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Thème

Boundary and initial value problem for nonlinier of Caputo-Katugampola type fractional differential equation

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Thanks

In the name of ALLAH the clément and the mercy

I would first like to thank God Almighty, who gave me the courage and the perseverance to realize this mode

I dee **Mr ARIOUA Yacine** who has alway available throughout the realization of this the

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Finally, I thank all tho

Dedications

In the name of ALLAH the clément and the mercy

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To the deare

My father and my mother who God guards them

To my dear sisters and my brothers

To all my friends

To all student

Contents

General introduction

Fractional calculus is a mathematical branch which investigates the properties of derivatives and integrals of non-integer orders (also known as fractional derivatives and integrals, briefly differ-integrals). The interested readers in the subject should refer to the books (Samko et al. 1993 [14], Podlubny 1999 [13], Kilbas et al. 2006 [10], Diethelm 2010 [5]).

The differential equations of fractional order are generalizations of classical differential equations of integer order; they are increasingly used in such fields as fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, probability and statistics... etc. The boundary value problem of fractional differential equations is recently approached by various researchers ([1], [3], [9], [11], [15]).

In chapter 1 we introduce the fundamental concepts and definitions of fractional calculus (Gamma and Beta functions, Riemann-Liouville, Caputo, Hadamard, Katugampola, Caputo-Katugampola fractional integral and derivative).

Next, in chapter 2, we will discuss the existence and uniqueness of the following boundary value problem of the nonlinear fractional differential equation using Caputo-Katugampola's fractional derivative ([7, 8])

$$\begin{cases} {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u\left(t\right)+f\left(t,u\left(t\right)\right)=0 \quad t\in[0,T] \\ u\left(0\right)=0, u\left(T\right)=0, \end{cases}$$

where ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u$ is the left Caputo-Katugampola fractional derivative, $1<\alpha\leq 2, \rho\in\mathbb{R}^{+}$, and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

In the three chapter, we are interested in the existence and uniqueness of initial value problem for nonlinear implicit fractional differential equation following :

$$\begin{cases} {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u\left(t\right)=f\left(t,u\left(t\right),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u\left(t\right)\right), \quad t\in[0,T] \\ u\left(0\right)=0, \end{cases}$$

where ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u$ is the left Caputo-Katugampola fractional derivative, $0<\alpha\leq 1, \rho\in\mathbb{R}^{+}$, and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Finally, in the last chapter we will study the initial value problem of a coupled system of nonlinear fractional differential equations with Caputo-Katugampola derivative following

$$\begin{cases} {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t) = f\left(t, u(t), v(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right) \\ {}^{\rho}_{*}\mathcal{D}^{\beta}_{0^+}v(t) = g\left(t, u(t), v(t), {}^{\rho}_{*}\mathcal{D}^{\beta}_{0^+}v(t)\right) \\ u(0) = 0, v(0) = 0 \end{cases}, t \in [0, T],$$

where $0 < \alpha, \beta \leq 1, \rho > 0$, The symbol ${}^{\rho}_{*}\mathcal{D}^{\delta}_{0^+}, \delta = \alpha, \beta$, is the Caputo-Katugampola fractional derivative of fractional order δ, and $f, g : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

Some existence and uniqueness results of solutions for the given problems are obtained by using the Banach contraction principle, Schauder's, Schaufer fixed point theorem. Several examples are presented to illustrate the usefulness of our results.

CHAPTER 1

BASIC DEFINITIONS OF FRACTIONAL CALCULUS

In this chapter we introduce the fundamental concepts, fundamental definitions and results of fractional calculus (Gamma and Beta functions, Riemann-Liouville, Caputo, Hadamard, Katugampola, Caputo-Katugampola fractional integral and derivative).

1.1 Special Functions

Definition 1.1 (Gamma Euler function [10]). We call the function Gamma the function defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, (z \in \mathbb{C}, \mathrm{Re}(z) > 0).$$

with $t^{z-1} = e^{(z-1)\ln t}$

properties 1.1. For all $z \in \mathbb{C}, Re(z) > 0, n \in \mathbb{N}$, we have

$$1.\, \Gamma(z+1) = z\Gamma(z)$$

$$2.\, \Gamma(n) = (n-1)!$$

$$3.\, \Gamma(n + \frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n\, n!}.$$

Definition 1.2 (Beta function [10]). The function of Beta is a type of integral of Euler defined by

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt \ \ (p, q \in \mathbb{C}, Re(p) > 0, Re(q) > 0).$$

For all $p, q \in \mathbb{C}$, with $Re(p) > 0, Re(q) > 0$, we have

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

1.2 Fractional integral and derivative

Definition 1.3 (Riemann-Liouville fractional integral [10]). The fractional integral of Riemann-Liouville to the left of order $\alpha > 0$ is defined by

$$\forall t \in [a,b]\,; (\mathcal{I}^{\alpha}_{a^+}\varphi)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\varphi(s)ds}{(t-s)^{1-\alpha}}, \qquad t > a \tag{1.1}$$

In the same way we define the fractional integral of Riemann-Liouville to the right of order $\alpha > 0$ by

$$\forall t \in [a,b]\,; (\mathcal{I}^{\alpha}_{b^-}\varphi)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{\varphi(s)ds}{(s-t)^{1-\alpha}}, \qquad t < b \tag{1.2}$$

Definition 1.4 (Riemann-Liouville fractional derivative [10]).
Let $\alpha > 0$, and $n = [\alpha] + 1$. The fractional derivative of Riemann-Liouville the left of order α is defined by

$$\begin{aligned}\forall t \in [a,b]\,, ({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)\,(t) &= \frac{1}{\Gamma\,(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{\varphi\,(s)\,ds}{(t-s)^{\alpha-n+1}}, \qquad t > a \\ &= \left(D^n \mathcal{I}^{n-\alpha}_{a^+}\varphi\right)(t)\,.\end{aligned} \tag{1.3}$$

In the same way we define the fractional derivative of Riemann-Liouville to the right of order $\alpha > 0$ by:

$$\begin{aligned}\forall t \in [a,b]\,, ({}_{RL}\mathcal{D}^{\alpha}_{b^-}\varphi)\,(t) &= \frac{1}{\Gamma\,(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b \frac{\varphi\,(s)\,ds}{(s-t)^{\alpha-n+1}}, \qquad t < b \\ &= (-1)^n \left(D^n \mathcal{I}^{n-\alpha}_{a^+}\varphi\right)(t)\end{aligned} \tag{1.4}$$

Definition 1.5 (Caputo fractional derivative [10]).
Let $\alpha > 0$, $D = \dfrac{d}{dt}$, and $n = [\alpha] + 1$. The fractional derivative of Caputo the left of order α is defined by

$$\begin{aligned}\forall t \in [a,b]\,, ({}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)\,(t) &= \frac{1}{\Gamma\,(n-\alpha)}\int_a^t \frac{\varphi^{(n)}\,(s)\,ds}{(t-s)^{\alpha-n+1}}, \qquad t > a \\ &= \left(\mathcal{I}^{n-\alpha}_{a^+} D^n \varphi\right)(t)\,.\end{aligned} \tag{1.5}$$

In the same way we define the fractional derivative of Caputo to the right of order $\alpha > 0$ by

$$\begin{aligned}\forall t \in [a,b]\,, ({}_{*}\mathcal{D}^{\alpha}_{b^-}\varphi)\,(t) &= \frac{(-1)^n}{\Gamma\,(n-\alpha)}\int_t^b \frac{\varphi^{(n)}\,(s)\,ds}{(s-t)^{\alpha-n+1}}, \qquad t < b \\ &= (-1)^n \left(\mathcal{I}^{n-\alpha}_{b^-} D^n \varphi\right)(t)\,.\end{aligned} \tag{1.6}$$

Definition 1.6 (Hadamard fractional integral [12]). The left and right-sided fractional integral of order $\alpha > 0$ of a function $\varphi : [a, b] \to \mathbb{R}$ is given by

$$\left(\mathcal{J}_{a+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\varphi(s)\frac{ds}{s}, \quad t > a \tag{1.7}$$

and

$$\left(\mathcal{J}_{b-}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\ln\frac{s}{t}\right)^{\alpha-1}\varphi(s)\frac{ds}{s}, \quad t < b \tag{1.8}$$

Definition 1.7 (Hadamard fractional derivative [12]). The Hadamard left and right-sided fractional derivatives are defined as

$$\begin{aligned}\left(\mathcal{D}_{a+}^{\alpha}\varphi\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\varphi(s)\frac{ds}{s}, \quad t > a \\ &= \delta^n\left(\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(t),\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\left(\mathcal{D}_{b-}^{\alpha}\varphi\right)(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_t^b \left(\ln\frac{s}{t}\right)^{n-\alpha-1}\varphi(s)\frac{ds}{s}, \quad t < b \\ &= (-\delta)^n\left(\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(t)\end{aligned} \tag{1.10}$$

where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α.

Definition 1.8 (Cputo-Hadamard fractional derivatives [12]). The left-sided and right-sided fractional derivatives in the Cputo-Hadamard sense are respectively given by

$$\begin{aligned}\left({}^C\mathcal{D}_{a+}^{\alpha}\varphi\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^n\varphi(s)\frac{ds}{s}, \quad t < a \\ &= \left(\mathcal{J}_{a^+}^{n-\alpha}\delta^n\varphi\right)(t),\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}\left({}^C\mathcal{D}_{b-}^{\alpha}\varphi\right)(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \left(\ln\frac{s}{t}\right)^{n-\alpha-1}\left(t\frac{d}{dt}\right)^n\varphi(s)\frac{ds}{s}, \quad t < b \\ &= (-1)^n\left(\mathcal{J}_{b^-}^{n-\alpha}\delta^n\varphi\right)(t).\end{aligned} \tag{1.12}$$

1.3 Fundamental theorem and property of fractional calculus

In this section we present the fundamental theorem and property of fractional calculus associated with the generalized fractional integral and the Caputo-type differential operator

Definition 1.9. (See [12]). The space $X_c^p(a, b)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ consists of those complex-valued Lebesgue measurable function on (a, b), for which $\|\varphi\|_{X_c^p} < \infty$ with

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |t^c\varphi(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}} \ (1 \leq p < \infty) \tag{1.13}$$

and

$$\|\varphi\|_{X_c^\infty} = \sup_{t\in(a,b)} \left[t^c \left|\varphi\left(t\right)\right|\right]. \tag{1.14}$$

Definition 1.10 (Katugampola fractional integral [12]**).**
Let $\alpha, \rho, c \in \mathbb{R}$ with $\alpha \notin \mathbb{N}$ and. The generalized fractional integrals $\left({}^\rho\mathcal{J}_{a^+}^\alpha\varphi\right)(t)$ (left-sided) and $\left({}^\rho\mathcal{J}_{b^-}^\alpha\varphi\right)(t)$ (left-sided), with $\varphi \in X_c^p\left(a,b\right)$, are defined by:

$$\left({}^\rho\mathcal{J}_{a^+}^\alpha\varphi\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_a^t \frac{s^{\rho-1}\varphi\left(s\right)}{\left(t^\rho - s^\rho\right)^{1-\alpha}}ds, \quad t > a \tag{1.15}$$

and

$$\left({}^\rho\mathcal{J}_{b^-}^\alpha\varphi\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_t^b \frac{s^{\rho-1}\varphi\left(s\right)}{\left(s^\rho - t^\rho\right)^{1-\alpha}}ds, \quad t < b \tag{1.16}$$

with $\rho > 0$.

Theorem 1.1. *(See [7]). Let $\alpha \in \mathbb{C}, Re\left(\alpha\right) \geq 0, n = \left[Re\left(\alpha\right)\right]$ and $\rho > 0$. Then, for $t > a$,*

$$1. \lim_{\rho\to 1}\left({}^\rho\mathcal{J}_{a^+}^\alpha\varphi\right)(t) = \frac{1}{\Gamma\left(\alpha\right)}\int_a^t \frac{\varphi\left(s\right)}{\left(t-s\right)^{1-\alpha}}ds \tag{1.17}$$

$$2. \lim_{\rho\to 0^+}\left({}^\rho\mathcal{J}_{a^+}^\alpha\varphi\right)(t) = \frac{1}{\Gamma\left(\alpha\right)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\varphi\left(s\right)\frac{d}{ds} \tag{1.18}$$

Definition 1.11 (Katugampola fractional derivative [12]**).**
Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha \notin \mathbb{N}, \alpha, \rho > 0$ and $n = [\alpha]+1$. The generalized left-sided and right-sided fractional derivatives, $\left({}^\rho\mathcal{D}_{a^+}^\alpha\varphi\right)(t)$ and $\left({}^\rho\mathcal{D}_{b^-}^\alpha\varphi\right)(t)$, are defined by

$$\begin{aligned}\left({}^\rho\mathcal{D}_{a^+}^\alpha\varphi\right)(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma\left(n-\alpha\right)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}\varphi\left(s\right)}{\left(t^\rho - s^\rho\right)^{1-n+\alpha}}ds, \quad t > a \\ &= \delta_\rho^n\left({}^\rho\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(t)\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}\left({}^\rho\mathcal{D}_{b^-}^\alpha\varphi\right)(t) &= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma\left(n-\alpha\right)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_t^b \frac{s^{\rho-1}\varphi\left(s\right)}{\left(s^\rho - t^\rho\right)^{1-n+\alpha}}ds, \quad t < b \\ &= (-1)^n\,\delta_\rho^n\left({}^\rho\mathcal{J}_{b^-}^{n-\alpha}\varphi\right)(t).\end{aligned} \tag{1.20}$$

If the integrals exist.

Theorem 1.2. *(See [7]). Let $\alpha \in \mathbb{C}, Re\left(\alpha\right) \geq 0, n = \left[Re\left(\alpha\right)\right]$ and $\rho > 0$. Then, for $t > a$,*

$$1. \lim_{\rho\to 1}\left({}^\rho\mathcal{D}_{a^+}^\alpha\varphi\right)(t) = \frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{\varphi\left(s\right)ds}{\left(t-s\right)^{\alpha-n+1}} \tag{1.21}$$

$$2. \lim_{\rho\to 0^+}\left({}^\rho\mathcal{D}_{a^+}^\alpha\varphi\right)(t) = \frac{1}{\Gamma\left(n-\alpha\right)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\varphi\left(s\right)\frac{d}{ds} \tag{1.22}$$

Definition 1.12 (Caputo-Katugampola fractional derivative [12]).
Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha \notin \mathbb{N}, \alpha, \rho > 0$ and $n = [\alpha] + 1$. The left-sided and right-sided Caputo-Katugampola fractional derivative are defined, for $0 \leq a < t < b \leq \infty$, by

$$\begin{aligned}\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} \left(s^{1-\rho}\frac{d}{ds}\right)^n \varphi(s)\, ds, \\ &= \left({}^{\rho}\mathcal{J}_{a^+}^{n-\alpha}\delta_\rho^n\varphi\right)(t),\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}\left({}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}\varphi\right)(t) &= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{1-n+\alpha}} \left(s^{1-\rho}\frac{d}{ds}\right)^n \varphi(s)\, ds, \\ &= (-1)^n \left({}^{\rho}\mathcal{J}_{b^-}^{n-\alpha}\delta_\rho^n\varphi\right)(t).\end{aligned} \tag{1.24}$$

Respectively, if the integrals exist.

Theorem 1.3. *(See [12]) Let $\alpha, \rho \in \mathbb{R}, \alpha \notin \mathbb{N}, \alpha, \rho > 0$ and $n = [\alpha] + 1$. Then, for $t > a$*

$$(a)\ \lim_{\rho\to 1}\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) = \left({}_{*}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\varphi^{(n)}(s)\, ds,$$

$$(b)\ \lim_{\rho\to 0^+}\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) = \left({}^{C}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1}\delta^n\varphi(s)\, ds.$$

The linearity of the differential operators ${}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}$ and ${}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}$ is ensured by the following theorem

Theorem 1.4. *(See [12]) Let $\alpha, \rho \in \mathbb{R}, \alpha, \rho > 0$ such that $\alpha \notin \mathbb{N}$. If $0 < a < b < \infty$, then*

$$\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}(\varphi + g)\right)(t) = \left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\varphi\right)(t) + \left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}g\right)(t).$$

The composition of the fractional integral operators ${}^{\rho}\mathcal{J}_{a^+}^{\alpha}$ and ${}^{\rho}\mathcal{J}_{b^-}^{\alpha}$ with the fractional differential operators ${}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}$ and ${}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}$, is given by the following result.

Theorem 1.5. *(See [12]) Let $\alpha, \rho \in \mathbb{R}, \alpha > 0$ and $\rho > 0$. If $0 < a < b < \infty$, then*

$$\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\,{}^{\rho}\mathcal{J}_{a^+}^{\alpha}\varphi\right)(t) = \varphi(t) \qquad \text{and} \qquad \left({}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}\,{}^{\rho}\mathcal{J}_{b^-}^{\alpha}\varphi\right)(t) = \varphi(t). \tag{1.25}$$

The next theorem yields the compositions of the fractional integral operators, ${}^{\rho}\mathcal{J}_{a^+}^{\beta}$ and ${}^{\rho}\mathcal{J}_{b^-}^{\beta}$ with the fractional differential operators, ${}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}$ and ${}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}$

Theorem 1.6. *(See [12]) Let $\alpha, \beta, \rho \in \mathbb{R}$ such that $\alpha > 0$, $\beta > \alpha$ and $\rho > 0$. If $0 < a < b < \infty$, then*

$$\left({}_{*}^{\rho}\mathcal{D}_{a^+}^{\alpha}\,{}^{\rho}\mathcal{J}_{a^+}^{\beta}\varphi\right)(t) = \left({}^{\rho}\mathcal{J}_{a^+}^{\beta-\alpha}\varphi\right)(t) \qquad \text{and} \qquad \left({}_{*}^{\rho}\mathcal{D}_{b^-}^{\alpha}\,{}^{\rho}\mathcal{J}_{b^-}^{\beta}\varphi\right)(t) = \left({}^{\rho}\mathcal{J}_{b^-}^{\beta-\alpha}\varphi\right)(t) \tag{1.26}$$

Exemple 1.1. Let $\alpha, \beta, \rho \in \mathbb{R}, \alpha, \rho > 0, (\beta - \alpha\rho) > 0$ and $\varphi(s) = s^{\beta}$. Taking the limit $a \to 0$, we get

$$\left({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+} s^{\beta}\right)(t) = \begin{cases} \dfrac{\Gamma\left(\dfrac{\beta}{\rho}+1\right)}{\Gamma\left(\dfrac{\beta}{\rho}-\alpha+1\right)}\rho^{\alpha} t^{\beta-\alpha\rho}, & \alpha > 0, \left(\alpha - \dfrac{\beta}{\rho}\right) \notin \mathbb{N} \\ \\ 0, & \alpha > 0, \left(\alpha - \dfrac{\beta}{\rho}\right) \in \mathbb{N}. \end{cases} \tag{1.27}$$

Proof. We consider Eq (1.23) with $a \longrightarrow 0$ and $\varphi(s) = s^{\beta}$. Hence, we can write

$$\left({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+} s^{\beta}\right)(t) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \lim_{a \to 0} \int_a^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-n+\alpha}} \left(s^{1-\rho}\frac{d}{ds}\right)^n s^{\beta} ds, \quad t > a, \tag{1.28}$$

with $n = [\alpha] + 1$ and $n = \{1, 2, ...\}$. Since

$$\begin{aligned}\left(s^{1-\rho}\frac{d}{ds}\right)^n s^{\beta} &= \beta(\beta-\rho)(\beta-2\rho)...(\beta-(n-1)\rho)s^{\beta-n\rho} \\ &= \rho^n \Gamma\left(\frac{\beta}{\rho}+1\right) = \frac{\beta}{\rho}\left(\frac{\beta}{\rho}-1\right)\left(\frac{\beta}{\rho}-2\right)...\left(\frac{\beta}{\rho}-n+1\right)\Gamma\left(\frac{\beta}{\rho}-n+1\right)s^{\beta-n\rho} \\ &= \rho^n \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)} s^{\beta-n\rho},\end{aligned}$$

we can substitute this expression into Eq (1.28) to obtain

$$\left({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+} s^{\beta}\right)(t) = \frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)} \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)} \int_0^t \frac{s^{\beta+(1-n)\rho-1}}{(t^{\rho}-s^{\rho})^{1-n+\alpha}} ds.$$

Further, with the change of variable $u = \dfrac{s^{\rho}}{t^{\rho}}$ we have

$$\begin{aligned}\left({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+} s^{\beta}\right)(t) &= \frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)} \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)} \int_0^1 \frac{(t^{\rho}u)^{\frac{\beta}{\rho}-n-\frac{1}{\rho}+1} u^{\frac{1}{\rho}-1}}{t^{\rho(1-n+\alpha)}(1-u)^{1-n+\alpha}} \rho^{-1} t du, \\ &= \frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)} \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)} \rho^{-1} t^{\beta-\alpha\rho} \underbrace{\int_0^1 u^{\frac{\beta}{\rho}-n+1-1}(1-u)^{n-\alpha-1} du,}_{B\left(\frac{\beta}{\rho}-n+1, n-\alpha\right)} \\ &= \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)\Gamma\left(\frac{\beta}{\rho}-n+1\right)\Gamma(n-\alpha)}{\Gamma(n-\alpha)\Gamma\left(\frac{\beta}{\rho}-n+1\right)\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)} \rho^{\alpha} t^{\beta-\alpha\rho}, \quad |B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.\end{aligned}$$

It follows that

$$\left({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}s^{\beta}\right)(t)=\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)}\rho^{\alpha}t^{\beta-\alpha\rho}.$$

□

Notice that, taking the limit $\rho\to 1$, Eq (1.30) becomes

$$\lim_{\rho\to 1}\left({}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}s^{\beta}\right)(t)=\left({}_{*}\mathcal{D}_{0+}^{\alpha}s^{\beta}\right)(t)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

We now present the semi group property of the Caputo-type fractional derivative ${}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}$. The result is also valid, for the operator ${}_{*}^{\rho}\mathcal{D}_{b-}^{\alpha}$.

Theorem 1.7. *(See [12]) Let $\alpha,\beta,\rho\in\mathbb{R}$ such that $\alpha,\beta>0$ and $\rho>0$. If $0<a<b<\infty$, then we have*

$$\left({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}\,{}_{*}^{\rho}\mathcal{D}_{a+}^{\beta}\varphi\right)(t)=\left({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha+\beta}\varphi\right)(t). \tag{1.29}$$

Theorem 1.8. *(See [12]) Let $\alpha,\rho\in\mathbb{R}$ such that $\alpha>0,\rho>0$ and $n=[\alpha]+1$. The relation between the generalized fractional derivatives and the Caputo-type fractional derivatives is given by the expressions*

$$\left({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}\varphi\right)(t)=\left({}^{\rho}\mathcal{D}_{a+}^{\alpha}\varphi\right)(t)-\sum_{k=0}^{n-1}\frac{\delta_{\rho}^{k}\varphi(a)}{\Gamma(k-\alpha+1)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{k-\alpha}, \tag{1.30}$$

and

$$\left({}_{*}^{\rho}\mathcal{D}_{b-}^{\alpha}\varphi\right)(t)=\left({}^{\rho}\mathcal{D}_{b-}^{\alpha}\varphi\right)(t)-\sum_{k=0}^{n-1}\frac{(-1)^{k}\delta_{\rho}^{k}\varphi(a)}{\Gamma(k-\alpha+1)}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{k-\alpha}. \tag{1.31}$$

In particular, when $0<\alpha<1$, Eq. (1.30) and Eq. (1.31) take the following form:

$$\left({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}\varphi\right)(t)=\left({}^{\rho}\mathcal{D}_{a+}^{\alpha}\varphi\right)(t)-\frac{\varphi(a)}{\Gamma(1-\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{-\alpha},$$
$$\left({}_{*}^{\rho}\mathcal{D}_{b-}^{\alpha}\varphi\right)(t)=\left({}^{\rho}\mathcal{D}_{b-}^{\alpha}\varphi\right)(t)-\frac{\varphi(b)}{\Gamma(1-\alpha)}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{-\alpha}.$$

Theorem 1.9. *(See [12]) Let $\alpha,\rho\in\mathbb{R}$ such that $\alpha>0$ and $\rho>0$ with $n=[\alpha]+1$. Consider $\varphi\in AC_{\delta}^{n}[a,b]$ with*

$$AC_{\delta}^{n}[a,b]=\left\{\varphi:[a,b]\to\mathbb{R}:\delta_{\rho}^{n-1}\varphi(t)\in AC[a,b],\delta=t\frac{d}{dt}\right\}.$$

(a) *If $\alpha\notin\mathbb{N}$ or $\alpha\in\mathbb{N}$ and $\Phi(t)=\left({}^{\rho}\mathcal{J}_{a+}^{\alpha}\varphi\right)(t)$ or $\Phi(t)=\left({}^{\rho}\mathcal{J}_{b-}^{\alpha}\varphi\right)(t)$, $\forall t\in[a,b]$, we obtain*

$$\left({}_{*}^{\rho}\mathcal{D}_{a+}^{\alpha}\Phi\right)(t)=\varphi(t)\quad \textit{and}\quad \left({}_{*}^{\rho}\mathcal{D}_{b-}^{\alpha}\Phi\right)(t)=\varphi(t). \tag{1.32}$$

(b) *If* $\left({}^{\rho}\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(t) \in AC_{\delta}^{n}[a,b]$ *, then*

$$\left({}^{\rho}\mathcal{J}_{a^+}^{\alpha}\,{}^{\rho}_{*}\mathcal{D}_{a^+}^{\alpha}\Phi\right)(t) = \Phi(t) - \sum_{k=0}^{[\alpha]} \frac{\delta_{\rho}^{k}\Phi(a)}{k!}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{k}, \tag{1.33}$$

and

$$\left({}^{\rho}\mathcal{J}_{b^-}^{\alpha}\,{}^{\rho}_{*}\mathcal{D}_{b^-}^{\alpha}\Phi\right)(t) = \Phi(t) - \sum_{k=0}^{[\alpha]} \frac{(-1)^{k}\delta_{\rho}^{k}\Phi(b)}{k!}\left(\frac{b^{\rho}-t^{\rho}}{\rho}\right)^{k}. \tag{1.34}$$

For $0 < \alpha < 1$*, we have*

$$\left({}_{a}^{\rho}\mathcal{J}_{b}^{\alpha}\,{}^{\rho}_{*}\mathcal{D}_{a^+}^{\alpha}\Phi\right)(t) = \Phi(t) - \Phi(a) \quad \text{and} \quad \left({}_{a}^{\rho}\mathcal{J}_{b}^{\alpha}\,{}^{\rho}_{*}\mathcal{D}_{b^-}^{\alpha}\Phi\right)(t) = \Phi(t) - \Phi(a).$$

1.4 Some fixed point theorems

The following fixed point theorems (Schauder, Shuafer, Banach and Ascoli-Arzela) are fundamental in the proofs of our results.

Definition 1.13. (Equicontinuous)
Let E be a Banach space. Call a part P in $C(E)$ equicontinuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall u, v \in E, \forall F \in P, \;\; \|u - v\| < \delta \Rightarrow \|F(u) - F(v)\| < \varepsilon.$$

Theorem 1.10. *(**Ascoli-Arzela.** See [4])*
Let E be a compact space. If F is an equicontinuous, bounded subset of $C(E)$, then F is relatively compact.

Definition 1.14. (Completely continuous)
We say $F : E \to E$ is completely continuous if for any bounded subset P of E, the set $F(P)$ is relatively compact.

Theorem 1.11. *(**Banach's fixed point.** See [4])*
Let P be a non-empty closed subset of a Banach space E, then any contraction mapping $F : P \to P$ has a unique fixed point.

Theorem 1.12. *(**Schauder's fixed point.** See [4])*
Let E be a Banach space, and P be a closed, convex and nonempty subset of E. Let $F : P \to P$ be a continuous mapping such that $F(P) \subset E$ is a relatively compact. Then F has at least one fixed point in P.

Theorem 1.13. ***(Schaefer's fixed point theorem.*** *See [6])*
Let E be a Banach space, and $F : E \to E$ a completely continuous operator. If the set

$$S = \{u \in E : u = \mu F u, \text{ for some } \mu \in (0,1)\},$$

is bounded, the F has fixed points.

CHAPTER 2

BOUNDARY VALUE PROBLEM OF CAPUTO-KAUGAMPOLA FDE

In this chapter we will discuss the existence and uniqueness of the boundary value problem of the nonlinear fractional differential equation using Caputo-Katugampola's fractional derivative following :

$$ {}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0 \quad t \in [0, T], \tag{2.1} $$

supplemented with the boundary conditions:

$$ u(0) = 0, u(T) = 0. \tag{2.2} $$

Where ${}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u$ is the left Caputo-Katugampola fractional derivative, $1 < \alpha \leq 2, \rho \in \mathbb{R}^+$, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Consider the space $X_c^p(0, T)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable function on $(0, T)$, for which $\|\varphi\|_{X_c^p} < \infty$, where the norm is defined by:

$$ \|\varphi\|_{X_c^p} = \left(\int_0^T |t^c \varphi(t)|^p \frac{ds}{s} \right)^{\frac{1}{p}} (1 \leq p < \infty), $$

for $1 \leq p < \infty, c \in \mathbb{R}$. For the case $p = \infty$;

$$ \|\varphi\|_{X_c^\infty} = \sup_{t \in (0,T)} [t^c |\varphi(t)|] \, (c \in \mathbb{R}). $$

By $C[0, T]$ we denote the Banach space of all continuous function from $[0, T]$ into $\mathbb{R}$ with the norm:

$$ \|\varphi\|_\infty = \sup_{0 \leq t \leq T} |\varphi(t)|. $$

Remark 2.1. Let $p, c, T \in \mathbb{R}_{+}^{*}$, be such that $p \geq 1, c > 0$ and $T \leq (pc)^{\frac{1}{pc}}$. It's clear that $\forall \varphi \in C[0,T]$

$$\|\varphi\|_{X_c^p} = \left(\int_0^T |s^c \varphi(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} \leq \left(\|\varphi\|_\infty^p \int_0^T s^{pc-1} ds\right)^{\frac{1}{p}} = \frac{T^c}{(pc)^{\frac{1}{p}}} \|\varphi\|_\infty,$$

and

$$\|\varphi\|_{X_c^\infty} = ess \sup_{0 \leq t \leq T} [t^c |\varphi(t)|] \leq T^c \|\varphi\|_\infty.$$

Which implies that $C[0,T] \hookrightarrow X_c^p[0,T]$, and

$$\|\varphi\|_{X_c^\infty} \leq \|\varphi\|_\infty, \text{ for all } T \leq (pc)^{\frac{1}{pc}}.$$

2.1 Fundamental lemmas

Lemma 2.1 (See [12]). *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha > 0$ and $\rho > 0$ with $n = [\alpha] + 1$. Consider $\varphi \in AC_\delta^n[a,b]$ with*

$$AC_\delta^n[a,b] = \left\{\varphi : [a,b] \to \mathbb{R} : \delta_\rho^{n-1}\varphi(t) \in AC[a,b], \delta = t\frac{d}{dt}\right\}.$$

If $\left({}^\rho\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(t) \in AC_\delta^n[a,b]$, then

$$\left({}^\rho\mathcal{J}_{0^+}^{\alpha}\, {}_*^\rho\mathcal{D}_{0^+}^{\alpha}\Phi\right)(t) = \Phi(t) - \sum_{k=0}^{[\alpha]} \frac{\delta_\rho^k \Phi(0)}{k!}\left(\frac{t^\rho}{\rho}\right)^k \tag{2.3}$$

Lemma 2.2. *Let $\alpha, \rho > 0, n = [\alpha] + 1$. If $u \in C[0,T]$, then*

(a) *The fractional deferential equation ${}_*^\rho\mathcal{D}_{0^+}^{\alpha} u(t) = 0$, has a unique solution:*

$$u(t) = C_0 + C_1 t^\rho + C_2 t^{2\rho} + ... + C_{n-1} t^{(n-1)\rho},$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2...n-1$.

(b) *If ${}_*^\rho\mathcal{D}_{0^+}^{\alpha} u(t) \in C[0,T]$ and $1 < \alpha \leq 2$, then:*

$${}^\rho\mathcal{J}_{0^+}^{\alpha}\left({}_*^\rho\mathcal{D}_{0^+}^{\alpha} u(t)\right) = u(t) + C_0 + C_1 t^\rho, \tag{2.4}$$

for some constant $C_0, C_1 \in \mathbb{R}$.

Lemma 2.3. *Let $\alpha, \rho \in \mathbb{R}$, be such that $1 < \alpha \leq 2$ and $\rho > 0$. We give ${}_*^\rho\mathcal{D}_{0^+}^{\alpha} u \in C[0,T]$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Then the boundary value problem (2.1)–(2.2), is equivalent to the fractional integral equation*

$$u(t) = \int_0^T G(t,s) f(s, u(s))\, ds, t \in [0,T]$$

where

$$G(t,s)=\begin{cases}\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})^{\alpha-1}-(t^{\rho}-s^{\rho})^{\alpha-1}\right], & ,0\leq s\leq t\leq T,\\ \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})^{\alpha-1}\right], & 0\leq t\leq s\leq T,\end{cases} \tag{2.5}$$

is the Green's function associated with the boundary value problem (2.1)–(2.2).

Proof. Let $\alpha,\rho\in\mathbb{R}^{+}$, be such that $1<\alpha\leq 2$. We apply lemma 2.2 to reduce the fractional equation (2.1) to an equivalent fractional integral equation.It is easy to prove the operator ${}^{\rho}\mathcal{J}_{0+}^{\alpha}$ has the linearity property for all $\alpha>0$ after direct integration. Then by applying ${}^{\rho}\mathcal{J}_{0+}^{\alpha}$ to equation (2.1), we get

$${}^{\rho}\mathcal{J}_{0+}^{\alpha}\,{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)+{}^{\rho}\mathcal{J}_{0+}^{\alpha}f(t,u(t))=0$$

From lemma 2.2, we find for $1<\alpha\leq 2$,

$${}^{\rho}\mathcal{J}_{0+}^{\alpha}\,{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)=u(t)+C_0+C_1t^{\rho},$$

for some $C_1,C_2\in\mathbb{R}$. Then, the integral solution of the equation (2.1)is:

$$u(t)=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{s^{\rho-1}f(s,u(s))}{(t^{\rho}-s^{\rho})^{1-\alpha}}ds-C_0-C_1t^{\rho}. \tag{2.6}$$

The condition (2.2) imply that:

$$\begin{cases}u(0)=0=0-C_0-0 & \Rightarrow C_0=0,\\ u(T)=0=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^T\frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds-C_1T^{\rho} & \Rightarrow C_1=-\frac{\rho^{1-\alpha}}{T^{\rho}\Gamma(\alpha)}\int_0^T\frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds.\end{cases}$$

The integral equation (2.6) is equivalent to:

$$u(t)=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\frac{s^{\rho-1}f(s,u(s))}{(t^{\rho}-s^{\rho})^{1-\alpha}}ds+\frac{t^{\rho}\rho^{1-\alpha}}{T^{\rho}\Gamma(\alpha)}\int_0^T\frac{s^{\rho-1}f(s,u(s))}{(T^{\rho}-s^{\rho})^{1-\alpha}}ds.$$

Therefore, the unique solution of problem (2.1)–(2.2) is:

$$\begin{aligned}u(t)&=\int_0^t\frac{\rho^{1-\alpha}s^{\rho-1}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})^{\alpha-1}-(t^{\rho}-s^{\rho})^{\alpha-1}\right]}{\Gamma(\alpha)}f(s,u(s))\,ds\\ &\quad+\int_t^T\frac{\rho^{1-\alpha}s^{\rho-1}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho}-s^{\rho})^{\alpha-1}\right]}{\Gamma(\alpha)}f(s,u(s))\,ds\\ &=\int_0^T G(t,s)f(s,u(s))\,ds.\end{aligned}$$

□

2.2 Existence and uniqueness results

Now, we will prove the first existence result for the problem (2.1)–(2.2) which is based on Banach's fixed point theorem 1.11.

We impose the following hypotheses:

(H1) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

(H2) For all $1 < \alpha \leq 2$, there exist a constant $\lambda > 0$ such that:

$$|f(t,u) - f(t,v)| \leq \lambda |u - v| \tag{2.7}$$

for any $u, v \in \mathbb{R}$ and $t \in [0,T]$.

(H3) there exist a constant $L > 0$ such that:

$$|f(t,u)| \leq L,$$

for all $t \in [0,T]$ and $u \in \mathbb{R}$.

Theorem 2.1. *Assume the hypotheses* **(H1)-(H2)** *hold. We give* $1 < \alpha \leq 2$*, and* $\rho > 0$*. If*

$$\frac{\lambda T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} < 1. \tag{2.8}$$

Then the problem (2.1)–(2.2) admits a unique solution on $[0,T]$*.*

Proof. To begin the proof, we will transform the problem (2.1)–(2.2) into a fixed point problem. Define the operator $F : C[0,T] \to C[0,T]$ by

$$Fu(t) = \int_0^T G(t,s) f(s, u(s))\, ds. \tag{2.9}$$

Because the problem (2.1)–(2.2) is equivalent to the fractional integral equation (2.10), the fixed points of F are solutions of the problem (2.1)–(2.2).

Let $u, v \in C[0,T]$, Then for all $t \in [0,T]$:

$$\begin{aligned} |Fu(t) - Fv(t)| &= \left| \int_0^T G(t,s) \left[f(s,u(s)) - f(s,v(s)) \right] ds. \right| \\ &\leq \int_0^T G(t,s) \left| f(s,u(s)) - f(s,v(s)) \right| ds \\ &\leq \int_0^T G(t,s) \left| u(s) - v(s) \right| ds, \end{aligned}$$

then

$$\begin{aligned}
\|Fu - Fv\|_\infty &\le \lambda \|u-v\|_\infty \int_0^T G\left(t,s\right) ds \\
&\le \lambda \|u-v\|_\infty \left[\int_0^t \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma\left(\alpha\right)} \left[\frac{t^\rho}{T^\rho} (T^\rho - s^\rho)^{\alpha-1} - (t^\rho - s^\rho)^{\alpha-1} \right] ds \right. \\
&\left. + \int_t^T \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma\left(\alpha\right)} \left[\frac{t^\rho}{T^\rho} (T^\rho - s^\rho)^{\alpha-1} \right] ds \right] \\
&\le \frac{\lambda \|u-v\|_\infty}{\rho^\alpha \Gamma\left(\alpha+1\right)} \left[\left[-\frac{t^\rho}{T^\rho} (T^\rho - s^\rho)^\alpha + (t^\rho - s^\rho)^\alpha \right]_0^t - \left[\frac{t^\rho}{T^\rho} (T^\rho - s^\rho)^\alpha \right]_t^T \right] \\
&\le \frac{\lambda \|u-v\|_\infty}{\rho^\alpha \Gamma\left(\alpha+1\right)} \left[\left[-\frac{t^\rho}{T^\rho} (T^\rho - t^\rho)^\alpha + \frac{t^\rho}{T^\rho} T^{\alpha\rho} - t^{\alpha\rho} \right] + \left[\frac{t^\rho}{T^\rho} (T^\rho - t^\rho)^\alpha \right] \right] \\
&\le \frac{\lambda \|u-v\|_\infty}{\rho^\alpha \Gamma\left(\alpha+1\right)} \left[t^\rho T^{\rho(\alpha-1)} - t^{\alpha\rho} \right] \\
&\le \frac{\lambda T^{\alpha\rho}}{\rho^\alpha \Gamma\left(\alpha+1\right)} \|u-v\|_\infty .
\end{aligned}$$

This implies that by (2.8), F is a contraction operator.

As a consequence of theorem 1.11, using Banach's contraction principle, we deduce that F has a unique fixed point which is the unique solution of the problem (2.1)–(2.2) on $[0,T]$. □

Theorem 2.2. *Assume that hypotheses **(H1)-(H3)** hold. We give $1 < \alpha \le 2$, and $\rho > 0$. If we put*

$$\frac{LT^{\alpha\rho}}{\rho^\alpha \Gamma\left(\alpha+1\right)} < 1,$$

then the problem (2.1)–(2.2) has at least one solution on $C\left[0,T\right]$.

Proof. In the previous theorem, we already transform the problem (2.1)–(2.2) into a fixed point problem

$$Fu\left(t\right) = \int_0^T G\left(t,s\right) f\left(s,u\left(s\right)\right) ds. \tag{2.10}$$

We demonstrate that F satisfies the assumption of Schauder's fixed point theorem 1.12.

This could be proved through three steps:

Step 1. F is a continuous operator.

Let $(u_n)_{n\in\mathbb{N}}$ be real sequence such that $\lim\limits_{n\to\infty} u_n = u$ in $C\left[0,T\right]$. Then for each $t \in C\left[0,T\right]$.

$$\begin{aligned}
\left|Fu_n\left(t\right) - Fu\left(t\right)\right| &\le \int_0^T G\left(t,s\right) \left|f\left(s,u_n\left(s\right)\right) - f\left(s,u\left(s\right)\right)\right| ds \\
&\le \left\|f\left(s,u_n\left(s\right)\right) - f\left(s,u\left(s\right)\right)\right\|_\infty \int_0^T G\left(t,s\right) ds \\
&\le \frac{T^{\alpha\rho}}{\rho^\alpha \Gamma\left(\alpha+1\right)} \left\|f\left(s,u_n\left(s\right)\right) - f\left(s,u\left(s\right)\right)\right\|_\infty .
\end{aligned}$$

For each $t \in C[0,T]$,the function $s \to G(t,s)$ is integrable on $[0,T]$,then the lebesgue dominated convergence theorem imply that:

$$|Fu_n(t) - Fu(t)| \to 0, \ \ as \ \ n \to \infty,$$

and hence:

$$\lim_{n\to\infty} \|Fu_n - Fu\|_\infty = 0.$$

Consequently, F is continuous.

Step 2. Let

$$r \geq \frac{LT^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)}.$$

We define:

$$P_r = \{u \in C[0,T] : \|u\|_\infty \leq r\}.$$

It is clear that P_r is a bounded, closed and convex subset of $C[0,T]$.
Let $u \in P_r$, and $F : P_r \to C[0,T]$ be the integral operator defined in (2.10), then $F(P_r) \subset P_r$.

Then:

$$\begin{aligned}
|Fu(t)| &\leq \left| \int_0^T G(t,s) f(s,u(s)\,ds \right| \\
&\leq \int_0^T G(t,s) |f(s,u(s))|\,ds \\
&\leq L \int_0^T G(t,s)ds \\
&\leq \frac{LT^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} \\
&\leq r.
\end{aligned}$$

Then $F(P_r) \subset P_r$.

Step 3. $F(P_r)$ relatively compact.
Let $t_1, t_2 \in [0,T], t_1 < t_2$, and $u \in P_r$. Then

$$\begin{aligned}
|Fu(t_2) - Fu(t_1)| &= \left| \int_0^{t_1} G(t_2,s) f(s,u(s))\,ds - \int_0^{t_1} G(t_1,s) f(s,u(s))\,ds \right| \\
&\leq \int_0^{t_1} |[G(t_2,s) - G(t_1,s)]\, f(s,u(s))|\,ds + \int_{t_1}^{t_2} G(t_2,s)\,|f(s,u(s))|\,ds \\
&\leq L\left[\int_0^{t_1} |(G(t_2,s) - G(t_1,s))|\,ds + \int_{t_1}^{t_2} G(t_2,s)ds \right].
\end{aligned}$$

For $0 \leq s \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned}
G(t_2, s) - G(t_1, s) &= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[{t_2}^\rho \left(\frac{(T^\rho - s^\rho)^{\alpha-1}}{T^\rho} \right) - ({t_2}^\rho - s^\rho)^{\alpha-1} \right] \\
&- \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[{t_1}^\rho \left(\frac{(T^\rho - s^\rho)^{\alpha-1}}{T^\rho} \right) - ({t_1}^\rho - s^\rho)^{\alpha-1} \right] \\
&= \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[({t_2}^\rho - {t_1}^\rho) \left(\frac{(T^\rho - s^\rho)^{\alpha-1}}{T^\rho} \right) - \left[({t_2}^\rho - s^\rho)^{\alpha-1} - ({t_1}^\rho - s^\rho)^{\alpha-1} \right] \right] \\
&\leq \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[({t_2}^\rho - {t_1}^\rho) \right] \frac{(T^\rho - s^\rho)^{\alpha-1}}{T^\rho} \\
&\leq \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[({t_2}^\rho - {t_1}^\rho) \right].
\end{aligned}$$

In the same way, for $0 \leq t_1 \leq s \leq t_2 \leq T$ or $0 \leq t_1 \leq t_2 \leq s \leq T$, we have:

$$G(t_2, s) - G(t_1, s) \leq \frac{\rho^{1-\alpha} s^{\rho-1}}{\Gamma(\alpha)} \left[({t_2}^\rho - {t_1}^\rho) \right].$$

Then

$$\begin{aligned}
|Fu(t_2) - Fu(t_1)| &\leq \left| \int_0^T [G(t_2, s) - G(t_1, s)] f(s, u(s))\, ds \right| \\
&\leq L \int_0^T |G(t_2, s) - G(t_1, s)| ds \\
&\leq L \int_0^T \frac{\rho^{1-\alpha} s^{1-\alpha}}{\Gamma(\alpha)} [{t_2}^\rho - {t_1}^\rho] ds \\
&\leq \frac{L \rho^{1-\alpha} s^{1-\alpha}}{\Gamma(\alpha)} [{t_2}^\rho - {t_1}^\rho] \left[\frac{1}{\rho} s^\rho \right]_0^T.
\end{aligned}$$

Finally

$$|Fu(t_2) - Fu(t_1)| \leq \frac{L T^\rho}{\rho^\alpha \Gamma(\alpha)} [{t_2}^\rho - {t_1}^\rho].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zeros.
As a consequence of steps 1 to 3 together, and by means of the Arzela-Ascoli theorem 1.11, we deduce that $F : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 2.6. Then F has a fixed point which is a solution of problem (2.1)–(2.2) on $[0, T]$. □

Theorem 2.3. *Assume that hypotheses **(H1)-(H2)** and **(H3)** hold. We give $1 < \alpha \leq 2$, and $\rho > 0$. If we put*

$$\frac{L T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha + 1)} < 1,$$

then the problem (2.1)–(2.2) has at least one solution on $C[0, T]$.

Proof. In the previous theorem, we already transform the problem (2.1)–(2.2) into a fixed point problem

$$Fu\left(t\right)=\int_{0}^{T}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds.$$

We demonstrate that F satisfies the assumption of Schaefer's fixed point theorem 2.3.
This could be proved through four steps:

Step 1. F is a continuous operator.
Let $\left(u_n\right)_{n\in\mathbb{N}}$ be real sequence such that $\lim\limits_{n\to\infty}u_n=u$ in $C\left[0,T\right]$. Then for each $t\in C\left[0,T\right]$.

$$\left|Fu_n\left(t\right)-Fu\left(t\right)\right|\leq\int_{0}^{T}G\left(t,s\right)\left|f\left(s,u_n\left(s\right)\right)-f\left(s,u\left(s\right)\right)\right|ds.$$

For each $t\in C\left[0,T\right]$,the function $s\to G\left(t,s\right)$ is integrable on $\left[0,T\right]$,then the lebesgue dominated convergence theorem imply that:

$$\lim_{n\to\infty}\left\|Fu_n-Fu\right\|_{\infty}=0.$$

Consequently, F is continuous.

Step 2. Let

$$r\geq\frac{LT^{\alpha\rho}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}.$$

We define:

$$P_r=\left\{u\in C\left[0,T\right]:\left\|u\right\|_{\infty}\leq r\right\}.$$

It is clear that P_r is a bounded, closed and convex subset of $C\left[0,T\right]$.
Let $u\in P_r$,and $F:\to C\left[0,T\right]$ be the integral operator defined in (2.10), then $F\left(P_r\right)\subset P_r$.

Then

$$\begin{aligned}\left|Fu\left(t\right)\right|&\leq\left|\int_{0}^{T}G\left(t,s\right)f(s,u\left(s\right)ds\right|\\&\leq\frac{LT^{\alpha\rho}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}\\&\leq r.\end{aligned}$$

and by following $F\left(P_r\right)$ is bounded.

Step 3. $F\left(P_r\right)$ relatively compact.

Let $t_1, t_2 \in [0,T], t_1 < t_2$, and $u \in P_r$. Then

$$\begin{aligned}|Fu(t_2) - Fu(t_1)| &= \left|\int_0^{t_2} G(t_2,s) f(s,u(s))\,ds - \int_0^{t_1} G(t_1,s) f(s,u(s))\,ds\right| \\ &\leq L\int_0^T |G(t_2,s) - G(t_1,s)|ds \\ &\leq L\int_0^T \frac{\rho^{1-\alpha}s^{1-\alpha}}{\Gamma(\alpha)}[{t_2}^{\rho} - {t_1}^{\rho}]ds \\ &\leq \frac{LT^{\rho}}{\rho^{\alpha}\Gamma(\alpha)}[{t_2}^{\rho} - {t_1}^{\rho}].\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zeros. Therefore, from Steps 1 to 3 together with the Arzela-Ascoli theorem, we can condition that $F : C[0,T] \to C[0,T]$ is completely continuous.

Step 4. A priori bounded . Now we need to show that set

$$\theta = \{u \in C[0,T] : u = \mu Fu \mu \in (0,1)\},$$

is bounded. Let $u \in \theta$, then $u = \mu F(u)$ for some $0 < \mu < 1$. Thus,for each $t \in [0,T]$, we have:

$$u(t) = \mu\int_0^T G(t,s)\,f(s,u(s))ds.$$

Then, we get,

$$\begin{aligned}|u(t)| &\leq L\left[\int_0^t \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho} - s^{\rho})^{\alpha-1} - (t^{\rho} - s^{\rho})^{\alpha-1}\right]ds \right.\\ &\left. + \int_t^T \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}\left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho} - s^{\rho})^{\alpha-1}\right]ds\right] \\ &\leq \frac{L}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\left[-\frac{t^{\rho}}{T^{\rho}}(T^{\rho} - s^{\rho})^{\alpha} + (t^{\rho} - s^{\rho})^{\alpha}\right]_0^t - \left[\frac{t^{\rho}}{T^{\rho}}(T^{\rho} - s^{\rho})^{\alpha}\right]_t^T\right] \\ &\leq \frac{L}{\rho^{\alpha}\Gamma(\alpha+1)}\left[t^{\rho}T^{\rho(\alpha-1)} - t^{\alpha\rho}\right] \\ &\leq \frac{LT^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}.\end{aligned}$$

Thus shows that the set θ is bounded.By Schaefer's fixed point theorem 1.13 ,we conclude that F has a fixed point which is a solution of the problem (2.1)–(2.2). □

2.3 Illustrative examples

Exemple 2.1. Let

$$\begin{cases} {}^{1}_{*}\mathcal{D}^{\frac{3}{2}}_{0^+} u(t) + \frac{\cos(t)[2+|u(t)|]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)[1+|u(t)|]} = 0, \quad t \in \left[0, \frac{\pi}{4}\right] \\ u(0) = u\left(\frac{\pi}{4}\right) = 0, \end{cases} \tag{2.11}$$

and

$$f(t,u) = \frac{\cos(t)\left[2+|u(t)|\right]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|\right]}, \quad t \in \left[0, \frac{\pi}{4}\right], u, v \in \mathbb{R}.$$

As $sin(t), cos(t)$ are continuous positive functions $\forall t \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq cos(t) \leq 1$, and $0 \leq sin(t) \leq \frac{\sqrt{2}}{2}$, then

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left| \frac{\cos(t)\left[2+|u|\right]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|\right]} - \frac{\cos(t)\left[2+|v|\right]}{\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|v|\right]} \right| \\ &= \left| \frac{\cos(t)}{\sqrt{2}\cos(t)+\sin(t)} \right| \left| \frac{2+|u|}{1+|u|} - \frac{2+|v|}{1+|v|} \right| \\ &\leq \left| |u| - |v| \right| \leq |u-v|. \end{aligned}$$

Hence, the condition (2.7) is satisfied with $\lambda = 1$. It remains to show that the condition (2.8)

$$\frac{\lambda T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} = \frac{\left(\frac{\pi}{4}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} = 0.523583... < 1,$$

is satisfied. It follows theorem 2.1 that the problem (2.11), has a unique solution.

Exemple 2.2. Consider the following boundary value problem
then :

$$\begin{cases} {}^{1}_{*}\mathcal{D}^{\frac{3}{2}}_{0^+} u(t) + \frac{\arctan(t)}{5(1+|u(t)|)} = 0, \quad t \in \left[0, \frac{\pi}{2}\right] \\ u(0) = u\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{2.12}$$

$$f(t,u) = \frac{\arctan(t)}{5(1+|u(t)|)} \quad t \in \left[0, \frac{\pi}{2}\right], u \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{2}\right]$ we have :

$$\begin{aligned}
|f(t,u) - f(t,v)| &= \left| \frac{\arctan(t)}{5(1+|u|)} - \frac{\arctan(t)}{5(1+|v|)} \right| \\
&= \left| \frac{\arctan(t)}{5} \left(\frac{1}{1+|u|} - \frac{1}{1+|v|} \right) \right| \\
&\leq \left| \frac{\arctan(t)}{5} \right| \left| \frac{|u|-|v|}{(1+|u|)(1+|v|)} \right| \\
&\leq \frac{\pi}{10} |u-v| .
\end{aligned}$$

Hence, the hypo these **(H2)** is satisfied with $\lambda = \frac{\pi}{10}$. Also we have:

$$\begin{aligned}
|f(t,u)| &= \left| \frac{\arctan(t)}{5(1+|u|)} \right| \\
&\leq \left| \frac{1}{5(1+|u|)} \right| |\arctan(t)| \\
&\leq \frac{\pi}{10}.
\end{aligned}$$

Thus hypo these **(H3)** is satisfied with $L = \frac{\pi}{10}$, and the condition

$$\frac{LT^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} = \frac{\left(\frac{\pi}{10}\right)\left(\frac{\pi}{2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \simeq 0.4651 < 1.$$

It follows from Theorem 2.2 that the problem (2.12) has at last one solution on $\left[0, \frac{\pi}{2}\right]$.

Exemple 2.3. Let

$$\begin{cases}
{}^{1}_{*}\mathcal{D}^{\frac{3}{2}}_{0^+} u(t) + \dfrac{\ln(1+t)}{1+|u(t)|} = 0 \quad t \in [0,1] \\
\\
u(0) = u(1) = 0.
\end{cases} \tag{2.13}$$

Her

$$f(t,u) = \frac{\ln(1+t)}{1+|u|} \quad t \in [0,1] \quad u \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u, v \in \mathbb{R}$ and $t \in [0,1]$ we have :

$$\begin{aligned}
|f(t,u) - f(t,v)| &= \left| \frac{\ln(1+t)}{1+|u|} - \frac{\ln(1+t)}{1+|v|} \right| \\
&= |\ln(1+t)| \left| \frac{1}{1+|u|} - \frac{1}{1+|v|} \right| \\
&\leq \ln(2) \left| \frac{|u|-|v|}{(1+|u|)(1+|v|)} \right| \\
&\leq \ln(2) |u-v| .
\end{aligned}$$

Hence, the hypo these **(H2)** is satisfied with $\lambda = \ln(2)$. Also we have

$$\begin{aligned} |f(t,u)| &= \left| \frac{\ln(1+t)}{1+u^2} \right| \\ &\leq \ln(1+t) \\ &\leq \ln(2). \end{aligned}$$

Thus hypo these **(H3)** is satisfied with $L = \ln(2)$, and the condition

$$\frac{LT^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} = \frac{\ln(2)}{\Gamma\left(\frac{5}{2}\right)} \simeq 0.5214 < 1.$$

It follows from Theorem 2.3 that the problem (2.13) has at last one solution on $[0,1]$.

CHAPTER 3

INITIAL VALUE PROBLEM FOR NONLINEAR IMPLICIT OF CAPUTO-KATUGAMPOLA FDE

In this chapter, we are interested in the existence and uniqueness of initial value problem for nonlinear implicit fractional differential equation following :

$$ {}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) = f\left(t, u(t), {}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)\right), \quad t \in [0, T] \tag{3.1} $$

with the initial condition:

$$ u(0) = 0. \tag{3.2} $$

Where ${}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u$ is the left Caputo-Katugampola fractional derivative, $0 < \alpha \leq 1, \rho \in \mathbb{R}^+$, $T \leq (pc)^{\frac{1}{pc}}$ for any $1 \leq p \leq \infty, c > 0$, and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

3.1 Fundamental lemmas

Lemma 3.1. *Let $\alpha, \rho > 0, n = [\alpha] + 1$. If $u \in C[0, T]$, then:*

(a) *The fractional deferential equation ${}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) = 0$, has a unique solution:*

$$ u(t) = C_0 + C_1 t^{\rho} + C_2 t^{2\rho} + \ldots + C_{n-1} t^{(n-1)\rho}, $$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2 \ldots n-1$.

(b) *If ${}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t) \in C[0, T]$ and $0 < \alpha \leq 1$, then:*

$$ {}^{\rho}\mathcal{J}_{0^+}^{\alpha}\left({}_{*}^{\rho}\mathcal{D}_{0^+}^{\alpha} u(t)\right) = u(t) + C, \tag{3.3} $$

for some constant $C \in \mathbb{R}$.

Lemma 3.2. *Let $\alpha, \rho \in \mathbb{R}$, be such that $0 < \alpha \leq 1$, and $\rho > 0$.*
We give $u, {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u \in [0, T]$, and $f(t, u, v)$ is continuous function. Then the problem (3.1)–(3.2) *is equivalent to the fractional integral equation:*

$$u(t) = \int_0^t G(t, s) f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right) ds, \tag{3.4}$$

where

$$G(t, s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-1}. \tag{3.5}$$

Proof. Let $0 < \alpha \leq 1$ and $\rho > 0$, we may apply lemma 3.1 to reduce the fractional equation (3.1) to equivalent fractional integral equation.
By applying ${}^{\rho}\mathcal{J}^{\alpha}_{0+}$ to equation (3.1) we obtain:

$${}^{\rho}\mathcal{J}^{\alpha}_{0+}{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t) = {}^{\rho}\mathcal{J}^{\alpha}_{0+} f\left(t, u(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right). \tag{3.6}$$

From lemma 3.1, we find easily:

$${}^{\rho}\mathcal{J}^{\alpha}_{0+}{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + C,$$

for some $C \in \mathbb{R}$. Then, the fractional integral equation (3.6), gives:

$$u(t) = {}^{\rho}\mathcal{J}^{\rho}_{0+} f\left(t, u(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right) - C. \tag{3.7}$$

If we use the condition (3.2) in equation (3.7) we find:

$$u(0) = 0 = -C \quad \Rightarrow C = 0.$$

Therefore,the problem (3.1)–(3.2) is equivalent to:

$$u(t) = \int_0^t G(t, s) f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right) ds, \tag{3.8}$$

where $G(t, s)$, which given by equality (3.5). □

3.2 Existence and uniqueness results

Now, we will prove our first existence result for the problem (3.1)–(3.2) which is based on Banach's fixed point theorem 1.11.

We impose the following hypotheses:

(K1) $f:[0,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function .

(K2) For all $0<\alpha\leq 1$, there exist two constant $\gamma,\beta>0$, where $\beta<1$ such that:

$$\left|f\left(t,u,v\right)-f\left(t,\tilde{u},\tilde{v}\right)\right|\leq\gamma\left|u-\tilde{u}\right|+\beta\left|v-\tilde{v}\right|,$$

for any $u,v,\tilde{u},\tilde{v}\in\mathbb{R}$ and $t\in[0,T]$.

(K3) There exist there positive function $a,b,c\in C\left[0,T\right]$ such that

$\left|f\left(t,u,v\right)\right|\leq a\left(t\right)+b\left(t\right)\left|u\right|+c\left(t\right)\left|v\right|$ for all $t\in[0,T]$ and $u,v\in\mathbb{R}$. We denote

$$M_0=\frac{a^*}{1-c^*},\text{ and }\ M_1=\frac{b^*}{1-c^*},$$

where

$$a^*=\sup_{t\in[0,T]}a\left(t\right),\ \ b^*=\sup_{t\in[0,T]}b\left(t\right),\ \ c^*=\sup_{t\in[0,T]}c\left(t\right),\text{ with } c^*<1.$$

In what follows, we present the principal theorems:

Theorem 3.1. *Assume the hypotheses **(K1)-(K2)** hold. We give $0<\alpha\leq 1$, and $\rho>0$. If*

$$\frac{\gamma T^{\rho\alpha}}{\left(1-\beta\right)\rho^{\alpha}\Gamma\left(\alpha+1\right)}<1. \tag{3.9}$$

Then the problem (3.1)–(3.2) *admits a unique solution on* $[0,T]$.

Proof. To begin the proof, we will transform the problem (3.1)–(3.2) into a fixed point problem. Define the operator $\mathcal{A}:C\left[0,T\right]\rightarrow C\left[0,T\right]$ by:

$$\mathcal{A}u\left(t\right)=\int_0^t G\left(t,s\right)f\left(s,u\left(s\right),{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right)ds. \tag{3.10}$$

Because the problem (3.1)–(3.2) is equivalent to the fractional integral equation (3.10), the fixed point of $\mathcal{A}$ are solution of the problem (3.1)–(3.2).
Let $u,v\in C\left[0,T\right]$, be such that:

$${}_*^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right),\quad {}_*^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)=f\left(t,u\left(t\right),{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)\right).$$

Which implies that:

$$\mathcal{A}u\left(t\right)-\mathcal{A}v\left(t\right)=\int_0^t G\left(t,s\right)\left[f\left(s,u\left(s\right),{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right)-f\left(s,v\left(s\right),{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(s\right)\right)\right]ds.$$

Then, for all $t\in[0,T]$

$$\left|\mathcal{A}u\left(t\right)-\mathcal{A}v\left(t\right)\right|\leq\int_0^t G\left(t,s\right)\left|{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)-{}_*^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(s\right)\right|ds. \tag{3.11}$$

By **(K2)** we have :

$$\begin{aligned}\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)-{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)\right| &= \left|f\left(t,u\left(t\right),{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)\right)-f\left(t,v\left(t\right),{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)\right)\right| \\ &\leq \gamma\left|u\left(t\right)-v\left(t\right)\right|+\beta\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)-{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)\right|.\end{aligned}$$

Thus

$$\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(t\right)-{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}v\left(t\right)\right|\leq\frac{\gamma}{1-\beta}\left|u\left(t\right)-v\left(t\right)\right|.$$

From (3.11) we have:

$$\left|\mathcal{A}u\left(t\right)-\mathcal{A}v\left(t\right)\right|\leq\frac{\gamma}{1-\beta}\int_{0}^{t}G\left(t,s\right)\left|u\left(s\right)-v\left(s\right)\right|ds.$$

Then:

$$\left\|\mathcal{A}u-\mathcal{A}v\right\|_{\infty}\leq\frac{\gamma T^{\alpha\rho}}{\left(1-\beta\right)\rho^{\alpha}\Gamma\left(\alpha+1\right)}\left\|u-v\right\|_{\infty}.$$

This implies that by (3.9) , $\mathcal{A}$ is a contraction operator.
As a consequence of theorem 1.11, using Banach's contraction, we deduce that $\mathcal{A}$ has a unique fixed point which is the unique solution of the problem (3.1)–(3.2) on $[0,T]$. □

Theorem 3.2. *Assume that hypotheses* ***(K1)-(K3)*** *hold. We give* $0<\alpha\leq 1$, *and* $\rho>0$. *If we put*

$$\frac{M_1 T^{\alpha\rho}}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}<1,$$

the problem (3.1)–(3.2) *has at last one solution on* $[0,T]$.

Proof. In the previous theorem, we already transform the problem (3.1)–(3.2) into a fixed point problem

$$\mathcal{A}u\left(t\right)=\int_{0}^{t}G\left(t,s\right)f\left(s,u\left(s\right),{}_{*}^{\rho}D_{0+}^{\alpha}u\left(s\right)\right)ds.$$

We demonstrate the $\mathcal{A}$ satisfies the assumption of Schauder's fixed point theorem 1.12. This could be proved though three steps:

Step 1. $\mathcal{A}$ is a continuous operator.
Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty}u_n=u$ in $C[0,T]$. Then for each $t\in[0,T]$,

$$\left|\mathcal{A}u_n\left(t\right)-\mathcal{A}u\left(t\right)\right|\leq\int_{0}^{t}G\left(t,s\right)\left|f\left(s,u_n\left(s\right),{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u_n\left(s\right)\right)-f\left(s,u\left(s\right),{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u\left(s\right)\right)\right|ds, \tag{3.12}$$

where

$$ {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t) = f\left(t, u_n(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t)\right), \text{ and } {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t) = f\left(t, u(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right). $$

As consequence of **(K2)**, we find easily ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n \to {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u$ in $C[0,T]$. In fact we have:

$$ \begin{aligned} \left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t) - {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right| &= \left|f\left(t, u_n(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t)\right) - f\left(t, u(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right)\right| \\ &\leq \gamma\left|u_n(t) - u(t)\right| + \beta\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t) - {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right|. \end{aligned} $$

Thus:

$$ \left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t) - {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right| \leq \frac{\gamma}{1-\beta}\left|u_n(t) - u(t)\right| $$

Since $u_n \to u$, then we get ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t) \to {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)$ as $n \to \infty$ for each $t \in [0,T]$.
Now let $K_0 > 0$, be such that for each $t \in [0,T]$, we have:

$$ \left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(t)\right| \leq K_0, \quad \left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(t)\right| \leq K_0. $$

Then, we have:

$$ \begin{aligned} \left|\mathcal{A}u_n(t) - \mathcal{A}u(t)\right| &\leq \int_0^t G(t,s)\left|f\left(s, u_n(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(s)\right) - f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right)\right| ds \\ &\leq \int_0^t G(t,s)\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(s) - {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right| ds \\ &\leq \int_0^t G(t,s)\left[\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u_n(s)\right| + \left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right|\right] ds \\ &\leq \int_0^t 2K_0 G(t,s) ds. \end{aligned} $$

For each $t \in [0,T]$, the function $s \to 2K_0 G(t,s)$ is integrable on $[0,T]$, then the Lebesgue dominated convergence theorem and (3.12) imply that:

$$ \left|\mathcal{A}u_n(t) - \mathcal{A}u(t)\right| \to 0 \ \ as \ \ n \to \infty, $$

and hence:

$$ \lim_{n\to\infty} \left\|\mathcal{A}u_n - \mathcal{A}u\right\|_\infty = 0. $$

Consequently, $\mathcal{A}$ is continuous.

Step 2. Let

$$ r \geq \frac{M_0 T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1) - M_1 T^{\alpha\rho}}. $$

We define:

$$ P_r = \left\{u \in C[0,T] : \|u\|_\infty \leq r\right\}. $$

It is clear that P_r is bounded, closed and convex subset of $C[0,T]$.
Let $u \in P_r$, and $\mathcal{A} : P_r \to C[0,T]$ be the integral operator defined in (3.12), then $\mathcal{A}(P_r) \subset P_r$.
In fact, for each $t \in [0,T]$, we have from **(K3)**:

$$\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right| = \left|f\left(t, u(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right)\right| \leq a(t) + b(t)|u(t)| + c(t)\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right|.$$

Then

$$\left|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right| \leq \frac{a^*}{1-c^*} + \frac{b^*}{1-c^*}r = M_0 + M_1 r. \tag{3.13}$$

Thus

$$\begin{aligned}
|\mathcal{A}u(t)| &\leq \int_0^t G(t,s)\left|f\left(s, u(s), {}^{\rho}_{*}D^{\alpha}_{0^+}u(s)\right)\right| ds \\
&\leq \int_0^t G(t,s)\left[M_0 + M_1 r\right] ds \\
&\leq \frac{M_0 T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{M_1 T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}r \\
&\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\alpha\rho}\right]\frac{M_0 T^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\alpha\rho}} + M_1 T^{\alpha\rho} r}{\rho^{\alpha}\Gamma(\alpha+1)} \\
&\leq \frac{\left[\rho^{\alpha}\Gamma(\alpha+1) - M_1 T^{\alpha\rho}\right] r + M_1 T^{\alpha\rho} r}{\rho^{\alpha}\Gamma(\alpha+1)} \\
&\leq r.
\end{aligned}$$

Then $\mathcal{A}(P_r) \subset P_r$.

Step 3. $\mathcal{A}(P_r)$ is relatively compact.
Let $t_1, t_2 \in [0,T], t_1 < t_2$, and $u \in P_r$. Then

$$\begin{aligned}
|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| &= \left|\int_0^{t_2} G(t_2,s) f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(s)\right) ds \right. \\
&\qquad \left. - \int_0^{t_1} G(t_1,s) f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(s)\right) ds\right| \\
&\leq \int_0^{t_1} \left|\left[G(t_2,s) - G(t_1,s)\right] f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(s)\right)\right| ds \\
&\quad + \int_{t_1}^{t_2} G(t_2,s)\left|f\left(s, u(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(s)\right)\right| ds \\
&\leq (M_0 + M_1 r)\left[\int_0^{t_1} |G(t_2,s) - G(t_1,s)| ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} G(t_2,s)\, ds\right].
\end{aligned} \tag{3.14}$$

We have:

$$\begin{aligned} G(t_2,s)-G(t_1,s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1}\left[(t_2^\rho - s^\rho)^{\alpha-1} - (t_1^\rho - s^\rho)^{\alpha-1}\right] \\ &= \frac{-1}{\alpha\rho^\alpha\Gamma(\alpha)}\frac{d}{ds}\left[(t_2^\rho - s^\rho)^{\alpha} - (t_1^\rho - s^\rho)^{\alpha}\right] \end{aligned}$$

then

$$\int_0^{t_1} |G(t_2,s)-G(t_1,s)|ds \leq \frac{1}{\rho^\alpha\Gamma(\alpha+1)}\left[(t_2^\rho - t_1^\rho)^\alpha + (t_2^{\alpha\rho} - t_1^{\alpha\rho})\right]$$

we have also

$$\begin{aligned} \int_{t_1}^{t_2} G(t_2,s)ds &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}(t_2^\rho - s^\rho)^{\alpha-1}ds \\ &= \frac{-1}{\alpha\rho^\alpha\Gamma(\alpha)}\left[(t_2^\rho - s^\rho)^\alpha\right]_{t_1}^{t_2} \\ &\leq \frac{1}{\rho^\alpha\Gamma(\alpha+1)}(t_2^\rho - t_1^\rho)^\alpha. \end{aligned}$$

Then (3.14) gives

$$|\mathcal{A}u(t_2) - \mathcal{A}u(t_1)| \leq \frac{M_0 + M_1 r}{\rho^\alpha\Gamma(\alpha+1)}\left[2(t_2^\rho - t_1^\rho)^\alpha + (t_2^{\alpha\rho} - t_1^{\alpha\rho})\right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.
As a consequence of steps 1 to 3 together, and by means of the Arzel-Ascoli theorem 1.10, we deduce that $\mathcal{A} : P_r \to P_r$ is continuous, compact and satisfies the assumption of Schauder's fixed point theorem 1.12. Then $\mathcal{A}$ has a fixed point which is a solution of the problem (3.1)–(3.2) on $[0,T]$. □

3.3 Illustrative examples

Exemple 3.1. Consider the following Cauchy problem:

$$\begin{cases} {}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+} u(t) = \dfrac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|+\left|{}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+}u(t)\right|\right]}, & t \in \left[0,\frac{\pi}{4}\right], \\ u(0) = 0. \end{cases} \tag{3.15}$$

Set:

$$f(t,u,v) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u|+|v|\right]}, \quad t \in \left[0,\frac{\pi}{4}\right], u,v \in \mathbb{R}.$$

Because $\sin(t), \cos(t)$ are continuous positive function $\forall t \in \left[0, \frac{\pi}{4}\right]$, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(t) \leq 1$ and $0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}$, then:

$$|f(t, u, v) - f(t, \tilde{u}, \tilde{v})| \leq \frac{1}{\pi}(|u - \tilde{u}| + |v - \tilde{v}|).$$

Hence, the condition **(K2)** is satisfied with:

$$\gamma = \beta = \frac{1}{\pi} \simeq 0.3183 < 1.$$

It remains to show the condition (3.9):

$$\frac{\gamma T^{\alpha\rho}}{(1-\beta)\rho^{\alpha}\Gamma(\alpha+1)} = \frac{\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\left(1-\frac{1}{\pi}\right)\Gamma\left(\frac{1}{2}+1\right)} = \frac{\sqrt{\pi}}{2(\pi-1)\Gamma\left(\frac{3}{2}\right)} \simeq 0.4669 < 1,$$

is satisfied. It follows from theorem 3.1 that the problem (3.15) has a unique solution.

Exemple 3.2.

$$\begin{cases} {}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+} u(t) = \dfrac{\cos(t)\left[2+|u(t)|+\left|{}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+} u(t)\right|\right]}{\pi\left(\sqrt{2}\cos(t)+\sin(t)\right)\left[1+|u(t)|+\left|{}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+} u(t)\right|\right]}, \quad t \in \left[0, \frac{\pi}{4}\right], \\ \\ u(0) = 0. \end{cases} \tag{3.16}$$

Set

$$f(t, u, v) = \frac{\cos(t)[2 + |u| + |v|]}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)[1 + |u| + |v|]}, \quad t \in \left[0, \frac{\pi}{4}\right], \quad u, v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}$ and $t \in \left[0, \frac{\pi}{4}\right]$, we have $\frac{\sqrt{2}}{2} \leq \cos(t) \leq 1$ and $0 \leq \sin(t) \leq \frac{\sqrt{2}}{2}$, then:

$$|f(t, u, v) - f(t, \tilde{u}, \tilde{v})| \leq \frac{1}{\pi}(|u - \tilde{u}| + |v - \tilde{v}|).$$

Hence, the condition **(K2)** is satisfied with $\gamma = \beta = \dfrac{1}{\pi} < 1$. Also, we have:

$$|f(t, u, v)| \leq \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}(2 + |u| + |v|).$$

Thus, the hypo-these **(K3)** is satisfies with

$$a(t) = \frac{2\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}, \quad \text{and} \quad b(t) = c(t) = \frac{\cos(t)}{\pi\left(\sqrt{2}\cos(t) + \sin(t)\right)}.$$

We have also

$$a^* = \frac{2}{\pi}, \text{ and } b^* = c^* = \frac{1}{\pi} < 1, \text{ and } M_0 = \frac{2}{\pi - 1}, \text{ and } M_1 = \frac{1}{\pi - 1}.$$

And the condition

$$\frac{M_1 T^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha + 1)} = \frac{\left(\frac{1}{\pi - 1}\right)\left(\frac{\pi}{4}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)} = \frac{\sqrt{\pi}}{2(\pi - 1)\Gamma\left(\frac{3}{2}\right)} \simeq 0.4669 < 1.$$

It follows from theorem 3.2, that the problem (3.16) has at least one solution.

INITIAL VALUE PROBLEM FOR A COUPLED SYSTEM OF COPUTO-KATUGAMPOLA TYPE FDE

In this chapter we will study the initial value problem of a coupled system of nonlinear fractional differential equations with Caputo-Katugampola derivative following :

$$\begin{cases} {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t) = f\left(t, u(t), v(t), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right) \\ {}^{\rho}_{*}\mathcal{D}^{\beta}_{0^+}v(t) = g\left(t, u(t), v(t), {}^{\rho}_{*}\mathcal{D}^{\beta}_{0^+}v(t)\right) \end{cases}, t \in [0, T], \tag{4.1}$$

with the initial condition

$$u(0) = 0, \quad v(0) = 0, \tag{4.2}$$

where $0 < \alpha, \beta \leq 1, \rho > 0, T \leq (pc)^{\frac{1}{pc}}$ for any $1 \leq p \leq \infty, c > 0$, is a finite positive constant. The symbol ${}^{\rho}_{*}\mathcal{D}^{\delta}_{0^+}, \delta = \alpha, \beta$, is the Caputo-Katugampola fractional derivative of fractional order δ, and $f, g : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

4.1 Fundamental lemmas

In follows, we present some significant lemmas to show the principal theorems, we have

Lemma 4.1. *Let $\alpha, \rho > 0, n = [\alpha] + 1$. If $u \in C[0, T]$, then:*
(a) *The fractional deferential equation ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t) = 0$, has a unique solution:*

$$u(t) = C_0 + C_1 t^{\rho} + C_2 t^{2\rho} + ... + C_{n-1} t^{(n-1)\rho},$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2...n - 1$.
(b) *If ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t) \in C[0, T]$ and $0 < \alpha \leq 1$, then:*

$$^{\rho}\mathcal{J}^{\alpha}_{0^+}\left({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0^+}u(t)\right) = u(t) + C, \tag{4.3}$$

for some constant $C \in \mathbb{R}$.

Lemma 4.2. *Let* $0 < \delta \leq 1$, *and* $\rho > 0$. *We give* $y, {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y \in C([0,T], \mathbb{R})$. *Then the solution of problem*

$$\begin{cases} {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t) = f\left(t, y(t), {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t)\right), \quad t \in [0,T], \\ y(0) = 0, \end{cases} \tag{4.4}$$

is equivalent to the fractional integral equation

$$y(t) = \int_0^t G_{\delta}(t,s) f\left(s, y(s), {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(s)\right) ds, \tag{4.5}$$

where

$$G_{\delta}(t,s) = \frac{\rho^{1-\delta}}{\Gamma(\delta)} s^{\rho-1}(t^{\rho} - s^{\rho})^{\delta-1}. \tag{4.6}$$

Proof. By applying ${}^{\rho}\mathcal{J}^{\delta}_{0+}$ to equation (4.4), we obtain

$${}^{\rho}\mathcal{J}^{\delta}_{0+}{}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t) = {}^{\rho}\mathcal{J}^{\delta}_{0+}f\left(t, y(t), {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t)\right). \tag{4.7}$$

From lemma 4.1, we fined easily

$${}^{\rho}\mathcal{J}^{\delta}_{0+}{}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t) = y(t) + C,$$

for some $C \in \mathbb{R}$. Then the fractional integral equation (4.4), gives

$$y(t) = {}^{\rho}\mathcal{J}^{\delta}_{0+}f\left(t, y(t), {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(t)\right) - C. \tag{4.8}$$

In view of the condition $y(0) = 0$, we get

$$y(0) = 0 = -C \quad \Rightarrow C = 0.$$

Therefore, the problem (4.4), is equivalent to

$$y(t) = \int_0^t G_{\delta}(t,s) f\left(s, y(s), {}^{\rho}_{*}\mathcal{D}^{\delta}_{0+}y(s)\right) ds, \tag{4.9}$$

where $G_{\delta}(t,s)$, which given by the equality (4.6). The proof is complete. □

Based on the previous lemma, we will define the integral solution of the problem (4.1)–(4.2).

Lemma 4.3 (See [2]). *Let* $0 < \alpha, \beta \leq 1$, *and* $\rho > 0$. *We give* $u, {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u, {}^{\rho}_{*}\mathcal{D}^{\beta}_{0+}u \in C([0,T], \mathbb{R})$. *Then the solution of problem* (4.1)–(4.2) *is equivalent to the coupled fractional integral equation*

$$u(t) = \int_0^t G_{\alpha}(t,s) f\left(s, u(s), v(s), {}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}u(s)\right) ds,$$
$$v(t) = \int_0^t G_{\beta}(t,s) g\left(s, u(s), v(s), {}^{\rho}_{*}\mathcal{D}^{\beta}_{0+}v(s)\right) ds,$$

with

$$G_{\alpha}(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1}(t^{\rho} - s^{\rho})^{\alpha-1}.$$
$$G_{\beta}(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\beta)} s^{\rho-1}(t^{\rho} - s^{\rho})^{\beta-1}.$$

Proof. In view of Lemma 4.2, for $\delta = \alpha$, and $\delta = \beta$, respectively. □

4.2 Existence and uniqueness results

Throughout this chapter, we assume the following hypotheses.

(L1) $f, g : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

(L2) There exist the constant $\lambda_i, \mu_i \ (i = 1.2.3)$, such that

$$|f(t, u, v, w) - f(t, \tilde{u}, \tilde{v}, \tilde{w})| \leq \lambda_1 |u - \tilde{u}| + \lambda_2 |v - \tilde{v}| + \lambda_3 |w - \tilde{w}|,$$
$$|g(t, u, v, w) - g(t, \tilde{u}, \tilde{v}, \tilde{w})| \leq \mu_1 |u - \tilde{u}| + \mu_2 |v - \tilde{v}| + \mu_3 |w - \tilde{w}|,$$

for any $u, v, w, \tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{R}$ and $t \in [0, T]$.

(L3) There exist three positive function $a_i, b_i, c_i, d_i \in [0, T], (i = 1.2)$, such that

$$|f(t, u, v, w)| \leq a_1(t) + b_1(t)|u| + c_1(t)|v| + d_1|w|,$$
$$|g(t, u, v, w)| \leq a_2(t) + b_2(t)|u| + c_2(t)|v| + d_2|w|,$$

for all $t \in [0, T]$ and $u, v, w \in \mathbb{R}$.

We denote

$$M_i = \frac{a_i^*}{1 - d_i^*},\ N_i = \frac{b_i^* + c_i^*}{1 - d_i^*},\ \text{and}\quad K_i = \frac{b_i^*}{1 - d_i^*},\ H_i = \frac{c_i^*}{1 - d_i^*},\ (i = 1.2).$$

where

$$a_i^* = \sup_{t \in [0,T]} a_i(t),\ b_i^* = \sup_{t \in [0,T]} b_i(t),\ c_i^* = \sup_{t \in [0,T]} c_i(t),\ d_i^* = \sup_{t \in [0,T]} d_i(t),\ \text{with } d_i^* < 1.$$

Throughout the remaining of this chapter T, p and c are real constants such that $p \geq 1, c > 0$, and $T \leq (pc)^{\frac{1}{pc}}$.

We will now prove a theorem about existence and uniqueness of solutions of initial value problem (4.1)–(4.2), which is based on Banach's fixed point theorem.

Let us introduce the space $E = X \times Y$, with the norm $\|(u, v)\| = \|u\| + \|v\|$, Obviously $(E, \|(u, v)\|)$, is a Banach space, where $X = \{u(t)/u(t) \in C([0, T], \mathbb{R})\}$, with the norm $\|u\| = \max_{0 \leq t \leq T} |u(t)|$, and $Y = \{v(t)/v(t) \in C([0, T], \mathbb{R})\}$, with the norm $\|v\| = \max_{0 \leq t \leq T} |v(t)|$.

We define an operator $F : E \to E$ by

$$F(u, v)(t) = \begin{pmatrix} F_1(u, v)(t) \\ \\ F_2(u, v)(t) \end{pmatrix}, \tag{4.10}$$

where

$$F_1(u,v)(t) = \int_0^t G_\alpha(t,s)\, f\left(s, u(s), v(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u(s)\right) ds,$$

$$F_2(u,v)(t) = \int_0^t G_\beta(t,s)\, g\left(s, u(s), v(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\beta} v(s)\right) ds,$$

with

$$G_\alpha(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}.$$

$$G_\beta(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\beta)} s^{\rho-1}(t^\rho - s^\rho)^{\beta-1}.$$

Theorem 4.1. *Assume the hypotheses* ***(L1)-(L2)*** *hold. If*

$$\frac{T^{\rho\alpha}(\lambda_1+\lambda_2)}{(1-\lambda_3)\rho^\alpha\Gamma(\alpha+1)} + \frac{T^{\rho\beta}(\mu_1+\mu_2)}{(1-\mu_3)\rho^\beta\Gamma(\beta+1)} < 1. \tag{4.11}$$

Then the problem (4.1)–(4.2) *has a unique solution on* $[0,T]$.

Proof. By the lemma 4.3, we will transform the problem (4.1)–(4.2) into a fixed point problem $F(u,v) = (u,v)$,where the operator F is defined by (4.10). Using the Banach contraction principle, we shall show that F has a fixed point.

Now for $(u_2,v_2), (u_1,v_1) \in E$ and $t \in [0,T]$, we get

$$\begin{aligned} |F_1(u_2,v_2)(t) - F_1(u_1,v_1)(t)| &= \int_0^t G_\delta(t,s)\, |\, f\left(s, u_2(s)\, v_2(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2(s)\right) \\ &\quad - f\left(s, u_1(s)\, v_1(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1(s)\right)|\, ds \\ &\le \int_0^t G_\alpha(t,s)\, |{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2 - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1|\, ds. \end{aligned} \tag{4.12}$$

By **(L2)**, we have

$$\begin{aligned} |{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2 - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1| &= |\, f\left(s, u_2(s)\, v_2(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2(s)\right) \\ &\quad - f\left(s, u_1(s)\, v_1(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1(s)\right)| \\ &\le \lambda_1 |u_2 - u_1| + \lambda_2 |v_2 - v_1| + \lambda_3 |{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2 - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1|, \end{aligned}$$

Thus

$$|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_2 - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha} u_1| \le \frac{1}{1-\lambda_3}\left[\lambda_1 |u_2 - u_1| + \lambda_2 |v_2 - v_1|\right].$$

From (4.12), we have

$$\begin{aligned} |F_1(u_2,v_2)(t) - F_1(u_1,v_1)(t)| &\le \frac{1}{1-\lambda_3}\int_0^t G_\alpha(t,s)\left[\, \lambda_1 |u_2(s) - u_1(s)| \right. \\ &\quad \left. + \lambda_2 |v_2(s) - v_1(s)|\right] ds \\ &\le \frac{T^{\rho\alpha}(\lambda_1+\lambda_2)}{(1-\lambda_3)\rho^\alpha\Gamma(\alpha+1)} \\ &\quad \times \left(\|u_2 - u_1\| + \|v_2 - v_1\|\right). \end{aligned}$$

Then

$$\|F_1(u_2,v_2)-F_1(u_1,v_1)\| \leq \frac{T^{\rho\alpha}(\lambda_1+\lambda_2)}{(1-\lambda_3)\rho^{\alpha}\Gamma(\alpha+1)}(\|u_2-u_1\|+\|v_2-v_1\|). \tag{4.13}$$

Similarly, one can find that

$$\|F_2(u_2,v_2)-F_2(u_1,v_1)\| \leq \frac{T^{\rho\beta}(\mu_1+\mu_2)}{(1-\mu_3)\rho^{\beta}\Gamma(\beta+1)}(\|u_2-u_1\|+\|v_2-v_1\|). \tag{4.14}$$

Thus is follows from (4.13) and (4.14), that

$$\begin{aligned}\|F(u_2,v_2)-F(u_1,v_1)\| \leq & \left[\frac{T^{\rho\alpha}(\lambda_1+\lambda_2)}{(1-\lambda_3)\rho^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\rho\beta}(\mu_1+\mu_2)}{(1-\mu_3)\rho^{\beta}\Gamma(\beta+1)}\right] \\ & \times(\|u_2-u_1\|+\|v_2-v_1\|).\end{aligned}$$

This implies that by (4.11), F is a contraction operator. As a consequence of Banach's contraction principle theorem 1.11, we deduce that F has a unique fixed point which is the unique solution of the problem (4.1)–(4.2) on $[0,T]$. The proof is complete. □

Theorem 4.2. *Assume that hypotheses **(L1)–(L3)** hold. If*

$$\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}N_1+\frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}N_2<1. \tag{4.15}$$

Then the problem (4.1)-(4.2) *has at last one solution on* $[0,T]$.

Proof. In the previous theorem, we already transform the problem (4.1)–(4.2) into a fixed point problem $F(u,v)=(u,v)$, where the operator F is defined by (4.10). We demonstrate that F satisfies the assumption of Schauder's fixed point theorem 1.12.

This could be proved though three steps:

Step 1. F is a continuous operator.

Let $(x_n,y_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim\limits_{n\to\infty}(x_n,y_n)=(x,y)$ in E. Then for each $t\in[0,T]$.

$$|F_1(x_n,y_n)(t)-F_1(x,y)(t)| \leq \int_0^t G_{\alpha}(t,s)\left|\begin{array}{l} f\left(s,x_n(s),y_n(s),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(s)\right) \\ -f\left(s,x(s),y(s),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(s)\right)\end{array}\right|ds,$$

where

$${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(t)=f\left(t,x_n(t),y_n(t),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(t)\right).$$

$${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(t)=f\left(t,x(t),y(t),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(t)\right).$$

As a consequence of **(L2)**, we find easily ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n\to{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x$ in $C[0,T]$. In fact we have

$$\begin{aligned}|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(t)-{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(t)| &= |f\left(t,x_n(t),y_n(t),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(t)\right) \\ &\quad -f\left(t,x(t),y(t),{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(t)\right)| \\ &\leq \lambda_1|x_n(t)-x(t)|+\lambda_2|y_n(t)-y(t)| \\ &\quad +\lambda_3|{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x_n(t)-{}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}x(t)|.\end{aligned}$$

Thus

$$\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n(t) - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x(t)\right| \leq \frac{1}{1-\lambda_3}\left(\lambda_1\left|x_n(t)-x(t)\right| + \lambda_2\left|y_n(t)-y(t)\right|\right).$$

Since $(x_n, y_n) \to (x, y)$, then we get ${}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n \to {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x$ as $n \to \infty$ for each $t \in [0, T]$.
Now let $K_0 > 0$, be such that for each $t \in [0, T]$, we have

$$\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n(t)\right| \leq K_0, \quad \left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x(t)\right| \leq K_0.$$

Then, we have

$$\begin{aligned}
\left|F_1(x_n, y_n)(t) - F_1(x, y)(t)\right| &\leq \int_0^t G_\alpha(t,s)\left|\begin{array}{l} f\left(s, x_n(s), y_n(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n(s)\right) \\ -f\left(s, x(s), y(s), {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x(s)\right) \end{array}\right| ds \\
&\leq \int_0^t G_\alpha(t,s)\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n(s) - {}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x(s)\right| ds \\
&\leq \int_0^t G_\alpha(t,s)\left[\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x_n(s)\right| + \left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}x(s)\right|\right] ds \\
&\leq \int_0^t 2K_0 G_\alpha(t,s)\, ds.
\end{aligned}$$

For each $t \in [0, T]$, the function $s \to 2K_0 G_\alpha(t, s)$ is integrable on $[0, T]$, then the Lebesgue dominated convergence theorem imply that

$$\left|F_1(x_n, y_n)(t) - F_1(x, y)(t)\right| \to 0 \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty}\left\|F_1(x_n, y_n) - F_1(x, y)\right\| = 0. \tag{4.16}$$

Similarly, one can find that

$$\lim_{n\to\infty}\left\|F_2(x_n, y_n) - F_2(x, y)\right\| = 0. \tag{4.17}$$

Thus is follows from (4.16) and (4.17), that

$$\lim_{n\to\infty}\left\|F(x_n, y_n) - F(x, y)\right\| = 0.$$

Consequently, F is continuous.

Step 2. let

$$r \geq \frac{\frac{T^{\alpha\rho}}{\rho^\alpha\Gamma(\alpha+1)}M_1 + \frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}M_2}{1 - \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}N_1 - \frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}N_2}.$$

We define

$$P_r = \{(u,v) \in E : \|(u,v)\| \leq r\}.$$

It is clear that P_r is bounded, closed and convex subset of E.

Let $(u,v) \in P_r$ and $F : P_r \to E$ be the integral operator defined in (4.10), then $F(P_r) \subset P_r$. In fact, for each $t \in [0,T]$, we have from **(L3)**

$$\begin{aligned}
\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right| &= \left|f\left(t,u(t),v(t),{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right)\right| \leq a_1(t) + b_1(t)|u(t)| + c_1(t)|v(t)| \\
&\quad + d_1(t)\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right|. \\
\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}v(t)\right| &= \left|g\left(t,u(t),v(t),{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}v(t)\right)\right| \leq a_2(t) + b_2(t)|u(t)| + c_2(t)|v(t)| \\
&\quad + d_2(t)\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}v(t)\right|.
\end{aligned}$$

Then

$$\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\alpha}u(t)\right| \leq \frac{a_1^*}{1-d_1^*} + \frac{b_1^*+c_1^*}{1-d_1^*}r = M_1 + N_1 r.$$

and

$$\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}v(t)\right| \leq \frac{a_2^*}{1-d_2^*} + \frac{b_2^*+c_2^*}{1-d_2^*}r = M_2 + N_2 r.$$

Thus

$$\begin{aligned}
|F_1(u,v)(t)| &\leq \int_0^t G_\alpha(t,s)\left|f\left(s,u(s),v(s),{}_{*}^{\rho}D_{0+}^{\alpha}u(s)\right)\right| ds \\
&\leq \int_0^t G_\alpha(t,s)\left|{}_{*}^{\rho}D_{0+}^{\alpha}u(s)\right| ds \\
&\leq \frac{T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}(M_1+N_1 r),
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
|F_2(u,v)(t)| &\leq \int_0^t G_\beta(t,s)\left|g\left(s,u(s),v(s),{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}v(s)\right)\right| ds \\
&\leq \int_0^t G_\beta(t,s)\left|{}_{*}^{\rho}\mathcal{D}_{0+}^{\beta}u(s)\right| ds \\
&\leq \frac{T^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}(M_2+N_2 r).
\end{aligned} \tag{4.19}$$

In consequence, we have

$$\|F(u,v)\| \leq r.$$

Then $F(P_r) \subset P_r$.

Step 3. $F(P_r)$ is relatively compact.

Let $t_1, t_2 \in [0, T], t_1 < t_2$ and $(u, v) \in P_r$. Then

$$\begin{aligned}|F_1(u,v)(t_2) - F_1(u,v)(t_1)| &= \bigg| \int_0^{t_2} G_\alpha(t_2, s) f(s, u(s), v(s), {}^\rho_*\mathcal{D}^\alpha_{0+} u(s))\, ds \\ &\quad - \int_0^{t_1} G_\alpha(t_1, s) f(s, u(s), v(s), {}^\rho_*\mathcal{D}^\alpha_{0+} u(s))\, ds \bigg| \\ &\leq \int_0^{t_1} |[G_\alpha(t_2, s) - G_\alpha(t_1, s)] \\ &\quad \times f(s, u(s), v(s), {}^\rho_*\mathcal{D}^\alpha_{0+} u(s))| \\ &\quad + \int_{t_1}^{t_2} G_\alpha(t_2, s) |f(s, u(s), v(s), {}^\rho_*\mathcal{D}^\alpha_{0+} u(s))|\, ds \\ &\leq (M_1 + N_1 r) \bigg[\int_0^{t_1} |G_\alpha(t_2, s) - G_\alpha(t_1, s)| ds \\ &\quad + \int_{t_1}^{t_2} G_\alpha(t_2, s)\, ds \bigg].\end{aligned} \tag{4.20}$$

We have

$$\begin{aligned}G_\alpha(t_2, s) - G_\alpha(t_1, s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left[(t_2^\rho - s^\rho)^{\alpha-1} - (t_1^\rho - s^\rho)^{\alpha-1} \right] \\ &= \frac{-1}{\alpha \rho^\alpha \Gamma(\alpha)} \frac{d}{ds} \left[(t_2^\rho - s^\rho)^\alpha - (t_1^\rho - s^\rho)^\alpha \right],\end{aligned}$$

then

$$\int_0^{t_1} |(G_\alpha(t_2, s) - G_\alpha(t_1, s))| ds \leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} \left[(t_2^\rho - t_1^\rho)^\alpha + (t_2^{\alpha\rho} - t_1^{\alpha\rho}) \right],$$

we have also

$$\begin{aligned}\int_{t_1}^{t_2} G_\alpha(t_2, s)\, ds &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} s^{\rho-1} (t_2^\rho - s^\rho)^{\alpha-1} ds \\ &= \frac{-1}{\alpha \rho^\alpha \mathrm{B}\Gamma(\alpha)} \left[(t_2^\rho - s^\rho)^\alpha \right]_{t_1}^{t_2} \\ &\leq \frac{1}{\rho^\alpha \Gamma(\alpha+1)} (t_2^\rho - t_1^\rho)^\alpha.\end{aligned}$$

Then (4.20), gives

$$|F_1(u,v)(t_2) - F_1(u,v)(t_1)| \leq \frac{M_1 + N_1 r}{\rho^\alpha \Gamma(\alpha+1)} \left[2(t_2^\rho - t_1^\rho)^\alpha + (t_2^{\alpha\rho} - t_1^{\alpha\rho}) \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

In the same way, we can obtain

$$|F_2(u,v)(t_2) - F_2(u,v)(t_1)| \leq \frac{M_2 + N_2 r}{\rho^\beta \Gamma(\beta+1)} \left[2(t_2^\rho - t_1^\rho)^\beta + \left(t_2^{\beta\rho} - t_1^{\beta\rho} \right) \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Therefore, the operator $F(u,v)$ is equicontinuous, and by means of the Arzela-Ascoli theorem 1.10, we deduce that $F : P_r \to P_r$ is continuous, relatively compact and satisfies the assumption of Schauder's fixed point theorem 1.12. Then F hasa fixed point which is a solution of the problem (4.1)–(4.2) on $[0,T]$. The proof is complete. □

4.3 Illustrative examples

Exemple 4.1. Consider the following coupled system of Caputo-Katugampola type fractional differential equations

$$\begin{cases} {}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u(t) = \dfrac{\arctan(1+t^2)}{\left[1+\frac{1}{5}|u(t)|+\frac{1}{50}|v(t)|+\frac{1}{100}\left|{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u(t)\right|\right]}, & \\ & t\in[0,1], \\ {}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v(t) = \dfrac{1}{10+|u(t)|}+\dfrac{1}{50+|v(t)|}+\dfrac{1}{100+t^2}\left({}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v(t)\right), & \\ u(0)=0, v(0)=0. \end{cases} \tag{4.21}$$

Let

$$f\left(t,u,v,{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u\right) = \frac{\arctan(1+t^2)}{\left[1+\frac{1}{5}|u(t)|+\frac{1}{50}|v(t)|+\frac{1}{100}\left|{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u(t)\right|\right]}.$$

$$g\left(t,u,v,{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v\right) = \frac{1}{10+|u(t)|}+\frac{1}{50+|v(t)|}+\frac{{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v(t)}{100+t^2}.$$

Clearly $f,g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions and we have

$$\begin{aligned} \left|f\left(t,u,v,{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u\right)-f\left(t,\tilde{u},\tilde{v},{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}\tilde{u}\right)\right| &\le \frac{\pi}{10}|u-\tilde{u}|+\frac{\pi}{100}|v-\tilde{v}| \\ &+\frac{\pi}{200}\left|{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}u-{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{2}}\tilde{u}\right|. \\ \left|g\left(t,u,v,{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v\right)-g\left(t,\tilde{u},\tilde{v},{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}\tilde{v}\right)\right| &\le \frac{1}{10}|u-\tilde{u}|+\frac{1}{50}|v-\tilde{v}| \\ &+\frac{1}{100}\left|{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}v-{}_{*}^{1}\mathcal{D}_{0+}^{\frac{1}{3}}\tilde{v}\right|. \end{aligned}$$

Thus the hypothesis **(L2)** is satisfied with $\lambda_1=\dfrac{\pi}{10}$, $\lambda_2=\dfrac{\pi}{100}$, $\lambda_3=\dfrac{\pi}{200}$, $\mu_1=\dfrac{1}{10}$, $\mu_2=\dfrac{1}{50}$, $\mu_3=\dfrac{1}{100}$ and

$$\frac{T^{\rho\alpha}(\lambda_1+\lambda_2)}{(1-\lambda_3)\rho^{\alpha}\Gamma(\alpha+1)}+\frac{T^{\rho\beta}(\mu_1+\mu_2)}{(1-\mu_3)\rho^{\beta}\Gamma(\beta+1)}\simeq 0.5318<1.$$

Therefore, (4.11) is satisfied. Hence, all condition of theorem 4.1 hold. thus the coupled system (4.21) has a unique solution on $[0,1]$.

Exemple 4.2. Consider the following coupled system of Caputo-Katugampola type fractional differential equations

$$\begin{cases} {}^{1}_{*}\mathcal{D}^{\frac{1}{2}}_{0^+}u(t) = \dfrac{1+u(t)+v(t)}{\sqrt{16+t^2}} + \dfrac{\arctan(t)}{2}\left({}^{1}_{*}\mathcal{D}^{\frac{3}{4}}_{0^+}u(t)\right), & t\in[0,1]. \\ {}^{1}_{*}\mathcal{D}^{\frac{1}{3}}_{0^+}v(t) = \dfrac{1+\left(\sqrt[3]{|u(t)|}+\sqrt[3]{|v(t)|}\right)^3}{4e^{t^2}} + \dfrac{\pi\ln\left(\left|{}^{1}_{*}\mathcal{D}^{\frac{4}{5}}_{0^+}v(t)\right|\right)}{8+t^2}, & \\ u(0)=0, v(0)=0, & \end{cases} \tag{4.22}$$

Let

$$f\left(t,u,v,{}^{1}_{*}\mathcal{D}^{\frac{3}{4}}_{0^+}u\right) = \frac{1+u(t)+v(t)}{\sqrt{16+t^2}} + \frac{\arctan(t)}{2}\left({}^{1}_{*}\mathcal{D}^{\frac{3}{4}}_{0^+}u(t)\right).$$
$$g\left(t,u,v,{}^{1}_{*}\mathcal{D}^{\frac{4}{5}}_{0^+}v\right) = \frac{1+\left(\sqrt[3]{|u(t)|}+\sqrt[3]{|v(t)|}\right)^3}{4e^{t^2}} + \frac{\pi\ln\left(\left|{}^{1}_{*}\mathcal{D}^{\frac{4}{5}}_{0^+}v(t)\right|\right)}{8+t^2}.$$

Clearly $f,g:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions and we have

$$\begin{aligned} \left|f\left(t,u,v,{}^{1}_{*}\mathcal{D}^{\frac{3}{4}}_{0^+}u\right)\right| &= \frac{1}{\sqrt{16+t^2}} + \frac{1}{\sqrt{16+t^2}}|u(t)| + \frac{1}{\sqrt{16+t^2}}|v(t)| \\ &\quad + \frac{\arctan(t)}{2}\left|{}^{1}_{*}\mathcal{D}^{\frac{3}{4}}_{0^+}u\right|. \\ \left|g\left(t,u,v,{}^{1}_{*}\mathcal{D}^{\frac{4}{5}}_{0^+}v\right)\right| &= \frac{1}{4}e^{-t^2} + \frac{1}{4}e^{-t^2}|u(t)| + \frac{1}{4}e^{-t^2}|v(t)| + \frac{\pi}{8+t^2}\left|{}^{1}_{*}\mathcal{D}^{\frac{4}{5}}_{0^+}v(t)\right|. \end{aligned}$$

Thus the hypothesis **(L3)** is satisfied with

$$\begin{aligned} a_1(t) = b_1(t) = c_1(t) &= \frac{1}{\sqrt{16+t^2}}, \quad & d_1(t) &= \frac{\arctan(t)}{2}, \\ a_2(t) = b_2(t) = c_2(t) &= \frac{1}{4}e^{-t^2}, & d_2(t) &= \frac{\pi}{8+t^2}, \\ a_1^* = b_1^* = c_1^* = a_2^* = b_2^* = c_2^* &= \frac{1}{4}, & d_1^* = d_2^* &= \frac{\pi}{8} < 1, \\ M_1 = M_2 &= \frac{2}{8-\pi}, & N_1 = N_2 &= \frac{4}{8-\pi}, \end{aligned}$$

and the condition (4.15),

$$\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}N_1 + \frac{T^{\rho\beta}}{\rho^{\beta}\Gamma(\beta+1)}N_2 \simeq 0.8753 < 1.$$

It follows from theorem 4.2, that the problem (4.22) has a last one solution on $[0,1]$.

General conclusion

In this memory we presented the existence and uniqueness of solution for the nonlinear boundary value problem, initial value problem of Caputo-Katugampola type fractional differential equation and initial value problem for a coupled system of Caputo-Katugampola type fractional differential equation have been discussed in Banach space $[0, T]$.
For our discussion we have used the some example are presented to illustrate the usefulness our results.

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ملخص:

الهدف من هذه المذكرة، هو دراسة وجود و وحدانية المعادلات التفاضلية الكسرية، باستعمال تقنيات النقطة الصامدة (شايفر، بناخ و شايدر)، سنتطرق في البداية الى دراسة معادلة تفاضلية كسرية من نوع كابوتو كاتوغامبولا، حيث تعرض بعض النتائج المتعلقة بوجود و وحدانية حلها. بعد ذلك سنعالج نظام معادلات تفاضلية كسرية من نوع كابوتو كاتوغامبولا حيث نثبت وجود الحل باستعمال نظرية النقطة الصامدة. وفي الأخير ترفق النتائج المحصل عليها بأمثلة توضيحية.

كلمات مفتاحية: نوع كابوتو، معادلة تفاضلية كسرية ، مسألة القيمة الحدية، الوجود والوحدانية.

Résumé:

L'objectif de cette mémoire est étudier l'existence et l'unicité des solutions de problème aux limites d'équations différentielles fractionnaires de type Caputo-Katugampola. Les résultats dérivés sont basés sur des théorèmes de points fixes (Banach, Schauder, Schaufer). Ensuite nous intéressons à l'étude d'un système d'équation différentielle fractionnaire de type Caputo-Katugampola. Nous établissons ainsi l'existence de la solution en utilisant la méthode du point fixe.
Nous concluons les résultats obtenus par des exemples illustratifs

Mots-Clés: Type Caputo, équation différentielle fractionnaire, problème aux limites, l'existence et l'unicité .

Abstract:

The aim of memory is to study the existence and uniqueness of solution to boundary value problem of nonlinear fractional differential equation with Caputo-Katugampola derivative in bounded domain. The derived results are based on fixed point theorems (Banach's, Schauder's, Schaufer's). Next, we establish the existence of solution to a certain coupled system of Caputo-Katugampola fractional differential equation. We conclude the results obtained by illustrative examples.

Keywords: Caputo type, fractional differential equation, boundary value problem, existence and uniqueness.