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On a conjecture of Heim and Neuhauser on some polynomials arising from modular forms and related to Fibonacci polynomials

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Abstract
Heim and Neuhauser investigated some polynomials related to the Dedekind function. They proved the log-concavity of these polynomials and conjectured that they have only real zeros. We prove this conjecture, and deduce some identities for Fibonacci numbers and determine the modes of another sequence related to Fibonacci polynomials.

Keywords Fibonacci number · Log–concave sequence · Polynomial · Zeros

Mathematics Subject Classification 5A19 · 11B39 · 11B65 · 26C10

1 Introduction

Let $\eta(t)$ be the Dedekind function, and $q = e^{2\pi i t}$, where $t \in H$, the upper half plane. For $z \in \mathbb{C}$, consider

$$\left(q^{-\frac{1}{24}}\eta(t)\right)^{-z} = \prod_{n=1}^{\infty}(1-q^n)^{-z} = \sum_{n=0}^{\infty} P_n(z)q^n. \tag{1}$$

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Writing the polynomials P_n as

$$P_n(z) = \frac{z}{n!}\sum_{k=0}^{n-1} N(n,k)z^k, \tag{2}$$

the coefficients $N(n,k)$ are positive integers. The values

$$N(n,0) = (n-1)!\sigma(n),\ N(n,n-2) = \frac{3n(n-1)}{2},\ N(n,n-1) = 1,\ n \in \mathbb{N}$$

where $\sigma(n)$ is as usual, the sum of the divisors of n, were calculated by Newman [28]. These polynomials may be also defined by

$$P_n(z) = \frac{z}{n}\sum_{k=1}^{n} \sigma(k)P_{n-k}(z). \tag{3}$$

For $n \le 10$, these $P_n(z)$ may be calculated recursively and have only real zeros (for $n \le 9$), see [28,33] . In addition, they were used by Serre [33] to prove a density theorem for the powers of the function η of Dedekind. However, because of the chaotic behaviour of the function $\sigma(n)$, little is known about $P_n(z)$ for general n. Note that $P_n(z)$ are related to many famous sequences, for example $(P_n(1))_{n=1}^{\infty} = (p_n)_{n=1}^{\infty}$ is the sequence of partition numbers and $(P_{n-1}(-24))_{n=1}^{\infty} = (\tau(n))_{n=1}^{\infty}$ is the sequence of Ramanujan numbers.

After calculating the first values of $\tau(n),\ 1 \le n \le 30$, Ramanujan conjectured that τ is a multiplicative function, a fact proved later by Mordell [27], using advanced tools of modular forms. According to [38], there is no known elementary proof of this result. For the importance of the η function in general, let us cite Atiyah [2]: *The Dedekind function is one of the most famous and well-studied function in mathematics, particularly in relation to elliptic curves and modular forms.* The Dedekind eta function is surprisingly almost everywhere! A good reference showing this quite unexpected presence is [12].

For its incursions in physics, see [5,32,37].

Lehmer [26] showed that $\tau(n) \neq 0$ for all $n < 3316799$, and that the least integer m where $\tau(m) = 0$ (if it exists) must be a prime number. He then conjectured that $\tau(n) \neq 0$ for all n. Due to the importance of the eta function (and what is around; its powers, the coefficients of certain of these powers such as $(\tau(n))_{n\ge1}$) the research to settle this conjecture is very active, especially in these last two decades. These studies are mainly in two directions: First, the extensions of the results of Newman and Serre (the vanishing properties of the rth power of the Dedekind eta function) see [20]. Second, a general approach towards the resolution of the conjecture, by considering sequences of polynomials generalizing the $P_n(z)$. For this type of results, see [14] and the references therein.

In a series of papers, Heim and his collaborators (mainly Neuhauser) developed an ingenious strategy to understand and eventually settle Lehmer conjecture. For this, let g and h be two arithmetic functions. In [17,21], the authors defined the following sequence of polynomials:

$$P_n^{g,h}(x) = \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{n-k}^{g,h}(x), \quad P_0^{g,h}(x) = 1 \tag{4}$$

Setting $h(n) = n$ and $g(n) = \sigma(n)$ in (4) yields (3). Varying h and g gives very interesting results; and may have important repercussions for equation (3).

For example Heim and Neuhauser [16], with $h(n) = n$, and the two functions $g(n) = \phi_1(n) = n$ and $g(n) = \phi_2(n) = n^2$, obtained the two sequences of polynomials $(P_n^{\phi_i}(x)),\ 1 \leq i \leq 2$ (for simplicity, we omitted g from the notation)

$$\begin{aligned} P_n^{\phi_i}(x) &= \frac{x}{n} \sum_{k=1}^{n} \phi_i(k) P_{n-k}^{\phi_i}(x) \\ &= \frac{x}{n!} \sum_{k=0}^{n-1} A_k^n(\phi_i) x^k, \ \ P_0^{\phi_i}(x) = 1. \end{aligned}$$

Since for every integer $n > 1$, $n < \sigma(n) < n^2$, the two polynomials interpolate well $P_n^{\sigma}(x)$, and thus furnishes many information about it.

In addition, they determined the coefficients $(A_k^n(\phi_i))_{k=0}^{n-1}$ and proved their log-concavity. Based on numerical evidence, they conjectured that the polynomials $P_n^{\phi_i}(x),\ i = 1,\ 2$ have only real zeros. The aim of this paper is to prove this conjecture and prove some results linked to the polynomials $P_n^{\phi_i}(x),\ i = 1,\ 2$. In the next section, we prove the conjecture. In the third section, we consider a polynomial related to Fibonacci polynomial and use it to prove some identities and find (new?) ones. The last section is devoted to the unimodality of the sequence considered in the third section.

2 The sequences $A_k^n(\phi_1)$ and $A_k^n(\phi_2)$

As noted in the previous section, in the case where the arithmetic functions are $\phi_1(n) = n$ and $\phi_2(n) = n^2$, with fixed $h(n) = n$ in (4), Heim and Neuhauser [16] obtained the following polynomials:

$$\widetilde{P}_n^{\phi_1}(x) = \sum_{k=0}^{n-1} A_k^n(n) x^k, \quad \widetilde{P}_n^{\phi_2}(x) = \sum_{k=0}^{n-1} A_k^n(n^2) x^k,$$

where

$$A_k^n(\phi_1) = \frac{n!}{(k+1)!}\binom{n-1}{k}, \ \ A_k^n(\phi_2) = \frac{n!}{(k+1)!}\binom{n+k}{2k+1}, \ 0 \leq k \leq n-1.$$

Note that in [16], the polynomials $P_n^{\phi_1}(x) = \frac{x}{n!}\widetilde{P}_n^{\phi_1}(x)$ and $P_n^{\phi_2}(x) = \frac{x}{n!}\widetilde{P}_n^{\phi_2}(x)$ are given explicitly for $1 \leq n \leq 7$. In this section, we prove the conjecture stated by the authors of [16]. The proof we give is based on the following classical theorem of Laguerre.

Theorem 1 (Laguerre) *Let $A(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial having only real zeros. Then the polynomial $\sum_{k=0}^{n} \frac{a_k}{k!} x^k$ has only real zeros .*

A proof of this result may be found in the book of Pólya and Szegö [29, Vol. II, Part V, problem 65]. For a very short and simple proof of Laguerre's Theorem, see [7]. There, the proof is based essentially on two facts: if a polynomial has real zeros then so does its reciprocal polynomial. And, if the polynomial $T(x)$ has only real zeros, the polynomial $T'(x) - \alpha T(x)$ also has this property for every real number α.

The following corollary is crucial for the proof of the main theorem, even it is a straightforward consequence from Laguerre Thorem.

Corollary 2 *Let $A(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial having only real zeros. Then for every $l \geq 0$, the polynomial $\sum_{k=0}^{n} \frac{a_k}{(k+l)!} x^k$ has only real zeros .*

Before starting the proof of the theorem, let us recall the definition of Jacobsthal-Fibonacci polynomials, $JF_n(x)$, (see ([22] p. 489), those are defined as follows

$$JF_n(x) = JF_{n-1}(x) + xJF_{n-2}(x),\ n \geq 2,$$

with the boundary conditions: $JF_0(x) = 0,\ JF_1(x) = 1$. The explicit formula of $JF_n(x)$ is

$$\begin{aligned} JF_n(x) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^k \\ &= \frac{1}{\sqrt{4x+1}} \left(\left(\frac{1+\sqrt{4x+1}}{2} \right)^n - \left(\frac{1-\sqrt{4x+1}}{2} \right)^n \right). \end{aligned}$$

Note that $JF_n(1) = F_n$, the Fibonacci number.
Now, we can prove our main result:

Theorem 3 *The polynomials $\widetilde{P}_n^{\phi_1}(x)$ and $\widetilde{P}_n^{\phi_2}(x)$ have only real zeros.*

Proof First, recall the definition of the unsigned Lah numbers $Lh(n,k)$ (this is sequence **A048854** in [34]); which count the total number of ways one can partition an $n-$element set into k nonempty totally ordered subsets. We have

$$Lh(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

Note that

$$P_n^{\phi_1}(x) = \frac{x}{n!} \sum_{k=1}^{n} Lh(n,k) x^{k-1},$$

is nothing but Lah polynomial $Lh_n(x)$ (up to a multiplicative factor), which is known to be real rooted. As it is stated in [19], the polynomial $\widetilde{P}_n^{\phi_1}(x) = (n-1)!L_{n-1}^1(-x)$ is a generalized Laguerre polynomial (these polynomials are orthogonal), so its zeros are real, and it is irreducible over $\mathbb{Z}$. For the sake of completeness, let us prove directly that $\widetilde{P}_n^{\phi_1}(x)$ has only real zeros: the polynomial $(1+x)^{n-1}$ has only real zeros and then by Theorem 2, the polynomial $\widetilde{P}_n^{\phi_1}(x) = \sum_{k=0}^{n-1} \frac{n!}{(k+1)!}\binom{n-1}{k}x^k$ has the same property too ($l = 1$).

For $\widetilde{P}_n^{\phi_2}(x)$, it suffices to prove that $\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k$ has only real zeros. But this is true. In fact, the last polynomial is just the reciprocal of $JF_{2n}(x)$, call it $JF_{2n}^r(x)$:

$$
\begin{aligned}
JF_{2n}^r(x) &= x^{n-1}JF_{2n}(1/x) = x^{n-1}\sum_{k=0}^{n-1}\binom{2n-k-1}{k}\frac{1}{x^k} \\
&= \sum_{k=0}^{n-1}\binom{2n-k-1}{k}x^{n-k-1} \\
&= \frac{x^{n-1}}{\sqrt{\frac{4}{x}+1}}\left(\left(\frac{1+\sqrt{\frac{4}{x}+1}}{2}\right)^{2n} - \left(\frac{1-\sqrt{\frac{4}{x}+1}}{2}\right)^{2n}\right) \\
&= \frac{(\sqrt{x})^{2n}}{x\sqrt{\frac{4}{x}+1}}\left(\left(\frac{1+\sqrt{\frac{4}{x}+1}}{2}\right)^{2n} - \left(\frac{1-\sqrt{\frac{4}{x}+1}}{2}\right)^{2n}\right) \\
&= \frac{1}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right).
\end{aligned}
$$

Now, the coefficient of x^k in $JF_{2n}^r(x)$ is

$$
\binom{2n-(n-1-k)-1}{n-1-k} = \binom{n+k}{n-1-k} = \frac{(n+k)!}{(n-k-1)!(2k+1)!} = \binom{n+k}{2k+1}.
$$

It follows then that

$$
\begin{aligned}
JF_{2n}^r(x) &= \sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k \\
&= \frac{1}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right). \qquad (5)
\end{aligned}
$$

Finally, by Corollary 2 applied to the polynomial $JF^r_{2n}(x)$ with $l = 1$, we deduce that $\widetilde{P}^{\phi_2}_n(x)$ has only real zeros. □

Remark 4 The polynomials $\widetilde{P}^{\phi_2}_n(x)$ are irreducible, this was proved in [11].
In [14, 20], the sequence of polynomials defined by (4) is investigated. Many interesting results are obtained, especially for particular arithmetic functions h and g. For example for $h(n) = 1$ and $g(n) = n$, $Q^g_n(x)$ in [13] has only real zeros.

The polynomials satisfying a three term recursion or more (see [13]) are generally real rooted. Thus, we can apply Laguerre theorem to these polynomials to obtain new real rooted polynomials (the kth coefficient is divided by $k!$).

In Example 2. 7. 1 of [13], the polynomial $Q^g_n(x) = xJF^r_{2n}(x) = xU(x/2+1)$, where $U(x)$ is the Chebyshev polynomial of the second kind. In fact since the discriminant $D = D(x) = x(x+4)$, then the roots of the characteristic equation are given by

$$\lambda_{1,2} = \frac{x+2 \pm \sqrt{D}}{2}.$$

It follows then that

$$Q^g_n(x) = \alpha \left(\frac{x+2+\sqrt{D}}{2}\right)^n + \beta \left(\frac{x+2-\sqrt{D}}{2}\right)^n, \ n \geq 1.$$

with $Q^g_0(x) = 1$, $Q^g_1(x) = x$, $Q^g_2(x) = x(x+2)$, we deduce α and β, so

$$Q^g_n(x) = \frac{x}{\sqrt{D}} \left(\left(\frac{x+2+\sqrt{D}}{2}\right)^n - \left(\frac{x+2-\sqrt{D}}{2}\right)^n\right).$$

But

$$\frac{x+2 \pm \sqrt{D}}{2} = \left(\frac{\sqrt{x} \pm \sqrt{x+4}}{2}\right)^2.$$

So,

$$\begin{aligned} Q^g_n(x) &= \frac{x}{\sqrt{x(x+4)}} \left(\left(\frac{\sqrt{x}+\sqrt{x+4}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{x+4}}{2}\right)^{2n}\right) \\ &= xJF^r_{2n}(x). \end{aligned}$$

The polynomials considered in this paper (Jacobsthal-Fibonacci, Lah) are closely linked to Laguerre and Chebyshev (orthogonal) polynomials, and may be this highlights the appearance of Fibonacci and Lucas numbers in the context of modular forms.

For the sake of completeness, we give without proof, the generating function of the sequence ${P_n}^{\phi_2}(x)$.

Theorem 5 *The generating function of the sequence $A_k^n(n^2)$ is given by*

$$\sum_{n\geq 0} P_n^{\phi_2}(x)z^n = \exp\left(\frac{xz}{(1-z)^2}\right).$$

Remark 6 Dilcher [10] deduced the formula

$$F_{4n} = 3\sum_{k=0}^{n-1}\binom{n+k}{2k+1}5^k \tag{6}$$

using the hypergeometric functions, and said that he was unable to find it in the literature. In fact this is known and proved in Swamy [32]. Note that (6) is easily deduced by setting $x = 5$ in (5), and remembering that

$$\left(\frac{\sqrt{5}\pm 3}{2}\right) = \left(\frac{1\pm\sqrt{5}}{2}\right)^2.$$

In the paper [3], we used the explicit formulas of the Fibonacci and Lucas polynomials to find some new identities (and rediscover some old ones) involving Fibonacci and Lucas numbers. In the next section, we use the same elementary technique i.e.; explicit formulas of Fibonacci and Lucas polynomials, their derivatives and antiderivatives to prove some identities, missed in [3].

3 Some identities involving Fibonacci and Lucas numbers

Let us start with

$$\begin{aligned} JF_{2n}^r(x) &= \sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k \\ &= \frac{1}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right). \end{aligned} \tag{7}$$

Integrating $JF_{2n}^r(x)$, we obtain

$$\begin{aligned} \int_0^x JF_{2n}^r(t)dt &= \sum_{k=0}^{n-1}\binom{n+k}{2k+1}\frac{x^{k+1}}{k+1} \\ &= \frac{1}{n}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right) - \frac{2}{n}. \end{aligned} \tag{8}$$

Using this, we prove

Proposition 7 *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \frac{1}{2(2n+1)} \left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2} \right)^{4n+2} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2} \right)^{4n+2} \right) \tag{9}$$

Proof We will prove the result using only (7) and (8). We have

$$\frac{\binom{2n+k+1}{2k}}{2n+k+1} = \frac{\binom{2n+k}{2k-1}}{2k}, \quad k \neq 0, \tag{10}$$

and then

$$\sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \frac{1}{2n+1} + \sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1},$$

by (10)

$$\sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \sum_{k=1}^{2n+1} \binom{2n+k}{2k-1} \frac{x^k}{2k}.$$

This last sum may be written as

$$\begin{aligned} \sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} &= \sum_{k=1}^{2n+1} \binom{2n+k}{2k-1} \frac{x^k}{2k} \\ &= \frac{1}{2} \sum_{k=0}^{2n} \binom{2n+k+1}{2k+1} \frac{x^{k+1}}{k+1}. \end{aligned}$$

Finally, using (7) we obtain

$$\begin{aligned} \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} &= \frac{1}{2n+1} + \frac{1}{2} \sum_{k=0}^{2n} \binom{2n+k+1}{2k+1} \frac{x^{k+1}}{k+1} \\ &= \frac{1}{2(2n+1)} \left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2} \right)^{4n+2} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2} \right)^{4n+2} \right). \end{aligned}$$

The proof is finished. ◻

Before giving the next result, recall the sequence of Lucas numbers, $(L_n)_{n\geq 0}$:

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2},\ n \geq 2.$$

Corollary 8 *For every $n \in \mathbb{N}$, we have the identities*

$$\text{(a)}\, F_{2n+1} = \pm\frac{2(2n+1)}{\sqrt{5}}\sum_{k=0}^{2n+1}\binom{2n+k+1}{2k}\frac{(\pm\sqrt{5}-2)^k}{2n+k+1},$$

$$\text{(b)}\, F_{2n+1} = (2n+1)\sum_{k=0}^{2n+1}(-1)^{k+1}\binom{2n+k+1}{2k}\frac{F_{3k}}{2n+k+1},$$

$$\text{(c)}\, F_{4n+2} = 2\sum_{k=1}^{2n+1}\frac{k}{2n+k+1}\binom{2n+k+1}{2k},$$

$$\text{(d)}\, L_{(4n+2)l} = 2(2n+1)\sum_{k=0}^{2n+1}(-1)^{l(2n+k+1)}\binom{2n+k+1}{2k}\frac{5^k F_l^{2k}}{2n+k+1},\quad l \geq 1,$$

$$\text{(e)}\, \sum_{k=0}^{2n+1}\binom{2n+k+1}{2k}\frac{(-4)^k}{2n+k+1} = -\frac{1}{2n+1},$$

$$\text{(f)}\, \sum_{k=0}^{2n+1}\binom{2n+k+1}{2k}\frac{(-2)^k}{2n+k+1} = 0.$$

Proof Relation (a) is relation (7.1) in Dilcher [10]. There, is it deduced via the hypergeometric representations of the Fibonacci numbers. Now, set $x = \pm\sqrt{5} - 2$, in the last relation in the proof of Proposition 7, and divide both sides by $\sqrt{5}$ to get (a). Relation (b) also appears in the paper [10], it is obtained by adding the two identities in (a). The identities (c) and (d) seem to be new. To obtain (c), differentiate (9), then set $x = 1$. For (d), remember the identity

$$5F_l^2(-1)^l + 4 = (-1)^l L_l^2,$$

then put $x = 5F_l^2(-1)^l = 5F_l^2 i^{2l}$ in (9). The remaining are obvious. □

4 On the modes of the sequences $A_k^n(\phi_2)$ and $\frac{\binom{2n+k+1}{2k}}{2n+k+1}$

The following facts will be needed in the sequel.

Definition 9 A sequence of positive real numbers $(a_k)_{k=0}^n$ is said to be unimodal, if there exist integers $k_0,\ k_1\ (k_0 \leq k_1)$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k_0} = a_{k_0+1} = \cdots = a_{k_1} \geq a_{k_1+1} \geq \cdots \geq a_n.$$

It is log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$, for $1 \leq k \leq n-1$. The integers k_0, $k_0+1, \ldots, k_1$ are the modes of the sequence.

A real sequence $(a_k)_{k=0}^n$ is said to be with no internal zeros (NIZ), if $i < j, a_i \neq 0$, $a_j \neq 0$ then $a_l \neq 0$ for every $l, i \leq l \leq j$. A (NIZ) log-concave sequence is obviously unimodal, but the converse is not true. The sequence 1, 1, 4, 2 is unimodal but not log-concave. Note the importance of (NIZ): the sequence 1, 0, 0, 1 is log-concave but not unimodal. A real polynomial is unimodal (log-concave, symmetric, respectively) provided that the sequence of its coefficients is unimodal (log-concave, symmetric, respectively).
If inequalities in the log-concavity definition are strict, then the sequence is called strictly log-concave (SLC for short), and in this case, it has at most two consecutive modes. The following result may be helpful in proving unimodality, this goes back to Newton:

Theorem 10 *If the polynomial $\sum_{k=0}^n a_k x^k$ associated with the sequence $(a_k)_{k=0}^n$ has only real zeros then*

$$a_k^2 \geq \frac{k+1}{k}\frac{n-k+1}{n-k} a_{k-1}a_{k+1} \text{ for } 1 \leq k \leq n-1. \tag{11}$$

By the previous theorem, if the positive sequence $(a_k)_{k=0}^n$, is such that its generating polynomial has only real zeros, then it is SLC, and it has at most two peaks. A result of Darroch [9] states that if the polynomial $\sum_{k=0}^n a_k x^k$, $a_k \geq 0$, has only negative real zeros, then every mode k_0 of the sequence $(a_k)_{k=0}^n$ satisfies the inequality

$$\left\lfloor \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rceil. \tag{12}$$

The explicit form of the coefficients of the polynomial $P_n^{\phi_1}(x)$ allows us to find the exact value of the modes; we can even see where a double maximum occurs. For the polynomial $P_n^{\phi_2}(x)$, the coefficients are also explicit, but it is not easy to find the exact value of the mode(s). Heim and Neuhauser [16] gave the following estimate of the mode:

$$k_0 \sim 2^{-\frac{2}{3}} n^{\frac{2}{3}}.$$

For the sake of simplicity, let $A_k^n(\phi_2) = A_{n,k}$. A double maximum occurs if and only if

$$A_{n,k} = A_{n,k+1},$$

which is equivalent to

$$n^2 = 4k^3 + 19k^2 + 28k + 13.$$

According to a classical theorem of Selberg, this equation has only a finite number of solutions. Next, we prove that the sequence $A_{n,k}$ has a unique mode for $n > 8$.

Theorem 11 *The equation*

$$n^2 = 4k^3 + 19k^2 + 28k + 13$$

has a unique solution in positive integers : $(n, k) = (8, 1)$, *thus; the sequence* $a_{n,k}$ *has a plateau for the single value* $n = 8$, *where* $a_{8,1} = a_{8,2}$.

Proof The equation

$$n^2 = 4k^3 + 19k^2 + 28k + 13,$$

is an elliptic curve. Using Magma, we have all the solutions of this equation, namely;

$$(n, k) = (8, 1), \ (n, k) = (-8, 1), \ (n, k) = (0, -1).$$

The only positive integer where the sequence has a plateau is $n = 8$, and in this case $k_0 = 1$. We can check that $A_{8,1} = A_{8,2}$. □

Now, let us consider the sequence $a_{n,k} = \frac{\binom{2n+k+1}{2k}}{2n+k+1}$. According to Proposition 7, this sequence is SLC and it has at most two consecutive modes. In fact we have the following result

Theorem 12 *The sequence* $(a_{n,k})_{k=0}^{2n+1}$ *is SLC and it has at most two consecutive maxima. Namely, if* k_n *is the integer where the maximum occurs, then*

$$k_n = \left\lceil \frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5} \right\rceil.$$

Proof Proposition 7 shows that the generating polynomial associated with the sequence $(a_{n,k})_{k=0}^{2n+1}$ has only real zeros. It follows then that the sequence $(a_{n,k})_{k=0}^{2n+1}$ is SLC. Let k_n be the integer satisfying

$$a_{n,k_n} > a_{n,k_n-1}, \text{ and } a_{n,k_n} \geq a_{n,k_n+1}.$$

We follow the steps of Tanny and Zuker [36]. The previous inequalities may be written as

$$5k_n^2 - 4k_n - 4n^2 - 4n < 0, \tag{13}$$
$$5k_n^2 + 6k_n + 2 - (2n+1)^2 \geq 0. \tag{14}$$

Let $k_n = (2n+1)\alpha,\ 0 < \alpha < 1$, then we have

$$5(\alpha(2n+1))^2 - 4\alpha(2n+1) - 4n^2 - 4n < 0$$

and

$$5(\alpha(2n+1))^2 + 6\alpha(2n+1) + 2 - (2n+1)^2 \geq 0.$$

Put

$$P(x) = 5(2n+1)^2 x^2 - 4(2n+1)x - 4n^2 - 4n$$

and

$$Q(x) = 5(2n+1)^2 x^2 + 6(2n+1)x + 2 - (2n+1)^2.$$

This yields $P(\alpha) < 0$ and $Q(\alpha) \geq 0$. So we deduce that α is between the roots of P, namely

$$\frac{2 \pm \sqrt{5(2n+1)^2 - 1}}{5(2n+1)},$$

and outside the roots of Q:

$$\frac{-3 \pm \sqrt{5(2n+1)^2 - 1}}{5(2n+1)}.$$

We deduce

$$\frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5(2n+1)} \leq \alpha < \frac{2 + \sqrt{5(2n+1)^2 - 1}}{5(2n+1)},$$

furthermore

$$\frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5} \leq k_n < \frac{2 + \sqrt{5(2n+1)^2 - 1}}{5},$$

The difference between the bounds is exactly one, hence the wanted result. □

Corollary 13 *The integer k_n satisfies*

$$k_n = \left\lfloor \frac{\sqrt{5}(2n+1)}{5} \right\rfloor \ or \ \left\lceil \frac{\sqrt{5}(2n+1)}{5} \right\rceil$$

Proof Note that

$$\frac{2+\sqrt{5(2n+1)^2-1}}{5} \leq \frac{2+\sqrt{5}(2n+1)}{5} = \frac{2}{5} + \frac{\sqrt{5}(2n+1)}{5}$$

and

$$\frac{-4}{5} + \frac{\sqrt{5}(2n+1)}{5} = \frac{-4+\sqrt{5}(2n+1)}{5} \leq \frac{-3+\sqrt{5(2n+1)^2-1}}{5(2n+1)}.$$

So, we get

$$\frac{-4}{5} + \frac{\sqrt{5}(2n+1)}{5} \leq k_n < \frac{2}{5} + \frac{\sqrt{5}(2n+1)}{5}.$$

Since $\frac{4}{5} < 1$ and $\frac{2}{5} < 1$, we get the result. □

In some cases, the sequence $a_{n,k}$ has a double maximum. For example, for $n = 0$, we have

$$a_{0,k} = \frac{\binom{k+1}{2k}}{k+1},$$

and $a_{0,0} = a_{0,1} = 1$. Next, we determine all the integers where a double maximum occurs for $a_{n,k},\ n \geq 1$. We have the following

Theorem 14 *The sequence $(a_{n,k})_{k=0}^{2n+1}$ has a double maximum if and only if*

$$n_j = \frac{F_{12j+9}-2}{4},$$

and in this case, the smallest mode k_j is given by

$$k_j = \frac{L_{12j+9}-6}{10}.$$

Proof We use relation (9)

$$a_{n,k} = \frac{\binom{2n+k}{2k-1}}{2k},\ \ k \geq 1.$$

The equation

$$a_{n,k} = a_{n,k+1}$$

is equivalent to

$$5k^2 + 6k + 2 - (2n+1)^2 = 0.$$

The reduced discriminant of this last equation is

$$d' = 5(2n+1)^2 - 1,$$

and must be a square. So, let $y = 2n+1$, our equation becomes $x^2 = 5y^2 - 1$, or

$$x^2 - 5y^2 = -1,$$

which is the so called negative Pell equation. The solutions of this equation are given by

$$\begin{aligned} x_n &= \frac{1}{2}\left[(2+\sqrt{5})^{2n+1} + (2-\sqrt{5})^{2n+1}\right] \\ y_n &= \frac{1}{2\sqrt{5}}\left[(2+\sqrt{5})^{2n+1} - (2-\sqrt{5})^{2n+1}\right]. \end{aligned}$$

More details are to be found in [1, Sect 3.6. p. 46] . Note that

$$2 \pm \sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^3,$$

so we obtain

$$x_n = \frac{1}{2}L_{6n+3}, \quad y_n = \frac{1}{2}F_{6n+3}.$$

The numbers

$$k_j = \frac{-3 + \frac{1}{2}L_{6j+3}}{5}, \quad n_j = \frac{F_{6j+3} - 2}{4},$$

are not always integers. It is easy to see that numbers k_j and n_j are integers if and only if $j = 2l + 1$. So, the wanted solutions are given by

$$k_j = \frac{-3 + \frac{1}{2}L_{12j+9}}{5}, \quad n_j = \frac{F_{12j+9} - 2}{4}.$$

In other words,

$$a_{n_j,k_j} = a_{n_j,k_j+1} \iff n_j = \frac{F_{12j+9} - 2}{4} \text{ and } k_j = \frac{-3 + \frac{1}{2}L_{12j+9}}{5}.$$

This proves the theorem. □

Corollary 15 *For every integer $j \geq 0$, we have*

$$\left\lfloor \frac{F_{12j+9}F_{F_{12j+9}}}{2L_{F_{12j+9}}} \right\rfloor = \frac{L_{12j+9}-6}{10},$$

$$\left\lfloor \frac{F_{4j+1}F_{F_{4j+1}-1}}{L_{F_{4j+1}}} \right\rfloor = F_{2j}^2,$$

$$\left\lfloor \frac{F_{4j+3}L_{F_{4j+3}-1}}{5F_{F_{4j+3}-1}} - \frac{2}{5} \right\rfloor = \frac{L_{4j+3}-4}{5}.$$

Proof If the positive sequence $(a_k)_{k=0}^n$ has a generating polynomial with only real zeros then, by Darroch Theorem [9], every mode k_0 of $(a_k)_{k=0}^n$ satisfies (12). By Proposition 7, the generating polynomial $B_n(x)$ of the sequence

$$a_{n,k} = \frac{\binom{2n+k+1}{2k}}{2n+k+1}$$

has only real zeros. So we have

$$\left\lfloor \frac{\sum_{j=1}^{n} ka_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum_{k=1}^{n} ka_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rceil.$$

In the case where we have two consecutive modes, $k_0,\ k_0+1$, necessarily

$$k_0 = \left\lfloor \frac{\sum_{k=0}^{n} ka_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rfloor.$$

Again, by Proposition 7 and Theorem 13,

$$k_j = \left\lfloor \frac{\sum_{k=0}^{n_j} ka_{n_j,k}}{\sum_{k=0}^{n_j} a_{n_j,k}} \right\rfloor = \left\lfloor \frac{B'_{n_j}(1)}{B_{n_j}(1)} \right\rfloor = \left\lfloor \frac{(2n_j+1)F_{4n_j+2}}{L_{4n_j+2}} \right\rfloor.$$

Replace n_j and k_j by their values yields the wanted result. For the second relation, we use the results of the paper [4], in which the properties of the sequence

$$c_{n,k} = \frac{n}{n-k}\binom{n-k}{k}$$

are studied. The generating polynomial of the sequence $c_{n,k}$ is given by

$$C_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k}x^k = \left(\frac{1+\sqrt{4x+1}}{2}\right)^n + \left(\frac{1-\sqrt{4x+1}}{2}\right)^n.$$

All the zeros of $C_n(x)$ are real. The integers where a double maximum occurs are $n_j = F_{4j+1}$, and in this case, the first mode is $k_j = F_{2j}^2$. Also,

$$\frac{C_n'(1)}{C_n(1)} = \frac{nF_{n-1}}{L_n}.$$

Finally

$$k_j = \left\lfloor \frac{C_{n_j}'(1)}{C_{n_j}(1)} \right\rfloor = \left\lfloor \frac{n_j F_{n_j-1}}{L_{n_j}} \right\rfloor.$$

Replacing n_j and k_j by their values yields the second relation. The last relation is obtained exactly as the previous one, using the results of [36].

5 Concluding remarks

- In [14,17,19,21], the polynomials defined by (4) are investigated. In the case $h(n) = 1$, the polynomial is denoted $Q_n^g(x)$:

$$Q_n^g(x) = x\sum_{k=1}^{n} g(k)Q_{n-k}^g(x), \quad Q_0^g(x) = 1. \tag{15}$$

  For a variety of functions g ($g(n)$= constant, n, n^2, etc), $Q_n^g(x)$ satisfies a recursion relation of k–terms ($k \geq 2$). In general, a polynomial satisfying such recursion is real rooted or at least log–concave. For $g(n) = n$, the obtained polynomial $Q_n^g(x)$ is what we denoted $xJF_{2n}(x)$. For $g(n) = 1$, $Q_n^g(x) = x(1+x)^{n-1}$.
  The polynomials $\widetilde{P}_n^{\phi_1}(x)$ and $\widetilde{P}_n^{\phi_2}(x)$ are obtained respectively, by dividing the coefficients of $(1+x)^{n-1}$ and of $U(x/2+1) = JF_{2n}(x)$ by $(k+1)!$. This is the conversion principle (Theorem 1 of [17]).

  It seems that the case $h(n) = 1$ is not only important in its own, but crucial for the study of the general sequence (4). Indeed, a polynomial $P(x)$ is obtained by dividing the coefficients of $Q_n^g(x)$ by $(k+1)!$, and, by Laguerre Theorem and other transformations (see [7]), the properties of real zeros (or log-concavity) of $Q_n^g(x)$ are transferred to $P(x)$.

  We may ask: given $Q_n^g(x)$, find the corresponding arithmetic functions $g_1(n)$ and $h_1(n)$ such that $P_n^{g_1,h_1}(x)$ is the polynomial whose coefficients are those of $Q_n^g(x)$ divided by $(k+l)!$, $l \geq 0$.

- A very interesting result of Kurtz [24] asserts that if the positive sequence $(a_k)_{k=0}^n$ satisfies

$$a_k^2 > 4a_{k+1}a_{k-1},\ 1 \le k \le n-1,$$

then $\sum_{k=0}^{n} a_k x^k$ has only real negative zeros. It would be nice to find special arithmetic functions $g,\ h$, (if they exist) such that

$$(A_{n,k}^{g,h})^2 > 4A_{n,k+1}^{g,h}A_{n,k-1}^{g,h},\ 1 \le k \le n-1.$$

- For the log-concavity, the results in [23,31] may be used (and may be improved and generalized) to prove log-concavity of the sequence $(A_{n,k}^{g,h})$.
- Finally, let us mention an important arithmetic function not considered in the series of papers of Heim and al., namely $g(n) = \frac{1}{(n-1)!},\ n \ge 1$. According to formula (2) of [17], we have

$$\sum_{n=0}^{\infty} P_n^{g,id}(x)z^n = \exp\left(x\sum_{n=1}^{\infty} g(n)\frac{z^n}{n}\right) = \exp\left(x\sum_{n=1}^{\infty}\frac{z^n}{n!}\right) = \exp\big(x(e^z-1)\big)$$

So, $P_n^{g,id}(x) = \sum_{k=1}^{n} S_2(n,k)x^k$, where $S_2(n,k)$ is the Stirling number of the second kind. Letting $g(n) = \frac{1}{n!}$ in formula (3) of [17], we obtain

$$\sum_{n=0}^{\infty} P_n^{g,1}(x)z^n = \frac{1}{1-x\sum_{n=1}^{\infty} g(n)z^n} = \frac{1}{1-x\sum_{n=1}^{\infty}\frac{z^n}{n!}} = \frac{1}{1-x\,(e^z-1)}$$

In this case, $P_n^{g,1}(x) = \sum_{k=1}^{n} k!S_2(n,k)x^k$. Both polynomials $P_n^{g,id}(x)$ and $P_n^{g,1}(x)$ have only real zeros.

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