(y, x)$ *Commutativity;*

(T2) $T(x, T(y, z)) = T(T(x, y), z))$ *Associativity;*

(T3) $T(x, y) \leq T(x, z))$ *if* $y \leq z$ *Monotonicity;*

(T4) $T(x, 1) = T(1, x) = x$,$T(x, 0) = T(0, x) = 0$ *Boundary condition.*

We say that a t-norm T has zero divisors if there exist $x, y > 0$ such that $T(x, y) = 0$. A t-norm T is Archimedean if $T(x, x) < x$. for every $x \in [0, 1]$.

There exist uncountable many t-norms. The following are four t-norms which are basic in some sense:

(T5) Minimum: $T_M(x, y) = \min\{x, y\}$

(T6) Product: $T_P(x, y) = x.y$

(T7) Lukasiewicz: $T_L(x, y) = \max\{x + y - 1, 0\}$

(T8) Drastic product:

$$T_D(x,y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{if } x, y < 1 \end{cases} \tag{1.1}$$

All t-norms coincide on the boundary of the unit square $[0, 1]^2$, see (T4). If for two t-norms T_1 and T_2 the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in [0, 1]^2$, then T_1 is called weaker than T_2 or, equivalently, T_2 is called stronger than T_1, and we write $T_1 \leq T_2$. T_D is the weakest, T_M is the strongest t-norm, i.e. for any t-norm it holds: (T9) $\mathrm{T_D \leq T \leq T_M}$. Since $T_L \leq T_P$ we have obviously: (T10) $\mathrm{T_D \leq T_L \leq T_P \leq T_M}$.

Definition 1.6. *[14] (The concept of dominance between t-norms)*

A t-norm T_1 is said to dominate another t-norm T_2 if and only if, for any quadruple $(x, y, u, v) in [0, 1]^4$, the following holds:

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T1(x, u), T_1(y, v))$$

Lemma 1.1. *(De Baets and Mesiar [25], Klement et al. [53])*

1. *Any t-norm T dominates itself.*
2. *The minimum t-norm T_M dominates any other t-norm.*
3. *If a t-norm T_1 dominates another t-norm T_2, then T_1 is stronger than T_2.*

Lemma 1.1 particularly implies that dominance is a reflexive and antisymmetric relation on the set of t-norms. Note that it still remains an open problem whether it is transitive.

Triangular conorms (t-conorms) are dual operations of t-norms, we recall the following definition of conorms.

Definition 1.7. *[60] A t-conorm is a function $S : [0, 1]^2 \to [0, 1]$ that for all $x, y, z \in [0, 1]$ satisfies* (T1)-(T3) and the following boundary condition (S4) $\mathrm{S(x, 0)} =$

$\mathrm{S(0,x) = x, S(x,1) = S(1,x) = 0}$

Remark 1.3. *Given a t-norm T, we find the associated dual t-conorm S by $S(x,y) = 1 - T(1-x, 1-y)$.*

The dual[60] t-conorms w.r.t. T_M, T_P, T_L and T_D are given by:

(S1) Maximum: $S_M(x,y) = \max\{x,y\}$

(S2) Probabilistic sum: $S_P(x,y) = x + y - s.y$

(T7) Lukasiewicz: $S_L(x,y) = \min\{x+y, 1\}$

(T8) Drastic sum:

$$S_D(x,y) = \begin{cases} 1 & \text{if } \ (x,y) \in [0,1)^2 \\ \max\{x,y\} & \text{otherwise} \end{cases} \tag{1.2}$$

Now, (T9) and (T10) change into (S9) $\mathrm{S_M \leq S \leq S_D}$ and (S10) $\mathrm{S_M \leq S_P \leq S_L \leq S_D}$. There are many parametric classes of t-norms and associated dual t-conorms. Let us mention one example:
Yager-familly:$\lambda \in [0, \infty]$ (See[60]).

$$T_\lambda^Y(x,y) = \begin{cases} T_D & \text{if } \ \lambda = 0 \\ T_M & \text{if } \lambda = \infty \\ \max\{1 - ((1-x)^\lambda + (1-y)^\lambda)^{1/\lambda}, 0\} & \text{otherwise} \end{cases} \tag{1.3}$$

The Yager-familly includes the strongest and weakest t-norms and t-conorms. The family is increasing/decreasing in the sense that: $\lambda_1 < \lambda_2 \Rightarrow T_{\lambda_1}^Y < T_{\lambda_2}^Y, S_{\lambda_2}^Y < S_{\lambda_1}^Y$.

1.4. L-subsets and L-relations

It is well known that fuzzy relations are of fundamental importance in almost all sub-fields of fuzzy logic and fuzzy set theory, play an important role in fuzzy modeling and fuzzy control, fuzzy inference, they also have important applications in relational databases, approximate reasoning, medical diagnosis. In many cases, fuzzy relations can handle real life problems better than the crisp ones. some examples needed in this thesis are fuzzy equivalence, fuzzy equality, fuzzy ordering used to investigate in the first part the compatibility of fuzzy relations, and in the second part for construction of fuzzy algebraic lattice.

In the further text, L will be a (complete) residuated lattice.

1.4.1. L-subsets

A fuzzy subset of a set X over L, or an L-subset of X simply, is any mapping from X into L. Ordinary crisp subset of X are considered as fuzzy subset of X (or an L-subset) of X, taking membership values in the set $\{0,1\} \subseteq L$. For later, L^X denotes the set of all L-subsets of X, i.e. the set of all mappings from X to L.
Let f and g be two L-subsets of X.
The equality of f and g is defined as the usual equality of mappings, i.e. $f = g$ if and only if $f(x) = g(x)$, for every $xinX$
The inclusion $f \leq g$ is also defined pointwise: $f \leq g$ if and only if $f(x) \leq g(x)$, for every $xinA$.
Endowed with this partial order the set $\pounds(X)$ of all L-subsets of X forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of L-subsets of X, are mappings from X into L defined by

$$(\bigwedge_{i\in I} f_i)(x) = \bigwedge_{i\in I} f_i(x), \ (\bigvee_{i\in I} f_i)(x) = \bigvee_{i\in I} f_i(x)$$

The product $f \otimes g$ is an L-subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $xinX$.
The crisp part of an L-subset f of X is a crisp subset $\hat{f} = \{x \in X \mid f(x) = 1\}$ of X. We will also consider $\hat{f}$ as a mapping $\hat{f} : X \to L$ defined by $\hat{f}(x) = 1$, if $f(x) = 1$, and $\hat{f}(x) = 0$, if $f(x) < 1$. For more properties of fuzzy subsets and L-subsets, see .....

1.4.2. L-relations

Basic definitions

A binary L-fuzzy relation (an L-relation, for short) on X is a mapping $R \in L^{X\times X}$, that is to say, any L-subsets of $XtimesX$. For every $x, y \in X$, the value $R(x, y)$ is called the degree of membership of (x, y) in R, and the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets. The transpose R^t of R is the L-relation on X defined by $R^t(y, x) = R(x, y)$. For crisp relations, we use the usual infix notation, e.g. we write $a \leq b$ instead of $\leq(a, b)$.
For a t-norm T and L-relations R, S on X, the T-composition of R and S denoted by $R \circ S$, is a fuzzy relation on X defined by

$$(R \circ S)(x, y) = \bigvee_{z\in X} T(R(x, z), S(z, y))$$

for all $x, y \in X$. For a fuzzy subset f of X and a fuzzy relation R on X, the T-compositions $f \circ R$ and $R \circ f$ are fuzzy subsets of X defined by

$$(f \circ R)(x) = \bigvee_{z \in X} T(f(z), R(z,x)), \ (R \circ f)(x) = \bigvee_{z \in X} T(R(x,z), f(x))$$

for any $xinX$. Finally, for fuzzy subsets f and g of X we write

$$f \circ g = \bigvee_{x \in X} T(f(x), g(x))$$

The value $f \circ g$ can be interpreted as the bdegree of overlapping" of f and g. We know that the T-composition of fuzzy relations is associative, and we can also easily verify that

$$(f \circ R) \circ S = f \circ (R \circ S), \ (f \circ R) \circ g = f \circ (R \circ g) \qquad (1)$$

for arbitrary fuzzy subsets f and g of X, and fuzzy relations R and S on X, and hence, the parentheses in (1) can be omitted. Note also that if X is a finite set with n elements, then R and S can be treated as $n \times n$ fuzzy matrices over L and $R \circ S$ is the matrix product, whereas $f \circ R$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix R, and $R \circ f$ as the product of an $n \times n$ matrix R and an $n \times 1$ matrix f^t (the transpose of f).

Consequently, an L-relation R is $*$-transitive if and only if $R \circ R \subseteq R$.
Moreover, for any L-relation R on a universe X, we will use the notation $R^{(i)} = R^{(i-1)} \circ R = R \circ R^{(i-1)}$, $i \geq 2$ where $R^{(2)} = R \circ R$. For a finite family $(R_i)_{1 \leq i \leq n}$ of L-relations on X, we will write $\circ_{i=1}^{n} R_i = R_1 \circ R_2 \circ ... \circ R_n$

Proposition 1.7. *[23] For any reflexive and $*$-transitive L-relation R on a universe X, it holds that $R^{(2)} = R$ and $R^{(i)} = R$, $i \geq 2$.*

Main properties

Let $R \in L^{X \times X}$ be a binary L-relation on X. We are interested in the following properties (see, for example [7, 14, 15, 17, 22, 34, 35, 48, 85]):

- Reflexivity: $R(x,x) = 1$ for any $x \in X$,
- Irreflexivity: $R(x,x) = 0$ for any a $x \in X$,
- Symmetry: $R(x,y) = R(y,x)$ for any $x, y \in X$,
- T-Asymmetric: $T(R(x,y), R(y,x)) = 0$ for any $x, y \in X$,
- T-Antisymmetric: $x \neq y$ implies $T(R(x,y), R(y,x)) = 0$,

- $*$-Transitivity: $R(x,y) * R(y,z) \leq R(x,z)$ for any $x,y,z \in X$,
- Separability: $R(x,y) = 1$ implies that $x = y$ for any $x,y \in X$.

Note that R is called Strongly complete, if $\max(R(x,y), R(y,x)) = 1$ for any $x,y \in$.

A binary L-relations that are reflexive and symmetric are called L-fuzzy tolerances, short L-tolerances. $*$-transitive L-tolerances are called L-equivalences. A separable L-equivalences are called L-equalities. Note that 2-equality on X is precisely the usual equality (identity) id_X, i.e. $id_X(x,y) = 1$ for $x = y$ and $id_X(x,y) = 0$ for $x \neq y$. The notion of L-equality is a natural generalization of the classical (bivalent) notion. First, it is near at hand to define fuzzy concept of ordering just by taking appropriate fuzzifications of the classical three ordering axiomes [42, 85]

Definition 1.8. *A mapping $R : X^2 \to [0,1]$ is called a fuzzy ordering on the non-empty crisp domain X with respect to a t-norm $*$, for brevity $*$-ordering, if and only if the following three axioms are satisfied:*

1. $\forall x \in X : \quad R(x,x) = 1 \quad (reflexivity)$
2. $\forall x,y \in X : \quad x \neq y \Rightarrow R(x,y) * R(y,x) = 0 \quad (* - antisymmetry)$
3. $\forall x,y,z \in X : \; R(x,y) * R(y,z) \leq R(x,z) \quad (* - transitivity)$

One would naturally expect fuzzy orderings to be able to fuzzify crisp linear orderings. Now consider an arbitrary linearly ordered set X. A natural requirement on a sound fuzzification of the original linear ordering would be the following monotonicity:

$$\forall x \in X : y \preceq z \Rightarrow R(x,y) \leq R(x,z) \qquad (2).$$

Using reflexivity and the above monotonicity, we obtain that, for an arbitrary $x \in X$,

$$\forall y \preceq x : R(y,x) = 1.$$

Linearity of $\preceq$ and $*$-antisymmetry, on the other hand, entail

$$\forall y \npreceq x : R(y,x) = 0,$$

and we have proved that the crisp ordering $\preceq$ itself is the only fuzzy ordering such that the property (2) is fulfilled.

Moreover, thinking of fuzzy orderings as mathematical models of concepts like "approximately smaller or equal" or approximately greater or equal" one immediately observes that there is an inherent component of indistinguishability.

This may entail the demand for bringing fuzzy orderings and indistinguishability together. In the following, a generalization which also takes indistinguishability into account. For proof details we refer to [11].

1.4.3. Linking orderings and indistinguishability relations

Definition 1.9. *[12] A mapping $E : X^2 \to [0,1]$ is called a fuzzy equivalence relation (indistinguishability or similarity relation) on X with respect to a t-norm $*$, short $*$-equivalence, if and only if has the following properties :*

- $\forall x \in X : \quad E(x,x) = 1 \quad (reflexivity)$
- $\forall x,y \in X : \quad E(x,y) = E(y,x) \quad (symmetry)$
- $\forall x,y,z \in X : \; E(x,y) * E(y,z) \leq E(x,z) \quad (* - transitivity)$

Definition 1.10. *[12] A function $R : X^2 \to [0,1]$ is called a fuzzy ordering on X with respect to a t-norm $*$, and a $*$-equivalence E, for brevity $*$-E-ordering if and only if it fulfills the following three axioms :*

- $\forall x,y \in X : \quad E(x,y) \leq R(x,y) \quad (E - reflexivity)$
- $\forall x,y \in X : \quad R(x,y) * R(y,x) \leq E(x,y) \quad (* - E - antisymmetry)$
- $\forall x,y,z \in X : \; R(x,y) * R(y,z) \leq R(x,z) \quad (* - transitivity)$

R is called strongly linear if, for every pair (x,y), either $R(x,y) = 1$ or $R(y,x) = 1$ holds.

Proposition 1.8. *Some basic properties [12]*

1. *Every $*$-equivalence is a $*$-E-ordering.*
2. *Every crisp ordering is a fuzzy ordering with respect to any t-norm and the crisp equality.*
3. *If R is a $*$-E-ordering, then its so-called inverse relation $G(x,y) = R(y,x)$ is a $*$-E-ordering as well.*
4. *A $*_1$-E-ordering is a $*_2$-E-ordering if $*_1$ is weaker than $*_2$.*
5. *Every $*$-ordering is a fuzzy ordering in the sense of Definition7 with respect to $*$ and the crisp equality.*

Remark 1.4. *Take E is a crisp equality in Definition7, we obtain Definition5.*

The concept of L-equivalence relation above mentioned has been introduced, named and studied in several different ways, it will be useful to give some other concepts related to compatibility that is the aim of this work.

Let $(L, *, \to)$ be a (complete) residuated lattice and E, F be L-equivalence relations on X and Y respectively. A fuzzy relation R on $L^{X\times Y}$ is called a perfect fuzzy function from X to Y w.r.t. E and F if and only if R satisfies the following four conditions:

(i) $R(x,y)*E(x,x') \leq R(x',y)$, for any $x,x' \in X$ and any $y \in Y$ (Extensionality w.r.t. E),

(ii) $R(x,y)*E(y,y') \leq R(x,y')$, for any $x \in X$ and any $y,y' \in Y$ (Extensionality w.r.t. F),

(iii) For each $x \in X$,$\exists y \in Y$ such that $R(x,y) = 1$,

(iv) $R(x,y) * R(x,y') \leq F(y,y')$, for any $x \in X$ and any $y,y' \in Y$.

A fuzzy relation R on $L^{(X\times X)\times X}$ satisfying the condition (iii) is said to be L-binary operation on X ([34]).

Theorem 1.1. *[12] Consider a reflexive and $*$-transitive binary fuzzy relation $R : X^2 \to [0,1]$(often called fuzzy preordering). The relation R is a $*$-E-ordering for some $*$-equivalence E if and only if, for all $x,y \in X$,*

$$R(x,y) * R(y,x) \leq E(x,y) \leq \min(R(x,y), R(y,x))$$

Moreover, the two bounds are $*$-equivalences themselves.
The next theorem shows how to define Cartesian products of fuzzy orderings.

Theorem 1.2. *[12] Let $X_1, ..., X_n$ be crisp sets and let $*$ be an arbitrary t-norm. If $(R_1, ..., R_n)$ and $(E_1, ..., E_n)$ are families of fuzzy relations such that, for all$i = 1, ..., n$,*

1. *R_i and E_i are binary fuzzy relations on X_i,*

2. *E_i is a $*$-equivalence on X_i,*

3. *R_i is a $*$-E_i-ordering on X_i.*

Then the fuzzy relation

$$\tilde{R} : (X_1 \times .... \times X_n)^2 \to [0,1]$$

,

$$((x_1, ..., x_n), (y_1, ..., y_n)) \mapsto *_{i=1,n} R_i(x_i, y_i)$$

is a fuzzy ordering with respect to $*$ and the fuzzy equivalence relation

$$\tilde{E}((x_1, ..., x_n), (y_1, ..., y_n)) = *_{i=1,n} E_i(x_i, y_i).$$

Definition 1.11. *(Compatibility in sense Bodenhofer [12])Let $\preceq$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called compatible with $\preceq$, if and only if the following implication holds for all $x,y,z \in X$:*

$$x \preceq y \preceq z \Rightarrow E(x,z) \leq \min(E(x,y), E(y,z))$$

Compatibility of a crisp order and a fuzzy equivalence relation can be understood *as follows: the two outer elements of an ascending three-element chain are at most as similar as any two elements of this chain.*

In analogy to the crisp case [15], given a non-empty domain X, a binary fuzzy relation $R : X^2 :\to [0,1]$ is a fuzzy weak order (or a weak $*$-order for short)if it has the following two properties for all $x, y, z \in X$, where $*$ denotes a left-continuous t-norm:

1. $R(x,y) * R(y,z) \leq R(x,z)$ ($*$ -transitivity),
2. $R(x,y) = 1$ *or* $R(y,x) = 1$ (strong completeness).

Note that (as a crisp case) a fuzzy weak orders are necessarily reflexive, i.e. $R(x,x) = 1$ holds for all $x \in X$.

One can state, the following theorem shows that a crisp weak order can always be decomposed into a crisp linear order and a crisp equivalence relation.

Theorem 1.3. *(Decompositions into crisp linear orders and $*$-equivalences [15]) Consider a binary fuzzy relation $R : X^2 \to [0,1]$. Then the following two statements are equivalent:*

(i) *R is a weak $*$-order.*

(ii) *There exists a crisp linear order $\preceq$ and a $*$-equivalence E that is compatible with $\preceq$ such that R can be represented as follows:*

$$R(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ E(x,y) & \text{otherwise.} \end{cases} \tag{1.4}$$

1.4.4. Strict Fuzzy Orderings

In the crisp case, strict orderings are defined as irreflexive and transitive relations. It is more than obvious how to translate this definition to fuzzy setting [39, 61]. In order to take the underlying fuzzy equivalence relation into account, we add extensionality.

Definition 1.12. *([16]) A binary fuzzy relation R on a universe X is called strict fuzzy ordering with respect to a t-norm T and a T-equivalence E, for brevity strict T-E-ordering, if it fulfills the following axioms for all $x, x', y, y', z \in X$:*

(i) *Irreflexivity: $R(x,x) = 0$,*

(ii) *T-transitivity: $T(R(x,y), R(y,z)) \leq R(x,z)$,*

(iii) *E-extensionality: $T(E(x,x'), E(y,y'), R(x,y)) \leq R(x',y')$.*

Note that, under the assumption of T-transitivity, irreflexivity implies T-asymmetry, i.e. that $T(R(x,y),R(y,x)) = 0$ for all $x,y \in X$, where the converse holds only if T does not have zero divisors. In other words, irreflexivity can be replaced equivalently by T-asymmetry if T does not have zero divisors. Furthermore, we can conclude that $T(E(x,y),R(x,y)) = 0$ holds for all $x,y \in X$ and any strict T-E-ordering R.

Example 1.2. *It is a well-known fact that* $E(x,y) = \max(1-|x-y|,0)$ *is a* T_L*-equivalence on* $\mathbb{R}$ *([24, 76]), with* $T_L(x,y) = \max(x+y-1,0)$ *being the Lukasiewicz t-norm. It is easy to show that* $L(x,y) = \max(\min(1-x+y,1),0)$ *is a strongly complete* T_L*-*E*-ordering ([13, 14]) and that* $R(x,y) = \max(\min(y-x,1),0)$ *is a strict* T_L*-*E*-ordering.*

E-extensionality as defined above is nothing else but a straightforward translation of the trivial crisp assertion ($x = x'$ and $y = y'$ and $x < y$) implies that $x' < y'$. In case that E is the classical crisp equality, E-extensionality is trivially fulfilled and we end up in the more traditional concept of a strict fuzzy ordering ([39, 61]).

1.4.5. fuzzy implications

Fuzzy implications have been widely studied in the fuzzy literature and they play important roles in different domains as approximate reasoning, fuzzy control, decision theory, control theory, expert systems, fuzzy mathematical morphology, image processing, etc. In Zadeh composition rule of fuzzy inference (fuzzy modus ponens, fuzzy modus tollens, etc.), a fuzzy implication represents the fuzzy relation between two variables [51]. In data mining, a fuzzy implication is used to express the relationship between two items in an association rule [78]. Also the set of axioms of fuzzy implications has been well studied [3, 73]. Originally, a fuzzy implication is a fuzzy operator which is used to represent IF-THEN rules in fuzzy logic. The truth value of the proposition "if p, then q" could be obtained through $I(v(p),v(q))$, in which I refers to a fuzzy implication and $v(p)$ and $v(q)$ refer to the truth value of p and q, respectively.
Fuzzy implications on the unit interval $[0,1]$ are defined as follows.

Definition 1.13. *(See* [4, 39] *A function* $I : [0,1]^2 \to [0,1]$ *is called a fuzzy implication if it satisfies, for any* $x, x_1, x_2, y, y_1, y_2 \in [0,1]$*, the following conditions:*

(i) *If* $x_1 \le x_2$*, then* $I(x_1,y) \ge I(x_2,y)$*, i.e.* $I(.,y)$ *is decreasing,*

(ii) *If* $y_1 \le y_2$*, then* $I(x,y_1) \ge I(x,y_2)$*, i.e.* $I(x,.)$ *is increasing,*

(ii) $I(0,0) = 1$*,* $I(1,1) = 1$*,* $I(1,0) = 0$*.*

From Definition 1.13, it is clear that a fuzzy implication, when restricted to $\{0,1\}$, coincides with the classical implication.

As examples, the following some individual definitions of fuzzy implications.

Name	Formula
Łukasiewicz [56]	$I_{LK}(x,y) = \min(1, 1-x+y)$
Gödel [45]	$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$
Reichenbach [70]	$I_{Re}(x;y) = 1 - x + xy$
Kleene-Dienes [36]	$I_{KD}(x,y) = \max(1-x, y)$
Goguen [43]	$I_{\triangle}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y/x, & \text{if } x > y \end{cases}$
Rescher [71]	$I_{RS}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y \end{cases}$
Yager [83]	$I_{YG}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$
Weber [77]	$I_{WB}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Fodor [41]	$I_{FD}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1-x, y), & \text{if } x > y \end{cases}$
Least FI [4]	$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$
Greatest FI [4]	$I_1(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$
Most Strict [54]	$I_D(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$
Early Zadeh [86]	$I_m(x,y) = \max(1-x, \min(x,y))$

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for any $x \in [0,1]$, whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition. Special interesting properties for fuzzy implication functions are:

(P_1) The exchange principle,

$$I(x, I(y,z)) = I(y, I(x,z), \text{ for any } x, y, z \in [0,1]\,.$$

(P_2) The law of importation with a t-norm T,

$$I(T(x,y), z) = I(x, I(y,z), \text{ for any } x, y, z \in [0,1]\,.$$

(P_3) The weak law of importation with a conjunctive, commutative and non-decreasing function $F : [0,1]^2 \to [0,1]$, (see [59])

$$I(F(x,y), z) = I(x, I(y,z), \text{ for any } x, y, z \in [0,1]\,.$$

(P_4) The left neutrality principle,

$$I(1, y) = y, \text{ for any } y?[0, 1].$$

(P_5) The ordering property,

$$x \leq y \Leftrightarrow I(x, y) = 1, \text{ for any } x, y \in [0, 1].$$

(P_6) The identity principle,

$$I(x, x) = 1, \text{ for any } x?[0, 1].$$

Remark 1.5. *(See [4]) If I is a fuzzy implication. Then the function N_I defined by $N_I(x) = I(x, 0)$, for any $x \in [0, 1]$ is called the natural negation of I.*

.

2 Characterization of compatibility in terms of traces

The notion of extensionality, introduced by Höhle and Blanchard, and the notion of compatibility, as coined by Bělohlávek, of a fuzzy relation w.r.t. a fuzzy equality are trivially equivalent. Here, this compatibility property is dissected into left and right compatibility, mimicking the original two-fold definition of extensionality, and studied in detail in the context of arbitrary fuzzy relations. Relying on the notions of left and right traces of a fuzzy relation, it is shown that compatibility can be characterized in terms of inclusions, shedding another light on the matter.

pose R^t of R is the L-relation on X defined by $R^t(y,x) = R(x,y)$. For crisp relations, we use

Let L be a residuated lattice, an L-relation E on a universe X is called an L-equivalence if it is reflexive, symmetric and $*$-transitive; a separable L-equivalence is called an L-equality. Note that the only $\{0,1\}$-equality on X is precisely the usual equality (identity) Id_X, i.e. $\mathrm{Id}_X(x,y) = 1$ if $x = y$ and $\mathrm{Id}_X(x,y) = 0$ if $x \neq y$.

Next, we recall the definition of a fuzzy order with respect to a t-norm $*$ and an L-equivalence Q.

Definition 2.1. [14] *An L-relation R on X is called an L-order with respect to a t-norm $*$ and an L-equivalence Q (a $*$-Q-order, for short) if it satisfies:*

(i) Q-reflexivity: $Q(x,y) \leq R(x,y)$, for any $x, y \in X$,

(ii) $*$-Q-antisymmetry: $R(x,y) * R(y,x) \leq Q(x,y)$, for any $x, y \in X$,

(iii) $*$-transitivity.

Example 2.1.

(i) *Every L-equivalence E is a $*$-E-order.*

(ii) *Every crisp order is an L-order with respect to any t-norm and the crisp equality.*

We recall the definition of the sup-$*$-composition of L-relations, [23].

Definition 2.2. *Let R and S be two L-relations on a universe X. The* sup-$*$-*composition of R and S is the L-relation $R \circ S$ on X defined by*

$$R \circ S(x,z) = \sup_{y \in X} R(x,y) * S(y,z)\,.$$

Consequently, an L-relation R on X is $*$-transitive if and only if $R \circ R \subseteq R$. Moreover, due to the adjointness property, $*$ is supremum-preserving and the sup-$*$-composition is associative [23]. We will further use the notation

$$R^{(i)} = R^{(i-1)} \circ R = R \circ R^{(i-1)} ,$$

for any $i \in \mathbb{N}$, $i \geq 2$, where $R^{(1)} = R$. For a finite family $(R_i)_{i=1}^n$ of L-relations on X, we will write

$$\circ_{i=1}^n R_i = R_1 \circ R_2 \circ \ldots \circ R_n .$$

Proposition 2.1. [23] *For any reflexive and $*$-transitive L-relation R on a universe X, it holds that $R^{(i)} = R$ for any $i \in \mathbb{N}_0$.*

Notation 2.1. *Let Ω_X be the set of all L-relations on a universe X. The following notations are used in this paper:*

(i) Ω_X^{ref}*: the set of reflexive elements of* Ω_X*,*

(ii) Ω_X^{tra}*: the set of $*$-transitive elements of* Ω_X*,*

(iii) $\Omega_X^{\text{ref,sym}}$*: the set of reflexive and symmetric elements of* Ω_X*, i.e. the set of L-tolerance relations on X,*

(iv) $\Omega_X^{\text{sym,tra}}$*: the set of symmetric and $*$-transitive elements of* Ω_X*,*

(v) $\Omega_X^{\text{ref,tra}}$*: the set of reflexive and $*$-transitive elements of* Ω_X*,*

(vi) $\Omega_X^{\text{ref,sym,tra}}$*: the set of reflexive, symmetric and $*$-transitive elements of* Ω_X*, i.e. the set of L-equivalences on X.*

2.1. Definition of compatibility of L-relations

The notion of extensionality of L-relations was first introduced by Höhle and Blanchard.

Definition 2.3. [47] *Let X be a universe equipped with an L-equality E and Y a universe equipped with an L-equality F. A mapping $M : X \times Y \to L$ is said to satisfy the extensionality property if*

(i) $M(x_1, y) * E(x_1, x_2) \leq M(x_2, y)$*, for any* $x_1, x_2 \in X$ *and* $y \in Y$*;*

(ii) $M(x, y_1) * F(y_1, y_2) \leq M(x, y_2)$*, for any* $x \in X$ *and* $y_1, y_2 \in Y$*.*

This notion applies in particular to L-relations on a universe X (i.e., $X = Y$) equipped with an L-equality E (i.e., $E = F$).

Bělohlávek introduced a similar definition of compatibility of an L-relation with respect to an L-equality E.

Definition 2.4. [7] *Let X be a universe equipped with an L-equality E. An L-relation R on a universe X is compatible with respect to E if*

$$R(x_1, y_1) * E(x_1, x_2) * E(y_1, y_2) \leq R(x_2, y_2),$$

for any $x_1, x_2, y_1, y_2 \in X$.

As we will show further on, both definitions are equivalent.

The notion of extensionality/compatibility appears in various contexts. For instance, it is used in the definition of strict fuzzy orderings by Bodenhofer and Demirci [16]. Also, several authors use it in their study of fuzzy functions [30, 66]. Furthermore, it is key to the fuzzy approach to concept lattices of Bělohlávek [7].

Inspired by the above definitions, we introduce the core definitions of this paper. They are based on conditions (i) and (ii) in Definition 2.3, but the main idea is to give up the focus on L-equalities.

Definition 2.5. *Let R_1 and R_2 be two L-relations on a universe X.*

(i) R_1 is called left compatible with R_2, denoted $R_1 \nabla_l R_2$, if it holds that

$$R_1(x, y) * R_2(x, z) \leq R_1(z, y) \ , \tag{2.1}$$

for any $x, y, z \in X$;

(ii) R_1 is called right compatible with R_2, denoted $R_1 \nabla_r R_2$, if it holds that

$$R_1(x, y) * R_2(y, t) \leq R_1(x, t) \ , \tag{2.2}$$

for any $x, y, t \in X$;

(iii) R_1 is called compatible with R_2, denoted $R_1 \nabla R_2$, if it holds that

$$R_1(x, y) * R_2(x, z) * R_2(y, t) \leq R_1(z, t) \ , \tag{2.3}$$

for any $x, y, z, t \in X$.

Note that this definition bears no relationship at all with the definitions of left and right extensionality as discussed by Daňková [20].

The following lemma is immediate.

Lemma 2.1. *For any two L-relations R_1 and R_2 on a universe X, the following equivalences hold:*

(i) $R_1 \nabla_l R_2$ if and only if $R_1^t \nabla_r R_2$;

(ii) $R_1 \nabla_r R_2$ if and only if $R_1^t \nabla_l R_2$;

(iii) $R_1 \nabla R_2$ if and only if $R_1^t \nabla R_2$.

In the following proposition, we show that if R_1 is both left and right compatible with R_2, then R_1 is compatible with R_2. The converse also holds if R_2 is reflexive.

Proposition 2.2. *Let R_1 and R_2 be two L-relations on a universe X. Then it holds that*

(i) If $R_1 \triangledown_l R_2$ and $R_1 \triangledown_r R_2$, then $R_1 \triangledown R_2$;

(ii) If $R_1 \triangledown R_2$ and R_2 is reflexive, then $R_1 \triangledown_l R_2$ and $R_1 \triangledown_r R_2$.

Proof.

(i) Suppose that $R_1 \triangledown_l R_2$ and $R_1 \triangledown_r R_2$. It then holds that

$$R_1(x, y) * R_2(x, z) \leq R_1(z, y) \,,$$

for any $x, y, z \in X$. Taking into account that $*$ is increasing and the fact that $R_1 \triangledown_r R_2$, it follows that

$$R_1(x, y) * R_2(x, z) * R_2(y, t) \leq R_1(z, y) * R_2(y, t) \leq R_1(z, t) \,,$$

for any $x, y, z, t \in X$, i.e. $R_1 \triangledown R_2$. Hence, assertion (i) follows.

(ii) Suppose that $R_1 \triangledown R_2$ and R_2 is reflexive. Setting $y = t$ in inequality (2.3) and taking into account that $R_2(y, y) = 1$, we obtain

$$R_1(x, y) * R_2(x, z) \leq R_1(z, y) \,,$$

for any $x, y, z \in X$, i.e. $R_1 \triangledown_l R_2$. Similarly, setting $x = z$ in inequality (2.3) and taking into account that $R_2(x, x) = 1$, we obtain

$$R_1(x, y) * R_2(y, t) \leq R_1(x, t) \,,$$

for any $x, y, t \in X$, i.e. $R_1 \triangledown_r R_2$. Hence, assertion (ii) follows.

□

This proposition expresses that the notions of extensionality and compatibility in Definitions 2.3 and 2.4 coincide.

Corollary 2.1. *For any reflexive L-relation R on a universe X, it holds that $R \triangledown R$ if and only if $R \triangledown_l R$ and $R \triangledown_r R$.*

Remark 2.1. *Note that Proposition 2.2(ii) states a sufficient condition under which compatibility implies both left and right compatibility. However, this implication does not hold in general. Indeed, let $X = \{a, b\}$, $L = [0, 1]$ and $* = \wedge$ (the minimum operation). Consider the L-relations R_1 and R_2 on X defined as:*

R_1	a	b
a	0	0
b	α	0

R_2	a	b
a	0	0
b	β	0

with $\alpha \neq 0$ and $\beta \neq 0$. It clearly holds that $R_1 \nabla R_2$ and $R_1 \nabla_r R_2$, but not $R_1 \nabla_l R_2$. Indeed, suppose that $R_1 \nabla_l R_2$. Then it holds that $R_1(b,a) * R_2(b,a) \leq R_1(a,a)$, which implies that $\alpha * \beta \leq 0$, i.e. $\alpha = 0$ or $\beta = 0$, a contradiction.

2.2. Compatibility in terms of traces of L-relations

In this section, we will provide an interesting characterization of compatibility of L-relations in terms of the notions of left and right traces introduced by Fodor [40].

2.2.1. Left and right trace of L-relations

In this subsection, we recall some basic definitions and results on traces of L-relations introduced by Fodor [40]. Further information on these traces can be found in [80].

Definition 2.6. [40] *Let R be an L-relation on a universe X.*

(i) The left trace R^l of R is the L-relation on X defined by

$$R^l(a,b) = \inf_{c \in X} R(c,a) \to R(c,b)\,.$$

(ii) The right trace R^r of R is the L-relation on X defined by

$$R^r(a,b) = \inf_{c \in X} R(b,c) \to R(a,c)\,.$$

For a symmetric L-relation R on X, it obviously holds that $R^l = (R^r)^t$ and $R^r = (R^l)^t$. In the following lemma and theorems, we recall some interesting links between the properties of an L-relation and those of its traces.

Lemma 2.2. [40] *Let R be an L-relation on a universe X. Then it holds that*

(i) R^l and R^r are reflexive;

(ii) R^l and R^r are $*$-transitive;

(iii) $(R^t)^l = (R^r)^t$ and $(R^t)^r = (R^l)^t$.

Theorem 2.1. [40] *Let R be an L-relation on a universe X. Then the following statements are equivalent*

(i) R is reflexive;

(ii) $R^l \subseteq R$;

(iii) $R^r \subseteq R$.

Theorem 2.2. [40] *Let R be an L-relation on a universe X. Then the following statements are equivalent*

(i) R *is* $*$*-transitive;*

(ii) $R \subseteq R^l$;

(iii) $R \subseteq R^r$.

Next, we recall the definitions of the sub- and super-composition of L-relations as introduced by Bandler and Kohout [1, 2].

Definition 2.7. [1, 2] *Let R and S be two L-relations on a universe X.*

(i) *The sub-composition $R \triangleleft S$ of R and S is the L-relation on X defined by*

$$(R \triangleleft S)(x, z) = \inf_{y \in X} R(x, y) \to S(y, z) .$$

(ii) *The super-composition $R \triangleright S$ of R and S is the L-relation on X defined by*

$$(R \triangleright S)(x, z) = \inf_{y \in X} S(y, z) \to R(x, y) .$$

A trivial, yet important observation is that left and right traces can be expressed in terms of sub- and super-compositions.

Proposition 2.3. *For any L-relation R on a universe X, it holds that $R^l = R^t \triangleleft R$ and $R^r = R \triangleright R^t$.*

These compositions were studied in detail by De Baets and Kerre [29]. Their results will be used to provide some new properties of traces of L-relations that will turn out to be useful technical tools to study various properties of the compatibility of L-relations, such as monotonicity w.r.t. inclusion and its relation with intersection, union and composition. To that end, the following properties of the sub- and super-composition of L-relations are needed.

Proposition 2.4. [29]

(i) *Monotonicity: If* $R_1 \subseteq R_2$, *then* $R_2 \triangleleft S \subseteq R_1 \triangleleft S$ *and* $S \triangleleft R_1 \subseteq S \triangleleft R_2$;

(ii) *Transposition:* $(R_1 \triangleleft R_2)^t = R_2^t \triangleright R_1^t$ *and* $(R_1 \triangleright R_2)^t = R_2^t \triangleleft R_1^t$;

(iii) *Intersection:* $\bigcap_{i \in I} (R_i \triangleleft S) = (\bigcup_{i \in I} R_i) \triangleleft S$ *and* $\bigcap_{i \in I} (S \triangleleft R_i) = S \triangleleft (\bigcap_{i \in I} R_i)$;

(iv) *Union:* $\bigcup_{i \in I} (R_i \triangleleft S) \subseteq (\bigcap_{i \in I} R_i) \triangleleft S$ *and* $\bigcup_{i \in I} (S \triangleleft R_i) \subseteq S \triangleleft (\bigcup_{i \in I} R_i)$.

Proposition 2.5. *Let R, S be L-relations on a universe X and Q be an L-equivalence relation on X. If R is Q-reflexive and $*$-transitive, then it holds that*

(i) *R is right compatible w.r.t. S if and only if $S \subseteq Q \triangleleft R$,*

(ii) *R is left compatible w.r.t. S if and only if $S \subseteq (R \triangleright Q)^t$.*

Proof.

(i) Suppose that R is right compatible w.r.t. S, i.e.

$$R(x,y) * S(y,z) \leq R(x,z), \text{ for any } x,y,z \in X\,.$$

From the adjointness property, it follows that

$$S(y,z) \leq R(x,y) \to R(x,z), \text{ for any } x,y,z \in X\,.$$

Since R is Q-reflexive, i.e. $Q(x,y) \leq R(x,y)$ for any $x,y \in X$, it follows that

$$R(x,y) \to R(x,z) \leq Q(x,y) \to R(x,z), \text{ for any } x,y,z \in X\,,$$

whence

$$S(y,z) \leq Q(x,y) \to R(x,z) \text{ for any } x,y,z \in X\,,$$

this implies that

$$S(y,z) \leq \inf_{x\in X} Q(x,y) \to R(x,z)\,.$$

Since Q is symmetric, it follows that

$$S(y,z) \leq \inf_{x\in X} Q(y,x) \to R(x,z) = (Q \triangleleft R)(y,z)\,,$$

for any $y,z \in X$. Hence, $S \subseteq Q \triangleleft R$.
Conversely, suppose that $S \subseteq Q \triangleleft R$, then

$$S(y,z) \leq \inf_{x\in X} Q(y,x) \to R(x,z) \text{ for any } y,z \in X\,.$$

It follows that

$$S(y,z) \leq Q(y,x) \to R(x,z) \text{ for any } x,y,z \in X\,.$$

Setting $x = y$ and taking into account that $Q(y,y) = 1$, it follows that

$$S(y,z) \leq Q(y,y) \to R(y,z) = 1 \to R(y,z) = R(y,z), \text{ for any } y,z \in X\,,$$

whence $S(y,z) \leq R(y,z)$, for any $y,z \in X$. Since $*$ is increasing and R is

$*$-transitive it follows that

$$R(x,y) * S(y,z) \leq R(x,y) * R(y,z) \leq R(x,z), \text{ for any } x,y,z \in X .$$

Hence, R is right compatible w.r.t. S.

(ii) Suppose that R is left compatible w.r.t. S, i.e.

$$R(x,y) * S(x,z) \leq R(z,y), \text{ for any } x,y,z \in X .$$

From the adjointness property, it follows that

$$S(x,z) \leq R(x,y) \to R(z,y), \text{ for any } x,y,z \in X .$$

Since R is Q-reflexive, i.e. $Q(x,y) \leq R(x,y)$ for any $x,y \in X$ it follows that

$$R(x,y) \to R(z,y) \leq Q(x,y) \to R(z,y), \text{ for any } x,y,z \in X .$$

Since Q is symmetric, it follows that

$$\begin{aligned} S(x,z) &\leq Q(x,y) \to R(z,y) \\ &\leq \inf_{y \in X} Q(x,y) \to R(z,y) \\ &= \inf_{y \in X} Q(y,x) \to R(z,y) \\ &= (R \triangleright Q)(z,x) \\ &= (R \triangleright Q)^t(x,z) , \end{aligned}$$

for any $y,z \in X$. Hence, $S \subseteq (R \triangleright Q)^t$.
Conversely, Suppose that $S \subseteq (R \triangleright Q)^t$, i.e.

$$S(x,z) \leq (R \triangleright Q)^t(x,z) = (R \triangleright Q)(z,x), \text{ for any } x,z \in X .$$

It follows that

$$S(x,z) \leq \inf_{y \in X} Q(y,x) \to R(z,y) \leq Q(y,x) \to R(z,y) ,$$

for any $x,y,z \in X$. Setting $x = y$ and taking into account that $Q(y,y) = 1$, it holds that

$$S(x,z) \leq Q(x,x) \to R(z,y) = 1 \to R(z,x) = R(z,x) \text{ for any } x,z \in X .$$

Since R is $*$-transitive, $*$ is commutative and increasing, it follows that

$$R(x,y) * S(x,z) \leq R(x,y) * R(z,x) \leq R(z,y), \text{ for any } x,y,z \in X .$$

Hence, R is left compatible w.r.t.S.

□

The final proposition in this subsection shows that the intersection of the left (resp. right) traces of a family of L-relations is included both in the left (resp. right) trace of the intersection as well as of the union of these L-relations.

Proposition 2.6. *For any family of L-relations $(R_i)_{i\in I}$ on a universe X, it holds that*

(i) $\bigcap_{i\in I} R_i^l \subseteq (\bigcap_{i\in I} R_i)^l$;

(ii) $\bigcap_{i\in I} R_i^l \subseteq (\bigcup_{i\in I} R_i)^l$;

(iii) $\bigcap_{i\in I} R_i^r \subseteq (\bigcap_{i\in I} R_i)^r$;

(iv) $\bigcap_{i\in I} R_i^r \subseteq (\bigcup_{i\in I} R_i)^r$.

Proof.

(i) According to Proposition 2.3, it holds that

$$\bigcap_{i\in I} R_i^l = \bigcap_{i\in I} (R_i^t \triangleleft R_i)\,.$$

From Proposition 2.4(i) and (iii), it follows that

$$\begin{aligned}\bigcap_{i\in I} R_i^l &\subseteq \bigcap_{i\in I}((\bigcap_{j\in I} R_j^t) \triangleleft R_i)\\ &= (\bigcap_{j\in I} R_j^t) \triangleleft (\bigcap_{i\in I} R_i)\\ &= (\bigcap_{j\in I} R_j)^t \triangleleft (\bigcap_{i\in I} R_i)\\ &= (\bigcap_{i\in I} R_i)^l\,.\end{aligned}$$

(ii) According to Proposition 2.3, it holds that

$$\bigcap_{i\in I} R_i^l = \bigcap_{i\in I} (R_i^t \triangleleft R_i)\,.$$

From Proposition 2.4(i) and (iii), it follows that

$$\begin{aligned}\bigcap_{i\in I} R_i^l &\subseteq \bigcap_{i\in I}(R_i^t \triangleleft (\bigcup_{j\in I} R_j))\\ &= (\bigcup_{i\in I} R_i^t) \triangleleft (\bigcup_{j\in I} R_j)\\ &= (\bigcup_{i\in I} R_i)^t \triangleleft (\bigcup_{j\in I} R_j)\\ &= (\bigcup_{i\in I} R_i)^l\,.\end{aligned}$$

(iii) Follows from (i) and Lemma 2.2(iii).

(iv) Follows from (ii) and Lemma 2.2(iii).

□

2.2.2. Compatibility in terms of left and right traces

In this subsection, we provide some characterizations of the compatibility of L-relations in terms of the notions of left and right traces. Subsequently, some interesting corollaries on the compatibility of L-relations are stated in contexts where the relations involved hold additional properties such as reflexivity, symmetry and transitivity.

The first important theorem expresses that left and right compatibility can be expressed as inclusions involving traces. Note that from here on, we usually start with right compatibility, as it characterizations are a bit more elegant than those of left compatibility.

Theorem 2.3. *For any two L-relations R and S on a universe X, the following equivalences hold*

(i) R is right compatible with S if and only if $S \subseteq R^l$;

(ii) R is left compatible with S if and only if $S \subseteq (R^r)^t$.

Proof.

(i) Suppose that R is right compatible with S, i.e.

$$R(x,y) * S(y,z) \leq R(x,z)\,,$$

for any $x, y, z \in X$. It follows that

$$S(y,z) \leq R(x,y) \to R(x,z) \leq \inf_{x\in X} R(x,y) \to R(x,z) = R^l(y,z)\,,$$

for any $y, z \in X$. Hence, $S \subseteq R^l$.

Conversely, if $S \subseteq R^l$, then

$$S(y,z) \leq \inf_{x \in X} R(x,y) \to R(x,z) \leq R(x,y) \to R(x,z)\,,$$

for any $x, y, z \in X$. From the adjointness property, it follows that

$$R(x,y) * S(y,z) \leq R(x,z)\,,$$

for any $x, y, z \in X$, i.e. R is right compatible with S.

(ii) Follows from Lemma 2.1, (i) and Lemma 2.2(iii).

□

Corollary 2.2. *For any L-relation R on a universe X, it holds that*

(i) R is right compatible with R^l: $R \nabla_r R^l$;

(ii) R is left compatible with $(R^r)^t$: $R \nabla_l (R^r)^t$.

Remarkably, the inclusions in Theorem 2.3 become extremely basic and no longer involve the traces in case the first relation is reflexive and $*$-transitive.

Corollary 2.3. *Let R and S be two L-relations on a universe X. If R is reflexive and $*$-transitive, then it holds that*

(i) R is right compatible with S if and only if $S \subseteq R$;

(ii) R is left compatible with S if and only if $S \subseteq R^t$;

(iii) If S is symmetric, then R is right compatible with S if and only if R is left compatible with S.

Proof. Suppose that R is reflexive and $*$-transitive, then it follows from Theorems 2.1 and 2.2 that $R = R^r = R^l$. It holds also that $R^t = (R^r)^t = (R^l)^t$.

(i) Follows from Theorem 2.3(i).

(ii) Follows from Theorem 2.3(ii).

(iii) If S is symmetric, then $S \subseteq R$ if and only if $S \subseteq R^t$. The equivalence in (iii) follows immediately from (i) and (ii).

□

In addition to Corollary 2.3(iii), the following corollary identifies a second case in which left and right compatibility are equivalent.

Corollary 2.4. *Let R and S be two L-relations on a universe X. If R is symmetric, then R is right compatible with S if and only if R is left compatible with S.*

Proof. If R symmetric, then $R^l = (R^t)^l$. From Lemma 2.2(iii), it follows that $R^l = (R^t)^l = (R^r)^t$. The equivalence then follows from Theorem 2.3. □

Corollary 2.5. *For any two L-relations R and S on a universe X, the following equivalences hold*

(i) $R \nabla_r S$ *if and only if* $R^l \nabla_r S$*;*

(ii) $R \nabla_l S$ *if and only if* $R^r \nabla_l S$*.*

Proof.

(i) From Theorem 2.3, it holds that $R \nabla_r S$ if and only if $S \subseteq R^l$. Since R^l is reflexive and $*$-transitive, it follows from Corollary 2.3(i) that $R^l \nabla_r S$ if and only if $S \subseteq R$. Hence, $R \nabla_r S$ if and only if $R^l \nabla_r S$.

(ii) Follows from Lemma 2.1 and (i).

□

The following proposition identifies several cases in which self-compatibility of an L-relation implies its $*$-transitivity.

Proposition 2.7. *For any L-relation R on a universe X, it holds that*

(i) If $R \nabla_r R$*, then R is* $*$*-transitive;*

(ii) If $R \nabla_l R$ *and R is symmetric, then R is* $*$*-transitive;*

(iii) If $R \nabla R$ *and R is reflexive, then R is* $*$*-transitive.*

Proof.

(i) If $R \nabla_r R$, then it follows from Theorem 2.3(i) that $R \subseteq R^l$. The $*$-transitivity of R then follows from Theorem 2.2(ii).

(ii) Follows from Lemma 2.1 and (i).

(iii) Follows from Corollary 2.1 and (i).

□

Proposition 2.8. *Let E be an L-equivalence on a universe X and R be a* $*$*-E-order on X. Then R is compatible with E.*

Proof. Since R is E-reflexive, i.e. $E \subseteq R$, and E is symmetric, it follows from Corollary 2.3(i)&(iii) that R is both left and right compatible with E. Proposition 2.2 then implies that R is compatible with E. □

Corollary 2.6. *Let R be a reflexive and* $*$*-transitive L-relation on a universe X. Then R is compatible with* $E = R * R^t$*.*

Proof. One easily verifies that E is an L-equivalence and that R is a $*$-E-order. Proposition 2.8 then implies that R is compatible with $E = R * R^t$. □

2.2.3. Some properties of the compatibility relations

In this subsection, we provide some properties of the compatibility relations ∇_r, ∇_r and ∇.

Proposition 2.9. *Consider a universe X. Then it holds that*

(i) ∇_r is reflexive on Ω_X^{tra};

(ii) ∇_l is reflexive on $\Omega_X^{\mathrm{sym,tra}}$;

(iii) ∇ is reflexive on $\Omega_X^{\mathrm{ref,sym,tra}}$.

Proof.

(i) Let $R \in \Omega_X^{\mathrm{tra}}$. Since R is $*$-transitive, it follows from Theorem 2.2(ii) that $R \subseteq R^l$. From Theorem 2.3(i), it follows that $R \nabla_r R$, i.e. ∇_r is reflexive on Ω_X^{tra}.

(ii) Follows from Lemma 2.1 and (i).

(iii) Follows from Corollary 2.1, (i) and (ii).

□

Proposition 2.10. *Consider a universe X. Then it holds that*

(i) ∇_r is antisymmetric on Ω_X^{ref};

(ii) ∇_l is antisymmetric on $\Omega_X^{\mathrm{ref,sym}}$;

(iii) ∇ is antisymmetric on Ω_X^{ref}.

Proof.

(i) Let $R_1, R_2 \in \Omega_X^{\mathrm{ref}}$. Suppose that $R_1 \nabla_r R_2$ and $R_2 \nabla_r R_1$, then it follows from Theorem 2.3(i) that $R_2 \subseteq R_1^l$ and $R_1 \subseteq R_2^l$. Since R_1 and R_2 are reflexive, it follows from Theorem 2.1 that $R_2 \subseteq R_1^l \subseteq R_1$ and $R_1 \subseteq R_2^l \subseteq R_2$. Hence, $R_1 = R_2$.

(ii) Follows from Lemma 2.1 and (i).

(iii) Follows from Proposition 2.2(ii) and (i).

□

Remark 2.2. *In general, the relations ∇_r, ∇_l and ∇ are not antisymmetric. Indeed, let $\leq$ be the usual order and $<$ be the usual strict order on $\mathbb{R}$, then we have that $\leq \nabla_r <$ and $< \nabla_r \leq$, but $\leq \neq <$.*

Remark 2.3. *The following is another proof of Proposition 2.10*

(i) *Let* $R_1, R_2 \in \Omega_X^{\text{ref}}$. *Suppose that* $R_1 \nabla_r R_2$ *and* $R_2 \nabla_r R_1$, *then*

$$R_1(x,y) * R_2(y,z) \leq R_1(x,z) \quad \textit{and} \quad R_2(x,y) * R_1(y,z) \leq R_2(x,z)\,,$$

for any $x, y, z \in X$. *Setting* $x = y$ *and taking into account that* $R_1(x,x) = 1$ *and* $R_2(x,x) = 1$, *it follows that*

$$R_2(x,z) \leq R_1(x,z) \quad \textit{and} \quad R_1(x,z) \leq R_2(x,z)\,,$$

for any $x, z \in X$, *hence* $R_1(x,z) = R_2(x,z)$, *for any* $x, z \in X$, *i.e.* $R_1 = R_2$.

(ii) *Let* $R_1, R_2 \in \Omega_{\text{ref,sym}}$. *Suppose that* $R_1 \nabla_l R_2$ *and* $R_2 \nabla_l R_1$, *then*

$$R_1(x,y) * R_2(x,z) \leq R_1(z,y) \textit{ and } R_2(x,y) * R_1(x,z) \leq R_2(z,y)\,,$$

for any $x, y, z \in X$. *Setting* $x = y$, *it follows that*

$$R_1(x,x) * R_2(x,z) \leq R_1(z,x) \quad \textit{and} \quad R_2(x,x) * R_1(x,z) \leq R_2(z,x)\,,$$

for any $x, z \in X$. *Since* $R_1(x,x) = 1$ *and* $R_2(x,x) = 1$, *it follows that*

$$R_2(x,z) \leq R_1(z,x) \quad \textit{and} \quad R_1(x,z) \leq R_2(z,x)\,,$$

for any $x, z \in X$. *Since* R_1 *and* R_2 *are symmetric, it follows that* $R_1(x,z) = R_2(x,z)$, *for any* $x, z \in X$, *i.e.* $R_1 = R_2$.

(iii) *The proof is similar to that of* (i).

Proposition 2.11. *Consider a universe* X. *Then it holds that*

(i) ∇_r *is transitive on* Ω_X^{ref};

(ii) ∇_l *is transitive on* $\Omega_X^{\text{ref,sym}}$;

(iii) ∇ *is transitive on* Ω_X^{ref}.

Proof.

(i) Let $R_1, R_2, R_3 \in \Omega_X^{\text{ref}}$. Suppose that $R_1 \nabla_r R_2$ and $R_2 \nabla_r R_3$, then it follows from Theorem 2.3(i) that $R_2 \subseteq R_1^l$ and $R_3 \subseteq R_2^l$. Since R_2 is reflexive, it follows from Theorem 2.1 that $R_2^l \subseteq R_2$. Since $R_2 \subseteq R_1^l$, it holds that $R_3 \subseteq R_2^l \subseteq R_2 \subseteq R_1^l$. Hence, $R_1 \nabla_r R_3$.

(ii) Follows from Lemma 2.1 and (i).

(iii) Follows from Proposition 2.2(ii) and (i).

□

Remark 2.4. *In general, the relations ∇_r, ∇_l and ∇ are not transitive. Indeed, let $X = \{a, b\}$ and $L = [0, 1]$. Consider the L-relations R_1, R_2 and R_3 on X defined as*

R_1	a	b
a	*1*	α
b	α	*1*

R_2	a	b
a	*0*	*0*
b	β	*0*

R_3	a	b
a	*0*	*0*
b	γ	*0*

with $\beta < \alpha < \gamma$.

It holds that $R_1 \nabla_r R_2$ and $R_2 \nabla_r R_3$, but not $R_1 \nabla_r R_3$. Indeed, suppose that $R_1 \nabla_r R_3$. Then it holds that $R_1(b, b) * R_3(b, a) \leq R_1(b, a)$, which implies that $1 * \gamma \leq \alpha$, a contradiction. It also holds that $R_1 \nabla R_2$ and $R_2 \nabla R_3$, but not $R_1 \nabla R_3$.

Remark 2.5. *The following is another proof of Proposition 2.11*

(i) *Let $R_1, R_2, R_3 \in \Omega_X^{\text{ref}}$. Suppose that $R_1 \nabla_r R_2$ and $R_2 \nabla_r R_3$. We have to show that $R_1 \nabla_r R_3$, i.e.*

$$R_1(x_1, y_1) * R_3(y_1, y_2) \leq R_1(x_1, y_2)\,,$$

for any $x_1, y_1, y_2 \in X$. Indeed, let $x_1, y_1, y_2 \in X$. Since $R_2(y_1, y_1) = 1$, it holds that

$$R_1(x_1, y_1) * R_3(y_1, y_2) = R_1(x_1, y_1) * R_2(y_1, y_1) * R_3(y_1, y_2)\,.$$

Since $R_2 \nabla_r R_3$, it holds that $R_2(y_1, y_1) * R_3(y_1, y_2) \leq R_2(y_1, y_2)$ and thus

$$R_1(x_1, y_1) * R_3(y_1, y_2) \leq R_1(x_1, y_1) * R_2(y_1, y_2)\,.$$

Since $R_1 \nabla_r R_2$, it holds that

$$R_1(x_1, y_1) * R_3(y_1, y_2) \leq R_1(x_1, y_1) * R_2(y_1, y_2) \leq R_1(x_1, y_2)\,.$$

Hence, $R_1 \nabla_r R_3$.

(ii) *Let $R_1, R_2, R_3 \in \Omega_X^{\text{ref,sym}}$. Suppose that $R_1 \nabla_l R_2$ and $R_2 \nabla_l R_3$. We have to show that $R_1 \nabla_l R_3$, i.e.*

$$R_1(x_1, y_1) * R_3(x_1, x_2) \leq R_1(x_2, y_1)\,,$$

for any $x_1, x_2, y_2 \in X$. Indeed, let $x_1, x_2, y_1 \in X$. Since $R_2(x_1, x_1) = 1$, it holds that

$$R_1(x_1, y_1) * R_3(x_1, x_2) = R_1(x_1, y_1) * R_2(x_1, x_1) * R_3(x_1, x_2)\,.$$

Since $R_2 \nabla_l R_3$*, it holds that* $R_2(x_1,x_1) * R_3(x_1,x_2) \leq R_2(x_2,x_1)$ *and thus*

$$R_1(x_1,y_1) * R_3(x_1,x_2) \leq R_1(x_1,y_1) * R_2(x_2,x_1)\,.$$

Since R_2 *is symmetric and* $R_1 \nabla_l R_2$*, it holds that*

$$R_1(x_1,y_1) * R_2(x_2,x_1) \leq R_1(x_2,y_1)\,.$$

Combining the above inequalities leads to

$$R_1(x_1,y_1) * R_3(x_1,x_2) \leq R_1(x_2,y_1)\,.$$

Hence, $R_1 \nabla_l R_3$*.*

(iii) *Let* $R_1, R_2, R_3 \in \Omega_X^{\text{ref}}$*. Suppose that* $R_1 \nabla R_2$ *and* $R_2 \nabla R_3$*. We have to show that* $R_1 \nabla R_3$*, i.e.*

$$R_1(x_1,y_1) * R_3(x_1,x_2) * R_3(y_1,y_2) \leq R_1(x_2,y_2)\,,$$

for any $x_1, x_2, y_1, y_2 \in X$*. Indeed, let* $x_1, x_2, y_1, y_2 \in X$*. Since*

$$R_2(x_1,x_1) = R_2(y_1,y_1) = R_3(x_1,x_1) = R_3(y_1,y_1) = 1\,,$$

we obtain

$$\begin{aligned} R_1(x_1,y_1) * R_3(x_1,x_2) * R_3(y_1,y_2) \quad = \quad & R_1(x_1,y_1) * \\ & [R_2(x_1,x_1) * R_3(x_1,x_1) * R_3(x_1,x_2)] * \\ & [R_2(y_1,y_1) * R_3(y_1,y_1) * R_3(y_1,y_2)]\,. \end{aligned}$$

Since $R_2 \nabla R_3$*, it holds that*

$$R_2(x_1,x_1) * R_3(x_1,x_1) * R_3(x_1,x_2) \leq R_2(x_1,x_2)$$

and

$$R_2(y_1,y_1) * R_3(y_1,y_1) * R_3(y_1,y_2) \leq R_2(y_1,y_2)\,,$$

whence

$$R_1(x_1,y_1) * R_3(x_1,x_2) * R_3(y_1,y_2) \leq R_1(x_1,y_1) * R_2(x_1,x_2) * R_2(y_1,y_2)\,.$$

Since $R_1 \nabla R_2$*, it holds that*

$$R_1(x_1,y_1) * R_2(x_1,x_2) * R_2(y_1,y_2) \leq R_1(x_2,y_2)\,.$$

Combining the above inequalities leads to

$$R_1(x_1, y_1) * R_3(x_1, x_2) * R_3(y_1, y_2) \leq R_1(x_2, y_2)\,.$$

Hence, $R_1 \nabla R_3$.

From Propositions 2.9, 2.10 and 2.11, we derive the following corollary.

Corollary 2.7. *Consider a universe X. Then it holds that*

(i) ∇_r *is an order relation on* $\Omega_X^{\text{ref,tra}}$;

(ii) ∇_l *is an order relation on* $\Omega_X^{\text{ref,sym,tra}}$;

(iii) ∇ *is an order relation on* $\Omega_X^{\text{ref,sym,tra}}$.

Together with Corollary 2.3 and Proposition 2.2, the above corollary implies that the compatibility relation coincides with the usual inclusion relation on the universe of reflexive, symmetric and $*$-transitive L-relations.

2.2.4. Properties of the compatibility relations on the basic operations

In this subsection, we discuss the interaction of the left and right compatibility relations with inclusion, intersection, union and composition.

Proposition 2.12. *For any L-relations R, S, S_1, S_2, R_1 and R_2 on a universe X, it holds that*

(i) If $S_1 \subseteq S_2$, then $R \nabla_r S_2$ implies $R \nabla_r S_1$ and $R \nabla_l S_2$ implies $R \nabla_l S_1$.

(ii) If R_1 is reflexive, R_2 is $*$-transitive and $R_1 \subseteq R_2$, then $R_1 \nabla_r S$ implies $R_2 \nabla_r S$ and $R_1 \nabla_l S$ implies $R_2 \nabla_l S$.

Proof.

(i) Suppose that $S_1 \subseteq S_2$ and $R \nabla_r S_2$. It follows from Theorem 2.3(i) that $S_1 \subseteq S_2 \subseteq R^l$, i.e. $R \nabla_r S_1$. Similarly, it follows that if $R \nabla_l S_2$, then $R \nabla_l S_1$.

(ii) Suppose that $R_1 \nabla_r S$ and $R_1 \subseteq R_2$ such that R_1 is reflexive and R_2 is $*$-transitive. It follows from Theorem 2.3(i) that $S \subseteq R_1^l$. Since R_1 is reflexive, it follows from Theorem 2.1(ii) that $R_1^l \subseteq R_1$. Since R_2 is $*$-transitive, it follows from Theorem 2.2(ii) that $R_2 \subseteq R_2^l$. Hence, $S \subseteq R_1^l \subseteq R_1 \subseteq R_2 \subseteq R_2^l$, i.e. $R_2 \nabla_r S$. Similarly, it follows that if $R_1 \nabla_l S$, then $R_2 \nabla_l S$.

□

The following propositions express that left and right compatibility are preserved by intersection and union.

Proposition 2.13. *For any two L-relations R and S on a universe X and any families $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ of L-relations on X, it holds that*

(i) If $R_i \nabla_r S$, for any $i \in I$, then $(\bigcap_{i\in I} R_i)\nabla_r S$.

(ii) If $R_i \nabla_l S$, for any $i \in I$, then $(\bigcap_{i\in I} R_i)\nabla_l S$.

(iii) If $R \nabla_r S_i$, for any $i \in I$, then $R\nabla_r(\bigcap_{i\in I} S_i)$.

(iv) If $R \nabla_l S_i$, for any $i \in I$, then $R\nabla_l(\bigcap_{i\in I} S_i)$.

Proof.

(i) Suppose that $R_i \nabla_r S$, for any $i \in I$, then it follows that $S \subseteq R_i^l$, for any $i \in I$. This implies that $S \subseteq \bigcap_{i\in I} R_i^l$. From Proposition 2.6(i), it follows that $S \subseteq (\bigcap_{i\in I} R_i)^l$. Hence, $(\bigcap_{i\in I} R_i)\nabla_r S$.

(ii) Follows from Lemma 2.1 and (i).

(iii) Suppose that $R\nabla_r S_i$, for any $i \in I$, then it follows that $S_i \subseteq R^l$, for any $i \in I$. This implies that $\bigcap_{i\in I} S_i \subseteq R^l$, i.e. $R\nabla_r(\bigcap_{i\in I} S_i)$.

(iv) Follows from Lemma 2.1 and (iii).

□

Proposition 2.14. *For any two L-relations R and S on a universe X and any two families of L-relations $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ on X, it holds that*

(i) If $R_i \nabla_r S$, for any $i \in I$, *then $(\bigcup_{i\in I} R_i)\nabla_r S$.*

(ii) If $R_i \nabla_l S$, for any $i \in I$, *then $(\bigcup_{i\in I} R_i)\nabla_l S$.*

(iii) $R\nabla_r S_i$, for any $i \in I$, *if and only if $R\nabla_r(\bigcup_{i\in I} S_i)$.*

(iv) $R\nabla_l S_i$, for any $i \in I$, *if and only if $R\nabla_l(\bigcup_{i\in I} S_i)$.*

Proof.

(i) Suppose that $R_i \nabla_r S$, for any $i \in I$, then it follows that $S \subseteq R_i^l$, for any $i \in I$. This implies that $S \subseteq (\bigcap_{i\in I} R_i^l)$. From Proposition 2.6(ii), it follows that $S \subseteq (\bigcup_{i\in I} R_i)^l$. Hence, $(\bigcup_{i\in I} R_i) \nabla_r S$.

(ii) Follows from Lemma 2.1 and (i).

(iii) Suppose that $R \nabla_r S_i$, for any $i \in I$, then it follows that $S_i \subseteq R^l$ for any $i \in I$. This implies that $(\bigcup_{i\in I} S_i) \subseteq R^l$, i.e. $R \nabla_r (\bigcup_{i\in I} S_i)$.
Conversely, suppose that $R \nabla_r (\bigcup_{i\in I} S_i)$, then it follows that $(\bigcup_{i\in I} S_i) \subseteq R^l$ for any $i \in I$. This implies that $S_i \subseteq R^l$, for any $i \in I$, i.e. $R \nabla_r S_i$.

(iv) Follows from Lemma 2.1 and (iii).

□

The final proposition in this subsection states that left and right compatibility are preserved by the sup-$*$ composition.

Proposition 2.15. *For any L-relation R on a universe X and any finite family $(S_i)_{i\in I}$ of L-relations on X, it holds that*

(i) *If $R \nabla_r S_i$,* for any $i \in I$, *then $R \nabla_r (\circ_{i\in I} S_i)$,*

(ii) *If $R \nabla_l S_i$,* for any $i \in I$, *then $R \nabla_l (\circ_{i\in I} S_i)$.*

Proof.

(i) Suppose that $R \nabla_r S_i$, for any $i \in I$, then it follows that $S_i \subseteq R^l$, for any $i \in I$. This implies that $\circ_{i\in I} S_i \subseteq R^l \circ R^l \circ \ldots$. Since R^l is reflexive and $*$-transitive, it follows from Proposition 2.1 that $R^l \circ R^l \circ \ldots = R^l$. Hence, $\circ_{i\in I} S_i \subseteq R^l$, i.e. $R \nabla_r (\circ_{i\in I} S_i)$.

(ii) Follows from Lemma 2.1 and (i).

□

Proposition 2.16. *For any L-relations R and S on a universe X, it holds that*

(i) *If R is right compatible w.r.t. S, then $R \circ R$ is right compatible w.r.t. $R^t \lhd (S \circ R)$,*

(ii) *If R is left compatible w.r.t. S, then $R \circ R$ is left compatible w.r.t. $R \lhd (S \circ R^t)$.*

Proof.

(i) Suppose that $R\nabla_r S$, it follows that $S \subseteq R^l = R^t \lhd R$, this implies that

$$S \circ R \subseteq (R^t \lhd R) \circ R = R^t \lhd (R \circ R)\,.$$

Hence,

$$\begin{aligned} R^t \lhd (S \circ R) &\subseteq R^t \lhd R^t \lhd (R \circ R) \\ &= (R^t \circ R^t) \lhd (R \circ R) \\ &= (R \circ R)^t \lhd (R \circ R) \\ &= (R \circ R)^l\,, \end{aligned}$$

i.e. $R \circ R$ is right compatible w.r.t. $R^t \lhd (S \circ R)$.

(ii) We now from Lemma 2.1, that for any L-relations R_1 and R_2 on a universe X, it holds that

$$R_1 \nabla_l R_2 \Leftrightarrow R_1^t \nabla_r R_2 \text{ and } R_1 \nabla_r R_2 \Leftrightarrow R_1^t \nabla_l R_2\,.$$

Suppose that R is left compatible w.r.t. S, this is equivalent to R^t is right compatible w.r.t. S. From (i) it follows that $R^t \circ R^t$ is right compatible w.r.t. $(R^t)^t \lhd (S \circ R^t)$, which implies that $(R \circ R)^t$ is right compatible w.r.t. $R \lhd (S \circ R^t)$, i.e. $R \circ R$ is left compatible w.r.t. $R \lhd (S \circ R^t)$.

□

Remark 2.6. *For any L-relation R on a universe X, the following subclasses of L-relations defined by:*

(i) $C_r(R) = \{S \mid R\nabla_r S\} =\downarrow R^l = [\emptyset, R^l]$ *and*

(ii) $C_l(R) = \{S \mid R\nabla_l S\} =\downarrow (R^r)^t = [\emptyset, (R^r)^t]$

are ideals of the lattice of L-relations closed under the composition of L-relations $\circ$.

2.2.5. Compatibility of $*$-E orders with L-equivalence relations

In this subsection, we investigate in particular case, the compatibility of an $*$-E orders with an L-equivalence relation. Combining Proposition 2.2 and Corollary 2.3, we obtain the following result.

Theorem 2.4. *Let E be an L-equivalence on a universe X and R be a $*$-E-order on X. The following statements are equivalent:*

(i) *R is right compatible w.r.t. E,*

(ii) *R is left compatible w.r.t. E,*

(iii) *R is compatible w.r.t. E,*

(iv) $E \subseteq R$,

(v) $E \subseteq R \cap R^t$.

Proof.

$(i) \Leftrightarrow (ii)$ The equivalence follows from Corollary 2.3 (iii).

$(ii) \Leftrightarrow (iii)$ Suppose that R is left compatible w.r.t. E. Since $(i) \Leftrightarrow (ii)$, it follows that R is also right compatible w.r.t. E. From Proposition 2.2 (i), it holds that that R is compatible w.r.t. E. Conversely, suppose that R is compatible w.r.t. E. Since E is reflexive, it follows from Proposition 2.2 (ii) that R is left compatible w.r.t. E.

$(i) \Leftrightarrow (iv)$ Follows from Corollary 2.3 (i)

$(iv) \Leftrightarrow (v)$ Since E is symmetric, it follows immediately that $E \subseteq R$ if and only if $E \subseteq R \cap R^t$.

□

Remark 2.7. *If $E = R * R^t$ (i.e. $E(x,y) = R(x,y) * R(y,x)$ for any $x, y \in X$), it follows that $E(x,y) \leq R(y,x) \wedge R(x,y) \leq R(x,y)$ for any $x, y \in X$, i.e. $E \subseteq R \cap R^t$ which implies that R is right compatible, left compatible and compatible w.r.t. $E = R * R^t$.*

Since every crisp order is an $*$-$=$-order, then it is an $*$-E-order for all L-equivalence E. The following corollary is an immediate result of Theorem 2.4.

Corollary 2.8. *Let $\leqslant$ be a crisp order on a universe X and E be an L-equivalence relation on X. Then, $\leqslant$ is compatible w.r.t. E if and only if E is a crisp equality on X.*

Indeed, Let $R = \leqslant$ be a crisp order on a universe X and E be an L-equivalence relation on X. Suppose that R is compatible w.r.t. E, then it follows from Theorem 2.4 that $E \subseteq R \cap R^t$. Since R is antisymmetric, it follows that $R(x,y) = 0$ or $R(y,x) = 0$ for any $x, y \in X$ with$x \neq y$. Hence, if $x \neq y$, then $E(x,y) = 0$. Also, from the reflexivity of E it follows that $E(x,x) = 1$ for all $x \in X$. Thus E is a crisp equality on X. Conversely, if E is the crisp equality, then it is obvious that $\leqslant$ is compatible w.r.t. E.

Example 2.2. *Let $X = \{a, b, c\}$. R and E be the L-relations on X defined as*

R	a	b	c
a	1	0.4	0.4
b	0.2	1	0.1
c	0.3	0.5	1

E	a	b	c
a	1	0.2	0.3
b	0.2	1	0.09
c	0.3	0.09	1

R is a $*$-=-order for the Lukasiewicz t-norm and E is an L-equivalence relation on X. Since $E \subseteq R$, it follows from Theorem 2.4 that R is right compatible, left compatible and compatible w.r.t. E.

Example 2.3. *Let $X = \{a, b, c\}$, R and E be the L-relations on X defined as*

R	a	b	c
a	1	0.5	0.3
b	0.4	1	0.3
c	0.4	0.6	1

E	a	b	c
a	1	0.4	0.3
b	0.4	1	0.3
c	0.3	0.3	1

R is a $\wedge$-=-order and E is an L-equivalence relation on X. Since $E \subseteq R$, it follows from Theorem 2.4 that R is right compatible, left compatible and compatible w.r.t. E. Note that $E(x, y) = R(y, x) * R(x, y)$.

Remark 2.8. *1. Note that the notion of $*$-E-order introduced by Bodenhofer* [14] *used in this chapter, is equivalent to the notion of L-order introduced by Bělohlávek [3, Definition 4.42], as an L-relation R on X equipped with an L-equality E and R satisfies the compatibility w.r.t. E, reflexivity, $\wedge$-E-antisymmetry and $*$-transitivity, such that the antisymmetry condition is formulated using $*$ in place of $\wedge$. Particular case, the definition of $*$-E-order is more general than the definition of L-order if we consider that the L-relation E in the definition of L-order is an L-equality. Indeed, since R is reflexive, $*$-transitive and E is symmetric, it follows from Corollary 2.3 that*

R is right compatible w.r.t. E if and only if R is left compatible w.r.t. E if and only if $E \subseteq R$.

Since E is reflexive it follows from proposition 2.2 that:

R is compatible w.r.t. E if and only if R is right compatible w.r.t. E and R is left compatible w.r.t. E.

Hence,

R is compatible w.r.t. E if and only if $E \subseteq R$.

Thus, the fact that R is compatible w.r.t E in the definition of L-order is equivalent to the condition of E-reflexivity in the definition of T-E-order. Therefore, the notion of $*$-E-order introduced by Bodenhofer [14] *is equivalent to the notion of L-order introduced by Bělohlávek (*[5] *such that the antisymmetry condition is formulated using $*$ in place of $\wedge$. Particular case, the definition of $*$-E-order is more general than the definition of L-order if we consider that the L-relation E*

in the definition of L-order is an L-equality. In this case also we can see that the results obtained by using the definition of $*$-E-order remain valid for L-order, the converse is true for the results not needs that E is an L-equality (needs only E is an L-equivalence).

F

3 The clone relation of a poset and its role in the compatibility relations

In this chapter, we recall the compatibility notion of Bělohlávek and show the negative result that the only reflexive fuzzy relation an order relation can be compatible with, is the crisp equality. The notion of clone relation is introduced in full detail. We will characterize the weak compatibility of a strict order with a fuzzy tolerance (resp. fuzzy equivalence) relation in terms of the clone tolerance relation. Whereas the clone relation is a tolerance relation in general, we show that it is intransitive for comparable clones, while it is transitive for incomparable clones. In order to provide deeper insights, we also study the clone relation of the union and linear sum of posets. Here, we will characterize the fuzzy tolerance (resp. equivalence) relations a strict order relation is weakly compatible with in terms of two fuzzy tolerance (resp. equivalence) relations, one restricted to comparable clones, the other to incomparable clones.

On the other hand, we show that although there exists no non-trivial (fuzzy) tolerance relation a partial order relation is compatible with (in the sense of Bělohlávek), the situation is quite different when considering its strict part. More specifically, we provide a characterization of all fuzzy tolerance (and, in particular, fuzzy equivalence) relations a strict order relation is weakly compatible with.

3.1. Compatibility of a crisp order with an L-equivalence relation

First, we recall some basic notions of a strict order relations that will be needed throughout this chapter.

Let $(X, \leq)$ be a poset.
Two elements x and y of X are called comparable, if $x \leqslant y$ or $y \leqslant x$ and we write $x \parallel y$, otherwise they are called incomparable and we write $x \not\parallel y$.
Consider a poset $(X, \leqslant)$. For any two elements $x, y \in X$, $\ll$ denote the crisp relation x is covered by y defined on X as: $x \ll y$ if and only if $x < y$ and there exists no $z \in X$ such that $x < z < y$, i.e. x and y are neighbors. If $x < y$ and there exists $z \in X$ such that $x < z < y$, then we write $x \not\ll y$.

A poset can be conveniently represented by a Hasse diagram, displaying the covering relation $<$. Note that $x < y$, if there is a sequence of connected lines upwards from x to y. For more details about order, strict order and equivalence relations see [21, 74].

Throughout this chapter, unless otherwise stated, L always denotes a complete residuated lattice.

For crisp relations, we use the usual infix notation, e.g. we write $a \leq b$ instead of $\leq(a, b)$. An L-relation R_1 is included in an L-relation R_2, denoted $R_1 \subseteq R_2$, if $R_1(x, y) \leq R_2(x, y)$, for any $x, y \in X$. The union of two L-relations R_1 and R_2 on X is the L-relation $R_1 \cup R_2$ on X defined by $R_1 \cup R_2(x, y) = R_1(x, y) \vee R_2(x, y)$, for any $x, y \in X$. An L-relation E on X is called an L-tolerance if it is reflexive and symmetric; an $*$-transitive L-tolerance is called an L-equivalence relation. A separable L-equivalence is called an L-equality. Note that the only $\{0, 1\}$-equality on X is precisely the usual equality (identity) Id_X, i.e. $\mathrm{Id}_X(x, y) = 1$ if $x = y$ and $\mathrm{Id}_X(x, y) = 0$ if $x \neq y$.

3.1.1. Compatibility of a crisp order with an L-relation reflexive

The notion of compatibility of an L-relation w.r.t. a fuzzy equality has been introduced by Bělohlávek in [7]. We generalize this notion to arbitrary L-relations.

Definition 3.1. [7] *Let R and S be L-relations on a universe X. Then R is called compatible with respect to (with, for short) S if*

$$R(x_1, y_1) * S(x_1, x_2) * S(y_1, y_2) \leq R(x_2, y_2)\,,$$

for any $x_1, x_2, y_1, y_2 \in X$.

Definition 3.2. *Let R and S be L-relations on a universe X. Then R is called weakly compatible with respect to (with, for short) S if*

$$R(x_1, y_1) * S(x_1, x_2) * S(y_1, y_2) \leq R(x_2, y_2)\,,$$

for any deferent elements $x_1, x_2, y_1, y_2 \in X$.

Let $\leqslant$ be a crisp order on a universe X and E be an L-tolerance relation on X. In this subsection, we will study the compatibility of a (crisp) order relation with an L-tolerance or L-equivalence relation. More specifically, the compatibility of a crisp order $\leqslant$ with an L-relation R states that

$$\tau(x \leqslant y) * R(x, z) * R(y, t) \leq \tau(z \leqslant t)\,, \tag{3.1}$$

for any $x, y, z, t \in X$. We will show that $\leqslant$ is compatible with E if and only if E is the crisp equality on X.

Note that throughout this chapter, we use the notation τ to refer to the characteristic mapping of a crisp relation. In particular, $\tau(x \leqslant y) = 1$ if $x \leqslant y$, while $\tau(x \leqslant y) = 0$ if $x \nleqslant y$. Let us emphasize a proper reading of Eq. (3.1): $\leqslant$ refers to the partial order relation on X, while $\leq$ refers to the partial order relation of the lattice L.

Theorem 3.1. *Let $(X, \leqslant)$ be a poset and E be a reflexive L-relation on X. Then it holds that $\leqslant$ is compatible with E if and only if E is the crisp equality on X.*

Proof. Suppose that $\leqslant$ is compatible with E. Since E is reflexive, it follows that

$$\underbrace{\tau(x \leqslant x)}_{=1} * \underbrace{E(x,x)}_{=1} * E(x,y) \leq \tau(x \leqslant y)\,,$$

for any $x, y \in X$, i.e. $E(x,y) \leq \tau(x \leqslant y)$. In the same way, it follows that

$$\underbrace{\tau(x \leqslant x)}_{=1} * E(x,y) * \underbrace{E(x,x)}_{=1} \leq \tau(y \leqslant x)\,,$$

for any $x, y \in X$, i.e. $E(x,y) \leq \tau(y \leqslant x)$. Therefore,

$$E(x,y) \leq \tau(x \leqslant y) \wedge \tau(y \leqslant x)\,.$$

The antisymmetry of $\leqslant$ implies that $\tau(x \leqslant y) = 0$ or $\tau(y \leqslant x) = 0$, for any $x, y \in X$ with $x \neq y$. Hence, if $x \neq y$, then it holds that $E(x,y) = 0$. Since E is reflexive, it then follows that E is the crisp equality on X.

Conversely, if E is the crisp equality on X, then it is obvious that $\leqslant$ is compatible with E. □

Corollary 3.1. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. Then it holds that $\leqslant$ is compatible with E if and only if E is the crisp equality on X.*

3.1.2. Link with the Bodenhofer's notion of compatibility

In this section, we will establish a link between the compatibility of a crisp order with an L-equivalence relation and the related notion of compatibility of an L-equivalence relation with a crisp order as introduced by Bodenhofer [12, 11, 14], (see also Demirci [34]). We will show that the compatibility of a crisp order with an L-equivalence relation implies the compatibility of this L-equivalence relation with this crisp order, but the converse does not hold in general.

Definition 3.3. [12] *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. Then E is called compatible with $\leqslant$ if and only if the following implication holds*

$$x \leqslant y \leqslant z \Longrightarrow E(x,z) \leq E(x,y) \wedge E(y,z)\,,$$

for any $x, y, z \in X$.

Theorem 3.2. *Let $(X, \leqslant)$ be a poset and E be an L-equivalence relation on X. If $\leqslant$ is compatible with E, then E is compatible (in the sense of Bodenhofer) with $\leqslant$. The converse does not hold in general.*

Proof.

Suppose that $\leqslant$ is compatible with E. According to Corollary 3.1, it holds that E is the equality relation. Obviously, if $x \leqslant y \leqslant z$, then $x = z$ implies that $x = y$ and $y = z$, and hence E is compatible with $\leqslant$.

Remark 3.1. *The following is another proof of Theorem 3.2 with a direct method.*

Suppose that $\leqslant$ is compatible with E. We will show that

$$x \leqslant y \leqslant z \Longrightarrow E(x,z) \leq \min(E(x,y), E(y,z)), \text{ for any x, y, z} \in \text{X}\,.$$

Let $x, y, z \in X$ with $x \leqslant y \leqslant z$.

(i) If $E(x,z) = 0$, then $E(x,z) \leq \min(E(x,y), E(y,z))$.

(ii) If $E(x,z) > 0$, then the compatibility of $\leqslant$ with E implies that

$$\tau(x \leqslant y) * E(x,z) * E(y,t) \leq \tau(z \leqslant t)\,,$$

for any $x, y, z, t \in X$. Setting $y = t$ and taking into account that $E(y,y) = 1$, it follows that

$$\tau(x \leqslant y) * E(x,z) \leq \tau(z \leqslant y)\,,$$

for any $x, y, z \in X$. Since $x \leqslant y$ (i.e. $\tau(x \leqslant y) = 1$) and $E(x,z) > 0$, it follows that $\tau(z \leqslant y) = 1$ (i.e. $z \leqslant y$). Since $y \leqslant z$, then it follows that $y = z$. This implies that $E(y,z) = 1$, i.e. $E(x,z) \leq E(y,z)$.

On the other hand, the compatibility of $\leqslant$ with E implies that

$$\tau(y \leqslant z) * E(z,x) * E(y,t) \leq \tau(t \leqslant x)\,,$$

for any $x, y, z, t \in X$. Setting $y = t$ and taking into account that $E(y,y) = 1$, *it follows that*

$$\tau(y \leqslant z) * E(z,x) \leq \tau(y \leqslant x)\,,$$

for any $x,y,z \in X$*. Since* $y \leqslant z$ *(i.e.* $\tau(y \leqslant z) = 1$*) and* $E(z,x) = E(x,z) > 0$*, then it follows that* $\tau(y \leqslant x) = 1$*, i.e.* $y \leqslant x$*. Since* $x \leqslant y$*, then it holds that* $x = y$*. This implies that* $E(x,y) = 1$*, i.e.* $E(x,z) \leq E(x,y)$*.*

Hence, $E(x,z) \leq \min(E(x,y), E(y,z))$*. Thus,* E *is compatible (in sense Bodenhofer) with* $\leqslant$*.*

The following example shows that the converse does not hold in general. Indeed, let $X = \{x,y,z\}$, $\leqslant$ be the crisp order on X and E be the L-equivalence relation on X defined as

$\leqslant$	z	y	x
z	1	0	0
y	1	1	0
x	1	1	1

E	x	y	z
x	1	α	α
y	α	1	α
z	α	α	1

with $\alpha \neq 0$.

It is easily verified that E is compatible with $\leqslant$ (in the sense of Bodenhofer). Clearly, it holds that $\underbrace{\tau(y \leqslant z)}_{=1} * \underbrace{E(y,y)}_{=1} * \underbrace{E(z,x)}_{=\alpha} = \alpha$, while $\tau(y \leqslant x) = 0$. Hence, $\leqslant$ is not compatible with E (in the sense of Bodenhofer). □

3.2. Clone relations of a poset

In this section, we introduce the notion of clone relation associated with a poset. Some basic properties of this relation and useful technical tools that will be used in the following sections are given.

3.2.1. Definitions and some examples

Definition 3.4. *The clone relation* $\approx$ *of a poset* $(X, \leqslant)$ *is the binary relation on* X *defined by*

$$x \approx y \quad \textit{if and only if} \quad \begin{cases} (\forall z \in X \setminus \{x,y\})(z < x \Leftrightarrow z < y) \\ \textit{and} \\ (\forall z \in X \setminus \{x,y\})(x < z \Leftrightarrow y < z)\,. \end{cases} \tag{3.2}$$

Proposition 3.1. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and $x, y \in X$. If $x \approx y$, then it holds that*

$$(\forall z \in X \setminus \{x, y\})(x \parallel z \Leftrightarrow y \parallel z)\,.$$

The following proposition is immediate.

Proposition 3.2. *The clone relation $\approx$ of a poset $(X, \leqslant)$ is a tolerance relation.*

In general, $\approx$ is not an equivalence relation, as can be seen from Example 3.6.

The name clone relation is appropriately chosen. If $x \approx y$, we say that x and y are clones of one another, meaning that they hold the same relation ($<$, $>$ or $\parallel$) with any other element.

Example 3.1. *For any 2-element poset $(\{x, y\}, \leqslant)$ it holds that $x \approx y$.*

Example 3.2. *For any chain $(X, \leqslant)$, the clone relation $\approx$ coincides with the covering relation $\ll$. For instance, for the set of integers Z equipped with the usual order $\leqslant$, the clone relation $\approx$ is given by*

$$\approx = \{(a, b) \in Z^2 \mid b = a + 1\}\,,$$

while the clone relation of the set of real numbers $\mathbb{R}$ equipped with the usual order $\leqslant$ is the empty relation.

Example 3.3. *In Figure 3.1, the Hasse diagrams of all possible 3-element posets $(\{x, y, z\}, \leqslant)$ with $x \approx y$ are depicted.*

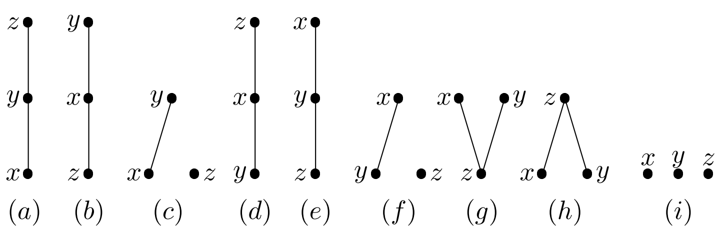

Figure 3.1: Hasse diagrams of all 3-element posets $(\{x, y, z\}, \leqslant)$ with $x \approx y$.

Given the clone relation $\approx$ of a poset $(X, \leqslant)$, we introduce the following binary relations $\triangleleft$, $\triangleright$ and $\diamond$ on X:

$$\begin{aligned} \triangleleft &= \approx \cap \ll \\ \triangleright &= \approx \cap \gg \\ \diamond &= \approx \cap \parallel\,. \end{aligned}$$

For instance, $x \triangleleft y$ if and only if x and y are clones and x is covered by y. If $x = y$, $x \triangleleft y$ or $x \triangleright y$, we say that x and y are comparable clones; similarly, if $x \diamond y$, we say that x and y are incomparable clones.

Corollary 3.2. *Let* $(X, \leqslant)$ *be a poset. Then it holds that* $\lhd^t = \rhd$ *and* $\diamond^t = \diamond$.

Example 3.4. *In Figure 3.2, the Hasse diagram of a poset* $(X, \leqslant)$ *with* $X = \{0, a, b, c, 1\}$ *and* $a \lhd c$ *is depicted. The blue segment is used to indicate comparable clones.*

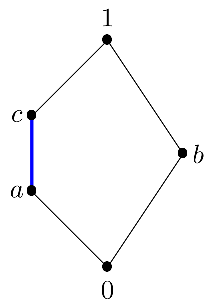

Figure 3.2: Hasse diagram of a poset $(X, \leqslant)$ with $X = \{0, a, b, c, 1\}$ and $a \lhd c$ (and hence $c \rhd a$).

Example 3.5. *In Figure 3.3, the Hasse diagram of a poset* $(X, \leqslant)$ *with* $X = \{0, a, b, c, 1\}$*,* $0 \lhd a$ *and* $b \diamond c$ *is depicted. The blue dashed line is used to indicate incomparable clones.*

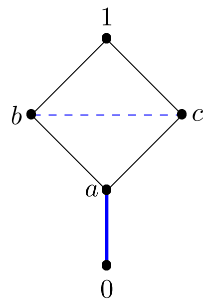

Figure 3.3: Hasse diagram of a poset $(X, \leqslant)$ with $X = \{0, a, b, c, 1\}$, $0 \lhd a$ and $b \diamond c$.

Example 3.6. *In Figure 3.4, the Hasse diagram of a poset* $(X, \leqslant)$ *with* $X = \{0, a, b, c, d, e, 1\}$ *is depicted. The blue segments are used to indicate comparable clones, while blue dashed lines are used to indicate incomparable clones. It holds that* $0 \lhd a$*,* $a \lhd b$*,* $c \diamond d$*,* $d \diamond e$ *and* $c \diamond e$*. Note that* $0 \approx a$*,* $a \approx b$*, but* $0 \not\approx b$*, illustrating that* $\approx$ *is not transitive in general.*

The tolerance classes $[x]_\approx = \{y \in X \mid y \approx x\}$ *for the different elements* x *of* X *are given by:* $[0]_\approx = \{0, a\}$*,* $[a]_\approx = \{0, a, b\}$*,* $[b]_\approx = \{a, b\}$*,* $[c]_\approx = [d]_\approx = [e]_\approx = \{c, d, e\}$ *and* $[1]_\approx = \{1\}$*.*

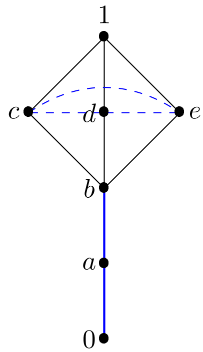

Figure 3.4: Hasse diagram of a poset $(X, \leqslant)$ with $X = \{0, a, b, c, d, e, 1\}$, $0 \lhd a$, $a \lhd b$, $c \diamond d$, $d \diamond e$ and $c \diamond e$.

3.2.2. Properties of the clone relation

Several useful properties of the clone relation are gathered in the following propositions.

Proposition 3.3. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and $x, y \in X$ with $x \neq y$. If $x \approx y$, then it holds that $x \ll y$, $y \ll x$ or $x \parallel y$.*

Proof. Let $x, y \in X$ with $x \approx y$ and $x \neq y$. If $x < y$, then $x \approx y$ implies that $x \ll y$. Indeed, suppose that there exists z such that $x < z < y$, then $x \not\approx y$, a contradiction. If $y < x$, then $x \approx y$ implies that $y \ll x$. Hence, $x \ll y$, $y \ll x$ or $x \parallel y$. □

Corollary 3.3. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. For any $x, y \in X$, exactly one of the following statements holds:*

(i) $x = y$,

(ii) $x \lhd y$,

(iii) $x \rhd y$,

(iv) $x \diamond y$,

(v) $x \not\approx y$.

In particular, this means that the clone relation can be partitioned as follows: $\approx = \lhd \cup \rhd \cup \diamond \cup =$.

The following corollary lists all cases in which $x \not\approx y$. The Hasse diagrams of the 3-element posets in which these cases occur are shown in Figure 3.5.

Corollary 3.4. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. For any $x, y \in X$, it holds that $x \not\approx y$ if and only if there exists $z \in X \setminus \{x, y\}$ such that exactly one of the following statements holds:*

(i) $x < z < y$ *or* $y < z < x$ *,*

(ii) $(x \ll y,\ z < y$ *and* $x \parallel z)$ *or* $(x \ll y,\ x < z$ *and* $y \parallel z)$ *,*

(iii) $(y \ll x,\ z < x$ *and* $y \parallel z)$ *or* $(y \ll x,\ y < z$ *and* $x \parallel z)$ *,*

(iv) $(x \parallel y,\ z < y$ *and* $x \parallel z)$ *or* $(x \parallel y,\ y < z$ *and* $x \parallel z)$ *,*

(v) $(x \parallel y,\ z < x$ *and* $y \parallel z)$ *or* $(x \parallel y,\ x < z$ *and* $y \parallel z)$ *.*

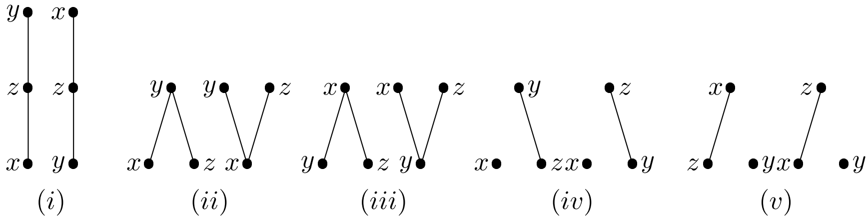

Figure 3.5: Hasse diagrams of all 3-element posets $(\{x, y, z\}, \leqslant)$ with $x \not\approx y$.

The following proposition discusses the (ir-)reflexivity and (anti-)symmetry of the relations $\lhd$, $\rhd$ and $\diamond$.

Proposition 3.4. *Let* $(X, \leqslant)$ *be a poset with clone relation* $\approx$*. Then it holds that*

(i) The relations $\lhd$ *and* $\rhd$ *are irreflexive and anti-symmetric, i.e., they are asymmetric.*

(ii) The relation $\diamond$ *is irreflexive and symmetric.*

Along the same line, the following proposition discusses the (anti-)transitivity of these relations. Recall that a binary relation R on a set X is called antitransitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \notin R$, for any $x, y, z \in X$.

Proposition 3.5. *Let* $(X, \leqslant)$ *be a poset with clone relation* $\approx$*. Then it holds that*

(i) The relation $\lhd$ *is antitransitive.*

(ii) The relation $\rhd$ *is antitransitive.*

(iii) The relation $\diamond$ *is transitive.*

Proof. Let $x, y, z \in X$.

(i) Suppose that $x \lhd y$ and $y \lhd z$, i.e. ($x \approx y$ and $x \ll y$) and ($y \approx z$ and $y \ll z$). It follows that $x < y < z$, i.e. $x \not\approx z$ and $x \not\ll z$, whence $x \ntriangleleft z$. Hence, $\lhd$ is antitransitive.

(ii) Follows immediately from (i) and the fact that $\rhd = \lhd^t$.

(iii) Suppose that $x \diamond y$ and $y \diamond z$, i.e. ($x \approx y$ and $x \parallel y$) and ($y \approx z$ and $y \parallel z$). We will show that $x \approx z$ and $x \parallel z$. Assume that $x \not\approx z$ or $x \nparallel z$.

(a) If $x \nparallel z$, then $x < z$ or $z < x$. Since $y \approx z$, it follows that $x < y$ or $y < x$, whence $x \nparallel y$, a contradiction. Therefore, $x \parallel z$.

(b) If $x \not\approx z$, then from Corollary 3.4 and the fact that $x \parallel z$, it follows that there exists $w \in X \setminus \{x, z\}$ such that ($w < z$ and $x \parallel w$), ($z < w$ and $x \parallel w$), ($w < x$ and $z \parallel w$) or ($x < w$ and $z \parallel w$). Since $x \approx y$ and $y \approx z$, it then follows that ($w < y$ and $x \parallel w$), ($y < w$ and $x \parallel w$), ($w < y$ and $z \parallel w$) or ($y < w$ and $z \parallel w$), respectively. From Corollary 3.4, it follows that $x \not\approx y$ or $y \not\approx z$, a contradiction. Hence, $\diamond$ is transitive.

□

Corollary 3.5. *The clone relation $\approx$ of a poset $(X, \leqslant)$ is an equivalence relation on the set of incomparable elements.*

The antitransitivity of $\lhd$ and $\rhd$ expresses that for a given poset $(X, \leqslant)$, the following cases are impossible: ($x \lhd y$, $y \lhd z$ and $x \lhd z$), as well as ($x \rhd y$, $y \rhd z$ and $x \rhd z$). The following proposition identifies a number of other impossible cases, which will be helpful in the proofs in Section 6. This proposition implies in particular that an element cannot have a comparable clone as well as an incomparable one.

Proposition 3.6. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. There cannot exist three elements $x, y, z \in X$ such that one of the following statements holds:*

(i) $x \lhd y$ *and* $y \rhd z$,

(ii) $x \lhd y$ *and* $y \diamond z$,

(iii) $y \lhd x$ *and* $y \lhd z$,

(iv) $y \lhd x$ *and* $y \diamond z$.

Proof.

(i) Suppose that ($x \lhd y$ and $y \rhd z$), i.e. ($x \approx y$ and $x \ll y$) and ($y \approx z$ and $z \ll y$). Since $x \ll y$ and $y \approx z$, it follows that $x < z$. On the other hand, since $z < y$ and $x \approx y$, it follows that $z < x$, a contradiction. Hence, ($x \lhd y$ and $y \rhd z$) is an impossible case.

(ii) Suppose that ($x \lhd y$ and $y \diamond z$), i.e. ($x \approx y$ and $x \ll y$) and ($y \approx z$ and $y \parallel z$). Since $x \ll y$ and $y \approx z$, it follows that $x < z$. On the other hand, since $x < z$ and $x \approx y$, it follows that $y < z$, a contradiction. Hence, ($x \lhd y$ and $y \diamond z$) is an impossible case as well.

(iii) The proof is similar to that of (i).

(iv) The proof is similar to that of (ii).

□

To conclude this subsection, we provide some insight into the structure of the tolerance classes of the clone relation by characterizing the intersection of two tolerance classes. In the following proposition, $[x]_\approx$ denotes the tolerance class of an element $x \in X$.

Proposition 3.7. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. For any two elements $x, y \in X$, it holds that*

(i) If $x \approx y$, then

$$[x]_\approx \cap [y]_\approx = \begin{cases} [x]_\approx & \text{, if } x = y \\ \{x, y\} & \text{, if } x \lhd y \text{ or } x \rhd y \\ \{x, y\} \cup \{z \in X \mid z \diamond x \text{ and } z \diamond y\} & \text{, if } x \diamond y \end{cases} .$$

(ii) If $x \not\approx y$ and $[x]_\approx \cap [y]_\approx \neq \emptyset$, then

$$[x]_\approx \cap [y]_\approx = \{z\} ,$$

where z is the unique element such that $x \ll z \ll y$ or $y \ll z \ll x$. Note that in this case $[z]_\approx = \{x, y, z\}$.

Proof. Let $x, y \in X$. If $x \approx y$, then it follows from Corollary 3.3 that $x = y$, $x \lhd y$, $x \rhd y$ or $x \diamond y$.

(a) Suppose that $x = y$, then it trivially holds that $[x]_\approx \cap [y]_\approx = [x]_\approx = [y]_\approx$.

(b) Suppose that $x \lhd y$ or $x \rhd y$, i.e. $x \approx y$ and ($x \ll y$ or $y \ll x$), and let $z \in [x]_\approx \cap [y]_\approx$. Since $x \neq z \neq y$ lead to a contradiction, it must hold that $z = x$ or $z = y$. Hence,
$$[x]_\approx \cap [y]_\approx = \{x, y\} .$$

(c) Suppose that $x \diamond y$, i.e. $x \approx y$ and $x \parallel y$ and let $z \in [x]_\approx \cap [y]_\approx$. From Propositions 3.5 and 3.6, it follows that the only possibility is $z \parallel y$ and $z \parallel x$, i.e. $z \diamond x$ and $z \diamond y$. Hence,

$$[x]_\approx \cap [y]_\approx = \{x, y\} \cup \{z \in X \mid z \diamond x \text{ and } z \diamond y\} .$$

Next, suppose that $x \not\approx y$ and $[x]_\approx \cap [y]_\approx \neq \emptyset$ and let $z \in [x]_\approx \cap [y]_\approx$. Since $x \neq z \neq y$, it follows from Proposition 3.3 that ($z \ll x$, $x \ll z$ or $z \parallel x$) and ($z \ll y$, $y \ll z$ or $z \parallel y$), i.e. there are nine cases to consider. From Proposition 3.6, it follows that the six cases ($x \ll z$ and $y \ll z$), ($x \ll z$ and $z \parallel y$), ($z \parallel x$ and $y \ll z$), ($z \ll x$ and $z \ll y$), ($z \ll x$ and $z \parallel y$) and ($z \parallel x$ and $z \ll y$) are impossible. Next, we will study the three remaining cases.

(a) Suppose that ($z \parallel x$ and $z \parallel y$). Since $z \approx x$ and $z \approx y$, it follows from

Proposition 3.5(iii) that ($x \approx y$ and $x \parallel y$), a contradiction. Hence, ($z \parallel x$ and $z \parallel y$) is also an impossible case.

(b) For the two cases ($z \ll x$ and $y \ll z$) and ($x \ll z$ and $z \ll y$), we will show that the intersection of $[x]_\approx$ and $[y]_\approx$ is the singleton containing the unique element z such that $y \ll z \ll x$ (first case) or $x \ll z \ll y$ (second case).

(α) Suppose that ($z \ll x$ and $y \ll z$). We will show next that $[x]_\approx \cap [y]_\approx = \{z\}$. Indeed, suppose that there exists another element t such that $t \approx x$ and $t \approx y$. Since $t \approx x$ and $z < x$, it follows that $z < t$. On the other hand, since $t \approx y$ and $y < z$, it follows that $t < z$, a contradiction. Hence, z is unique.

Next, we will show that $[z]_\approx = \{x, y, z\}$. Let $u \in [z]_\approx$ such that $u \neq z$. If $x \neq u$ and $y \neq u$, then it follows from $u \approx z$ and $y \ll z \ll x$ that $y < u < x$. This implies that $z \parallel u$. Hence, ($z \approx x$ and $z \ll x$) and ($z \approx u$ and $z \parallel u$), and according to Proposition 3.6(iv) this is an impossible case. Therefore $[z]_\approx = \{x, y, z\}$.

(β) Suppose that ($x \ll z$ and $z \ll y$), then it follows that $x < z < y$. In the same way as in (a), it follows that $[x]_\approx \cap [y]_\approx = \{z\}$, and $[z]_\approx = \{x, y, z\}$.

□

Example 3.7. *Let $(X, \leqslant)$ be the poset in Example 3.6. It holds that*

$$\begin{aligned}
[0]_\approx \cap [a]_\approx &= \{0, a\} \\
[a]_\approx \cap [b]_\approx &= \{a, b\} \\
[0]_\approx \cap [b]_\approx &= \{a\} \\
[c]_\approx \cap [d]_\approx &= [c]_\approx \cap [e]_\approx = [d]_\approx \cap [e]_\approx = \{c, d, e\}\,.
\end{aligned}$$

3.3. Clone relation of the union and linear sum of posets

In this section, we will characterize the clone relation of the union and linear sum of posets. First, we recall the definition of a subposet of a poset and of the linear sum of posets.

Definition 3.5. *A poset $p = (P, \leqslant_P)$ is called a subposet of a poset $q = (Q, \leqslant_Q)$ if*

(i) $P \subseteq Q$,

(ii) For any $x, y \in P$, it holds that $x \leqslant_P y$ if and only if $x \leqslant_Q y$.

Definition 3.6. [21] *Let $p = (P, \leqslant_P)$ and $q = (Q, \leqslant_Q)$ be two disjoint posets.*

(i) The union $p \cup q$ of p and q is the poset $p \cup q = (P \cup Q, \leqslant)$ with

$$\leqslant = \leqslant_P \cup \leqslant_Q \ .$$

In other words, $x \leqslant y$ if and only if ($x, y \in P$ and $x \leqslant_P y$) or ($x, y \in Q$ and $x \leqslant_Q y$).

(ii) The linear sum $p \oplus q$ of p and q is the poset $p \oplus q = (P \cup Q, \leqslant)$ with

$$\leqslant = \leqslant_P \cup \leqslant_Q \cup (P \times Q)\,.$$

In other words, $x \leqslant y$ if and only if one of the following cases holds: ($x, y \in P$ and $x \leqslant_P y$), ($x, y \in Q$ and $x \leqslant_Q y$) or ($x \in P$ and $y \in Q$).

In case P and Q are finite, a Hasse diagram of $p \oplus q$ is obtained by placing a Hasse diagram of p below a Hasse diagram of q and adding a line segment from every maximal element of p to every minimal element of q.

Proposition 3.8. *Let $p = (P, \leqslant_P)$ be a subposet of a poset $q = (Q, \leqslant_Q)$ with clone relations $\approx_P$ and $\approx_Q$. For any $x, y \in P$, it holds that $x \approx_Q y$ implies that $x \approx_P y$.*

Proof. Let $x, y \in P$. Suppose that $x \approx_Q y$, then ($z <_Q x \Leftrightarrow z <_Q y$) and ($x <_Q z \Leftrightarrow y <_Q z$), for any $z \in Q \setminus \{x, y\}$. Since p is a subposet of q, it follows that ($z <_Q x \Leftrightarrow z <_P x$, for any $z \in P \setminus \{x, y\}$) and ($z <_Q y \Leftrightarrow z <_P y$, for any $z \in P \setminus \{x, y\}$), whence $z <_P x \Leftrightarrow z <_P y$, for any $z \in P \setminus \{x, y\}$. In the same way, it follows that $x <_P z \Leftrightarrow y <_P z$, for any $z \in P \setminus \{x, y\}$. Hence, $x \approx_P y$. □

Remark 3.2. *Note that the converse of the statement in Proposition 3.8 does not hold: if $x \approx_P y$, for some $x, y \in P$, then it does not necessarily hold that $x \approx_Q y$. Indeed, let $p = (P, \leqslant_P)$ be the poset with $P = \{0, a, b, 1\}$ and $q = (Q, \leqslant_Q)$ be the poset with $Q = \{0, a, b, c, 1\}$ represented by the Hasse diagrams in Figure 3.6. Clearly, $(P, \leqslant_P)$ is a subposet of $(Q, \leqslant_Q)$, $a \approx_P b$, while $a \not\approx_Q b$.*

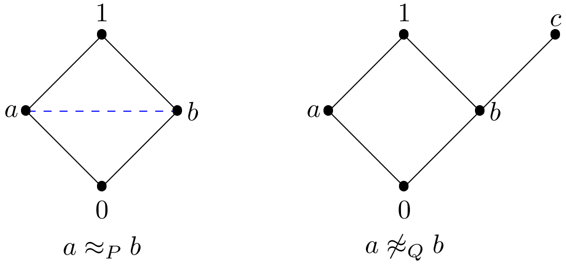

Figure 3.6: The poset $(P, \leqslant_P)$ is a subposet of $(Q, \leqslant_Q)$, $a \approx_P b$ and $a \not\approx_Q b$.

Let $p = (P, \leqslant_P)$ and $q = (Q, \leqslant_Q)$ be two posets. Consider the subset $P_{\|}$ of P and $Q_{\|}$ of Q defined as

$$\begin{aligned} P_{\|} &= \{x \in P \mid (\forall z \in P \setminus \{x\})(x \parallel z)\} \\ Q_{\|} &= \{y \in Q \mid (\forall z \in Q \setminus \{y\})(y \parallel z)\} \,. \end{aligned}$$

Proposition 3.9. *Let $p = (P, \leqslant_P)$ and $q = (Q, \leqslant_Q)$ be two disjoint posets with clone relations $\approx_P$ and $\approx_Q$. The clone relation $\approx$ of the union $p \cup q = (P \cup Q, \leqslant)$ is given by*

$$\approx \, = \, \approx_P \cup \approx_Q \cup (P_{\|} \times Q_{\|}) \cup (Q_{\|} \times P_{\|}) \,.$$

Proof. Since P and Q are disjoint, it follows immediately that

$$\approx_P \cup \approx_Q \cup (P_{\|} \times Q_{\|}) \cup (Q_{\|} \times P_{\|}) \subseteq \approx \,.$$

We will now show the converse inclusion. Let $x, y \in P \cup Q$ such that $x \approx y$. There are four cases to consider: ($x \in P$ and $y \in P$), ($x \in Q$ and $y \in Q$), ($x \in P$ and $y \in Q$) or ($x \in Q$ and $y \in P$).

(i) Suppose that $x, y \in P$. Since $(P, \leqslant_P)$ is a subposet of $(P \cup Q, \leqslant)$, it follows from Proposition 3.8 that $x \approx y$ implies $x \approx_P y$.

(ii) If $x, y \in Q$, then again from Proposition 3.8 it follows that $x \approx y$ implies $x \approx_Q y$.

(iii) If $x \in P$ and $y \in Q$, then one of the following cases holds: ($x \in P \setminus P_{\|}$ and $y \in Q$), ($x \in P$ and $y \in Q \setminus Q_{\|}$) or ($x \in P_{\|}$ and $y \in Q_{\|}$).

   (a) Suppose that $x \in P \setminus P_{\|}$ and $y \in Q$, then there exists $z \in P$ such that $z <_P x$ or $x <_P z$, and hence $z < x$ or $x < z$. Since $y \in Q$, it follows that $z \parallel y$. Hence, $x \not\approx y$, a contradiction.

   (b) Suppose that $x \in P$ and $y \in Q \setminus Q_{\|}$, then as in (a), it follows that $x \not\approx y$, a contradiction.

   (c) Since the previous cases lead to a contradiction, it must hold that $(x, y) \in P_{\|} \times Q_{\|}$.

(iv) If $x \in Q$ and $y \in P$, then it follows from (iii) and the symmetry of $\approx$ that $(x, y) \in Q_{\|} \times P_{\|}$.

□

Proposition 3.10. *Let $p = (P, \leqslant_P)$ and $q = (Q, \leqslant_Q)$ be two disjoint posets with clone relations $\approx_P$ and $\approx_Q$. The clone relation $\approx$ of the linear sum $p \oplus q$ is given by* *(i) If* $p = (P, \leqslant_P)$ *has greatest element* 1_P *and* $q = (Q, \leqslant_Q)$ *has smallest element* 0_Q*, then*

$$\approx = \approx_P \cup \approx_Q \cup \{(1_P, 0_Q), (0_Q, 1_P)\}\,.$$

(ii) If $p = (P, \leqslant_P)$ *has no greatest element or* $q = (Q, \leqslant_Q)$ *has no smallest element, then*

$$\approx = \approx_P \cup \approx_Q \ .$$

Proof. We prove the equality in statement (i), the proof of (ii) being a simple adaptation to the absence of a greatest element of p or a smallest element of q. Suppose that $p = (P, \leqslant_P)$ has greatest element 1_P and $q = (Q, \leqslant_Q)$ has smallest element 0_Q. Since P and Q are disjoint, it follows immediately that $\approx_P \cup \approx_Q \cup \{(1_P, 0_Q), (0_Q, 1_P)\} \subseteq \approx$. We will now show the converse inclusion. Let $x, y \in P \cup Q$ such that $x \approx y$. Since $1_P \ll 0_Q$, there are six cases to consider: ($x \in P$ and $y \in P$), ($x \in Q$ and $y \in Q$), ($x \in P$ and $y \in Q \setminus \{0_Q\}$), ($x \in P \setminus \{1_P\}$ and $y \in Q$), ($x = 1_P$ and $y = 0_Q$) or ($x = 0_Q$ and $y = 1_P$).

(a) Suppose that $x, y \in P$. Since p is a subposet of $p \oplus q$, it follows from Proposition 3.8 that $x \approx y$ implies $x \approx_P y$.

(b) Similarly, if $x, y \in Q$, then it follows that $x \approx_Q y$.

(c) Suppose that $x \in P$ and $y \in Q \setminus \{0_Q\}$. Since $x \approx y$ and $0_Q < y$, it follows that $0_Q < x$. Hence, $0_Q < 1_P$, a contradiction. Hence, this is an impossible case.

(d) Similarly, $x \in P \setminus \{1_P\}$ and $y \in Q$ is an impossible case.

(e) Suppose that $x = 1_P$ and $y = 0_Q$. Let $z \in P \cup Q \setminus \{1_P, 0_Q\}$. If $z < 1_P$, then it holds that $z < 0_Q$. Also, if $z < 0_Q$, then it follows that $z \in P$, and hence $z < 1_P$. If $1_P < z$, then it follows that $z \in Q$, and hence $0_Q < z$. Also, if $0_Q < z$, then it holds that $1_P < z$. We conclude that $z < 1_P$ if and only if $z < 0_P$, and $1_P < z$ if and only if $0_Q < z$. Thus, $1_P \approx 0_Q$.

(f) Suppose that $x = 0_Q$ and $y = 1_P$, then it follows from the symmetry of $\approx$ and (e) that $0_Q \approx 1_P$.

□

3.4. Weak compatibility of a strict order with L-tolerance relations

In this section, we will study the weak compatibility of a strict order relation with an L-tolerance or L-equivalence relation. Let $(X, \leqslant)$ be a poset and R be an

L-relation on X, then the weak compatibility of $<$ with R states that

$$\tau(x < y) * R(x,z) * R(y,t) \leq \tau(z < t)\,, \tag{3.3}$$

for any deferent elements $x, y, z, t \in X$.

3.4.1. Weak compatibility of a strict order with an L-tolerance relation

In this subsection, we will study the weak compatibility of a strict order relation with an L-tolerance relation. First we need to show the following key results.

Proposition 3.11. *The strict order relation $<$ of a poset $(X, \leqslant)$ is weakly compatible with the clone relation $\approx$.*

Proof. Let $x, y, z, t \in X$ a deferent elements, then we need to prove that

$$\tau(x < y) * \tau(x \approx z) * \tau(y \approx t) \leq \tau(z < t)\,.$$

Since ($x < y$ and $x \approx z$) implies $z < y$, it follows that $\tau(x < y) * \tau(x \approx z) \leq \tau(z < y)$. Due to the monotonicity of $*$, it holds that

$$\tau(x < y) * \tau(x \approx z) * \tau(y \approx t) \leq \tau(z < y) * \tau(y \approx t)\,.$$

Also, since ($z < y$ and $y \approx t$) implies $z < t$, it follows that $\tau(z < y) * \tau(y \approx t) \leq \tau(z < t)$. Hence,

$$\tau(x < y) * \tau(x \approx z) * \tau(y \approx t) \leq \tau(z < y) * \tau(y \approx t) \leq \tau(z < t)\,.$$

Thus, $<$ is weakly compatible with $\approx$. ☐

In the following theorem we show that the weak compatibility of the strict order relation of a poset with an L-tolerance relation is equivalent with the inclusion of the L-tolerance relation in the clone relation of that poset.

Theorem 3.3. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-tolerance relation on X. Then it holds that $<$ is weakly compatible with E if and only if $E \subseteq \approx$.*

Proof. Suppose that $<$ is weakly compatible with E, i.e. $\tau(x < y) * E(x,z) * E(y,t) \leq \tau(z < t)$, for any deferent elements $x, y, z, t \in X$. Let $a, b \in X$. If $E(a,b) = 0$, then it trivially holds that $E(a,b) \leq \tau(a \approx b)$. If $E(a,b) > 0$, then we need to show that $\tau(a \approx b) > 0$, i.e. $a \approx b$. Let $c \in X \setminus \{a, b\}$, then we need to consider four subcases.

(i) If $a < c$, then since E is reflexive and $<$ is weakly compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(a < c)}_{=1} * \underbrace{E(a,b)}_{>0} * \underbrace{E(c,c)}_{=1} \leq \tau(b < c)\,.$$

This implies $\tau(b < c) > 0$. Hence, $\tau(b < c) = 1$, i.e. $b < c$.

(ii) If $b < c$, then since E is reflexive, symmetric and $<$ is weakly compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(b < c)}_{=1} * \underbrace{E(b,a)}_{=E(a,b)>0} * \underbrace{E(c,c)}_{=1} \leq \tau(a < c)\,.$$

This implies $\tau(a < c) > 0$. Hence, $\tau(a < c) = 1$, i.e. $a < c$.

(iii) If $c < a$, then since E is reflexive and $<$ is weakly compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(c < a)}_{=1} * \underbrace{E(c,c)}_{=1} * \underbrace{E(a,b)}_{>0} \leq \tau(c < b)\,.$$

This implies $\tau(c < b) > 0$. Hence, $\tau(c < b) = 1$, i.e. $c < b$.

(iv) If $c < b$, then since E is reflexive, symmetric and $<$ is weakly compatible with E, it follows that

$$0 < E(a,b) = \underbrace{\tau(c < b)}_{=1} * \underbrace{E(c,c)}_{=1} * \underbrace{E(b,a)}_{=E(a,b)>0} \leq \tau(c < a)\,.$$

This implies $\tau(c < a) > 0$. Hence, $\tau(c < a) = 1$, i.e. $c < a$.

Hence, if $E(a,b) > 0$, then $a \approx b$. We conclude that $E \subseteq \approx$.

Conversely, if $E \subseteq \approx$, then due to the monotonicity of $*$, it holds that

$$\tau(x < y) * E(x,z) * E(y,t) \leq \tau(x < y) * \tau(x \approx z) * \tau(y \approx t)\,,$$

for any $x, y, z, t \in X$. From Proposition 3.11, it follows that

$$\tau(x < y) * E(x,z) * E(y,t) \leq \tau(z < t)\,,$$

for any $x, y, z, t \in X$. Hence, $<$ is weakly compatible with E. □

The above theorem implies that the clone relation $\approx$ (resp. the crisp equality) is the greatest (resp. smallest) L-tolerance relation on X such that $<$ is weakly compatible with it.

The following lemma will be instrumental in the proof of our representation theorems.

Lemma 3.1. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. Let α and β be two L-tolerance relations on X such that $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$, then the union $E = \alpha \cup \beta$ is the L-tolerance relation on X given by*

$$\alpha \cup \beta(x,y) = \begin{cases} 1 & \text{, if } x = y, \\ \alpha(x,y) & \text{, if } x \lhd y \text{ or } x \rhd y, \\ \beta(x,y) & \text{, if } x \diamond y, \\ 0 & \text{, if } x \not\approx y. \end{cases}$$

Theorem 3.4. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-tolerance relation on X. Then it holds that $<$ is weakly compatible with E if and only if there exists an L-tolerance relation α on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and an L-tolerance relation β on X with $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$.*

Proof. Suppose that $<$ is weakly compatible with E. Consider the L-relations α and β on X defined by

$$\alpha(x,y) = \begin{cases} E(x,y) & \text{, if } (x,y) \in \lhd \cup \rhd \cup = , \\ 0 & \text{, otherwise} , \end{cases}$$

$$\beta(x,y) = \begin{cases} E(x,y) & \text{, if } (x,y) \in \diamond \cup = , \\ 0 & \text{, otherwise} . \end{cases}$$

Note that if $(x,y) \in \lhd \cup \rhd \cup =$, then it also holds that $(y,x) \in \lhd \cup \rhd \cup =$. Similarly, if $(x,y) \in \diamond \cup =$, then also $(y,x) \in \diamond \cup =$. Hence, α and β are L-tolerance relations on X. Since $<$ is weakly compatible with E, it follows from Theorem 3.3 that $E \subseteq \approx$, whence $E(x,y) = 0$ if $x \not\approx y$. Using Lemma 3.1, it follows that $E = \alpha \cup \beta$.

Conversely, let $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$ be two L-tolerance relations on X. Lemma 3.1 implies that $E = \alpha \cup \beta$ is an L-tolerance relation on X. Since $E \subseteq \approx$, Theorem 3.3 guarantees that $<$ is weakly compatible with E. ☐

As corollary, we obtain the following representation of the crisp tolerance relations a strict order relation is weakly compatible with.

Corollary 3.6. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be a crisp tolerance relation on X. Then it holds that $<$ is weakly compatible with E if and only if there exists a crisp tolerance relation α on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and a crisp tolerance relation β on X with $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$.*

Remark 3.3. *In the setting of Corollary 3.6, we have:*

(i) If α and β are the crisp equality on X, then so is E.

(ii) If $\alpha = \lhd \cup \rhd \cup =$ and $\beta = \diamond \cup =$, then $E = \approx$.

3.4.2. Weak compatibility of a strict order with an L-equivalence relation

In this subsection, we will study the weak compatibility of a strict order with an L-equivalence relation. First, we mention the following proposition.

Proposition 3.12. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$. Let α and β be two L-equivalence relations on X such that $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$, then the union $E = \alpha \cup \beta$ is an L-equivalence relation on X.*

Proof. Due to Lemma 3.1, E is an L-tolerance relation. It remains to show that E is transitive. Let $x, y, z \in X$, then we need to show that

$$E(x, y) * E(y, z) \leq E(x, z)\,.$$

We consider the following cases.

(i) If $x = y$ or $y = z$, then the inequality trivially holds.

(ii) Due to Proposition 3.6, the cases ($x \lhd y$ and $y \rhd z$), ($x \lhd y$ and $y \diamond z$), ($x \rhd y$ and $y \lhd z$) or ($x \rhd y$ and $y \diamond z$) are impossible.

(iii) If $x \lhd y$ and $y \lhd z$, then $E(x, y) = \alpha(x, y)$ and $E(y, z) = \alpha(y, z)$. The anti-transitivity of $\lhd$ implies that $(x, z) \notin \lhd \cup \rhd \cup =$. Since also $(x, z) \notin \diamond \cup =$, it follows that $\alpha(x, z) = \beta(x, z) = 0$ and thus $E(x, z) = 0$. Since α is $*$-transitive, it holds that $\alpha(x, y) * \alpha(y, z) \leq \alpha(x, z)$, i.e. $E(x, y) * E(y, z) \leq E(x, z)$.

(iv) If $x \rhd y$ and $y \rhd z$, then the proof is similar to that of (iii).

(v) If $x \diamond y$ and $y \diamond z$, then $E(x, y) = \beta(x, y)$ and $E(y, z) = \beta(y, z)$. The transitivity of $\diamond$ implies that $E(x, z) = \beta(x, z)$. Since β is $*$-transitive, it holds that $\beta(x, y) * \beta(y, z) \leq \beta(x, z)$, i.e. $E(x, y) * E(y, z) \leq E(x, z)$.

(vi) If $x \not\approx y$ or $y \not\approx z$, then $E(x, y) = 0$ or $E(y, z) = 0$ and it trivially holds that $E(x, y) * E(y, z) \leq E(x, z)$.

We conclude that $E = \alpha \cup \beta$ is an L-equivalence relation on X. □

Combining Theorem 3.4 and Proposition 3.12 easily leads to the following theorem.

Theorem 3.5. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-equivalence relation on X. Then it holds that $<$ is weakly compatible with E if and only if there exist two L-equivalence relations α and β on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$.*

Proof. Theorem 3.4 states that $<$ is weakly compatible with E if and only if there exist two L-tolerance relations α and β on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$, where

$$\alpha(x,y) = \begin{cases} E(x,y) & \text{, if } (x,y) \in \lhd \cup \rhd \cup = , \\ 0 & \text{, otherwise} , \end{cases}$$

$$\beta(x,y) = \begin{cases} E(x,y) & \text{, if } (x,y) \in \diamond \cup = , \\ 0 & \text{, otherwise} . \end{cases}$$

Let $x, y, z \in X$, then we need to show that

$$\alpha(x,y) * \alpha(y,z) \leq \alpha(x,z)$$

and

$$\beta(x,y) * \beta(y,z) \leq \beta(x,z) .$$

We consider the following cases.

(i) If $x = y$ or $y = z$, then these inequalities trivially hold.

(ii) Due to Proposition 3.6, the cases ($x \lhd y$ and $y \rhd z$), ($x \lhd y$ and $y \diamond z$), ($x \rhd y$ and $y \lhd z$) or ($x \rhd y$ and $y \diamond z$) are impossible.

(iii) If $x \lhd y$ and $y \lhd z$, then $\alpha(x,y) = E(x,y)$ and $\alpha(y,z) = E(y,z)$. The anti-transitivity of $\lhd$ implies that $(x,z) \notin \lhd \cup \rhd \cup =$, and hence $\alpha(x,z) = 0$. Since also $(x,z) \notin \diamond \cup =$, it follows that $\beta(x,z) = 0$ and thus $E(x,z) = 0$. Since E is $*$-transitive, it holds that $E(x,y) * E(y,z) \leq E(x,z)$, i.e. $E(x,y) * E(y,z) = 0$. Thus $\alpha(x,y) * \alpha(y,z) = 0 \leq \alpha(x,z)$. Also, in this case it holds that $\beta(x,y) = 0$ and $\beta(y,z) = 0$, which means that $\beta(x,y) * \beta(y,z) \leq \beta(x,z)$.

(iv) If $x \rhd y$ and $y \rhd z$, then the proof is similar to that of (iii).

(v) If $x \diamond y$ and $y \diamond z$, then $\beta(x,y) = E(x,y)$ and $\beta(y,z) = E(y,z)$. The transitivity of $\diamond$ implies that $\beta(x,z) = E(x,z)$. Since E is $*$-transitive, it holds that $E(x,y) * E(y,z) \leq E(x,z)$, i.e. $\beta(x,y) * \beta(y,z) \leq \beta(x,z)$. Also, in this case it holds that $\alpha(x,y) = 0$ and $\alpha(y,z) = 0$. Thus, it trivially holds that $\alpha(x,y) * \alpha(y,z) \leq \alpha(x,z)$.

(vi) If $x \not\approx y$ or $y \not\approx z$, then ($\alpha(x,y) = 0$ and $\beta(x,y) = 0$) or ($\alpha(y,z) = 0$ and $\beta(y,z) = 0$) and it trivially holds that $\alpha(x,y) * \alpha(y,z) \leq \alpha(x,z)$ and $\beta(x,y) * \beta(y,z) \leq \beta(x,z)$.

We conclude that α and β are L-equivalence relations on X.

For the converse, Proposition 3.12 guarantees that $E = \alpha \cup \beta$ is an L-equivalence relation on X. □

As corollary, we obtain the following representation of the L-equality relations or crisp equivalence relations a strict order relation is weakly compatible with.

Corollary 3.7. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be an L-equality relation on X. Then it holds that $<$ is weakly compatible with E if and only if there exist two L-equality relations α and β on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$.*

Corollary 3.8. *Let $(X, \leqslant)$ be a poset with clone relation $\approx$ and E be a crisp equivalence relation on X. Then it holds that $<$ is weakly compatible with E if and only if there exists two crisp equivalence relations α and β on X with $\alpha \subseteq \lhd \cup \rhd \cup =$ and $\beta \subseteq \diamond \cup =$ such that $E = \alpha \cup \beta$.*

3.4.3. Some examples

Example 3.8. *In Figure 3.7, the Hasse diagram of a chain $(\{0, a, 1\}, \leqslant)$ is depicted. The only clonal relations are $0 \lhd a$ and $a \lhd 1$. Hence, any L-tolerance relation E the strict order relation $<$ is weakly compatible with, is determined by two arbitrary values α_0 and α_1 in L such that E is given by*

$E(.,.)$	0	a	1
0	1	α_0	0
a	α_0	1	α_1
1	0	α_1	1

If additionally it holds that $\alpha_0 * \alpha_1 = 0$, then we obtain the general representation of an L-equivalence relation the strict order relation $<$ is weakly compatible with.

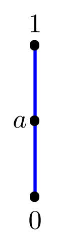

Figure 3.7: Hasse diagram of the poset in Example 3.8.

Example 3.9. *In Figure 3.8, the Hasse diagram of a poset $(X, \leqslant)$ with $X = \{0, a, b, c, 1\}$ is depicted. The only clonal relations are $a \diamond b$, $b \diamond c$ and $a \diamond c$. Hence, any L-tolerance relation E the strict order relation $<$ is weakly compatible with, is determined by three arbitrary values β_0, β_1 and β_2 in L such that E is given by*

$E(.,.)$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	β_0	β_1	0
b	0	β_0	1	β_2	0
c	0	β_1	β_2	1	0
1	0	0	0	0	1

Also, E is an L-equivalence relation on X if and only if

$$\beta_0 * \beta_1 \leq \beta_2$$
$$\beta_0 * \beta_2 \leq \beta_1$$
$$\beta_1 * \beta_2 \leq \beta_0 .$$

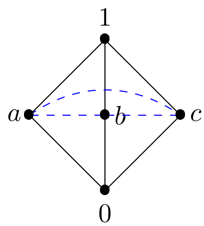

Figure 3.8: Hasse diagram of the poset in Example 3.9.

Example 3.10. *Let $X = \{0, a, b, c, d, 1\}$, E be an L-equivalence relation on X and $<$ be a strict order on X given by the following Hasse diagram:*

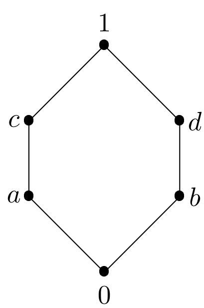

Figure 3.9: Hasse diagram of the poset in Example 3.10.

Suppose that $<$ is weakly compatible with E. Since $(a \lhd c)$, $(b \lhd d)$, then from Theorem 3.4, it follows that $E(a,c) = E(c,a) = \alpha(a,c)$, $E(b,d) = E(d,b) = \alpha(b,d)$, $E(x,x) = 1$, for any $x \in X$ and $E(x,y) = 0$, otherwise.

$E(.,.)$	*0*	a	b	c	d	*1*
0	*1*	*0*	*0*	*0*	*0*	*0*
a	*0*	*1*	*0*	$\alpha(a,c)$	*0*	*0*
b	*0*	*0*	*1*	*0*	$\alpha(b,d)$	*0*
c	*0*	$\alpha(a,c)$	*0*	*1*	*0*	*0*
d	*0*	*0*	$\alpha(b,d)$	*0*	*1*	*0*
1	*0*	*0*	*0*	*0*	*0*	*1*

Example 3.11. *Let $X = \{0, a, b, c, d, 1\}$ and $<$ be the strict crisp order on X given by the following Hasse diagram (Fig. 5).*

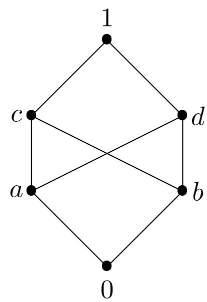

Figure 3.10: Hasse diagram of the poset in Example 3.11.

Note that $(X, <)$ is a poset.
In this example, $(a \diamond b)$. Hence $E(a, b) = \beta(a, b)$.
$(c \diamond d)$, hence $E(c, d) = \beta(c, d)$.
E is symmetric then also we have $E(b, a) = \beta(a, b)$ and $E(d, c) = \beta(c, d)$.
$E(x, x) = 1$, $\forall x \in \{0, a, b, c, d, 1\}$ and $E(x, y) = 0$, otherwise.
We obtain following table of E in which the strict order is weakly compatible with E:

E	*0*	a	b	c	d	*1*
0	*1*	*0*	*0*	*0*	*0*	*0*
a	*0*	*1*	$\beta(a,b)$	*0*	*0*	*0*
b	*0*	$\beta(a,b)$	*1*	*0*	*0*	*0*
c	*0*	*0*	*0*	*1*	$\beta(c,d)$	*0*
d	*0*	*0*	*0*	$\beta(d,c)$	*1*	*0*
1	*0*	*0*	*0*	*0*	*0*	*1*

Example 3.12. *Let $X = \{0, a, b, c, d, 1\}$ and $<$ the strict order on X given by the following Hasse diagram (Fig. 6).*

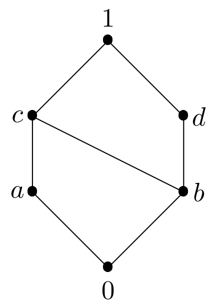

Figure 3.11: Hasse diagram of the poset in Example 3.12.

In this example, there exists no $x, y \in X$ *in which* $x \lhd y$ *or* $x \diamond y$. *Therefore* $E(x, x) = 1 \ \forall x \in X$ *and* $E(x, y) = 0$ *otherwise.*
Thus the $L-$*equivalence* E *in this example is the crisp equality.*
We obtain following table of E in which the strict order is weakly compatible with E:

E	*0*	*a*	*b*	*c*	*d*	*1*
0	*1*	*0*	*0*	*0*	*0*	*0*
a	*0*	*1*	*0*	*0*	*0*	*0*
b	*0*	*0*	*1*	*0*	*0*	*0*
c	*0*	*0*	*0*	*1*	*0*	*0*
d	*0*	*0*	*0*	*0*	*1*	*0*
1	*0*	*0*	*0*	*0*	*0*	*1*

4 Right and left compatibility of a strict order with L-relations

In chapter 2, we have investigated the notion of the left compatibility, right compatibility and compatibility of arbitrary fuzzy relations and we characterized them in terms of left and right traces introduced by Fodor. In this chapter, we will again investigate these notions, but this time we pay the attention to the weak left and weak right compatibility of a strict order relations with fuzzy tolerance and fuzzy equivalence relations.

In this chapter, we provide characterizations of the right and left weak compatibility of a crisp strict order with fuzzy tolerance or fuzzy equivalence relations. The characterization of both the left and the right weak compatibility of a strict order relations with fuzzy tolerance or fuzzy equivalence relations leads to the same characterization of the weak compatibility obtained in Chapter 3.

4.1. Right and left compatibility of an order relation with L-equivalences

In this section, we first characterize crisp equivalence and L-equivalence relations in which crisp order is compatible with respect to them. Second, we characterize the same relations in which crisp strict order is compatible with respect to them.

4.1.1. Right and left compatibility of an order with equivalence relations

Theorem 4.1. *Let $\leqslant$, E be respectively a crisp order and a crisp equivalence relations on a nonempty domain X. Then*

(i) *$\leqslant$ is right compatible w.r.t. E if and only if E is a crisp equality on X.*

(ii) *$\leqslant$ is left compatible w.r.t. E if and only if E is a crisp equality on X.*

Proof. (i) If E is the crisp equality $=$, then it is obvious that $\leqslant$ is right compatible w.r.t. E.

For the other direction, assume that $\leqslant$ is right compatible w.r.t. E and we shall show that E is a crisp equality.

We know that a crisp equivalence relation E on X such that

$$\forall\, x, y \in X : E(x, y) = 1 \Longleftrightarrow x = y$$

is a crisp equality.
Suppose that E is not a crisp equality, i.e.; there exists $z, y \in X$ such that: $E(z, y) = 1$ and $z \neq y$.
Since $\leqslant$ is right compatible w.r.t. E then for all $x, y, z \in X$:

$$\tau(x \leqslant y) * E(y, z) \leq \tau(x \leqslant z).$$

If $z \neq y$ then we have three case to consider $z < y$, $y < z$ or y and z are incomparable.

(1) Assume that $z < y$
The right compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(y \leqslant y) * E(z, y) \leq \tau(y \leqslant z).$$

We have $\tau(y \leqslant z) = 0$ (since $z < y$ and $\leqslant$ is antisymmetric). Hence $1 * 1 \leq 0$, and this is a contradiction. Thus $z \nless y$.

(2) Assume that $y < z$
The right compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(z \leqslant z) * E(z, y) \leq \tau(z \leqslant y).$$

$\tau(z \leqslant y) = 0$ (since $y < z$ and $\leqslant$ is antisymmetric). Hence $1 * 1 \leq 0$, and this is a contradiction. Thus $y \nless z$.

(3) Assume that $y \parallel z$
The right compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(z \leqslant z) * E(z, y) \leq \tau(z \leqslant y).$$

$\tau(z \leqslant y) = 0$ (since $y \parallel z$). Hence $1 * 1 \leq 0$, and this is also a contradiction. Thus $y \nparallel z$.

If $z \nless y$, $y \nless z$ and $y \nparallel z$ then the only possibility is $y = z$.
Thus, E is the crisp equality.

(ii) If E is the crisp equality $=$, then it is obvious that $\leqslant$ is left compatible w.r.t. E.

For the other direction, suppose that $\leqslant$ is left compatible w.r.t. E and we shall show that E is a crisp equality.
Suppose that E is not a crisp equality, i.e.; there exists $z, y \in X$ such that: $E(z, y) = 1$ and $z \neq y$.
Since $\leqslant$ is left compatible w.r.t. E then for any $x, y, z \in X$:

$$\tau(x \leqslant y) * E(x, z) \leq \tau(z \leqslant y).$$

If $z \neq y$ then we have three case to consider $z < y$, $y < z$ or y and z are incomparable.

(1) Assume that $z < y$
The left compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(z \leqslant z) * E(z, y) \leq \tau(y \leqslant z).$$

We have $\tau(y \leqslant z) = 0$ (since $z < y$ and $\leqslant$ is antisymmetric). Hence $1 * 1 \leq 0$, and this is a contradiction. Thus $z \nless y$.

(2) Assume that $y < z$
The left compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(y \leqslant y) * E(y, z) \leq \tau(z \leqslant y).$$

$\tau(z \leqslant y) = 0$ (since $y < z$ and $\leqslant$ is antisymmetric). Hence $1 * 1 \leq 0$, and this is a contradiction. Thus $y \nless z$.

(3) Assume that $y \parallel z$
The left compatibility of $\leqslant$ w.r.t. E implies that

$$\tau(z \leqslant z) * E(z, y) \leq \tau(y \leqslant z).$$

$\tau(y \leqslant z) = 0$ (since $y \parallel z$). Hence $1 * 1 \leq 0$, and this is also a contradiction. Thus $y \nparallel z$.

If $z \nless y$, $y \nless z$ and $y \nparallel z$ then the only possibility is $y = z$.
Thus, E is the crisp equality.

□

4.1.2. Right and left compatibility of an order with L-equivalence relations

Theorem 4.2. *Let $\leqslant$ be a crisp order on X and E be an L- equivalence relation on X, then*

(i) *$\leqslant$ on X is left compatible w.r.t. E if and only if E is a crisp equality.*

(ii) *$\leqslant$ is right compatible w.r.t. E if and only if E is a crisp equality.*

Proof. (i) Suppose that $\leqslant$ is left compatible w.r.t. E. Then for all $x, y, z \in X$:

$$\tau(x \leqslant z) * E(x, y) \leq \tau(y \leqslant z) \tag{4.1}$$

We shall show that if $x \neq y$ then $E(x, y) = 0$. Indeed, suppose that $x \neq y$ then we have three cases to consider $x < y$ or $y < x$ or $x \parallel y$.

(a) If $x < y$, take $z = x$ the inequality (4.2) implies that $\tau(x \leqslant x)*E(x, y) \leq \tau(y \leqslant x)$. Hence $1 * E(x, y) \leq 0$, this implies that $E(x, y) = 0$.

(b) If $y < x$, set $x' = y, y' = x, z' = y$. From the inequality (4.2), it holds that $\tau(x' \leqslant z')*E(x', y') \leq \tau(y' \leqslant z')$, i.e. $\tau(y \leqslant y)*E(y, x) \leq \tau(x \leqslant y)$. Hence $1 * E(y, x) = 0$ and this implies that $E(x, y) = 0$.

(c) If $x \parallel y$, take $x = z$ and from the inequality (4.2) we obtain that $\tau(x \leqslant x) * E(x, y) \leq \tau(y \leqslant x)$. Hence $1 * E(x, y) \leq 0$, and thus $E(x, y) = 0$. Finally, we have if $x \neq y$ then $E(x, y) = 0$. Also since E is reflexive $x = y \Longrightarrow E(x, y) = 1$. Hence E is a crisp equality on X.

For the converse, it is obvious that if E is a crisp equality $=$, then the crisp order $\leqslant$ is left compatible w.r.t. $=$.

(ii) Suppose that $\leqslant$ is right compatible w.r.t. E. Then for any $x, y, z \in X$:

$$\tau(z \leqslant x) * E(x, y) \leq \tau(z \leqslant y) \tag{4.2}$$

We shall show that if $x \neq y$ then $E(x, y) = 0$. Indeed, suppose that $x \neq y$ then we have three cases to consider $x < y$ or $y < x$ or $x \parallel y$.

(a) If $y < x$, take $z = x$ the inequality (4.2) implies that $\tau(x \leqslant x)*E(x, y) \leq \tau(x \leqslant y)$. Hence $1 * E(x, y) \leq 0$, this implies that $E(x, y) = 0$.

(b) If $x < y$, set $x' = y, y' = x, z' = y$. From the inequality (4.2), it holds that $\tau(z' \leqslant x')*E(x', y') \leq \tau(z' \leqslant y')$, i.e. $\tau(y \leqslant y)*E(y, x) \leq \tau(y \leqslant x)$. Hence $1 * E(y, x) = 0$ and this implies that $E(x, y) = 0$.

(c) If $x \parallel y$, take $x = z$ and from the inequality (4.2) we obtain that $\tau(x \leqslant x) * E(x, y) \leq \tau(x \leqslant y)$. Hence $1 * E(x, y) \leq 0$, and thus

$E(x,y) = 0$. Finally, we have if $x \neq y$ then $E(x,y) = 0$. Also since E is reflexive $x = y \Longrightarrow E(x,y) = 1$. Hence E is a crisp equality on X.

For the converse, it is obvious that if E is a crisp equality $=$, then the crisp order $\leqslant$ is right compatible w.r.t. $=$.

□

4.2. Right and left compatibility of a total strict order with an equivalence relation

In this subsection, we characterize the crisp equivalence relations in which a total crisp strict order is compatible with respect to it.

Theorem 4.3. *Let $<$ a total crisp strict order on X, and E a crisp equivalence relation on X. Then*

(i) *$<$ is right compatible w.r.t. E if and only if E is a crisp equality on X.*

(ii) *$<$ is left compatible w.r.t. E if and only if E is a crisp equality on X.*

Proof. (i) If E is a crisp equality $=$, then it is obvious that $<$ is right compatible w.r.t. E.
For the converse, suppose that $<$ is right compatible.w.r.t. E and we show that E is a crisp equality $=$. It is known that a crisp equivalence relation E such that

$$\forall\ x, y \in X : E(x,y) = 1 \Longleftrightarrow x = y$$

is a crisp equality. Suppose that E is not a crisp equality, that is means that there $\exists z, y \in X$ such that $z \neq y$ and $E(z,y) = 1$.
Since $<$ is right compatible w.r.t. E, i.e.

$$\forall x, y, z \in X : \tau(x < y) * E(y,z) \leq \tau(x < z)$$

If $z \neq y$, then we have two cases to consider: $z < y$, $y < z$ (since $<$ is total).

(1) Suppose that $z < y$. The right compatibility of $<$ w.r.t. E implies that

$$\tau(z < y) * E(y,z) \leq \tau(z < z).$$

We have $\tau(z < z) = 0$ (since $<$ is a crisp strict order, i.e. $<$ is not reflexive). Hence $1 * 1 \leq 0$ and this is a contradiction. Thus $z \not< y$.

(2) Suppose that $y < z$. The right compatibility of $<$ w.r.t. E implies that

$$\tau(y < z) * E(y,z) \leq \tau(y < y).$$

This is a contradiction for the same reason as (1). Thus $y \nless z$.
Therefore, if $z \nless y$, $y \nless z$ and $<$ is total, then the only possibility is $y = z$. Thus E a crisp equality $=$.

(ii) If E is a crisp equality $=$, then it is obvious that $<$ is left compatible w.r.t. E.
For the converse, suppose that $<$ is left compatible.w.r.t. E and we show that E is a crisp equality $=$. Suppose that E is not a crisp equality, that is means that there $\exists z, y \in X$ such that $z \neq y$ and $E(z, y) = 1$.
Since $<$ is left compatible w.r.t. E, i.e.

$$\forall x, y, z \in X : \tau(y < x) * E(y, z) \leq \tau(z < x)$$

If $z \neq y$, then we have two cases to consider: $z < y$, $y < z$ (since $<$ is total).

(1) Suppose that $z < y$. The left compatibility of $<$ w.r.t. E implies that

$$\tau(z < y) * E(z, y) \leq \tau(y < y).$$

We have $\tau(y < y) = 0$ (since $<$ is a crisp strict order, i.e. $<$ is not reflexive). Hence $1 * 1 \leq 0$ and this is a contradiction. Thus $z \nless y$.

(2) Suppose that $y < z$. The right compatibility of $<$ w.r.t. E implies that

$$\tau(y < z) * E(y, z) \leq \tau(z < z).$$

This is a contradiction for the same reason as (1). Thus $y \nless z$.
Therefore, if $z \nless y$, $y \nless z$ and $<$ is total, then the only possibility is $y = z$. Thus E a crisp equality $=$.

□

Remark 4.1. *If the crisp strict order $<$ is not total, then we show latter that E is not necessarily the crisp equality.*

4.3. Right and left clone relations

First, we introduce some crisp relations (right clone and left clone), which shall be used to characterize the compatibility of strict orders with L-tolerance and L-equivalence relations.

Definition 4.1. *Let $(X, \leqslant)$ be a poset.*

(i) *The right clone relation* $\approx^r_<$ *associated with* $\leqslant$ *is the binary relation on* X *defined by*

$$x \approx^r_< y \text{ if and only if } x < z \Leftrightarrow y < z, \text{ for any } z \in X \setminus \{x, y\},$$

(i) *The left clone relation* $\approx^l_<$ *associated with* $\leqslant$ *is the binary relation on* X *defined by*

$$x \approx^l_< y \text{ if and only if } z < x \Leftrightarrow z < y, \text{ for any } z \in X \setminus \{x, y\};$$

(iii) *The clone relation* $\approx_<$ *associated with* $\leqslant$ *is the binary relation on* X *defined by*

$$x \approx_< y \text{ if and only if } x \approx^l_< y \text{ and } x \approx^r_< y.$$

Remark 4.2. *Note that* $\approx^l_< = \approx^r_{<^t}$ and $\approx^r_< = \approx^l_{<^t}$.

Useful properties of the relations $\approx^r_<$, $\approx^l_<$ and $\approx_<$ are given in the following propositions.

Proposition 4.1. *Let* $(X, \leq)$ *be a poset and* $x, y \in X$ *such that* $x \neq y$*. Then, it holds that*

(a) *If* $x \approx^r_< y$*, then* $x \ll y$*,* $y \ll x$ *or* $x \parallel y$,

(b) *If* $x \approx^l_< y$*, then* $x \ll y$*,* $y \ll x$ *or* $x \parallel y$.

Proof.

Let $x, y \in X$ such that $x \neq y$.

(a) Suppose that $x \approx^r_< y$. Since $x \neq y$, it follows that $x < y$ or $y < x$ or $x \parallel y$. Using the fact that $x \approx^r_< y$, it follows that $x \ll y$ or $y \ll x$ or $x \parallel y$. Thus, one of the statements (i) or (ii) or (iii) is satisfied. Conversely, It is clear that if one of the statements (i) or (ii) or (iii) is satisfied, then $x \approx^r_< y$.

(b) The proof is similar that of (a).

□Throughout this chapter, the following crisp relations $\lhd_r$, $\rhd_r$, $\diamond_r$, $\lhd_l$, $\rhd_l$ and $\diamond_l$ on X, are used.

$$x \lhd_r y \text{ if and only if } x \approx^r_< y \text{ and } x \ll y,$$

$$x \rhd_r y \text{ if and only if } x \approx^r_< y \text{ and } y \ll x,$$

$$x \diamond_r y \text{ if and only if } x \approx^r_< y \text{ and } x \parallel y.$$

$$x \lhd_l y \text{ if and only if } x \approx^l_< y \text{ and } x \ll y,$$

$$x \rhd_l y \text{ if and only if } x \approx^l_< y \text{ and } y \ll x\,,$$

$$x \diamond_l y \text{ if and only if } x \approx^l_< y \text{ and } x \parallel y\,.$$

Proposition 4.2. *Let $(X, \leq)$ be a poset and $x, y \in X$. Then, it holds that*

(a) *Exactly one of the following statements holds:*

(i) $x = y\,,$

(ii) $x \lhd_r y\,,$

(iii) $x \rhd_r y\,,$

(iv) $x \diamond_r y\,,$

(v) $x \not\approx^r_< y\,.$

(b) *Exactly one of the following statements holds:*

(i) $x = y\,,$

(ii) $x \lhd_l y\,,$

(iii) $x \rhd_l y\,,$

(iv) $x \diamond_l y\,,$

(v) $x \not\approx^l_< y\,.$

Proof.

(a) For any $x, y \in X$, $x = y$ or $x \neq y$.
Suppose that $x \neq y$, it follows that $x \approx^r_< y$ or $x \not\approx^r_< y$.
If $x \approx^r_< y$, then from Proposition 4.1, it follows that ($x \approx^r_< y$ and $x \ll y$) or ($x \approx^r_< y$ and $y \ll x$) or ($x \approx^r_< y$ and $x \parallel y$).

(b) The proof is similar to that of(a).

□

Proposition 4.3. *Let $(X, \leq)$ be a poset. Then it holds that $\approx^r_<$ and $\approx^l_<$ are tolerance relations.*

Proof.

It is obvious that $\approx^r_<$ and $\approx^l_<$ are reflexive and symmetric. □

Proposition 4.4. *Let $(X, \leq)$ be a poset. Then, it holds that*

(i) *The crisp relation $\diamond_r$ is transitive,*

(ii) *The crisp relation $\diamond_l$ is transitive.*

Proof.

Let $x, y, z \in X$ three distinct elements.

(i) Assume that $x \diamond_r y$ and $y \diamond_r z$, i.e. ($x \approx^r_< y$ and $x \parallel y$) and ($y \approx^r_< z$ and $y \parallel z$). We will show that $x \approx^r_< z$ and $x \parallel z$.

(a) Suppose that $x \not\parallel z$, it follows that $x < z$ or $z < x$. If $x < z$, then by the fact that $x \approx^r_< y$, it follows that $y < z$, i.e. $y \not\parallel z$, a contradiction. If $z < x$, then by the fact that $y \approx^r_< z$, it follows that $y < x$, i.e. $x \not\parallel y$, a contradiction. Hence, $x \parallel z$.

(b) Suppose that $x \not\approx^r_< z$, since $x \parallel z$, it follows that there exists $w \in X \setminus \{x, z\}$ such that ($z < w$ and $x \parallel w$) or ($x < w$ and $z \parallel w$). Since $x \approx^r_< y$ and $y \approx^r_< z$, it follows that ($y < w$ and $x \parallel w$) or ($y < w$ and $z \parallel w$), i.e. $x \not\approx^r_< y$ or $y \not\approx^r_< z$, a contradiction. Hence, $\diamond_r$ is transitive.

(ii) The proof is similar to that of(i).

□

Remark 4.3. *In general, the relations $\approx^r_<$ and $\approx^l_<$ are not transitive. Indeed, suppose that $x \approx^r_< y$ and $y \approx^r_< z$ such that $x \ll y$ and $y \ll z$, it then follows that $x < y < z$, i.e. $x \not\approx^r_< z$. In the same way, we obtain that $\approx^l_<$ are not transitive.*

From Propositions 4.3 and 3.5, we derive the following corollary.

Corollary 4.1. *Let $(X, \leq)$ be a poset. Then, it holds that*

(i) *$\approx^r_<$ and $\approx^l_<$ are tolerance relations on X,*

(ii) *$\approx^r_<$ and $\approx^l_<$ are equivalence relations on the set of incomparable elements of $(X, \leq)$.*

4.4. Left and right weak compatibility of a strict orders with L-equivalences

In this section, we shall characterize the L-tolerance and L-equivalence such that a crisp strict order is weakly compatible with them.
First, we need the following notation.

Notation 4.1. *The symbol τ stands for truth value.*

Let $(X, \leq)$ be a poset and E be an L-relation on X, consider the right weak compatibility, the left weak compatibility and the weak compatibility of the crisp strict order $<$ with the L-relation E as follows:

(i) $<$ is called weakly right compatible with E, if it holds that

$$\tau(x < y) * E(y, z) \leq \tau(x < z) \ , \tag{4.3}$$

for any deferent elements $x, y, z \in X$;

(ii) $<$ is called weakly left compatible with E, if it holds that

$$\tau(x < y) * E(x, z) \leq \tau(z < y)\,, \tag{4.4}$$

for any deferent elements $x, y, z \in X$;

(iii) $<$ is called weakly compatible with E, if it holds that

$$\tau(x < y) * E(x, z) * E(y, t) \leq \tau(z < t)\,, \tag{4.5}$$

for any deferent elements $x, y, z \in X$.

The following lemma is immediate.

Lemma 4.1. *Let $(X, \leq)$ be a poset. Then it holds that*

(i) *$<$ is weakly right compatible with $\approx^l_<$;*

(ii) *$<$ is weakly left compatible with $\approx^r_<$.*

In the following theorem, we will show that the weak right (resp. left) compatibility can be expressed in terms of $\approx^l_<$ and $\approx^r_<$ introduced in the above section.

Theorem 4.4. *Let $(X, \leq)$ be a poset and E be an L-tolerance relation on X. Then the following equivalences hold*

(i) *$<$ is weakly right compatible with E if and only if $E \subseteq \approx^l_<$;*

(ii) *$<$ is weakly left compatible with E if and only if $E \subseteq \approx^r_<$.*

Proof.

(i) Suppose that $<$ is weakly right compatible with E, i.e.

$$\tau(x < y) * E(y, z) \leq \tau(x < z), \text{ for any deferent elements } x, y, z \in X\,.$$

Let $a, b \in X$, two cases to consider:

(1) If $E(a, b) = 0$, then $E(a, b) \leq \tau(a \approx^l_< b)$. Hence, $E \subseteq \approx^l_<$.

(2) If $E(a, b) > 0$, then for any $c \in X \setminus \{a, b\}$, it follows that:

- If $c < a$, then the weak right compatibility of $<$ with E implies that

$$\underbrace{\tau(c < a)}_{=1} * \underbrace{E(a, b)}_{>0} \leq \tau(c < b)\,,$$

it then follows that $\tau(c < b) > 0$, i.e. $\tau(c < b) = 1$. Hence, $c < b$.

- If $c < b$, then the weak right compatibility of $<$ with E and the symmetry of E implies that

$$\underbrace{\tau(c < b)}_{=1} * \underbrace{E(b,a)}_{=E(a,b)>0} \leq \tau(c < a)\,,$$

  it then follows that $\tau(c < a) = 1$. Hence, $c < a$.

Thus, if $E(a,b) > 0$, then $a \approx^l_< b$. Therefore, $E \subseteq \approx^l_<$.

Conversely, if $E \subseteq \approx^l_<$, then for any $x, y, z \in X$:

$$\tau(x < y) * E(y,z) \leq \tau(x < y) * \tau(y \approx^l_< z)\,.$$

It follows from Lemma 4.1 that

$$\tau(x < y) * E(y,z) \leq \tau(x < z)\,,$$

for any $x, y, z \in X$. Hence, $<$ is weakly right compatible with E.

(ii) Follows from Lemma 4.1, (i) and Remark 4.2.

□

Remark 4.4. *Let $(X, \leq)$ be a poset.*

(i) *$\approx^l_<$ is the greatest L-tolerance relation on X such that $<$ is weakly right compatible with it,*

(ii) *$\approx^r_<$ is the greatest L-tolerance relation on X such that $<$ is weakly left compatible with it.*

Theorem 4.5. *Let $(X, \leq)$ be a poset and E be an L-tolerance relation on X . Then, it holds that*

(i) *$<$ is weakly right compatible with E if and only if there exist two L-tolerance relations α_r and β_r on X such that for any $x, y \in X$:*

$$E(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha_r(x,y) & \text{if } (x \approx^l_< y \text{ and } x \ll y) \\ & \quad \text{or } (x \approx^l_< y \text{ and } y \ll x) \\ \beta_r(x,y) & \text{if } x \approx^l_< y \text{ and } x \parallel y \\ 0 & \text{if } x \not\approx^l_< y \end{cases} \tag{4.6}$$

(ii) *$<$ is weakly left compatible with E if and only if there exist two L-tolerance* *relations α_l and β_l on X such that for any $x, y \in X$:*

$$E(x,y) = \begin{cases} 1 & \text{if } x = y \\ \alpha_l(x,y) & \text{if } (x \approx^r_< y \text{ and } x \ll y) \\ & \quad \text{or } (x \approx^r_< y \text{ and } y \ll x) \\ \beta_l(x,y) & \text{if } x \approx^r_< y \text{ and } x \parallel y \\ 0 & \text{if } x \not\approx^r_< y \end{cases} \tag{4.7}$$

Proof.

(i) Let us define the L-tolerance relations α_r and β_r on X as

$$\alpha_r(x,y) = \begin{cases} E(x,y) & , \text{ if } (x \approx^r_< y \text{ and } x \ll y) \\ & \quad \text{or } (x \approx^r_< y \text{ and } y \ll x) \\ 0 & , \text{ otherwise} \end{cases}$$

$$\beta_r(x,y) = \begin{cases} E(x,y) & \text{if } x \approx^r_< y \text{ and } x \parallel y \\ 0 & \text{if } x \not\approx^r_< y \end{cases}$$

Suppose that $<$ is weakly right compatible with E. Based on the behavior of any two elements $x, y \in X$ given in Proposition 4.2, we will characterize the L-tolerance E.

(a) If $x = y$, then since E is is reflexive, it follows that $E(x,x) = 1$, for any $x \in X$.

(b) If $x \approx^l_< y$ and $x \ll y$, then it follows that $x < y$ and for any $z \in X \setminus \{x, y\}$, ($z < x$ if and only if $z < y$). This implies that ($z < x < y$), ($x < y < z$), ($x < y$, $x < z$ and $y \parallel z$), ($x < y$, $z < y$ and $x \parallel z$) or ($x < y$, $x \parallel z$ and $y \parallel z$), for any $z \in X \setminus \{x, y\}$.

- Suppose that $z < x < y$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=1} * E(x,y) \leq \underbrace{\tau(z < y)}_{=1} .$$

Hence, $1 * E(x,y) \leq 1$. This implies that $E(x,y) = \alpha_r(x,y) \in L$.

- Suppose that $x < y < z$. Since $<$ is weakly right compatible with

E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \alpha_r(x, y) \in L$.

- Suppose that $x < y$, $x < z$ and $y \parallel z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \alpha_r(x, y) \in L$.

- Suppose that $x < y$, $z < y$ and $x \parallel z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=1} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \alpha_r(x, y) \in L$.

- Suppose that $x < y$, $x \parallel z$ and $y \parallel z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \alpha_r(x, y) \in L$.

(c) If $x \approx^l_< y$ and $y \ll x$, by similar proof, we obtain $E(y, x) = \gamma_r(x, y) \in L$. Since E is symmetric, it follows that $E(x, y) = \gamma_r(x, y) = \alpha_r(x, y)$.

(d) If $x \approx^l_< y$ and $x \parallel y$, then it follows that $x \parallel y$ and for any $z \in X \setminus \{x, y\}$, ($z < x$ if and only if $z < y$). This implies that ($x \parallel y$, $z < x$ and $z < y$), ($x \parallel y$, $x < z$ and $y < z$), ($x \parallel y$, $x < z$ and $y \parallel z$), ($x \parallel y$, $x \parallel z$ and $y < z$) or ($x \parallel y$, $x \parallel z$ and $y \parallel z$).

- Suppose that $x \parallel y$, $z < x$ and $z < y$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=1} * E(x, y) \leq \underbrace{\tau(z < y)}_{=1} .$$

Hence, $1 * E(x, y) \leq 1$. This implies that $E(x, y) = \beta_r(x, y) \in L$.

- Suppose that $x \parallel y$, $x < z$ and $y < z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \beta_r(x, y) \in L$.

- Suppose that $x \parallel y$, $x < z$ and $y \parallel z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \beta_r(x, y) \in L$.

- Suppose that $x \parallel y$, $x \parallel z$ and $y < z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \beta_r(x, y) \in L$.

- Suppose that $x \parallel y$, $x \parallel z$ and $y \parallel z$. Since $<$ is weakly right compatible with E, then it follows that

$$\underbrace{\tau(z < x)}_{=0} * E(x, y) \leq \underbrace{\tau(z < y)}_{=0} .$$

Hence, $0 * E(x, y) \leq 0$. This implies that $E(x, y) = \beta_r(x, y) \in L$.

(e) Suppose that $x \not\approx^l_< y$. Since $<$ is weakly right compatible with E, it follows from Theorem 4.4 that $E \subseteq \approx^l_<$. Thus, $E(x, y) = 0$.

By combining the above cases, it follows that if $<$ is weakly right compatible with E, then

$$E(x,y)=\begin{cases}1 & \text{if } x=y\\ \alpha_r(x,y) & \text{if } \ (x\approx^l_< y \text{ and } x\ll y)\\ & \ \text{ or } (x\approx^l_< y \text{ and } y\ll x)\\ \beta_r(x,y) & \text{if } x\approx^l_< y \text{ and } x\parallel y\\ 0 & \text{if } x\not\approx^l_< y\end{cases}$$

Note that it is easy to verify that E is an L-tolerance relation.

Conversely, suppose that E takes the form (4.6). Since $E\subseteq\approx^l_<$, then it follows from Theorem 4.4 (i) that $<$ is weakly right compatible with E.

(ii) Follows from Lemma 4.1, (i) and Remark 4.2.

□

Remark 4.5. (i) *If $\alpha_r(x,y)=\beta_r(x,y)=0$, for any $x,y\in X$, then E is the crisp equality.*

(ii) *If $\alpha_r(x,y)=\beta_r(x,y)=1$, for any $x,y\in X$, then $E=\approx^l_<$.*

In the following proposition, we will show that given additional conditions on $\alpha_r(x,y)$, $\beta_r(x,y)$, $\alpha_l(x,y)$ and $\beta_l(x,y)$, the L-tolerance relations (4.6) and (4.7) given in the above Theorem 4.5 are L-equivalence relations.

Proposition 4.5. *Let $(X,\leq)$ be a poset. α_r, α_l, β_r and β_l be L-tolerance relations on X such that α_r, β_r satisfy the following conditions:*

(1) $\alpha_r(x,y)*\alpha_r(y,z)\leq\beta_r(x,z)$, if $x\rhd_l y$ and $y\lhd_l z$.
And $\alpha_r(x,y)*\alpha_r(y,z)=0$, otherwise.

(2) $\alpha_r(x,y)*\beta_r(y,z)\leq\alpha_r(x,z)$, if $x\lhd_l y$. And $\alpha_r(x,y)*\beta_r(y,z)=0$, if $x\rhd_l y$;

(3) $\beta_r(x,y)*\beta_r(y,z)\leq\beta_r(x,z)$.

And α_l, β_l satisfy the following conditions:

(1) $\alpha_l(x,y)*\alpha_l(y,z)\leq\beta_l(x,z)$, if $x\lhd_r y$ and $y\rhd_r z$.
And $\alpha_l(x,y)*\alpha_l(y,z)=0$, otherwise.

(2) $\alpha_l(x,y)*\beta_l(y,z)\leq\alpha_l(x,z)$, if $x\rhd_r y$. And $\alpha_l(x,y)*\beta_l(y,z)=0$, if $x\lhd_r y$;

(3) $\beta_l(x,y)*\beta_l(y,z)\leq\beta_l(x,z)$.

Then, it holds that

(i) *The L-relation E defined by:*

$$E(x,y)=\begin{cases}1 & \text{if } x=y\\ \alpha_r(x,y) & \text{if } (x\approx^l_< y \text{ and } x\ll y)\\ & \quad\text{or } (x\approx^l_< y \text{ and } y\ll x)\\ \beta_r(x,y) & \text{if } x\approx^l_< y \text{ and } x\parallel y\\ 0 & \text{if } x\not\approx^l_< y\end{cases} \tag{4.8}$$

is an L-equivalence relation on X.

(ii) *The L-relation E' defined by:*

$$E'(x,y)=\begin{cases}1 & \text{if } x=y\\ \alpha_l(x,y) & \text{if } (x\approx^r_< y \text{ and } x\ll y)\\ & \quad\text{or } (x\approx^r_< y \text{ and } y\ll x)\\ \beta_l(x,y) & \text{if } x\approx^r_< y \text{ and } x\parallel y\\ 0 & \text{if } x\not\approx^r_< y\end{cases} \tag{4.9}$$

is an L-equivalence relation on X.

Proof.

From Theorem 4.5, E is an L-tolerance relations. It remain to show that E is transitive, i.e. we show that

$$E(x,y)*E(y,z)\le E(x,z),\ \text{for any } x,y,z\in X\,.$$

Let $x,y,z\in X$. Using the conditions on α_r, β_r, α_l and β_l for any $x,y,z\in X$ and the possible values of $E(x,y)$ and $E(y,z)$, we obtain that

$$E(x,y)*E(y,z)\le E(x,z),\ \text{for any } x,y,z\in X\,,$$

and

$$E'(x,y)*E'(y,z)\le E'(x,z),\ \text{for any } x,y,z\in X\,.$$

□

From Theorem 4.5 and Proposition 4.5, we obtain the following characterization theorem.

Theorem 4.6. *Let $(X,<)$ be a poset and E be an L-equivalence relation on X. Then, it holds that*

(i) *$<$ is weakly right compatible with E if and only if there exist two L-equivalence relations α and β on X such that E takes the form (4.8) given in Proposi-*

tion 4.5.

(ii) $<$ *is weakly left compatible with* E *if and only if there exist two* L*-equivalence relations* α *and* β *on* X *such that* E *takes the takes the form (4.9) given in Proposition 4.5.*

General conclusions

In this thesis, we have shown that the properties of extensionality and compatibility w.r.t. a fuzzy equality are equivalent. By alleviating the restriction to fuzzy equalities, we have introduced the more general notions of left and right compatibility of fuzzy relations. We have shown that Fodors notions of traces of a fuzzy relation largely aid the understanding of these notions. In particular, in one way or another, compatibility always boils down to some inclusion. We have also introduced the interesting notion of the clone relation associated with a poset and we have made a distinction between comparable clones (for which the clone relation is anti-transitive) and incomparable clones (for which the clone relation is transitive). In view of its prominent role in this work, we have scrutinized the key aspects of this notion, thereby contributing in some sense also to classical poset theory. The notion of a clone relation has greatly facilitated a transparent characterization of the L-tolerance or L-equivalence relations a given strict order relation is compatible with, positively complementing the negative result that the only L-tolerance relation a given order relation is compatible with is the crisp equality relation. Future work is anticipated in multiple directions. First of all, we think it makes sense to study the notion of a clone relation for other types of crisp relations. Secondly, we intend to extend this notion to fuzzy relations, and to characterize the L-tolerance or L-equivalence relations a fuzzy strict order relation is compatible with. On the other hand, with the notion of the right (resp. left) compatibility of a crisp strict order with fuzzy tolerance (resp. equivalence) relations, we arrived to characterize these tolerance and equivalence relations by four values. As open question, how to characterize these fuzzy relations using strict fuzzy orders instead of crisp strict orders. In future work, we will again restrict the attention, but this time to fuzzy orders.

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