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Nehari manifold for a class of elliptic problems

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in front of the jury :

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Notation

We introduce the necessary notations and definitions that are used

$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}^*$	Set of nonzero natural numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of positive real numbers.
$\mathbb{R}^N$	Euclidean space of dimension N.
Ω	Open of $\mathbb{R}^N$ equipped with the measure of Lebesgue.
u	Defined measurable function of Ω in $\mathbb{R}$.
∇u	gradient of u, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_N}\right)$.
Δu	Laplacian of u, $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}$.
$C(\Omega)$	Space of continuous functions on Ω.
$C^k(\Omega)$	space of continuous functions on Ω whose partial derivatives of order k are continuous on Ω,
$C^1(\Omega, \mathbb{R})$	The set of differentiable functions and the derivative is continuous.
$C^\infty(\Omega)$	space $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$.
$\mathcal{D}(\Omega)$	Space of indefinitely differentiable functions in Ω, with compact support in Ω.
$W^{1.p}(\Omega)$	Sobolev space of functions of $L^p(\Omega)$ whose derivatives partial in the sense of first-order distri
$W_0^{1.p}(\Omega)$	The closing of $\mathcal{D}(\Omega)$ in $W^{1.p}(\Omega)$, $W_0^{1.p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1.p}}$.
a.e	Almost everywhere.
$\rightharpoonup$	Weakly converges.
$\to$	Converges strongly.
$\langle \cdot, \cdot \rangle$	the scalar product.
$H^1(\Omega)$	$u \in L^2(\Omega)$ et $\nabla u \in L^2(\Omega)$.
$H_0^1(\Omega)$	Is $\overline{\mathcal{D}(\Omega)}^{H^1(\Omega)}$.
$E \hookrightarrow F$	E is continuously injected into F.
$E \hookrightarrow_c F$	E is injected in a compact way into F.
p^*	The sobolev exponent, such that $p^* = \frac{N_p}{N-p}$.
p'	Hölder's conjugate of p, $p' = \frac{p}{p-1}$, if $p > 1$ and $p' = \infty$ if $p = 1$.

Introduction

The objective of our thesis is to apply Nehari's method, this method introduce a new approach to finding solutions for a class of semilinears elliptic problems which is based on constrained minimization, this approach is slightly more complicated than direct minimization on Spheres.

Nehari manifold goes back to Nehari's work [17], [18]. The most important articles that used this method [6, 22, 14, 9, 12].

To approach this class of problem, today, we use some different classes of approaches, variational methods (for problems in variational form) and critical point theory, this two techniques are very useful and popular for proving the existence of solutions to boundary value problems in a variety of contexts, including ordinary differential equations, differential equations partial, impulsive problems, etc.

In the first chapter, we have collected some of the concepts that we need to study the problems in the next two chapters, and we mentioned the concept of critical points and some spaces such as Lebegue space and Sobolev space, and some related properties.

In the second chapter, we have presented two methods Sphere of $L^p(\Omega)$ method and Nehari Manifold, to prove the existence of a nontrivial solution of superlinear problem, and apply them to a problem of the following form

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in} \;\; \Omega, \\ u = 0 & \text{on} \;\; \partial\Omega. \end{cases}$$

and p a real number such that

$$2 < p < 2^* = \frac{2N}{N-2},$$

this kind of problems their associated functionals are generally unbounded from below and they present a lack of convexity or concavity properties.

for this problem we are going to use the minimization techniques: minimization on Sphere by using the set, for every $\beta > 0$

$$\sum_{\beta} = \left\{ u \in H_0^1(\Omega) \;\; | \;\; \int_{\Omega} |u|^p \, dx = \beta \right\}$$

we minimize the function whose critical points are the (weak) solution on $\sum_{\beta}$.

The second technique is minimizing on Nehari Manifold by using the set

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \;\; | \;\; u \neq 0, \;\; \langle I'(u), u \rangle = 0 \right\},$$

the minimization on Nehari Manifold can be extended to cover more general problems, such as the perturbed problem that has a fixed function $h \in L^2(\Omega)$ and $p \in]2, 2^*[$, the Dirichlet problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

in this chapter by considering h is small we will prove that this problem admits at least two nontrivial solutions.
For the third chapter, We study semilinear elliptic problem with Neumann condition, that were presented in the article [22], posted in (2013), let the following form

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on} \quad \partial\Omega. \end{cases}$$

Using the Nehari manifold and fibering maps, we prove the existence theorem of this nonlinear boundary problem. Here the exponents p and q satisfy

$$1 < q < 2 < p < 2^* = \frac{2N}{N-2}.$$

This is a developed model of some previously studied models of this format in the references [11, 8, 13, 19, 21].

Chapter 1

Preliminary

1.1 Functional spaces

1.1.1 L^p Spaces

Definition 1.1 *[4].Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N ; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{\frac{1}{p}}.$$

Definition 1.2 *[4]. We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N ; f \text{is measurable and } |f(x)| \leq c \ \text{ a.e.on } \Omega \right\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = inf\left\{ c : |f(x)| \leq c \ \text{ a.e.on } \Omega \right\},$$

with c is a constant.

Theorem 1.1 *[4].$L^p(\Omega)$ is a Banach space, for any $1 \leq p \leq \infty$, and reflexive for $1 \leq p < \infty$, and separable for $1 < p < \infty$.*

Notation 1.2 *.Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.3 (Hölder's inequality) *[4].Assume that $f \in L^p$ and $g \in L^p$ with $1 \leq p \leq \infty$, then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

1.1.2 $W^{1.p}(\Omega)$ Spaces

Let $\Omega \in \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.3 *[4].We denote by $\mathcal{D}(\Omega)$ the set of function of class $\mathcal{C}^\infty(\Omega)$ with support compact include in Ω. The Sobolev space $W^{1.p}(\Omega)$ is defined by*

$$W^{1.p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g_i \in L^p(\Omega) \text{such that:} \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = - \int_\Omega g_i \varphi dx, \forall \varphi \in \mathcal{D}(\Omega), \forall i = 1, 2, ..., N \right\}.$$

We set

$$H^1(\Omega) = W^{1.2}(\Omega).$$

For $u \in W^{1.p}(\Omega)$ we set $\frac{\partial u}{\partial x_i} = g_i$, and we write

$$\nabla u = grad\ u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}).$$

The space $W^{1.p}$ is equipped with the norm $\|u\|_{W^{1.P}} = \|u\|_{L^P} + \|\nabla u\|_{L^P}$, or sometimes with the equivalent norm $\|u\|_{W^{1.P}} = (\|u\|^p_{L^p} + \|\nabla u\|^p_{L^p})^{\frac{1}{p}}$ if $(1 \leq p < \infty)$.
The space $H^1(\Omega)$ is equipped with the scalar product

$$\int_\Omega uv dx + \int_\Omega \nabla u \nabla v dx.$$

The associated norm $\|u\|_{H^1} = (\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2})^{\frac{1}{2}}$ is equivalent to the $W^{1.2}$ norm.

Proposition 1.1 *[4].$W^{1.p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p < \infty$.*
In particular $H^1(\Omega)$ is reflexive, separable and Hilbert space.

1.1.3 Sobolev Injections and Inequalities

Let $(E, \|.\|_E), (F, \|.\|_F)$ Banach spaces

Notation 1.4

1. E is injected continuously into F, means that the canonical injection $j : E \to F$ is continuous i.e, $\ni c > 0, \forall x \in E : \|x\|_F \leq c \|x\|_E$, and we denote by $E \hookrightarrow F$.

2. E is injected in compact into F means that the canonical injection $j : E \to F$ is compact i.e for all sequence bounded u_n in E we can extract subsequence u_{nk} convergent in F, and we denote by $E \hookrightarrow_c F$.
   If $1 \leq p < \infty$, the sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Theorem 1.5 *[4].Let $1 \leq p \leq \infty$, we suppose that Ω is on open set of class $\mathcal{C}^1$ a bounded frontier, and we take $\Omega = \mathbb{R}^N_+$*

1. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*[\quad if \;\; p < N.$

2. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty[\quad if \;\; p = N.$

3. $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega) \;\; if \;\; p > N.$

Theorem 1.6 (Rellich-Kondrachon) *[4].Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class c^1*

1. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1.p^*[\quad if \;\; p < N.$

2. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p.\infty[\quad if \;\; p = N.$

3. $W^{1.p}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right) \;\; if \;\; p > N.$

1.1.4 $W_0^{1.p}(\Omega)$ Space

Definition 1.4 *.Let $1 \leq p \leq \infty$, $W_0^{1.p}$ means the closing of $\mathcal{D}(\Omega)$ in $W^{1.p}$, we notice*

$$\begin{aligned} W_0^{1.p}(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1.p}} \\ &= \left\{u \in W^{1.P}(\Omega) : u = 0 \; sur \; \partial\Omega\right\}, \end{aligned}$$

and

$$H_0^1(\Omega) = W_0^{1.2}(\Omega).$$

The space $W_0^{1.p}(\Omega)$ provided with norm induced by $W^{1.p}$, H_0^1 is a Hilbert space for the scalar product of H^1 we put $\|u\|_{H_0^1} = \|u\|$.

Remark 1.1 *.when $\Omega = \mathbb{R}^N$, we know that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1.p}(\mathbb{R}^N)$, and there for*

$$W_0^{1.p}(\mathbb{R}^N) = W^{1.p}(\mathbb{R}^N).$$

Proposition 1.2 (Poincaré's inequality) *[4].Let $\Omega \subset \mathbb{R}^N$ on open set, Then there exists a constant C such that*

$$\|u\|_{W^{1.p}(\Omega)} \leq c\,\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1.p}(\Omega).$$

In other words, on $W_0^{1.p}$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1.p}$ norm.

Theorem 1.7 (Young's inequality) *. For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

1.2 Differentiability of Functionals and Critical Points

In what follows, we introduce some notion of derivatives for functions defined on Banach spaces, we start with the directional derivative.

1.2.1 Derivative in the sense of Gateau

Definition 1.5 *.Let E be a Banach space, $\Omega \subseteq E$ an open set, and let $I : \Omega \longrightarrow \mathbb{R}$ a functional, we say that I is ad differentiable in the sense of Gateau (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$, where $I(u+tv)$ exists for $t > 0$ small enough, the directional derivative $DI(u)$ exists i.e:*

$$\lim_{t \longrightarrow 0} \frac{I(u+tv) - I(u)}{t} = \langle A, v \rangle$$

if I is differentiable in the sense of Gateau in u, there exists only one verified linear functional.

Example 1.1 *.Let $I : L^p(\Omega) \to \mathbb{R}$, $I(u) = \int_\Omega |u|^p dx$. I is G-differentiable and we have $\langle I'_G(u), v \rangle = p \int_\Omega |u|^{p-2} uvdx$, indeed, let $x \in \Omega$, t sufficient small fixed and we define $g_{u.v}(s) = |u+tv|^p$. $s \in [0,t]$ let $g_{u.v} : [0,t] \to \mathbb{R}$, we have $g_{u.v}$ is continuous on the closed interval $[0,t]$ and differentiable on the interval open $]0,t[$, then according to the mean value theorem, there exists a real $c_t \in]0,t[$ and when $t \to 0$, we have $c_t \to 0$*

$$\begin{aligned} \lim_{t \to 0} \frac{|u+tv|^p - |u|^p}{t} &= \lim_{c_t \longrightarrow 0} p|u+c_t v|^{p-2}(u+c_t v)v \\ &= p|u|^{p-2}uv. \end{aligned}$$

According to Lebesgue's convergence theorem, we have

$$\langle I'(u), v \rangle = p \int_\Omega |u|^{p-2} uvdx.$$

and we define $A : L^p \to \mathbb{R}$

$$v \to A(v) = p \int_\Omega |u|^{p-2} uvdx$$

we prove that $A \in (L^p(\Omega))' = L^{p'}(\Omega)$, i.e A is linear and continuous
A is linear, in effect, let $v_1, v_2 \in L^p(\Omega)$ and let $\alpha.\beta \in \mathbb{R}$

$$\begin{aligned} A(\alpha v_1 + \beta v_2) &= p \int_\Omega |u|^{p-2} u(\alpha v_1 + \beta v_2) dx \\ &= p[\int_\Omega |u|^{p-2} u\alpha v_1 dx + \int_\Omega |u|^{p-2} u\beta v_2 dx] \\ &= \alpha p \int_\Omega |u|^{p-2} uv_1 dx + \beta p \int_\Omega |u|^{p-2} uv_2 dx \\ &= \alpha A(v_1) + \beta A(v_2). \end{aligned}$$

A is linear. A is continuous, in effect, let $u, v \in L^p(\Omega)$

$$\begin{aligned} |p \int_\Omega |u|^{p-2} uvdx| &\leq p \int_\Omega |u|^{p-1} |v| dx \\ &\leq p(\int_\Omega |u|^{(p-1)p'} dx)^{\frac{1}{p'}} (\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &\leq p(\int_\Omega |u|^{(p-1)(\frac{p}{p-1})} dx)^{\frac{p-1}{p}} (\int_\Omega |v|^p dx)^{\frac{1}{p}} \\ &= \|u\|_{L^p}^{p-1} \|v\|_{L^p}. \end{aligned}$$

Then A is a continuous, so the function is G-differentiable.

1.2.2 Derivative in the sense of Frechet

Definition 1.6 *.Let E be a Banach space ,$\Omega \subseteq E$ an open set and let $I : \Omega \longrightarrow \mathbb{R}$ a functional, we say that I is ad differentiable in the sense of Frechet, at $u \in \Omega$, if there exists $A \in E'$ such as*

$$\lim_{\|v\|\to 0} \frac{I(u+tv) - I(v) - Av}{\|v\|} = 0$$

or

$$I(u+tv) - I(u) = Av + o(\|v\|).$$

if I is differentiable, then A is unique and we denote $I'(u) = A$, the set of differentiable function, we will be denote $\mathcal{C}^1(\Omega.\mathbb{R})$.

Proposition 1.3 *.Suppose that $\Omega \subseteq E$ is on open set, such that I G-differentiable in Ω and that I'_G is continuous at $\Omega \in E$, then I is also differentiable at u, and of cours $I'_G = I'(u)$.*

Remark 1.2 *.The importance of Proposition 1.3 reside in the fact that it is often technically easier to calculate the derivative in the sense of Gateau an then to prove that it is continuous, rather than to directly prove the differentiability in the sense of Frechet.*

Exercice 1.8 *.We prove that the function $J : L^p(]0,1[) \to \mathbb{R}$ $(p > 2)$ and by using Holder inequality we get*

$$u \to J(u) = \int_0^1 |u|^p dx.$$

is a Frechet differentiable on $L^2(]0,1[)$.
we already prove that J is G-differentiable, with

$$J'_G : L^p(\Omega) \to (L^p(\Omega))' = L^q(\Omega)$$

$$u \to J'_G(u) : L^p(\Omega) \to \mathbb{R}$$

$$v \to \langle J'(u), v\rangle = p\int_0^1 |u|^{p-1}uvdx$$

it remains to prove that $J'_G : L^p(\Omega) \to (L^p(\Omega))' = L^q(\Omega)$

$$u \to J'_G(u)$$

is continuous.
Let $(u_n) \subset L^p(\Omega)$ such as $u_n \to u$ in $L^p(\Omega)$ we prove that $J'_G(u_n) \to J'_G(u)$ in $(L^p(\Omega))'$

$$\|J'_G(u_n) - J'_G(u)\|_{(L^p(\Omega))'} = \sup_{v\in L^p(\Omega)} |\langle J'_G(u_n) - J'_G(u), v\rangle|$$

let $v \in L^p(\Omega)$, such as $\|v\|_{L^p} = 1$, we have

$$\begin{aligned} |\langle J'_G(u_n) - J'_G(u), v\rangle| &= |\langle J'_G(u_n), v\rangle - \langle J'_G(u), v\rangle| \\ &= p|\int_0^1 |u_n|^{p-2}u_n v dx - \int_0^1 |u|^{p-2}uvdx| \\ &\le p[\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}}dx](\int_0^1 |v|^p dx)^{\frac{1}{p}}. \end{aligned}$$

We pose $u_n = |u_n|^{p-2}\, u_n, u = |u|^{p-2}\, u$, *we know that* $u_n \to u$ *in* $L^p(0,1)$, *there is a subsequence* $u_{nk} = u_n$, *such that* $u_n(x) \to u(x)$ *in* $]0,1[$.
moreover there exists $g \in L^P$ *such that*

$$|u_n(x)| \leq g(x) \quad p.p \ in \ ]0,1[.$$

so

$$|u_n(x)|^{p-1} \leq |g(x)|^{p-1} \quad \in L^{\frac{p}{p-1}}$$

according to $u_n \to u$ *in* $\frac{p}{p-1}$ *we recall that* $\|J'_G(u_n) - J'(u)\| \leq \int_\Omega |u_n|^{p-2}u_n - |u|^{p-2}udx \to 0$ *and then* $\lim_{n\to+\infty} \|J'_G(u_n) - J'(u)\|_{(L^p(\Omega))'} = 0$
as J *is G-differentiable and* J'_G *is continuous, so* J *is Frechet differentiable and* $J'_G = J'$.

1.2.3 Critical points

Definition 1.7 *[15].Let* Ω *an open set of Banach space* E, *suppose that* $I \in \mathcal{C}^1(\Omega.\mathbb{R})$, *we say that* $u \in \Omega$ *is critical point of* I, *if*

$$I'(u) = 0$$

if u *is not critical point, then we say that* u *is regular point of* I.
if $c \in \mathbb{R}$, *then we say that* c *is a critical value of* I, *if there exists* $u \in \Omega$ *such as*

$$I(u) = c \ \ and \ \ I'(u) = 0.$$

if c *is not a critical value, then* c *is said to be a regular value of* I.

1.3 Some Theorems used

Definition 1.8 *[4](Minimizing sequence). A minimizing sequence of a functional* $J : E \longrightarrow]-\infty, +\infty]$ *is a sequence* (u_n), *such that*

$$\lim_{n\longrightarrow+\infty} J(u_n) = \inf_{u\in E} J(u).$$

Definition 1.9 *Let* $J : E \longrightarrow \mathbb{R}$ *be a functional, we say that* J *is weakly lower semi-continuous if* $\forall u_n \subset E : u_n \rightharpoonup u_0$ *in* E, *we have*

$$J(u_0) \leq \lim_{n\longrightarrow+\infty} \inf J(u_n).$$

Definition 1.10 *[4](coercive) Let* $J : E \longrightarrow \mathbb{R} \cup +\infty$ *a functional on* E . *We say that* J *is coercive if and only if*

$$\lim_{\|u\|_E\longrightarrow+\infty} \frac{J(u)}{\|u\|} = +\infty.$$

Theorem 1.9 *[4]any functional continuous convex on* E, *then* J *is weakly lower semi-continuous.*

Theorem 1.10 *[4](Lebesgue's dominated convergence theorem) Let* (f_n) *be a sequence of functions of* L^1. *We suppose that*

1. $f_n(x) \longrightarrow f(x)$ *a.e.on* Ω,

2. *there exists a function* $g \in L^1$ *such that for every* n, $|f_n(x)| \leq g(x)$ *a.e.in* Ω.

We have $f \in L^1(\Omega)$ *and* $\|f_n - f\|_{L^1} \longrightarrow 0$.

Theorem 1.11 *[4](Lebesgue's dominated convergence inverse theorem)Let* (f_n) *be a sequence in* L^p *and let* $f \in L^p$ *be such that* $\|f_n - f\|_{L^p} \longrightarrow 0$.
Then, there exist a subsequence (f_{n_k}) *and a function* $h \in L^p$ *such that*

1. $f_{n_k}(x) \longrightarrow f(x)$ *a.e.on* Ω,

2. $|f_{n_k}(x)| \leq h(x) \quad \forall k$, *a.e.in* Ω.

Chapter 2

Constrained Minimization and Applications

2.1 Introduction

In the case of superlinear problems, the functionals associated to these problems are generally unbounded from below and they present a lack of convexity or concavity properties. Actually for rather general f and h only partial results are known, in this section we begin to present here some of the simplest cases, in which h is identically zero and f is a power. We take throughout this section a real number p such that

$$2 < p < 2^* = \frac{2N}{N-2},$$

and we search a function u that satisfies the super-linear and subcritical problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This problem admits the zero solution, we will prove the existence of a nontrivial solution by two different methods, which can be extended to cover more general problems.
In this section, the general framework is specified by the assumption
(**h1**) $\Omega \subset \mathbb{R}^N$ is bounded and open, $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ a.e in Ω.

Theorem 2.1 *[3].Let $p \in]2, 2^*[$ and assume that (**h1**) holds. Then the problem(2.1) admits at least one non-negative and nontrivial solution.*

The functional $I : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega q(x)u^2 dx - \frac{1}{p}\int_\Omega |u|^p dx = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_{L^p}^p, \tag{2.2}$$

is Fréchet differentiable using the same method with Example 1.8, we remark that the critical points of I are the weak solutions of (2.1). We notice immediately that I is unbounded from below, since for every $u \neq 0$, we have

$$I(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}\|u\|_{L^p}^p \longrightarrow -\infty.$$

as $t \longrightarrow +\infty$, because $p > 2$. Notice also that I is the difference of two strictly convex functionals.

2.2 Minimization on Spheres

The main property in this method is the homogeneity of the two terms in I; indeed the first term is positively homogeneous of degree 2,while the second is positively homogeneous of degree p. We cannot use minimization on the whole space H_0^1 since the functional I is unbounded from below, we get rid of this unboundedness by restricting this functional on a set where it becomes bounded below. In this method we use the sphere of L^p.

For every $\beta > 0$, we set

$$\Sigma_\beta = \left\{ u \in H_0^1\Omega : \int_\Omega |u|^p \, dx = \beta \right\}.$$

In this set, we notice that I takes the form

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p}\beta,$$

so that it is bounded from below, now by minimizing I on Σ_β, we get

$$m_\beta = \inf_{u \in \Sigma_\beta} \|u\|^2.$$

We are going to show that m_β is attained by some function, and that this function leads to beginning of appearance to the solution of problem (2.1).

Lemma 2.1 *[3].For every $\beta > 0$, the level m_β is attained by a nonnegative function; namely there exists $u_0 \in \Sigma_\beta$, $u_0(x) \geq 0$ a.e in Ω, such that*

$$\|u_0\|^2 = m_\beta.$$

Proof. .Let $\{u_k\}_k \subset \Sigma_\beta$ be a minimizing sequence for $\|u\|^2$. Obviously the sequence $\{|u_k|\}_k$ is still a minimizing sequence in Σ_β and therefore we can assume from the beginning that $u_k(x) \geq 0$ a.e. in Ω and for all $k \in \mathbb{N}$. This minimizing sequence is of course bounded in $H_0^1(\Omega)$ which is reflexive space, so that, up to subsequences there exist $u_{k_l} = u_k$ and $u_0 \in H_0^1(\Omega)$, such that

$$u_k \rightharpoonup u_0 \text{ in } H_0^1(\Omega)$$

and for $2 < p < 2^*$ the embedding of $H_0^1(\Omega)$ into L^p is compact. Then

$$u_k \longrightarrow u_0 \text{ in } L^p$$

$$\text{and} \qquad \exists u_{k_{l'}} = u_k \qquad u_k(x) \longrightarrow u_0(x) \text{ a.e in } \Omega.$$

Since

$$\|u_0\|_{L^p} = \lim_{k \longrightarrow +\infty} \|u_k\|_{L^p} = \lim_{k \longrightarrow +\infty} \beta = \beta,$$

then $u_0 \in \Sigma_\beta$.

By weak lower semicontinuity of the norm, we obtain

$$\|u_0\|^2 \leq \lim_{k \longrightarrow \infty} \inf_{u \in \Sigma_\beta} \|u_k\|^2 = \inf_{u \in \Sigma_\beta} \|u\|^2 = m_\beta.$$

On the other hand, we have

$$\|u_0\|^2 \geq \inf_{u \in \Sigma_\beta} \|u\|^2 = m\beta,$$

we get $\|u_0\|^2 = m_\beta$, and $u_0 \in \Sigma_\beta$ implies that u_0 does not vanish identicaly. ■

Remark 2.1 *The preceding proof should retain that the key assumption is the subcritical growth condition $p < 2^*$, the embedding of $H_0^1(\Omega)$ into L^p is compact, eventually that what allows that $u \in \Sigma_\beta$. We conclude that u is a minimum in Σ_β.*
The compactness of the embedding fails in the critical problems and the argument used so far do not work when $p = 2^*$, some problems may have no nontrivial solution. This is the starting point of a line of research, begun with[5].

Lemma 2.2 *[3].Let u_0 be a minimizer found with the previous lemma. Then u_0 satisfies*

$$\int_\Omega \nabla u_0.\nabla v dx + \int_\Omega q(x) u_0 v dx = \frac{m_\beta}{\beta} \int_\Omega |u_0|^{p-2} u_0 v dx \tag{2.3}$$

for all $v \in H_0^1(\Omega)$.

Proof. We proofed that u_0 minimizes the functional $N(u) = \|u\|^2$ on Σ_β, but this not mean we can conclude that the differential of N vanishes at u_0, also $u_0 + v$ is not belong to Σ_β there for Σ_β is not a vector space, this means that we cannot compare the values of $N(u_0)$ and of $N(u_0 + v)$. In general, we have to form small variations of u_0 that lie on Σ_β.
We fix $v \in H_0^1(\Omega)$, for s small enough, say $s \in]-\epsilon, \epsilon[$, the function $u_0 + sv$ is not identically zero. Therefore there exists $t :]-\epsilon, \epsilon[\longrightarrow]0, +\infty[$, such that

$$\int_\Omega |t(s)(u_0 + sv)|^p \, dx = \beta,$$

precisely,

$$t(s) = \left(\frac{\beta}{\int_\Omega |u_0 + sv|^p \, dx} \right)^{\frac{1}{p}} = \beta^{\frac{1}{p}} \left(\int_\Omega |u_0 + sv|^p \, dx \right)^{-\frac{1}{p}}.$$

Indeed, we have

$$\begin{aligned} \int_\Omega |t(s)(u_0 + sv)|^p \, dx &= \int_\Omega \frac{\beta}{\int_\Omega |u_0 + sv|^p \, dx} |u_0 + sv|^p \, dx \\ &= \frac{\beta}{\int_\Omega |u_0 + sv|^p \, dx} \int_\Omega |u_0 + sv|^p \, dx \\ &= \beta. \end{aligned}$$

then we conclude that $t(s)(u_0 + sv) \in \Sigma_\beta$. The function t is differentiable on $]-\epsilon, \epsilon[$, we have

$$t'(s) = -\beta^{\frac{1}{p}} \left(\int_\Omega |u_0 + sv|^p \, dx \right)^{-\frac{1}{p}-1} \int_\Omega |u_0 + sv|^{p-2} (u_0 + sv) v dx.$$

So $t(0) = 1$ and $t'(0) = -\beta^{-1} \int_\Omega |u_0|^{p-2} uv dx$.
When $s = 0$ the curve of the application $s \longmapsto t(s)(u_0 + sv)$ on Σ_β passes through u_0, we provide this application with norm, such that

$$\gamma(s) = N(t(s)(u_0 + sv)) = \|t(s)(u_0 + sv)\|^2,$$

$$\gamma(s) = \int_\Omega |\nabla [t(s)(u_0 + sv)]|^2 \, dx + \int_\Omega q(x)(t(s)(u_0 + sv))^2 dx.$$

and we remark that $\langle u, v \rangle_{H_0^1} = \int_\Omega \nabla u \nabla v dx + \int_\Omega q(x) uv dx$.
Since $t(s)(u_0 + sv) \in \Sigma_\beta$, for every $s \in]-\epsilon, \epsilon[$ the point $s = 0$ is a local minimum for γ, the function γ is differentiable and

$$\gamma'(s) = 2\int_\Omega \nabla\left[t(s)(u_0+sv)\right].\nabla\left[t'(s)(u_0+sv)+t(s)v\right]dx$$
$$+2\int_\Omega q(x)t(s)(u_0+sv)(t'(s)(u_0+sv)+t(s)v)dx$$
$$= 2\left\langle t(s)(u_0+sv), t'(s)(u_0+sv)+t(s)v\right\rangle.$$

so that

$$\begin{aligned} 0=\gamma'(0) &= 2\left\langle t(0)u_0, t'(0)u_0+t(0)v\right\rangle \\ &= 2t(0)t'(0)\|u_0\|^2 + 2t^2(0)\left\langle u_0, v\right\rangle \\ &= -2\frac{m_\beta}{\beta}\int_\Omega |u_0|^{p-2}u_0 v dx + 2\left\langle u_0, v\right\rangle \end{aligned}$$

and this shows that

$$\left\langle u_0, v\right\rangle = \frac{m_\beta}{\beta}\int_\Omega |u_0|^{p-2}u_0 v dx,$$

for every $v \in H_0^1(\Omega)$, namely (2.3).
end of the proof. ∎

2.3 Minimization on the Nehari Manifold

In this section we are going to present a second method to prove the existence of a nontrivial solution to (2.1), and this approach is still based on constrained minimization but more complicated and has the advantage that it does not require the nonlinearity to be homogeneous, which can be extended to cover more general problems, we use the problem (2.1) to explain it more.
We consider the associated functional as (2.2), we know that I is unbownded from below, to get rid of this problem we restrict I to suitable set, we consider the following set

$$\begin{aligned} \mathcal{N} &= \left\{u \in H_0^1(\Omega) \;\mid\; u \neq 0, \langle I'(u), u\rangle = 0\right\} \\ &= \left\{u \in H_0^1(\Omega) \;\mid\; u \neq 0, \|u\|^2 = \int_\Omega |u|^p dx\right\}. \end{aligned} \tag{2.4}$$

This set is called the Nehari manifold.

Remark 2.2 *By definition, the Nehari manifold contains all the nontrivial critical points of I. we have*

$$\langle I'(u), v\rangle = \int_\Omega \nabla u \nabla v dx + \int_\Omega q(x)uv dx - \int_\Omega |u|^{p-2}uv dx.$$

Notice that

$$\begin{aligned} \langle I'(u), u\rangle = 0 &\iff \int_\Omega |\nabla u|^2 dx + \int_\Omega q(x)u^2 dx - \int_\Omega |u|^p dx = 0 \\ &\iff \|u\|^2 = \|u\|_{L^p}^p. \end{aligned}$$

and this leads us that the function I reads

$$I(u) = \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 = \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_{L^p}^p dx = \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega |u|^p dx.$$

This shows at once that I is coercive on $\mathcal{N}$, in the sense that, if $\{u_k\}_k \subset \mathcal{N}$ satisfies $\|u_k\| \longrightarrow \infty$, then $I(u_k) \longrightarrow \infty$. We define

$$m = \inf_{u \in \mathcal{N}} I(u).$$

we are going to show that m is attained by some $u \in \mathcal{N}$ which is a critical point of I on the whole space $H_0^1(\Omega)$, and there for a solution to problem (2.1).
We begin with some basic properties of $\mathcal{N}$ and I.

Lemma 2.3 *[3] The Nehari manifold is not empty.*

Proof. For every $u \in H_0^1(\Omega)$, $u \neq 0$, for $t > 0$, we see that $tu \in \mathcal{N}$ and it is equivalent to

$$\|tu\|^2 = \int_\Omega |tu|^p \, dx. \tag{2.5}$$

we obtien

$$t^{p-2} = \frac{\|u\|^2}{\int_\Omega |u|^p \, dx},$$

so

$$t = \left(\frac{\|u\|^2}{\int_\Omega |u|^p \, dx} \right)^{\frac{1}{p-2}} > 0.$$

and this means that there exists $t > 0$, such that $tu \in \mathcal{N}$, and from it we conclude that $\mathcal{N} \neq \emptyset$. ■

Lemma 2.4 *[3] we have*

$$m = \inf_{u \in \mathcal{N}} I(u) > 0.$$

Proof. Let $2 < p < 2^*$ we pose $tu \in \mathcal{N}$, by the continuous embedding $H_0^1(\Omega) \hookrightarrow L^p$, there exists $c > 0$ such that

$$\|u\|^2 = \|u\|_{L^p}^p \leq c \|u\|^p .$$

since $\|u\| \neq 0$, then we obtain

$$\|u\| \geq \left(\frac{1}{c} \right)^{\frac{1}{p-2}} > 0, \tag{2.6}$$

so that, for every $tu \in \mathcal{N}$ we have

$$m = \inf_{u \in \mathcal{N}} I(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \inf_{u \in \mathcal{N}} \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{1}{c} \right)^{\frac{2}{p-2}} > 0.$$

■

Lemma 2.5 *[3] The level m is attained by a nonnegative function, namely there exists $u_* \in \mathcal{N}, u_*(x) \geq 0$ a.e.in Ω, such that*

$$I(u_*) = m.$$

Proof. Let$\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I, that is $\lim_k I(u_k) = \inf I(u) = m$, obviously $|u_k| \in \mathcal{N}$ and $I(|u_k|) = I(u_k)$, the sequence $\{|u_k|\}_k$ is still a minimizing sequence in $\mathcal{N}$, therefore we can assume straight away that $u_k(x) \geq 0$ a.e in Ω for all k, we have already spotted that I is coercive on $\mathcal{N}$, and since $I(u_k) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2$, if we suppose that $\{u_k\}_k$ is not bounded so $\left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|^2 \longrightarrow +\infty$, consequently $I(u_k)$ converge to $+\infty$, which is contradiction with the fact that $I(u_k)$ converge to m, this implies that the sequence $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$ which is reflexif space, so that up to subsequences, there exists $u_{k_l} = u_k$ and $u_* \in H_0^1(\Omega)$ such that

$$u_k \rightharpoonup u_* \text{ in } H_0^1(\Omega)$$

and for $2 < p < 2^*$, the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is compact, then $u_k \longrightarrow u_*$ in $L^p(\Omega)$ and $\quad \exists u_{k_{l'}} = u_k \quad u_k(x) \longrightarrow u_*(x)$ a.e in Ω
then we have $u_* \geq 0$, and by weak lower semicontinuity

$$\begin{aligned} I(u_*) &= \frac{1}{2}\|u_*\|^2 - \frac{1}{p}\|u_*\|_{L^p}^p \leq \liminf_k \left(\frac{1}{2}\|u_k\|^2 - \frac{1}{p}\|u_k\|_{L^p}^p\right) \\ &= \liminf_k I(u_k) = m \end{aligned} \tag{2.7}$$

Since $u_k \in \mathcal{N}$, we have $\|u_k\| = \int_\Omega |u_k|^p\, dx$, by (2.6) it cannot be $\|u_k\| \longrightarrow 0$ and therefore $\int_\Omega |u_k|^p\, dx$ cannot tend to zero, thus the strong convergence in $L^p(\Omega)$, implies that

$$\|u_k\|_{L^p}^p \longrightarrow \|u_*\|_{L^p}^p .$$

Consequently $\int_\Omega |u_*|^p\, dx \neq 0$, which shows that $u_* \neq 0$. Passing to the limit we obtain

$$\begin{aligned} \|u_*\|^2 \leq \liminf_{k\longrightarrow+\infty} \|u_k\|^2 &= \liminf_{k\longrightarrow+\infty} \int_\Omega |u_*|^p\, dx \\ &= \int_\Omega |u_*|^p\, dx \neq 0 \end{aligned}$$

we obtian

$$\|u_*\|^2 \leq \int_\Omega |u_*|^p\, dx \tag{2.8}$$

if $\|u_*\|^2 = \int_\Omega |u_*|^p\, dx$, then $u_* \in \mathcal{N}$ and (2.7) shows that u_* is the required minimizer. Since (2.8) holds, we only have to treat the case where

$$\|u_*\|^2 < \int_\Omega |u_*|^p\, dx. \tag{2.9}$$

take $t > 0$ such that $tu_* \in \mathcal{N}$, namely

$$t = \left(\frac{\|u_*\|^2}{\int_\Omega |u_*|^p\, dx}\right)^{\frac{1}{p-2}}$$

Since we are assuming (2.9) holds, we deduce that $0 < t < 1$, but $tu_* \in \mathcal{N}$, so that

$$\begin{aligned} 0 < m &\leq I(tu_*) = \left(\frac{1}{2} - \frac{1}{p}\right)\|tu_*\|^2 = t^2\left(\frac{1}{2} - \frac{1}{p}\right)\|u_*\|^2 \\ &\leq t^2 \liminf_{k\longrightarrow+\infty} \left(\frac{1}{2} - \frac{1}{p}\right)\|u_k\|^2 = t^2 \liminf_{k\longrightarrow+\infty} I(u_k) = t^2 m < m. \end{aligned}$$

contradiction, and this leads us to conclude that

$$I(u_*) = m.$$

this is the end of the proof. ∎

Lemma 2.6 *[3] u_* is a solution of problem (2.3).*

Proof. Let $v \in H_0^1(\Omega)$, for s small enough, say $s \in]-\epsilon, \epsilon[$ the function $u_* + sv$ is not identically zero, there for, there exists $t(s) > 0$ a function such that $t(s)(u_* + sv) \in \mathcal{N}$, precisely

$$t(s) = \left(\frac{\|u_* + sv\|^2}{\int_\Omega |u_* + sv|^p \, dx} \right)^{\frac{1}{p-2}}.$$

the function t is differentiable because t is a composition of differentiable functions we dont need t' here, we have $t(0) = 1$, the application $s \mapsto t(s)(u_* + sv)$ defines a curve on $\mathcal{N}$ passe through u_* when $s = 0$ along which we evaluate I. We define $\gamma :]-\epsilon, \epsilon[\longrightarrow \mathbb{R}$ as

$$\begin{aligned}
\gamma(s) &= I(t(s)(u_* + sv)) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \|t(s)(u_* + sv)\|^2 \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \left[\int_\Omega \left(\nabla \left[t(s)(u_* + sv)\right]\right)^2 + \int_\Omega q(x) \left(t(s)(u_* + sv)\right)^2 dx \right]
\end{aligned}$$

and

$$\begin{aligned}
\gamma'(s) &= 2\left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega \nabla \left[t(s)(u_* + sv)\right] . \nabla \left[t'(s)(u_* + sv) + t(s)v\right] dx \\
&+ 2\left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega q(x) t(s)(u_* + sv) \left(t'(s)(u_* + sv) + t(s)v\right) dx.
\end{aligned}$$

So that

$$\begin{aligned}
0 = \gamma'(0) &= 2\left(\frac{1}{2} - \frac{1}{p}\right) \left[\int_\Omega \nabla \left[t(0)u_*\right] . \nabla \left[t'(0)u_* + t(0)v\right] dx + \int_\Omega q(x) t(0) u_* \left(t'(0)u_* + t(0)v\right) dx \right] \\
&= \langle I'(t(0)u_*), t'(0)u_* + t(0)v \rangle \\
&= \langle I'(u_*), t'(0)u_* \rangle + \langle I'(u_*), v \rangle \\
&= t'(0) \langle I'(u_*), u_* \rangle + \langle I'(u_*), v \rangle
\end{aligned}$$

we have $\langle I'(u_*), u_* \rangle = 0$,
then we obtain for every $v \in H_0^1(\Omega)$ $\langle I'(u_*), v \rangle = 0$, which proved that u_* is a solution of problem (2.1). ∎

Remark 2.3 *The last part of the proof shows that a minimum of I constrained on the Nehari manifold $\mathcal{N}$ is actually a free critical point of I, on the whole space $H_0^1(\Omega)$. This remarkable fact is expressed by saying that the Nehari manifold is a natural constraint for I.*

2.4 A Perturbed Problem

As we saw from the previous section that minimization on the Nehari manifold can be prolonged to cover more general problems, throughout this section, we deal with perturbed problem which we begin with an existence result, when we consider a fixed function $h \in L^2(\Omega)$ and a power nonlinearity. The result contained in this section is quite delicate, and can be omitted upon

first reading with under suitable assumptions we are going to prove it.
Let $h \in L^2(\Omega)$ and let $p \in]2, 2^*[$, we consider the following the Derichlet problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

The unperturbed problem when $(h = 0)$ admits two solution, the one found with Theorem (2.1) and the trivial solution $u \equiv 0$, and if $\|h\|_{L^2}$ is sufficiently small, we are going to show that (2.10) admits at least two nontrivial solutions.
If h does not vanish identicaly, the trivial solution $u \equiv 0$ exchanged by a nonzero solution.
the most important thing here, is to focuse on finding the analogue of the solution found in Theorem (2.1) by minimization on the Nehari manifold, while in this scheme the trivial solution of the unperturbed case can be found very easily.
The functional associated to (2.10) is

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p \, dx - \int_\Omega hudx = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|^p_{L^p} - \int_\Omega hudx,$$

which is differentiable on $H_0^1(\Omega)$. The main result is the following.

Theorem 2.2 *[3]Let $p \in]2, 2^*[$ and assume that (**h1**) holds. Then there exists ϵ^* such that for every $h \in L^2(\Omega)$, with $\|h\|_{L^2} \le \epsilon^*$, Problem (2.10) admits at least two solutions.*

We tacitly assume that the hypothesis (**h1**) holds, and going through a series of lemmas, we will prove the existence of two solutions, one of these solutions is found in Theorem (2.1).
From the previous section, we take the same constant m, such that

$$m = \left(\frac{1}{2} - \frac{1}{p}\right) \inf \|u\|^2,$$

we know that the Nehari manifold defined in (2.4) is associated to the function I, corresponding "unperturbed problem", so, we need to define a new Nehari manifold associated to the function J.
By embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, we denote S_p the best constant, such that

$$S_p = \inf\left\{c > 0 \;\mid\; \|u\|_{L^p} \le c\|u\| \quad \forall u \in H_0^1(\Omega)\right\}.$$

We define the positive numbers

$$d_1 = \left(\frac{1}{(p-1)S_p^p}\right)^{\frac{1}{p-2}}, \qquad d_2 = \left(\frac{1}{2}d_1^2\right)^{\frac{1}{p}},$$

$$d_3 = \frac{1}{2}min\left\{d_1, \frac{d_2}{S_p}\right\}.$$

we notice that $d_3 < d_2$.

Lemma 2.7 *[3] There exists $\epsilon_1 > 0$ such that for every $h \in L^2(\Omega)$ with $\|h\|_{L^2} \leq \epsilon_1$ and for every $u \in H_0^1(\Omega)$, if*

$$\|u\|^2 = \int_\Omega |u|^p \, dx + \int_\Omega hudx = \|u\|_{L^p}^p + \int_\Omega hudx. \tag{2.11}$$

then either

$$\|u\| < d_3 \quad or \quad \|u\| > d_1. \tag{2.12}$$

if the second case occurs, we have also

$$\|u\|_{L^p} \geq d_2. \tag{2.13}$$

Remark 2.4 *We will introduce later the Nehari manifold that relative to J, and for the fact that the condition (2.11) expresses that u belongs to this set, the meaning of the previous lemma is that for $\|h\|_{L^2}$ is small, then, from expression (2.12), we will get either $\|u\| < d_3$ or $\|u\| > d_1$. The region $\|u\| \in [d_3, d_1]$ is forbidden.*

Proof. We start by assuming the condition (2.11) and by using the Holder's inequality for $p = p' = 2$ and the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we obtain

$$\begin{aligned} \|u\|^2 &\leq S_p^p \|u\|^p + \|h\|_{L^2} \|u\|_{L^2} \\ &\leq S_p^p \|u\|^p + c \|h\|_{L^2} \|u\| . \end{aligned}$$

where $c = \left(\frac{1}{\lambda_1}\right)^{\frac{1}{2}}$, then if $u \neq 0$, we have

$$\frac{\|u\|^2}{\|u\|} \leq S_p^p \frac{\|u\|^p}{\|u\|} + c \frac{\|h\|_{L^2} \|u\|}{\|u\|}$$

i.e,

$$\|u\| - S_p^p \|u\|^{p-2} - c \|h\|_{L^2} \leq 0. \tag{2.14}$$

Setting $t = \|u\|$, and considering the function $\varphi : [0, +\infty[\longrightarrow \mathbb{R}$ defined by

$$\varphi(t) = t - S_p^p t^{p-1} - c \|h\|_{L^2}$$

the function φ is differentiable, then

$$\varphi'(t) = 1 - (p-1) S_p^p t^{p-2}.$$

we obtain

$$\varphi'(t) = 1 - (p-1) S_p^p t^{p-2} = 0$$

$$\Longleftrightarrow \quad t = \left(\frac{1}{(p-1) S_p^p}\right)^{\frac{1}{p-2}} = d_1,$$

we conclude that $t = d_1$ is a unique global maximum point because the function φ is strictly increasing on $[0, d_1[$ and strictly decreasing on $]d_1, +\infty[$.

when $t=0$ then $\varphi(0)=-c\|h\|_{L^2}<0$, and $\lim\limits_{t\longrightarrow+\infty}\varphi(t)=-\infty$
when $t=d_1$ we get

$$\begin{aligned}
\varphi(d_1) &= \left(\frac{1}{(p-1)S_p^p}\right)^{\frac{1}{p-2}} - S_p^p\left(\frac{1}{(p-1)S_p^p}\right)^{\frac{p-1}{p-2}} - c\|h\|_{L^2}\\
&= \frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} - \frac{1}{S_p^{-1}S_p^{\frac{p(p-1)}{p-2}}}\left(\frac{1}{p-1}\right)^{1+\frac{1}{p-2}} - c\|h\|_{L^2}\\
&= \frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}} - \frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(\frac{1}{p-1}\right) - c\|h\|_{L^2}\\
&= \frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\left(1-\frac{1}{p-1}\right) - c\|h\|_{L^2}\\
&= \frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\frac{p-2}{p-1} - c\|h\|_{L^2}.
\end{aligned}$$

we put

$$\alpha=\frac{1}{S_p^{\frac{p}{p-2}}}\left(\frac{1}{p-1}\right)^{\frac{1}{p-2}}\frac{p-2}{p-1}$$

then

$$\varphi(d_1)=\alpha-c\|h\|_{L^2}$$

since $p>2$, then $\alpha>0$, if we take $\|h\|_{L^2}\leq\frac{\alpha}{2c}$
then

$$\varphi(d_1)=\alpha-c\|h\|_{L^2}\geq\alpha-c\frac{\alpha}{2c}=\frac{\alpha}{2}>0$$

when we assumed that

$$\|h\|_{L^2}\leq\frac{\alpha}{2c}$$

that means the function φ has exactly two zeros t_1,t_2 then $\varphi(t_1)=\varphi(t_2)=0$ such that $t_1<d_1<t_2$, and $\varphi(t)>0$ in $]t_1,t_2[$, while $\varphi(t)<0$ in $[0,t_1[\cup]t_2,+\infty[$.
we have $\varphi(t_1)=0$ means that

$$t_1-S_p^pt_1^{p-1}-c\|h\|_{L^2}=0$$

i.e,

$$c\|h\|_{L^2}=t_1\left(1-S_p^pd_1^{p-2}\right).$$

Since $t_1<d_1$, we deduce from this and the definition of d_1 that

$$\begin{aligned}
c\|h\|_{L^2} &\geq t_1\left(1-S_p^pt_1^{p-2}\right)\\
&= t_1\left(1-\frac{1}{p-1}\right)\\
&= t_1\frac{p-2}{p-1},
\end{aligned}$$

so that

$$t_1\leq\frac{p-1}{p-2}c\|h\|_{L^2}.$$

But, if we take

$$\|h\|_{L^2} < \frac{p-2}{p-1}\frac{d_3}{c}$$

we get $t_1 < d_3$.
Summary of the above, if we choose

$$\|h\|_{L^2} < min\left\{\frac{p-2}{p-1}\frac{d_3}{c}, \frac{\alpha}{2c}\right\} \tag{2.15}$$

we deduce that $\varphi(t) \leq 0$ implies $t < d_3$ or $t > d_1$.
From (2.15),we have that the inequality (2.14) implies (2.12), and thus also (2.11) implies (2.12).
Which completed the proof of the first part.

Now, we proof the second part by checking (2.13) if (2.11) holds, since $\|u\| > d_1$, using the Holder's inequality for $p = p' = 2$, we obtain

$$\begin{aligned} \|u\|_{L^p}^p = \|u\|^2 - \int_\Omega hudx &\geq d_1^2 - \|h\|_{L^2}\|u\|_{L^2} \\ &\geq d_1^2 - d\|h\|_{L^2}\|u\|_{L^p}, \end{aligned}$$

where $d = |\Omega|^{\frac{p-2}{2p}}$, then

$$\|u\|_{L^p}^p + d\|h\|_{L^2}\|u\|_{L^p} - d_1^2 \geq 0, \tag{2.16}$$

Consider the function $\phi : [0, +\infty[\longrightarrow \mathbb{R}$ defined by

$$\phi(t) = t^p + d\|h\|_{L^2} t - d_1^2,$$

the function ϕ is differentiable, we have

$$\phi'(t) = pt^{p-1} + d\|h\|_{L^2},$$

we can see that $\phi'(t) > 0$ for all $t > 0$. When $t = d_2$, we have
and when $t = d_2$ we have

$$\begin{aligned} \phi(d_2) &= \frac{1}{2}d_1^2 + d\|h\|_{L^2} d_2 - d_1^2 \\ &= d\|h\|_{L^2} d_2 - \frac{1}{2}d_1^2. \end{aligned}$$

So,

$$\|h\|_{L^2} < \frac{d_1^2}{2dd_2}$$

we get

$$\phi(d_2) = d\|h\|_{L^2} d_2 - \frac{1}{2}d_1^2 < d\frac{d_1^2}{2dd_2}d_2 - \frac{1}{2}d_1^2 = 0,$$

since $\phi(d_2) < 0$ and $\phi'(t) > 0$ for all $t > 0$, and for $t \in [0, d_2]$ we deduce that $\phi(t) < 0$, then the inequality (2.16) implies that $\|u\|_{L^p} \geq d_2$
this finishes the proof with the choice

$$\epsilon_1 = min\left\{\frac{p-2}{p-1}\frac{d_3}{c}, \frac{\alpha}{2c}, \frac{d_1^2}{2dd_2}\right\}.$$

■

From now, we assume $\|h\|_{L^2} < \epsilon_1$. We define the set

$$\begin{aligned} \mathcal{N}_h &= \left\{ u \in H_0^1(\Omega) \mid \langle J'(u), u \rangle = 0 \ , \ \|u\| > d_1 \right\} \\ &= \left\{ u \in H_0^1(\Omega) \mid \|u\|^2 = \int_\Omega |u|^p \, dx + \int_\Omega hudx, \|u\| > d_1 \right\}. \end{aligned}$$

and let the value

$$m_h = \inf_{u \in \mathcal{N}_h} J(u).$$

Notice that $\mathcal{N}_h$ is not the complete Nehari manifold associated to J, but only a subset of it, on $\mathcal{N}_h$, the functional J takes the form

$$J(u) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u\|^2 - \left(1 - \frac{1}{p} \right) \int_\Omega hudx.$$

First, we investigate under which conditions $\mathcal{N}_h$ is not empty.

Lemma 2.8 *[3] There exists $\epsilon_2 \in]0, \epsilon_1]$, such that for every $h \in L^2(\Omega)$ with $\|h\|_{L^2} \leq \epsilon_2$, there results $\mathcal{N}_h \neq 0$, $\mathcal{N}_h$ is not empty.*

Proof. For every not identically zero $u \in H_0^1(\Omega)$, we study the behaviour of the function

$$\begin{aligned} t \longmapsto \langle J'(tu), tu \rangle &= \|tu\|^2 - \int_\Omega |tu|^p - \int_\Omega htudx \\ &= t^2 \|u\|^2 - t^p \int_\Omega |u|^p - t \int_\Omega hudx \\ &= t \left[t \|u\|^2 - t^{p-1} \int_\Omega |u|^p - \int_\Omega hudx \right]. \end{aligned}$$

for $t > 0$, we analyse the function

$$\eta(t) = t \|u\|^2 - t^{p-1} \int_\Omega |u|^p - \int_\Omega hudx.$$

the function $\eta(t)$ is differentiable and

$$\eta'(t) = \|u\|^2 - (p-1) t^{p-2} \|u\|_{L^p}^p,$$

when $\eta'(t_*) = 0$ we get

$$\eta'(t_*) = \|u\|^2 - (p-1) t_*^{p-2} \|u\|_{L^p}^p = 0,$$

then

$$t_* = \left(\frac{\|u\|^2}{(p-1) \|u\|_{L^p}^p} \right)^{\frac{1}{p-2}}.$$

The function η has a global maximum at the point t_*, with an easy computation, we get

$$\begin{aligned}
\eta(t_*) &= t_* \|u\|^2 - t_*^{p-1} \int_\Omega |u|^p - \int_\Omega hudx \\
&= \left(\frac{\|u\|^2}{(p-1)\|u\|_{L^p}^p}\right)^{\frac{1}{p-2}} \|u\|^2 - \left(\frac{\|u\|^2}{(p-1)\|u\|_{L^p}^p}\right)^{\frac{p-1}{p-2}} \|u\|_{L^p}^p - \int_\Omega hudx \\
&= \frac{\|u\|^{2\frac{p-1}{p-2}}}{(p-1)^{\frac{1}{p-2}} \|u\|_{L^p}^{\frac{p}{p-2}}} - \frac{\|u\|^{2\frac{p-1}{p-2}}}{(p-1)^{\frac{p-1}{p-2}} \|u\|_{L^p}^{\frac{p}{p-2}}} - \int_\Omega hudx \\
&= \frac{\|u\|^{2\frac{p-1}{p-2}}}{\|u\|_{L^p}^{\frac{p}{p-2}}} \frac{1}{(p-1)^{\frac{1}{p-2}}} \left(1 - \frac{1}{p-1}\right) - \int_\Omega hudx \\
&= \frac{\|u\|^{2\frac{p-1}{p-2}}}{\|u\|_{L^p}^{\frac{p}{p-2}}} \frac{1}{(p-1)^{\frac{1}{p-2}}} \frac{p-2}{p-1} - \int_\Omega hudx.
\end{aligned}$$

We put

$$\alpha = \frac{1}{(p-1)^{\frac{1}{p-2}}} \frac{p-2}{p-1},$$

since $p > 0$, then $\alpha > 0$.Using the embedding $H_0^1(\Omega)$ into L^p, we get

$$\begin{aligned}
\eta(t_*) &\geq \frac{\|u\|^{2\frac{p-1}{p-2}}}{\|u\|_{L^p}^{\frac{p}{p-2}} S_p^{\frac{p}{p-2}}} \alpha - \int_\Omega hudx \\
&= \|u\| \frac{1}{S_p^{\frac{p}{p-2}}} \alpha - \|h\|_{L^2} \|u\|_{L^2} \\
&\geq \|u\| \frac{1}{S_p^{\frac{p}{p-2}}} \alpha - c\|h\|_{L^2} \|u\| = \|u\| \left(\frac{\alpha}{S_p^{\frac{p}{p-2}}} - c\|h\|_{L^2}\right).
\end{aligned}$$

So, if we assume that

$$\|h\|_{L^2} \leq \frac{\alpha}{2cS_p^{\frac{p}{p-2}}}, \tag{2.17}$$

then, we have $\eta(t_*) > 0$. From (2.17), the function η is strictly increasing in $]0, t_*[$, and strictly decreasing in $]t_*, +\infty[$, and we have $\eta(t_*) > 0$ and $\lim_{t \longrightarrow +\infty} \eta(t) = -\infty$, this shows that the function η has at least one zero $t_1 \in]t_*, +\infty[$ such that $\eta(t_1) = 0$, which implies that the function $v = t_1 u$ satisfait (2.11). Moreover,

$$\begin{aligned}
\|v\| = t_1 \|u\| &> t_* \|u\| = \left(\frac{\|u\|^2}{(p-1)\|u\|_{L^p}^p}\right)^{\frac{1}{p-2}} \|u\| \\
&\geq \frac{1}{(p-1)^{\frac{1}{p-2}} S_p^{\frac{p}{p-2}}} = d_1.
\end{aligned}$$

then $\|v\| > d_1$, and this shows that $v \in \mathcal{N}_h$. Which finishes the proof with the choice

$$\epsilon_2 = min\left\{\epsilon_1, \frac{\alpha}{2cS_p^{\frac{p}{p-2}}}\right\}.$$

■

Now, we prove that for $\|h\|_{L^2}$ is small, the levels m_h are uniformly bounded from above, the uniform bound from below will be obtained in Lemma 2.11.

Lemma 2.9 *[3]Let $\epsilon_3 = \min\{1, \epsilon_2\}$. There exists $c_2 > 0$ such that for every $\|h\|_{L^2} < \epsilon_3$ there results $m_h \leq c_2$.*

Proof. We denote $u^* \in \mathcal{N}$ the solution of the unperturbed problem (2.1) when ($h \equiv 0$) solved in the preceding section, and m^* the level of the solution

$$m^* = \min_{u \in N} I(u) = I(u^*).$$

We extract from the previous lemma that $\mathcal{N}_h$ is not empty and this means that if $\|h\|_{L^2} < \epsilon_3$, there exists $t_h > 0$ such that $t_h u^* \in \mathcal{N}_h$. We know that if $u^* \in \mathcal{N}$, then, it satisfies $\|u^*\|^2 = \|u^*\|_{L^p}^p$, thus the condition $t_h u^* \in \mathcal{N}_h$ is equivalent to

$$\left\|t^h u^*\right\|^2 - \|t_h u^*\|_{L^p}^p = \int_\Omega t_h h u^* dx$$

$$\left(t_h^2 - t_h^p\right)\|u^*\|^2 = t_h \int_\Omega h u^* dx$$

then

$$\left(t_h - t_h^{p-1}\right)\|u^*\|^2 = \int_\Omega h u^* dx$$

We know that $\int_\Omega h u^* dx \leq \|h\|_{L^2}\|u^*\|_{L^2} \leq c\|h\|_{L^2}\|u^*\|$, this implies that
the condition implies

$$\left(t_h - t_h^{p-1}\right)\|u^*\|^2 \geq -c\|h\|_{L^2}\|u^*\|,$$

i.e.

$$\left(t_h - t_h^{p-1}\right) \geq -\frac{c\|h\|_{L^2}}{\|u^*\|}.$$

if $\|h\|_{L^2} < \epsilon_3 \leq 1$ we obtain

$$\left(t_h - t_h^{p-1}\right) \geq -\frac{c}{\|u^*\|}. \tag{2.18}$$

when $t \longrightarrow +\infty$ the function $t \longmapsto t - t^{p-1}$ tends to $-\infty$, the inequality (2.18) shows that there exists $c_3 > 0$ independent of h, such that $t_h \leq c_3$.
Since $t_h u^* \in \mathcal{N}_h$ and by using the Holder's inequality we have

$$\begin{aligned} m_h &\leq J(t_h u^*) = \left(\frac{1}{2} - \frac{1}{p}\right)\|t_h u^*\|^2 - \left(1 - \frac{1}{p}\right)\int_\Omega h t_h u^* dx \\ &\leq \left(\frac{1}{2} - \frac{1}{p}\right) c_3^2 \|u^*\|^2 + \left(1 - \frac{1}{p}\right) c_3 c\|h\|_{L^2}\|u^*\| \end{aligned}$$

and since $\|h\|_{L^2} < \epsilon_3 \leq 1$ then

$$\begin{aligned} m_h &\leq \left(\frac{1}{2} - \frac{1}{p}\right) c_3^2 \|u^*\|^2 + \left(1 - \frac{1}{p}\right) c_3 c\|h\|_{L^2}\|u^*\| \\ &= c_2. \end{aligned}$$

We have $c_2 > 0$ and c_2 does not depend on h, then we obtain $m_h \leq c_2$.
this the end of the proof. ■

We go on with a uniform bound on minimizing sequences.

Lemma 2.10 *[3]There exists $c_4 > 0$ independent of h, such that if $\|h\|_{L^2} < \epsilon_3$, then there exists a minimizing sequence $\{u_l\}_l$ for m_h such that $\|u_l\| \leq c_4$ for all l, and also $\|u_l\|_{L^p} \leq S_p c_4$ for all l.*

Proof. Let $\{v_l\}_l$ be a minimizing sequence for m_h and $v_l \in \mathcal{N}_h$, and let $\|h\|_{L^2} < \epsilon_3 \leq 1$, namely

$$\lim_{l \longrightarrow +\infty} J(v_l) = \inf J(v).$$

we have

$$J(v_l) = \left(\frac{1}{2} - \frac{1}{p}\right) \|v_l\|^2 - \left(1 - \frac{1}{p}\right) \int_\Omega h v_l dx,$$

then, $J(v_l) \longrightarrow m_h$, as$l \longrightarrow +\infty$.
From previous lemma we have $m_h \leq c_2$, so, there exists $\tilde{l}$ such that, for every $l \geq \tilde{l}$, we get $J(v_l) \leq 2c_2$ and this implies that

$$\begin{aligned} 2c_2 &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v_l\|^2 - \left(1 - \frac{1}{p}\right) \int_\Omega h v_l dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v_l\|^2 - \left(1 - \frac{1}{p}\right) c \|h\|_{L^2} \|v_l\| \end{aligned}$$

we put $r_1 = \left(\frac{1}{2} - \frac{1}{p}\right)$ and $r_2 = \left(1 - \frac{1}{p}\right) c$, then

$$2c_2 \geq r_1 \|v_l\|^2 - r_2 \|v_l\|,$$

from this inequality, we obtain

$$\left(\sqrt{r_1} \|v_l\| - \frac{r_2}{2\sqrt{r_1}}\right)^2 - \frac{r_2^2}{4a_1} \leq 2c_2.$$

This implies that

$$\|v_l\| \leq \frac{r_2 + \sqrt{r_2^2 8 r_1 c_2}}{2r_1}.$$

we set $c_4 = \frac{r_2 + \sqrt{r_2^2 8 r_1 c_2}}{2r1}$, and $u_l = v_{\tilde{l}+l}$ then we have $\|u_l\| \leq c_4$. ■

The next lemma will complete the estimate of Lemma 2.9.

Lemma 2.11 *[3]There exists $\epsilon_4 \leq \epsilon_3$ such that if $\|h\|_{L^2}$, then $m_h \geq \frac{1}{2} m^* > 0$.*

Proof. We notice from preceding section that the minimization on the Nehari manifold dealt with the unperturbed problem (2.1) that it is associated to the function I, and in this proof we denote by J_h the functional I associated with the perturbed problem (2.10).
For every h with $\|h\|_{L^2} < \epsilon_3$, we consider the minimizing sequence $\{u_l\}_l$ obtained in Lemma 2.10, let $u_l \in \mathcal{N}_h$ and let t_l be such that $t_l u_l \in \mathcal{N}$ (notice that both u_l and t_l depend on h).
As we know from the preceding section

$$t_l = \left(\frac{\|u_l\|^2}{\|u_l\|_{L^p}^p}\right)^{\frac{1}{p-2}}.$$

Since $t_l u_l \in \mathcal{N}$, then $\|u_l\|^2 = \|u_l\|_{L^2}^p + \int_\Omega h u_l dx$, we have

$$t_l = \left(1 + \frac{\int_\Omega h u_l dx}{\|u_l\|_{L^p}^p}\right)^{\frac{1}{p-2}}, \tag{2.19}$$

then we obtain

$$\begin{aligned} m^* &\leq I(t_l u_l) = \left(\frac{1}{2} - \frac{1}{p}\right) \|t_l u_l\|^2 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) t_l^2 \|u_l\|^2 - \left(1 - \frac{1}{p}\right) t_l^2 \int_\Omega h u_l dx + \left(1 - \frac{1}{p}\right) t_l^2 \int_\Omega h u_l dx \\ &= t_l^2 J(u_l) + \left(1 - \frac{1}{p}\right) t_l^2 \int_\Omega h u_l dx. \end{aligned} \tag{2.20}$$

Now we begin from estimate t_l from above, to this aim, by the estimates that obtained in Lemmas 2.7 and 2.10 we have

$$\begin{aligned} \frac{|\int_\Omega h u_l dx|}{\|u_l\|_{L^p}^p} &\leq \|h\|_{L^2} \frac{c\|u_l\|}{\|u_l\|_{L^p}^p} \\ &\leq \|h\|_{L^2} \frac{cc_4}{d_2^p}. \end{aligned}$$

Now we set the condition

$$\|h\|_{L^2} \leq \frac{d_2^p}{cc_4} \left(\left(\frac{4}{3}\right)^{\frac{p-2}{2}} - 1\right),$$

by this condition we obtain

$$\frac{|\int_\Omega h u_l dx|}{\|u_l\|_{L^p}^p} \leq \left(\frac{4}{3}\right)^{\frac{p-2}{2}} - 1,$$

then from (2.19) we will have

$$\begin{aligned} t_l = \left(1 + \frac{|\int_\Omega h u_l dx|}{\|u_l\|_{L^p}^p}\right)^{\frac{1}{p-2}} &\leq \left(1 + \left(\frac{4}{3}\right)^{\frac{p-2}{2}} - 1\right)^{\frac{1}{p-2}} \\ &= \left(\frac{4}{3}\right)^{\frac{1}{2}}. \end{aligned}$$

we notice that t_l is bound from above, also we want to obtain a uniform bound from below on $J_h(u_l)$.

Next we analyse the last term in (2.20), from Lemma 2.20 we have $\|u_l\| \leq c_4$ we see immediately that

$$\begin{aligned} \left|\left(1 - \frac{1}{p}\right) t_l^2 \int_\Omega h u_l dx\right| &\leq \left(1 - \frac{1}{p}\right) t_l^2 \|h\|_{L^2} c \|u_l\| \\ &\leq \left(1 - \frac{1}{p}\right) \frac{4}{3} \|h\|_{L^2} cc_4. \end{aligned}$$

and now we choose the condition

$$\|h\|_{L^2} < \frac{m^*}{4\left(1 - \frac{1}{p}\right) cc_4},$$

we obtain

$$\begin{aligned} \left|\left(1-\frac{1}{p}\right)t_l^2\int_\Omega hu_l dx\right| &\le \left(1-\frac{1}{p}\right)\frac{4}{3}\|h\|_{L^2}cc_4 \\ &\le \frac{m^*}{3}. \end{aligned}$$

From this and from (2.20), we obtain

$$\begin{aligned} m^* &\le t_l^2 J_h(u_l) + \left(1-\frac{1}{p}\right)t_l^2\int_\Omega hu_l dx \\ &\le t_l^2 J_h(u_l) + \frac{m^*}{3} \end{aligned}$$

we deduce that

$$t_l^2 J_h(u_l) \ge \frac{2}{3}m^*.$$

Since $t_l > 0$ we conclude that $J_h(u_l) > 0$, and we also have $t_l^2 \le \frac{4}{3}$, we eventually arrive to

$$\frac{2}{3}m^* \le t_l^2 J_h(u_l) \le \frac{4}{3}J_h(u_l),$$

finally we get $\frac{1}{2}m^* \le J_h(u_l)$, and when $l \longrightarrow +\infty$, we obtain

$$m_h \ge \frac{1}{2}m^* > 0.$$

and this is the end of the proof by choosing the bounds of $\|h\|_{L^2}$ with

$$\epsilon_4 = \left\{\epsilon_3, \frac{d_2^p}{cc_4}\left(\left(\frac{4}{3}\right)^{\frac{p-2}{2}} - 1\right), \frac{m^*}{4\left(1-\frac{1}{p}\right)cc_4}\right\}.$$

■

Now by next Lemma we show that the minimum on $\mathcal{N}_h$ is attained.

Lemma 2.12 *[3]There exists $\epsilon_5 \le \epsilon_4$ such that for every $\|h\|_{L^2} < \epsilon_5$ the infimum m_h is attained by some $u \in \mathcal{N}_h$.*

Proof. Let $\|h\|_{L^2} < \epsilon_4$, and let $\{u_l\}_l$ be a minimizing sequence for m_h and $u_l \in \mathcal{N}_h$ to be specific $\lim_{l\longrightarrow+\infty} J(u_l) = \inf J(u) = m_h$, constructed as in Lemma 2.10, and with the fact that $J(u_l)$ converge to m_h it implies that the sequence $\{u_l\}_l$ is bounded in $H_0^1(\Omega)$ which is reflexif space, we can assume that, up to subsequence there exist$u_{l_k} = u_l$ and $u_1 \in H_0^1(\Omega)$ such that

$$u_l \rightharpoonup u_1$$

and for $2 < p < 2^*$ the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is compact then $u_l \rightharpoonup u_1$ in $L^p(\Omega)$ and

$$\exists u_{l_k} = u_{l'} \qquad u_l(x) \longrightarrow u_1(x) \text{ a.e in } \Omega$$

and by weak lower semicontinuity

$$\begin{aligned} J(u_1) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_1\|^2-\left(1-\frac{1}{p}\right)\int_\Omega hu_1dx \\ &\leq \lim_{l\to+\infty}\inf\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|u_l\|^2-\left(1-\frac{1}{p}\right)\int_\Omega hu_ldx\right] \\ &= \lim_{l\to+\infty}\inf J(u_l)=m_h. \end{aligned} \tag{2.21}$$

and that

$$\|u_1\|^2\leq\|u_1\|^p+\int_\Omega hu_1dx. \tag{2.22}$$

if we suppose that

$$\|u_1\|^2=\|u_1\|^p+\int_\Omega hu_1dx. \tag{2.23}$$

That means the arguments in Lemma 2.7 holds, then either $\|u_1\|>d_1$ or $\|u_1\|<d_3$. In the first case, $u\in\mathcal{N}_h$, and then (2.21) implies that u_1 is the minimum we are looking for.
If $\|u_1\|<d_3$, by the definition of d_3, we obtain

$$\|u_1\|_{L^p}\leq S_p\|u_1\|\leq S_pd_3\leq S_p\frac{d_2}{S_p}=d_2.$$

This is a contradiction in what we saw in Lemma 2.7 that says that $\|u_l\|_{L^p}\geq d_2$, by the strong convergence in $L^p(\Omega)$ we obtain $\|u_1\|_{L^p}\geq d_2$, as $l\to+\infty$.
This means that, if (2.23) holds, it cannot be $\|u_1\|<d_3$, then, as we have seen u_1 is a minimum.
We suppose that in (2.22) the strict inequality hold, namely

$$\|u_1\|^2<\|u_1\|^p+\int_\Omega hu_1dx. \tag{2.24}$$

From Lemma 2.7, we know that there exist $t_1>0$ such that $t_1u_1\in\mathcal{N}_h$ and

$$t_1>\tilde{t}=\left(\frac{\|u_1\|^2}{(p-1)\|u_1\|_{L^p}^p}\right)^{\frac{1}{p-2}}.$$

Since (2.22), we obtain

$$\tilde{t}\leq\left(\frac{\|u_1\|_{L^p}^p+\int_\Omega hu_1dx}{(p-1)\|u_1\|_{L^p}^p}\right)^{\frac{1}{p-2}}=\left(\frac{1}{p-1}+\frac{\int_\Omega hu_1dx}{(p-1)\|u_1\|_{L^p}^p}\right)^{\frac{1}{p-2}}.$$

and by using the Holder's inequality, and $\|u_1\|\leq c_4$ and $\|u_1\|\geq d_2$ we will have

$$\begin{aligned} \frac{|\int_\Omega hu_1dx|}{(p-1)\|u_1\|_{L^p}^p} &\leq \|h\|_{L^2}c\frac{\|u_1\|}{(p-1)\|u_1\|_{L^p}^p} \\ &\leq \|h\|_{L^2}cc_4\frac{1}{(p-1)d_2^p}, \end{aligned}$$

then we obtain

$$\tilde{t}\leq\frac{1}{p-1}+\|h\|_{L^2}cc_4\frac{1}{(p-1)d_2^p},$$

Since $\frac{1}{p-1}<1$, we can choose $\|h\|_{L^2}<\frac{(p-2)(p-1)d_2^p}{2cc_4}$, then we have the choice

$$\epsilon_5=\min\left\{\frac{(p-2)(p-1)d_2^p}{2cc_4},\epsilon_4\right\}.$$

if $\|h\|_{L^2} < \epsilon_5$, then $\widetilde{t} < 1$.
We consider the function that we have already studied

$$\eta(t) = t\|u_1\|^2 - t^{p-1}\|u_1\|_{L^p}^p - \int_\Omega hu_1 dx$$

we know that η is decreasing in $]\widetilde{t}, +\infty[$. Since $t_1 u_1 \in \mathcal{N}_h$ we have $\eta(t_1) = 0$. The inequality (2.24) is equivalent to $\eta(1) < 0$. Since $\widetilde{t} < 1$ and $\widetilde{t} < t_1$, we must necessarily have $t_1 < 1$, since $t_1 u_1 \in \mathcal{N}_h$, we obtain

$$\begin{aligned} m_h &\leq J(t_1 u_1) = (t_1)^2 \left(\frac{1}{2} - \frac{1}{p}\right)\|u_1\|^2 - t_1\left(1 - \frac{1}{p}\right)\int_\Omega hu_1 dx \\ &\leq (t_1)^2 \lim_{l\to+\infty}\inf\left(\frac{1}{2} - \frac{1}{p}\right)\|u_l\|^2 - t_1 \lim_l \left(1 - \frac{1}{p}\right)\int_\Omega hu_l dx \\ &\leq t_1 \lim_{l\to+\infty}\inf\left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_l\|^2 - \left(1 - \frac{1}{p}\right)\int_\Omega hu_l dx\right] \\ &= t_1 \lim_{l\to+\infty}\inf J(u_l) = t_1 m_h < m_h. \end{aligned}$$

The last strict inequality holds because $t_1 \in]0,1[$ and $m_h > 0$, and it is contradiction and this shows that the inequality (2.12) cannot hold, and this means (2.23) is true. In this case, as we have seen, u_1 is the required minimum. ■

The last step consists in proving that the minimum is a critical point of J on the whole space $H_0^1(\Omega)$.

Lemma 2.13 *[3]There exists $\epsilon_6 \in]0, \epsilon_5[$ such that if $\|h\|_{L^2} < \epsilon_6$, then the minimum u_1 of J on $\mathcal{N}_h$ satisfies $\langle J'(u), v\rangle = 0$ for all $v \in H_0^1(\Omega)$.*

Proof. We fix $v \in H_0^1(\Omega)$ and consider the function $\sigma : \mathbb{R}\times]0,+\infty[\longrightarrow \mathbb{R}$ defined by

$$\sigma(s,t) = t^2\|u + sv\|^2 - t^p\|u + sv\|_{L^p}^p - t\int_\Omega h(u + sv)dx.$$

we put $s = 0$ and $t = 1$, and we have that (2.23) holds and since $u_1 \in \mathcal{N}_h$, we obtain

$$\sigma(0,1) = \|u_1\|^2 - \|u_1\|_{L^p}^p - \int_\Omega hu_1 dx = 0$$

then $\sigma(0,1) = 0$.
The function σ is of class C^1, since

$$\|u_1\|^2 = \|u_1\|_{L^p}^p + \int_\Omega hu_1 dx,$$

we obtain

$$\begin{aligned} \frac{\partial\sigma}{\partial t}(0,1) &= 2\|u_1\|^2 - p\|u_1\|_{L^p}^p - \int_\Omega hu_1 dx \\ &= (2-p)\|u_1\|^2 + (p-1)\int_\Omega hu_1 dx. \end{aligned}$$

if we suppose $\frac{\partial\sigma}{\partial t}(0,1) = 0$, we would have

$$(2-p)\|u_1\|^2 + (p-1)\int_\Omega hu_1 dx = 0$$

so,

$$\|u_1\|^2 = \frac{p-1}{p-2}\int_\Omega hu_1dx,$$

by Holder's inequality and the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ we get

$$\|u_1\| \leq \frac{p-1}{p-2}\|h\|_{L^2} c\|u_1\|. \tag{2.25}$$

We know that $\|u_1\| > d_1$, if we take

$$\|h\|_{L^2} < \frac{p-2}{c(p-1)}d_1$$

then, the inequality (2.25) will be $\|u_1\| < d_1$, a contradiction. Then it must be $\frac{\partial\sigma}{\partial t}(0,1) \neq 0$ for such choices of h, Therefore, obtaining that there exist a number $\delta > 0$ and a C^1 function $t(s) :]-\delta,\delta[\longrightarrow \mathbb{R}$ such that $\sigma(s,t(s)) = 0$ for every $s \in]-\delta,\delta[$, and $t(0) = 1$. Since $\|u_1\| > d_1$, we can also take δ so small that $t(s)(u+sv) > d_1$. We have thus constructed a differentiable curve $s \mapsto \mathcal{N}_h$ passing through u_1 when $s = 0$. We now evaluate J along this curve, now we consider the function

$$\gamma(s) = J(t(s)(u_1+sv))$$

The function γ is differentiable and

$$\gamma'(s) = [t'(s)(u_1+sv) + t(s)v]\, J'(t(s)(u_1+sv))$$

the function γ has also a local minimum at $s = 0$ and

$$\begin{aligned} 0 = \gamma'(0) &= [t'(0)u_1 + t(0)v]\, J'(t(0)u_1) \\ &= [t'(0)u_1 + v]\, J'(u_1) \\ &= J'(u_1)\, u_1 + t'(0)J'(u_1)\, u_1, \end{aligned}$$

we have $\langle J'(u_1), u_1\rangle = 0$, then, we obatin $\langle J'(u_1), v\rangle = 0$
because $u_1 \in \mathcal{N}_h$. Since this holds for every $v \in H_0^1(\Omega)$, it is enough to take a positive ϵ_6 such that

$$\epsilon_6 < \min\left\{\epsilon_5, \frac{(p-2)d_1}{c(p-1)}\right\}.$$

we have obtained $\langle J'(u_1), v\rangle = 0$, proving that u_1 is a solution of problem (2.10). ∎

We now complete the proof of the theorem by constructing a second solution.

Lemma 2.14 *[3] For every $\epsilon > 0$ there exists $\delta > 0$ such that if $\|h\|_{L^2} < \delta$, then Problem (2.10) admits a solution u_h satisfying $\|u_h\| < \epsilon$.*

Proof. We have from previous section $I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_{L^p}^p$, by using the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, we obtain

$$I(u) \geq \frac{1}{2}\|u\|^2 - \frac{S_p^p}{p}\|u\|^p$$

consider the function

$$K(t) = \frac{1}{2}t^2 - \frac{S_p^p}{p}t^p$$

the function K is continuous and satisfies $K(0) = 0$, and is strictly increasing in a right neighbourhood of 0, therefore there certainly exists $\epsilon' \leq \epsilon$ such that for all $t \in]0, \epsilon'[$ there results $k(t) > 0$.
We fix $\alpha \in]0, \epsilon'[$, we obtain $I(u) \geq K(\alpha) > 0$ if $\|u\| = \alpha$, we deduce that by using Holder's inequality and the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$

$$\begin{aligned} J(u) = I(u) - \int_\Omega hudx &\geq K(\alpha) - \|h\|_{L^2} c \|u\| \\ &= K(\alpha) - \|h\|_{L^2} c\alpha. \end{aligned}$$

choosing

$$\|h\|_{L^2} < \frac{K(\alpha)}{2c\alpha} = \delta$$

we conclude that, for $\|u\| = \alpha$.

$$J(u) \geq \frac{K(\alpha)}{2} > 0.$$

Let now

$$B_\alpha = \left\{ u \in H_0^1(\Omega) \;\; | \;\; \|u\| \leq \alpha \right\}$$

and

$$n_\alpha = \inf_{u \in B_\alpha} J(u).$$

Clearly $n_\alpha \neq \pm\infty$. Moreover $n_\alpha \leq J(0) = 0$. the value n_α is attained by some $u_h \in B_\alpha$ and this can be proofed by using the arguments that used many times in the first part of this chapter, since $J(u_h) = n_\alpha \leq 0$ that means that $\|u_h\| \neq \alpha$, and therefore u_h lies in the interior of the ball B_α. The function u_h is thus a local minimum for J and solves the problem (2.10). Moreover, $\|u_h\| < \epsilon$. ■
Now, we proved that Theorem 2.2 holds. Thanks to Lemma 2.13, we know that for every $\|h\|_{L^2} < \epsilon_6$, there exists a solution u_1 of Problem (2.10) with $\|u_1\| > d_1$. By Lemma 2.14, choosing $\epsilon = d_1$, we can fix $\delta > 0$ such that for every $\|h\|_{L^2} < \delta$ there exists a solution u_h of Problem (2.10) with $\|u_h\| < d_1$. If

$$\|h\|_{L^2} < \epsilon^* = \min \{\epsilon_6, \delta\}.$$

we obtain two distinct solutions of Problem (2.10). The proof is complete.

Chapter 3

The Nehari manifold for a semilinear elliptic problem with the nonlinear boundary condition

3.1 Main results

In this chapter, we will consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

Where $\lambda > 0$ is a parameter, and Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and with a smooth boundary $\partial\Omega$. $\frac{\partial}{\partial n}$ denotes the derivative along the outer normal; that is

$$\frac{\partial u}{\partial n} = \nabla u.\vec{n} = \sum_{n}^{i=1} \frac{\partial u}{\partial x_i}.n_i.$$

We notice that the problem (3.1) has two nonlinear terms. One is from the equation, namely, the term $|u|^{p-2}u$, and another is from the boundary condition, i.e., $|u|^{q-2}u$. Here the exponents p and q satisfy

$$1 < q < 2 < p < 2^* = \frac{2N}{N-2}. \tag{3.2}$$

Definition 3.1 *We say that $u \in H^1(\Omega)$ is a weak solution of Problem (3.1) if and only if*

$$\int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx - \lambda \int_{\partial\Omega} |u|^{q-2} uv ds - \int_\Omega |u|^{p-2} uv dx = 0,$$

for any $v \in H^1(\Omega)$.

For solving Problem (3.1), we use the variational method to study the functional

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |u|^2\right) dx - \frac{1}{p}\int_\Omega |u|^p dx - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q ds \tag{3.3}$$

where ds is the measure on the boundary. It is easy to see that J is well-defined and C^1 in $H^1(\Omega)$ if p and q verify condition (3.2). We have

$$\langle J'(u), v\rangle = \int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx - \int_\Omega |u|^{p-2} uv dx - \lambda \int_{\partial\Omega} |u|^{q-2} uv ds. \tag{3.4}$$

The weak solution of (3.1) corresponds to the critical point of the functional J on $H^1(\Omega)$, where $H^1(\Omega)$ is a standard Sobolev space on Ω equipped with the norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + |u|^2\right) dx.$$

In the following we present the main results of this section.

Theorem 3.1 *[22]Assume that the condition (3.2) holds. If λ satisfies $0 < \lambda < \lambda_0$, then Problem (3.1) has at least two positive solutions u_1 and u_2, where $\lambda_0 > 0$ is a constant such that $\mathcal{N}^0 = \emptyset$ (we will introduce it latter).*

Next, we consider the condition

$$q = 2, \quad 1 < p < 2, \tag{3.5}$$

under this condition we get a bifurcation problem from the first eigenvalue of the related eigenvalue problem. This eigenvalue problem is known as the Steklov problem [16]

$$(S) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \ \Omega, \\ \frac{\partial u}{\partial n} = \lambda u & \text{on } \ \partial\Omega. \end{cases}$$

It is well-known that the first eigenvalue λ_{1S} of problem (S) is just the best constant in the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ in the sense that

$$\lambda_{1S} \|u\|^2_{L^2(\partial\Omega)} \le \|u\|^2.$$

We will use the Nehari manifold methods to solve Problem (3.1) where (3.5) is satisfies.

Theorem 3.2 *[22] If λ satisfies $0 < \lambda < \lambda_{1S}$ and $q = 2$, $1 < p < 2$, then Problem (3.1) has at least one solutions u_λ such that $u_\lambda > 0$ in Ω, and $\lim\limits_{\lambda \longrightarrow \lambda_{1s}^-} \|u_\lambda\| \longrightarrow +\infty$, where λ_{1S} is the first eigenvalue of the Steklov problem.*

3.2 Corresponding Nehari manifold

We denote $\mathcal{N}$ to be the Nehari manifold

$$\mathcal{N} = \left\{u \in H^1(\Omega) \quad | \quad u \neq 0, \langle J'(u), u\rangle = 0\right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between $H^1(\Omega)$ and $H^1(\Omega)^*$. It is clear that any critical points of J must lie on $\mathcal{N}$. As we will see below, local minimizers on $\mathcal{N}$ are usually critical points of J. $u \in \mathcal{N}$ if and only if

$$\langle J'(u), u\rangle = 0 \iff \|u\|^2 - \int_\Omega |u|^p\, dx - \int_{\partial\Omega} |u|^q\, ds = 0. \tag{3.6}$$

by definition

$$\phi_u(t) = J(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}\int_\Omega |u|^p\, dx - \lambda \frac{t^q}{q}\int_{\partial\Omega} |u|^q\, ds.$$

If $u \in H^1(\Omega)$, we have

$$\begin{aligned}
\phi'_u(t) &= \langle J'(tu), u\rangle \\
&= \int_\Omega \left(\nabla(tu)\,|\nabla u| + u^2\right) dx - \int_\Omega |tu|^p\,dx - \lambda \int_{\partial\Omega} |tu|^q\,ds. \\
&= t\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - t^{p-1}\int_\Omega |u|^p\,dx - \lambda t^{q-1}\int_{\partial\Omega} |u|^q\,ds \\
&= t\,\|u\|^2 - t^{p-1}\int_\Omega |u|^p\,dx - \lambda t^{q-1}\int_{\partial\Omega} |u|^q\,ds. \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
\phi''_u(t) &= \int_\Omega \left(|\nabla u|^2 + u^2\right) dx - (p-1)t^{p-2}\int_\Omega |u|^p\,dx - \lambda(q-1)t^{q-2}\int_{\partial\Omega} |u|^q\,ds. \\
&= \|u\|^2 - (p-1)t^{p-2}\int_\Omega |u|^p\,dx - \lambda(q-1)t^{q-2}\int_{\partial\Omega} |u|^q\,ds.
\end{aligned}$$

From (3.6) and (3.7), we can see that $\phi'_u(1) = 0$ if and only if $u \in \mathcal{N}$, more generally, $tu \in \mathcal{N}$ if and only if $\phi'_u(1) = 0$, i.e., the elements in $\mathcal{N}$ correspond to stationary points of fibering maps ϕ_u. Hence for $u \in \mathcal{N}$, by (3.6) we get

$$\begin{aligned}
\phi''_u(1) &= \|u\|^2 - (p-1)\int_\Omega |u|^p\,dx - \lambda(q-1)\int_{\partial\Omega} |u|^q\,ds \\
&= (2-p)\int_\Omega |u|^p\,dx + \lambda(2-q)\int_{\partial\Omega} |u|^q\,ds \\
&= (2-q)\,\|u\|^2 - (p-q)\int_\Omega |u|^p\,dx \\
&= (2-p)\,\|u\|^2 + \lambda(2-q)\int_{\partial\Omega} |u|^q\,ds.
\end{aligned}$$

Therefore we can split the Nehari manifold $\mathcal{N}$ into three parts, i.e.,

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} \;:\; \phi''_u(1) > 0\} \\
\mathcal{N}^- &= \{u \in \mathcal{N} \;:\; \phi''_u(1) < 0\} \qquad (3.8) \\
\mathcal{N}^0 &= \{u \in \mathcal{N} \;:\; \phi''_u(1) = 0\}.
\end{aligned}$$

We will research the existence of minimizers of functional J on $\mathcal{N}$ to prove the existence of solutions of Problem (3.1) corresponding to local minima, local maxima and points of inflection. The following Lemma shows that the local minimizers on Nehari manifold $\mathcal{N}$ are usually critical points of J.

Lemma 3.1 *[22] Suppose that u_0 is a local minimizer of J on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then u_0 is a critical point of J.*
For proof we can refer to Binding et al.[7] and Brown–Zhang [6].

3.3 Convex–concave subcritical result: the case $1 < q < 2 < p < 2^*$

In this section we use the fibering methods and the technique of the Nehari manifold decomposition to study Problem (3.1) with $1 < q < 2 < p < 2^* = \frac{2N}{N-2}$.

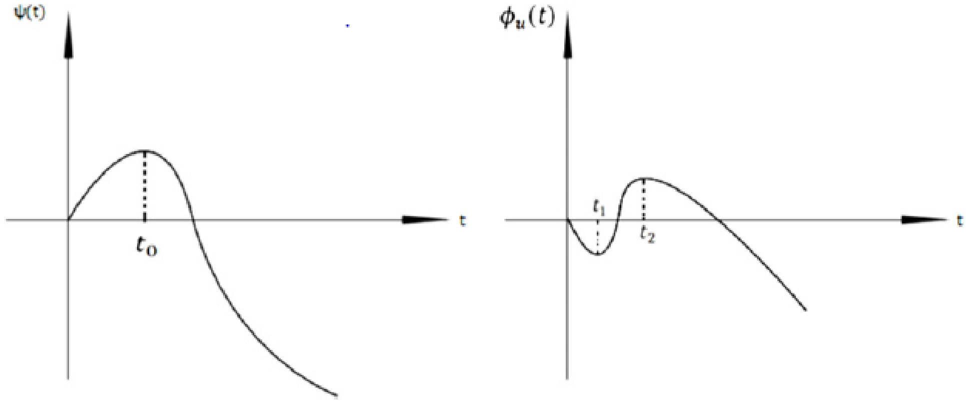

Figure 3.1: possible graphs of function $\psi(t)$ and $\phi_u(t)$

From (3.4), letting $v = tu$, we get

$$\begin{aligned}\langle J'(tu), tu\rangle &= \int_\Omega t^2(|\nabla u|^2 + |u|^2)dx - \int_\Omega t^p |u|^p \, dx - \lambda \int_{\partial\Omega} t^q |u|^q \, ds, \\ &= t^2 \|u\|^2 - t^p \int_\Omega |u|^p \, dx - t^q \lambda \int_{\partial\Omega} |u|^q \, ds.\end{aligned}$$

Let $tu \in \mathcal{N}$, then

$$\langle J'(tu), tu\rangle = 0 \Longleftrightarrow t^2 \|u\|^2 - t^p \int_\Omega |u|^p \, dx - t^q \lambda \int_{\partial\Omega} |u|^q \, ds. = 0. \tag{3.9}$$

From (3.9), we can consider the function

$$\psi(t) = t^{2-q} \|u\|^2 - t^{p-q} \int_\Omega |u|^p \, dx.$$

Clearly, for $t > 0$, $tu \in \mathcal{N}$ if and only if t is the solution of the equation

$$\psi(t) = \lambda \int_{\partial\Omega} |u|^q \, dx, \tag{3.10}$$

we analyse the function $\psi(t)$, since

$$\psi'(t) = (2-q)t^{1-q} |u|^2 - (p-q)t^{p-q-1} \int_\Omega |u|^p \, dx,$$

the function ψ has a global maximum at t_0 (see Figure 3.1), such that

$$t_0 = \left[\frac{(2-q)\|u\|^2}{(p-q)\int_\Omega |u|^p \, dx}\right]^{\frac{1}{p-2}},$$

we deduce

$$\psi'(t) > 0 \Longleftrightarrow 0 < t < t_0,$$

then the function ψ initially increasing and eventually decreasing because of the condition $p - q > 2 - q > 0$.

From Figure 3.1 and the fact $\lambda \int_{\partial\Omega} |u|^q \, dx > 0$ and that the equation(3.10) has no solution if λ satisfies the following condition

$$\lambda \int_{\partial\Omega} |u|^q \, dx > \psi(t_0) = \left[\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} - \left(\frac{2-q}{p-q}\right)^{\frac{p-q}{p-2}}\right] \frac{\|u\|^{\frac{2(p-q)}{p-2}}}{(\int_\Omega |u|^p \, dx)^{\frac{2-q}{p-2}}}, \tag{3.11}$$

then we have the inequality

$$t_0^{2-q}\|u\|^2 - t_0^{p-q}\int_\Omega |u|^p\,dx < \lambda\int_{\partial\Omega}|u|^q\,dx \tag{3.12}$$

multiplying (3.12) by t_0^{q-1}, we get

$$t_0\|u\|^2 - t_0^{p-1}\int_\Omega |u|^p\,dx - \lambda t_0^{q-1}\int_{\partial\Omega}|u|^q\,dx < 0$$

Moreover, (3.7) and (2.2) implies that $\phi_u'(t) < 0$. Hence we have $tu \notin \mathcal{N}$ for any $t > 0$. On the other hand, if λ satisfies

$$0 < \lambda\int_{\partial\Omega}|u|^q\,dx < \psi(t_0),$$

we can verify that t is solution of equation

$$\psi(t) = \lambda\int_{\partial\Omega}|u|^q\,dx$$

if and only if t is solution of

$$\phi_u'(t) = 0$$

then from Figure 3.1, we know that equation (3.10) has two solutions t_1, t_2 with $t_1 < t_0 < t_2$ such that $\phi_u'(t_1) = \phi_u'(t_2) = 0$ and $\psi'(t_1) > 0, \psi'(t_1) < 0$. Therefore, we have

$$\begin{aligned}\phi_u''(t_1) &= (2-q)\|u\|^2 - (p-q)t_1^{p-2}\int_\Omega |u|^p\,dx\\ &= t_1^{q-1}\left((2-q)t_1^{1-q}\|u\|^2 - (p-q)t_1^{p-q-1}\int_\Omega |u|^p\,dx\right)\\ &= t_1^{q-1}\psi'(t_1),\end{aligned}$$

and

$$\begin{aligned}\phi_u''(t_2) &= (2-q)\|u\|^2 - (p-q)t_2^{p-2}\int_\Omega |u|^p\,dx\\ &= t_2^{q-1}\left((2-q)t_2^{1-q}\|u\|^2 - (p-q)t_2^{p-q-1}\int_\Omega |u|^p\,dx\right)\\ &= t_2^{q-1}\psi'(t_2).\end{aligned}$$

These imply that $\phi_u''(t_1) > 0$ and $\phi_u''(t_2) < 0$, and so fibering maps ϕ_u have local minimum at t_1 and a local maximum at t_2 such that $t_1 \in \mathcal{N}^+$ and $t_2 \in \mathcal{N}^-$. (See graph of ϕ_u as shown in Figure 3.1).
Now, we consider functions ψ and ϕ_u at the point t_0. Since $\psi'(t_0) = 0$ and

$$\phi_u''(t_0) = (2-q)\|u\|^2 - (p-q)t_0^{p-2}\int_\Omega |u|^p\,dx = t_0^{q-1}\psi'(t_0).$$

we have $\phi_u''(t_0) = 0$, so t_0 is a point of inflection of function ϕ_u. Then, we obtain the following result.

Lemma 3.2 *[22]There exists a constant $\lambda_0 > 0$ such that $\mathcal{N}^0 = \emptyset$ for $0 < \lambda < \lambda_0$.*

Proof. By contradiction, Suppose that there exists $0 < \lambda < \lambda_0$, such that $\mathcal{N}^0 \neq \emptyset$. Then, for $u \in \mathcal{N}^0$, we have

$$\begin{aligned} 0 &= \phi_u''(1) = (2-p)\|u\|^2 + \lambda(p-q)\int_{\partial\Omega}|u|^q\,ds \\ &= (2-q)\|u\|^2 - (p-q)\int_\Omega |u|^p\,dx. \end{aligned} \tag{3.13}$$

Let $C_b, C > 0$ be the constant of Sobolev embedding, by the Sobolev embedding theorem, we have

$$\begin{aligned} \|u\|^2 &= \frac{\lambda(p-q)}{(p-2)}\int_{\partial\Omega}|u|^q\,ds \\ &= \frac{\lambda(p-q)}{(p-2)}\|u\|^q_{L^q(\partial\Omega)} \\ &\leq \frac{\lambda(p-q)}{(p-2)}c_b^q\|u\|^q, \end{aligned}$$

i.e,

$$\|u\| \leq \left(\frac{\lambda(p-q)c_b^q}{(p-2)}\right)^{\frac{1}{2-q}}. \tag{3.14}$$

Using (3.13) and Sobolev embedding, we get

$$\begin{aligned} \|u\|^{-2} &= \frac{(2-q)}{(p-q)\int_\Omega |u|^p\,dx} \\ &= \frac{(2-q)}{(p-q)\|u\|^p_{L^p(\Omega)}} \\ &\geq \frac{(2-q)}{(p-q)c^p\|u\|^p}, \end{aligned}$$

i.e,

$$\|u\| \geq \left(\frac{(2-q)}{(p-q)c^p}\right)^{\frac{1}{p-2}}. \tag{3.15}$$

the inequalities (3.14) and (3.15) imply that

$$\left(\frac{\lambda(p-q)c_b^q}{(p-2)}\right)^{\frac{1}{2-q}} \geq \left(\frac{(2-q)}{(p-q)c^p}\right)^{\frac{1}{p-2}}$$

then

$$\lambda \geq \left(\frac{(2-q)}{c^p}\right)^{\frac{2-q}{p-2}}\left(\frac{1}{(p-q)}\right)^{\frac{p-q}{p-2}}\left(\frac{(p-2)}{c_b^q}\right) = \lambda_0.$$

which is a contradiction, Thus we conclude that there exists a constant $\lambda_0 > 0$ such that $\mathcal{N}^0 = \emptyset$ for $0 < \lambda < \lambda_0$. ■

Lemma 3.3 *[22]The functional J is coercive and bounded from below on $\mathcal{N}$.*

Proof. By the Sobolev embedding theorem and since $1 < q < 2 < p$, we have that

$$\begin{aligned} J(u) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\partial\Omega}|u|^q\,ds \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{p}\right)\|u\|^q_{L^q(\partial\Omega)} \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 - \lambda\left(\frac{1}{q}-\frac{1}{p}\right)c_b^q\|u\|^q. \end{aligned}$$

Where c_b be a positive constant then.
We obtain that the functional J is coercive and bounded from below on $\mathcal{N}$. ■
By Lemmas 3.2 and 3.3, for $0<\lambda<\lambda_0$, and we obtain that $\mathcal{N}=\mathcal{N}^-\cup\mathcal{N}^+$ and J is bounded from below on $\mathcal{N}^-$ and $\mathcal{N}^+$. Therefore, we may define

$$\beta_0^+=\inf_{u\in\mathcal{N}^+}J(u) \text{ and } \beta_0^-=\inf_{u\in\mathcal{N}^-}J(u).$$

We have the following results.

Proposition 3.1 *[22]If $0<\lambda<\lambda_0$, then the functional J has a minimizer u_1 in $\mathcal{N}^+$ and satisfies*

1. $J(u_1)=\inf_{u\in\mathcal{N}^+}J(u)<0,$
2. *u_1 is a positive solution of problem (3.1).*

Proof. Since J is bounded from below on $\mathcal{N}^+$, there exists a minimizing sequence $\{u_n\}_n\subseteq\mathcal{N}^+$ such that

$$\lim_{n\longrightarrow\infty}J(u_n)=\inf_{u\in\mathcal{N}^+}J(u).$$

Then, by Lemma 3.3 it implies that the sequence $\{u_n\}_n$ is bounded in $H^1(\Omega)$, so that up to subequence there exist $u_{n_l}=u_n$ and $u_1\in H^1(\Omega)$ such that

$$u_n\rightharpoonup u_1 \text{ in } H^1(\Omega).$$

from (3.2), the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ and $L^q(\Omega)$ are compact, then

$$u_n\longrightarrow u_1 \text{ in } L^p(\Omega) \text{ and } L^q(\Omega)$$

i.e,

$$\int_\Omega |u_n|^p\,dx\longrightarrow\int_\Omega|u_1|^p\,dx \text{ and } \int_{\partial\Omega}|u_n|^q\,ds\longrightarrow\int_{\partial\Omega}|u_1|^q\,ds.$$

as $n\longrightarrow\infty$. There exists t_1 such that $t_1u_1\in\mathcal{N}^+$ and $J(t_1u_1)<0$ (see Figure 3.1). Hence we have $\inf_{u\in\mathcal{N}^+}J(u)<0$.
Now we prove that $u_n\longrightarrow u_1$ strongly in $H^1(\Omega)$. If not, then $\|u_1\|<\lim_{n\longrightarrow\infty}\inf\|u_n\|$. Thus, for $u_n\in\mathcal{N}^+$, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}\phi'_{u_n}(t_1) &= \lim_{n\longrightarrow\infty}\left(t_1\|u_n\|^2-t_1^{p-1}\int_\Omega|u_n|^p\,dx-\lambda t_1^{q-1}\int_{\partial\Omega}|u_n|^q\,ds\right)\\ &> t_1\|u_1\|^2-t_1^{p-1}\int_\Omega|u_1|^p\,dx-\lambda t_1^{q-1}\int_{\partial\Omega}|u_1|^q\,ds\\ &= \phi'_{u_1}(t_1)=0.\end{aligned}$$

i.e, $\phi'_{u_n}(t_1)>0$ for n sufficiently large. Since $u_n=1.u_n\in\mathcal{N}^+$, From Figure 3.1 it is easy to see that $\phi'_{u_n}(t)<0$ for $t\in]0,1[$ and $\phi'_{u_n}(1)=0$ for all n. Hence we must have $t_1>1$. On the other hand, $\phi'_{u_1}(t)$ is decreasing on $]0,t_1[$, and so

$$J(t_1u_1)\le J(u_1)<\lim_{n\longrightarrow\infty}J(u_n)=\inf_{u\in\mathcal{N}^+}J(u),$$

which is a contradiction. Hence $u_n \longrightarrow u_1$ strongly in $H^1(\Omega)$. This implies

$$J(u_n) \longrightarrow J(u_1) = \inf_{u\in\mathcal{N}^+} J(u) \text{ as } n \longrightarrow \infty.$$

Thus u_1 is a minimizer of J on $\mathcal{N}^+$. Since $J(u_1) = J(|u_1|)$ and $|u_1| \in \mathcal{N}^+$, using Lemma 3.1 we have that u_1 is a positive solution of problem (3.1). ■

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$.

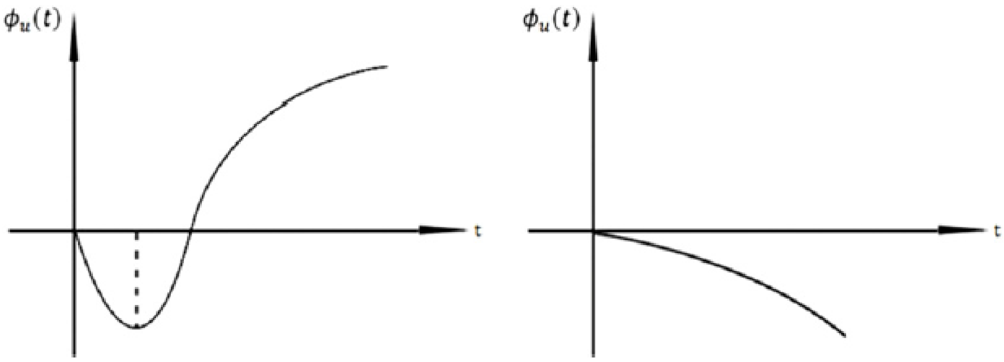

Figure 3.2: possible graphs of $\phi_u(t)$

Proposition 3.2 *[22]For $0 < \lambda < \lambda_0$, then the functional J has a minimizer u_2 in $\mathcal{N}^-$ and satisfies*

1. $J(u_2) = \inf_{u\in\mathcal{N}^-} J(u)$,
2. *u_2 is a positive solution of problem (3.1).*

Proof. Analogous to the proof of Proposition 3.1, we have J is bounded from below on $\mathcal{N}^-$. Then there exists a minimizing sequence $\{u_n\}_n \subseteq \mathcal{N}^-$ such that

$$\lim_{n\longrightarrow\infty} J(u_n) = \inf_{u\in\mathcal{N}^-} J(u).$$

Then, by Lemma 3.3 it implies that the sequence $\{u_n\}_n$ is bounded in $H^1(\Omega)$, so that up to subequence there exist $u_{n_l} = u_n$ and $u_2 \in H^1(\Omega)$ such that

$$u_n \rightharpoonup u_2 \text{ in } H^1(\Omega),$$

and for the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ and $L^q(\Omega)$ then

$$\int_\Omega |u_n|^p \, dx \longrightarrow \int_\Omega |u_2|^p \, dx \text{ and } \int_{\partial\Omega} |u_n|^q \, ds \longrightarrow \int_{\partial\Omega} |u_2|^q \, ds.$$

As $n \longrightarrow \infty$. From the analysis for fibering maps ϕ_u and Figure 3.2, we know that there exists t_1, t_2 with $t_1 < t_0 < t_2$ such that

$$t_1u \in \mathcal{N}^-, t_2u \in \mathcal{N}^- \text{ and } J(t_1u) \leq J(tu) \leq J(t_2u).$$

Now we prove that $u_n \longrightarrow u_2$ strongly in $H^1(\Omega)$. If not, then $\|u_2\| < \lim_{n\longrightarrow\infty} \inf \|u_n\|$. Since $u_n \in \mathcal{N}^-$, we have $J(u_n) \geq J(tu_n)$ for all $t \geq t_0$, and

$$\begin{aligned} J(t_2u_2) &= \frac{t_2^2}{2}\|u_2\|^2 - \frac{t_2^p}{p}\int_\Omega |u_2|^p\,dx - \lambda\frac{t_2^q}{q}\int_{\partial\Omega}|u_2|^q\,ds \\ &< \lim_{n\longrightarrow\infty}\left(\frac{t_2^2}{2}\|u_n\|^2 - \frac{t_2^p}{p}\int_\Omega |u_n|^p\,dx - \lambda\frac{t_2^q}{q}\int_{\partial\Omega}|u_n|^q\,ds\right) \\ &= \lim_{n\longrightarrow\infty} J(t_2u_n) \leq \lim_{n\longrightarrow\infty} J(u_n) = \inf_{u\in\mathcal{N}^-} J(u). \end{aligned}$$

which is a contradiction. Hence $u_n \longrightarrow u_2$ strongly in $H^1(\Omega)$. This implies

$$J(u_n) \longrightarrow J(u_2) \;\text{ as }\; n \longrightarrow \infty.$$

Thus u_2 is a minimizer of J on $\mathcal{N}^-$. Since $J(u_2) = J(|u_2|)$ and $|u_2| \in \mathcal{N}^-$, using Lemma 3.1 we have that u_2 is a positive solution of problem (3.1). ■

Now we can give the proof of Theorem 3.1

Proof of Theorem 3.1. By Propositions 3.1 and 3.2 and Lemma 3.1, we get that Problem (3.1) has two positive solutions $u_1 \in \mathcal{N}^+$ and $u_2 \in \mathcal{N}^-$ in $H^1(\Omega)$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, then those two solutions are distinct. This completes the proof. ■

Remark 3.1 *Our method can be applied to the condition* $1 < p < 2 < q < 2_b^* = \frac{2(N-1)}{N-2}$ *i.e., p stands for the concave term, and q stands for the convex one. The proof of the existence result is similar to the one performed for the problem (3.1).*

3.4 Bifurcation result: the case $q = 2$, $1 < p < 2$

In a recent paper, Brown and Zhang [6] have studied a subcritical semilinear elliptic equation with a sign-changing weight function for a bifurcation real parameter in the case $q = 2$ and Dirichlet boundary conditions, i.e.,

$$-\Delta u = \lambda a(x)u + b(x)|u|^{p-2}u \;\text{ for }\; x\in\Omega;\;\; u = 0 \;\text{ for }\; x \in \partial\Omega,$$

where $2 < p < 2^*$ and $a, b : \Omega^- \longrightarrow \mathbb{R}$ are smooth and sign-change functions. It is well-known that, when $p > 2$, a curve of positive solution bifurcates from the zero solution at $\lambda = \lambda_1(a)$, where $\lambda_1(a)$ is the principal eigenvalue of the weight linear problem

$$-\Delta u = \lambda a(x)u \;\text{ for }\; x\in\Omega;\;\; u = 0 \;\text{ for }\; x \in \partial\Omega.$$

In that paper, they gave an interesting explanation of a well-known bifurcation result from the relationship between the Nehari manifold and fibering maps. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value.

In this section, we fix p such that $1 < p < 2$ and regard λ as a real parameter. Therefore, we still consider the sublinear perturbation of nonlinear boundary problem, i,e,

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

The problem (3.16) is asymptotically linear when $1 < p < 2$, and then in this case the bifurcation at infinity occurs when $\lambda = \lambda_{1S}$, where λ_{1S} is the principal eigenvalue of the Steklov problem

$$S(\Omega) \quad \begin{cases} -\Delta u + u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = \lambda u & \text{on} \quad \partial\Omega. \end{cases}$$

the eigenvalue λ_{1S} of problem $S(\Omega)$ can be defined as follows

$$\lambda_{1S} = \inf_{u \in H^1(\Omega)} \left\{ \int_\Omega \left(|\nabla u|^2 + |u|^2 \right) dx \quad : \quad \int_{\partial\Omega} |u|^2 \, ds = 1 \right\}.$$

For the problem (3.16), the corresponding energy function $J(u)$ and fibering maps $\phi_u(t)$ are given by

$$J(u) = \frac{1}{2} \left(\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds \right) - \frac{1}{p} \int_\Omega |u|^p \, dx,$$

$$\phi_u(t) = J(tu) = \frac{t^2}{2} \left(\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds \right) - \frac{t^p}{p} \int_\Omega |u|^p \, dx.$$

We define

$$\begin{aligned} B_+ &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \int_\Omega |u|^p \, dx > 0 \right\}, \\ B_- &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \int_\Omega |u|^p \, dx < 0 \right\}, \\ B_0 &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \int_\Omega |u|^p \, dx = 0 \right\}. \end{aligned}$$

and

$$\begin{aligned} L_+ &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds > 0 \right\}, \\ L_- &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds < 0 \right\}, \\ L_0 &= \left\{ u \in H^1(\Omega) \quad : \quad \|u\| = 1 \text{ and } \|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds = 0 \right\}. \end{aligned}$$

then we get the following two cases:
case(I) $\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds > 0$,
case(II) $\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds < 0$.
We note that, in Case (I), $\phi_u(t) < 0$ for t small enough, and $\phi_u(t) \longrightarrow \infty$ as $t \longrightarrow \infty$ for $u \in B_+ \cap L_+$, thus, the fibering maps $\phi_u(t)$ has local minimum at point

$$t_{min} = \left(\frac{\int_\Omega |u|^p \, dx}{\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds} \right)^{\frac{1}{2-q}},$$

and satisfies $t_{min} u \in \mathcal{N}$. The possible graph of ϕ_u can be seen in Figure 3.2.
In case (II), we obtain that $u \in B_+ \cap L_+$ and $\phi_u(t)$ is strictly decreasing for all $t > 0$, thus, ϕ_u has no turning point and so no multiples of u lies in $\mathcal{N}$. The possible graph of ϕ_u can be seen in Figure 3.2.
Suppose $0 < \lambda < \lambda_{1S}$. It is well-known that

$$\|u\|^2 - \lambda \int_{\partial\Omega} |u|^2 \, ds > \|u\|^2 - \lambda_1 \int_{\partial\Omega} |u|^2 \, ds \geq 0$$

for all $u \in H^1(\Omega) - \{0\}$, L_- and L_0 are empty. Then we have the following result.

Proposition 3.3 *[22] Suppose $0 < \lambda < \lambda_{1S}$. Then, $\mathcal{N}^+$ is bounded and J is bounded from below on $\mathcal{N}^+$.*

Proof. We suppose that $\{u_n\} \subseteq \mathcal{N}^+$ a sequence unbounded, so that up to subsequence there exist$u_{n_k} = u_n$ and $\{u_n\} \subseteq \mathcal{N}^+$, such that $\|u\| \longrightarrow +\infty$ as $n \longrightarrow +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, up to subsequence, we assume that $\exists v_0 \in H^1(\Omega)$, such that

$$v_n \rightharpoonup v_0 \ \text{ in } \ H^1(\Omega), \ \text{ and } \ v_n \longrightarrow v_0 \ \text{ in } \ L^2(\partial\Omega) \ \text{ and } \ L^p(\Omega),$$

where $1 < p < 2$. Since $u_n \in \mathcal{N}^+$, we get

$$\begin{aligned} \|v_n\|^2 - \lambda \int_{\partial\Omega} |v_n|^2 \, ds &= \left(\frac{\|u_n\|^2}{\|u_n\|^2} - \lambda \int_{\partial\Omega} \frac{|u_n|^2}{\|u_n\|^2} ds \right) \\ &= \frac{1}{\|u_n\|^2} \left(\|u_n\|^2 - \lambda \int_{\partial\Omega} |u_n|^2 \, ds \right) \\ &= \int_\Omega \frac{|u_n|^p}{\|u_n\|^2} dx = \int_\Omega \frac{|v_n|^p}{\|u_n\|^{2-p}} dx \\ &\longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \end{aligned}$$

Now we prove that $v_n \longrightarrow v_0$ strongly in $H^1(\Omega)$. Suppose the contrary. Then $\|v_0\| < \liminf_{n\longrightarrow\infty} \|v_n\|$ and so

$$\|v_0\|^2 - \lambda \int_{\partial\Omega} |v_0|^2 \, ds < \lim \inf_{n\longrightarrow\infty} \|v_n\|^2 - \lambda \int_{\partial\Omega} |v_n|^2 \, ds = 0.$$

Thus, we obtain $\frac{v_0}{\|v_0\|} \in L_-$ which is impossible since $L_- = \emptyset$. Hence $v_n \longrightarrow v_0$ strongly in $H^1(\Omega)$. Therefore, $\|v_0|$ and

$$\|v_0\|^2 - \lambda \int_{\partial\Omega} |v_0|^2 \, ds = 0,$$

i.e., $v_0 \in L_0$ which is impossible since $L_0 = \emptyset$. Hence $\mathcal{N}^+$ is bounded.
Now we prove J is bounded from below on $\mathcal{N}^+$. For $u_n \in \mathcal{N}^+$, by the Sobolev embedding theorem and the fact that $\mathcal{N}^+$ is bounded, and let c_1 be a Sobolev embedding constant, we have

$$\begin{aligned} J(u_n) &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_\Omega |u_n|^p \, dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p} \right) c_1^p \|u_n\|^p . \end{aligned}$$

then, J is bounded from below on $\mathcal{N}^+$. This completes the proof. ■

Proposition 3.4 *[22] Suppose $0 < \lambda < \lambda_{1S}$. Then there exists a minimizer for J on $\mathcal{N}^+$.*

Proof. Since J is bounded from below on $\mathcal{N}^+$, we can take $\{u_n\} \subseteq \mathcal{N}^+$ a minimizing sequence, such that

$$\inf_{u\in\mathcal{N}^+} J(u) = \lim_{n\longrightarrow+\infty} J(u_n).$$

From previous Proposition we know that the sequence $\{u_n\}$ is bounded. so up to subsequence, we may assume that $\exists u_0$, such that

$$u_n \rightharpoonup u_0 \ \text{ in } \ H^1(\Omega) \ \text{ and } \ u_n \longrightarrow u_0 \ \text{ in } \ L^2(\partial\Omega) \ \text{ and } \ L^p(\Omega),$$

where $1 < p < 2$. From Figure 3.2 there exists t_0 such that $t_0 u_0 \in \mathcal{N}^+$ and $J(t_0 u_0) < 0$. Now we prove that $u_n \longrightarrow u_0$ strongly in $H^1(\Omega)$. Suppose the contrary. Then $\|u_0\| < \lim_{n\longrightarrow\infty} \inf \|u_n\|$. the function $\phi_u(t)$ is differentiable and

$$\phi'_{u_n}(t) = t\,\|u\|^2 - t^{p-1}\int_\Omega |u|^p\,dx - \lambda t \int_{\partial\Omega} |u|^2\,ds.$$

Thus, for $u_n \in \mathcal{N}^+$,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \phi'_{u_n}(t_0) &= \lim_{n\longrightarrow\infty}\left(t_0\,\|u_n\|^2 - t_0^{p-1}\int_\Omega |u_n|^p\,dx - \lambda t_0 \int_{\partial\Omega} |u_n|^2\,ds\right)\\
&> t_0\,\|u_0\|^2 - t_0^{p-1}\int_\Omega |u_0|^p\,dx - \lambda t_0 \int_{\partial\Omega} |u_0|^2\,ds\\
&= \phi'_{u_0}(t_0) = 0,
\end{aligned}$$

that is, $\phi'_{u_n}(t_0) > 0$ for n sufficiently large. Since $u_n = 1\cdot\, u_n \in \mathcal{N}^+$, from Figure 3.2, it is easy to see that $\phi'_{u_n}(t_0) < 0$ for $t \in]0,1[$ and $\phi'_{u_n}(t_0) = 0$ for all n. Hence we must have $t_0 > 1$. On the other hand, $\phi'_{u_n}(t_0) = 0$ is decreasing on $]0, t_0[$, and so

$$J(t_0 u_0) < J(u_0) < \lim_{n\longrightarrow\infty} J(u_n) = \inf_{u_n\in\mathcal{N}^+} J(u),$$

which is a contradiction. Hence $u_n \longrightarrow u_0$ strongly in $H^1(\Omega)$. This implies

$$J(u_n) \longrightarrow J(u_0) = \inf_{u_n\in\mathcal{N}^+} J(u) \quad \text{as} \quad n \longrightarrow \infty.$$

Thus, u_0 is a minimizer of J on $\mathcal{N}^+$. The assertion is proved. ■

Remark 3.2 *For $1 < p < 2$, we know that the problem (3.16) is asymptotically linear with corresponding nonlinear boundary Steklov problem $S(\Omega)$. Since $L_- = \emptyset$ for $\lambda < \lambda_{1S}$, it follows from Proposition 3.4 that J has a minimizer on $\mathcal{N}^+$ whenever $\lambda < \lambda_{1S}$.*

Our next result corresponds to the fact that a branch of positive solutions bifurcates from infinity to the left at $\lambda = \lambda_{1S}$ when $\int_\Omega |\varphi_1|^p\,dx > 0$, where φ_1 is the eigenfunction corresponding to the principle eigenvalue λ_{1S}.
Now we can give the proof of Theorem 3.2,
Proof of Theorem 3.2. By Propositions 3.3 and 3.4 and Lemma 3.1, we get that Problem (3.1) has at least one nontrivial solution $u_\lambda \in \mathcal{N}^+$. By $J(u_\lambda) = J(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}^+$, u_λ is a positive solution of problem (3.1) with $1 < p < 2$ and $q = 2$.
when $\lambda = \lambda_{1S}$ and $u = \varphi_1$ we obtain the problem

$$\begin{cases} -\Delta\varphi_1 + \varphi_1 = 0 & \text{in} \ \ \Omega,\\ \frac{\partial\varphi_1}{\partial n} = \lambda_{1S}\varphi_1 & \text{on} \ \ \partial\Omega. \end{cases}$$

then

$$\int_\Omega \nabla\varphi_1\nabla v dx - \lambda_{1S}\int_{\partial\Omega}\varphi_1 v ds + \int_\Omega \varphi_1 v dx = 0,$$

we put $v = \varphi_1$

$$\int_\Omega |\nabla\varphi_1|^2\,dx - \lambda_{1S}\int_{\partial\Omega}\varphi_1^2 ds + \int_\Omega \varphi_1^2 dx = 0,$$

$$\|\varphi_1\|^2 - \lambda_{1S} \int_{\partial\Omega} \varphi_1^2 ds = 0,$$

On the other hand, since $0 < \lambda < \lambda_{1S}$, we have

$$\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds = (\lambda_{1S} - \lambda) \int_{\partial\Omega} |\varphi_1| ds > 0,$$

and then $\varphi_1 \in L_+ \cap B_+$, namely, $\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds$ and $\int_\Omega |\varphi_1|^p dx$ have the same sign. Hence, from the analysis of Case(I), the fibering map ϕ_{φ_1} has a unique turning point at

$$t_{\varphi_1} = \left(\frac{\int_\Omega |\varphi_1|^p dx}{\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds} \right)^{\frac{1}{2-p}},$$

such that $t_{\varphi_1}\varphi_1 \in \mathcal{N}^+$ and

$$\begin{aligned}
J(t_{\varphi_1}\varphi_1) &= \left(\frac{1}{2} - \frac{1}{p}\right) t_{\varphi_1}^2 \left(\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds \right) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) t_{\varphi_1}^2 \left[\frac{\int_\Omega |\varphi_1|^p dx}{\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds} \right]^{\frac{2}{2-p}} \left(\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds \right) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \frac{\left(\int_\Omega |\varphi_1|^p dx\right)^{\frac{2}{2-p}}}{\left(\|\varphi_1\|^2 - \lambda \int_{\partial\Omega} |\varphi_1|^2 ds \right)^{\frac{p}{2-p}}} \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{1}{\lambda_{1S} - \lambda} \right)^{\frac{p}{2-p}} \frac{\left(\int_\Omega |\varphi_1|^p dx\right)^{\frac{2}{2-p}}}{\left(\int_{\partial\Omega} |\varphi_1|^2 ds \right)^{\frac{p}{2-p}}}.
\end{aligned}$$

This means

$$\lim_{\lambda \longrightarrow \lambda_{1S}^-} \inf_{u_\lambda \in \mathcal{N}^+} J(u_\lambda) \longrightarrow -\infty.$$

Since $u_\lambda \in \mathcal{N}^+$, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) c_2^p \|u_\lambda\|^p \leq \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |u_\lambda|^p dx = J(u_\lambda) \longrightarrow -\infty$$

as $\lambda \longrightarrow \lambda_{1S}^-$ for $c_2 > 0$. Hence for $u_\lambda \in \mathcal{N}^+$ we get $\|u_\lambda\| \longrightarrow \infty$ as $\lambda \longrightarrow \lambda_{1S}^-$. This completes the proof. ■

CONCLUSION

In this work, we have studied some minimization techniques to solve semilinear elliptic equation, posed on a bounded open set $\Omega \subset \mathcal{R}^N$. In the first chapter, we gave some basic concepts which are necessary for our work, In the second chapter, we proved the existence of a nontrivial solution by two different methods, Minimization on Sphere and Minimization on the Nehari Manifold. And by using the method of Minimization on the Nehari Manifold we proved that the perturbed problem admits at least two nontrivial solution. As for the third chapter, we have studied the existence theorem of the nonlinear boundary problem by using the Nehari manifold and fibering maps.

There are some cases that we have not studied, so we look forward to be studied by our colleagues in the coming years, such as, critical case for the semilinear elliptic problem and the case when $q = 2$ and $2 < p < 2^*$.

Abstract

In this work, we study some minimization techniques to solve semilinear elliptic equation, posed on a bounded open set $\Omega \subset \mathcal{R}^N$. the first problem considered is

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove the existence of a nontrivial solution by two different methods: Minimization on Sphere and Minimization on the Nehari Manifold. Also, we consider the perturbed problem, for $h \in L^2(\Omega)$ and $p \in]2, 2^*[$, the Dirichlet problem

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the method of Minimization on the Nehari Manifold we proved that this problem admits at least two nontrivial solution. Using the Nehari manifold and fibering maps, we prove the existence theorem of the nonlinear boundary problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^N$. We discuss how the Nehari manifold changes as λ changes, and show how the existence of solutions depends on the properties of the Nehari manifold.

Résumé

Dans ce travail, nous étudions quelques techniques de minimisation pour résoudre l'équation elliptique semi-linéaire, posée sur un ensemble ouvert borné $\Omega \subset \mathcal{R}^N$. Le premier problème considéré est

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Nous prouvons l'existence d'une solution non triviale par deux méthodes différentes : la minimisation sur la sphère et la minimisation sur la variété de Nehari. Aussi, nous considérons le problème perturbé, pour $h \in L^2(\Omega)$ et $p \in]2, 2^*[$

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

En utilisant la méthode de minimisation sur la variété de Nehari nous avons prouvé que ce problème admet au moins deux solutions nontriviales. En utilisant la variété de Nehari et les cartes de fibrage, nous prouvons le théorème d'existence du problème de frontière non linéaire

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

sur un borné domaine $\Omega \subseteq \mathbb{R}^N$, Nous discutons de la façon dont la variété de Nehari change lorsque λ change, et montrer comment l'existence des solutions dépend des propriétés de la variété de Nehari.

ملخص

في هذا العمل. نقوم بدراسة بعض تقنيات التصغير لحل مسائل ناقصية نصف خطية مطروحة على مفتوح محدود من $\mathbb{R}^N$. المسألةالأولى المعتبرة هي كالتالي:

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

نثبت وجود حل غير تافه باستعمال طريقتين مختلفتين. طريقة التصغير على سطح كرة وطريقة التصغير على منوعة نهاري. في هذه المذكرة. ندرس كذلك مسألة مضطربة. وهي كالتالي

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

حيث $h \in L^2(\Omega)$ ، $p \in]2, 2^*[$ باستعمال التصغير على منوعة نهاري، نبرهن أن المسألة تقبل على الاقل حلين غير تافهين .في الجزء الاخير تم تدعيم المذكرة بمسألة اخرى. حيث وبتطبيق مزج بين طريقة التصغير على منوعة نهاري وطريقة تطبيقات الألياف نثبت نظرية وجود للمسألة الحدية غير الخطية

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

حيث Ω ميدان محدود من $\mathbb{R}^N$. نناقش من خلالها كيفية تغير منوعة نهاري بتغير الوسيط الحقيقي λ و نظهر كيفية تعلق وجود الحلول بخواص منوعة نهاري.

قائمة المصطلحات

FRANÇAIS	ENGLISH	عربية
Points Critique	Critical points	النقاط الحرجة
Faiblement semi-continue inférieurement	Weakly lower semi-semi continuous	نصف مستمر من الأسفل بضعف
Minimisation locale	Locale minimization	تصغير محلي
Minoré	Bounded below	محدود من الأدنى
Majoré	Bounded above	محدود من الأعلى
Variété	Manifold	منوعة
Compact	Compact	متراص
Convexe	Convexe	محدب
Concave	Concave	مقعر
Problem semiliniéar elliptique	Semilineair elliptic Problem	مسألة ناقصية نصف خطية
Contrainte	Constrained	مقيد
Injection continue	Injection continuous	تباين مستمر
Injection compact	Injection compact	تباين متراص
Suite de minimisation	Minimizing sequence	تصغيير متتالية
Problem pertrubé	Perturbed problem	المسألة المضطربة
Espace vectorial	Vector space	فضاء شعاعي
Dérivie partielle	Partial derivative	مشتق جزئي
Conjugué	Conjugate	مرافق

Solution nontrivial	Nontrivial solution	حل غير بديهي
Global	Global	شامل
Décroissant	Decreasing	متناقض
Contradiction	Contradiction	تناقص
Gradient	Gradient	تدرج
Dérivable	Differentiable	قابل للمفاضلة

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