UNIVERSITE MOHAMED BOUDIAFDE M'SILA
Faculté des Mathématiques et de l'Informatique
Département de Mathématiques

MEMOIRE DE FIN D'ETUDE

Présenté pour l'obtention du Diplôme de **MASTER**
Domaine : Mathématiques et Informatique
Filière : Mathématiques
Option : Analyse numérique et mathématique

Par

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Sujet

Natural and Cardinal splines collocation method for the resolution of Fredholm integral equation of the second kind

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Promotion : 2018 / 2019

Dedication

I dedicate this modest work:

- To my parents

-To my brothers

- To my sisters

-To my familly

I want to thank all the students of my class

In the end I dedicate this thesis to my colleagues and all those who are dears to me.

Acknowledgements

I would like to sincerely thank my supervisor, Prof.Lakehali Belkacem, for his guidance and support throughout this study, and especially for his confidence in me. I would also like to thank Prof.Nadir Mostefa, I learned from his insight a lot, Also, I would like to thank, Prof.Khirani Amina, in a special way, I express my heartfelt gratefulness for her guide and support that I belived I learned from the best.

To all my friends, thank you for your understanding and encouragement in my many moments of crisis. Your friendship makes my life a wonderful experience. I cannot list all the names here, but you are always on my mind.

Thank you, Lord, for always being there for me.

This thesis is only a beginning of my journey.

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Introduction

The Fredholm's linear integral equation stem from several areas of scientific research. Or, by mathematical modeling of the various problems arising from mathematical physics. Of course there's a lot of methods to solve this equation, our study will be focusing on the cubic splines collocation methods. Due to it's important in our today computing data world.

In this thesis, we present two cubic splines collocations methds, in order to find a good approximation solution to this integral equation.

So our thesis consists of four chapters:

The first chapter: we highlight some of the major terms needed in our chosen study space, which is the Banach space.

The second chapter: we defines The Fredholm integral equation. Then we study the existence and the uniqueness of the solution

The third chapter: we present the cubic splines and the two types (natural and cardinal splines).

The fourth chapter: we compare between the two collocation methods. Then we conclude The most effective one.

CHAPTER I

REVIEW OF FUNCTIONNAL ANALYSIS

1.1 Preliminary Definitions

In this paper, our study will be assumed in the case of continuous functions with the standard of uniform convergence $C\left([a,b],\|.\|_{\infty}\right)$

Definition 1 *We call norm on the space* $C([a,b])$ *all the application noted* $\|.\|$ *defined on* $C[(a,b)]$ *values in* $\mathbb{R}^{+}$ *by :*

$$x \in \mathbb{R}^n \rightarrow \|x\| \in \mathbb{R}^{+}$$

Verifies that for all x and y in $C([a,b])$ and α in $\Bbbk$ the followers conditions:

1. Separation: $\|x\| = 0 \Longleftrightarrow x = 0.$
2. Homogeneity: $\|\lambda x\| = |\lambda| \|x\|.$
3. Triangular inequality: $\|x+y\| \leq \|x\| + \|y\|.$

any vector space with a norme $\|.\|$, we say it's normed vector space.

Definition 2 *The space of continuous functions defined by:*

$$C([a,b]) = \{f : [a,b] \rightarrow \mathbb{R}/f : \textit{continued}\}$$

Equipped with the norme

$$\|f\|_{\infty} = \sup_{x\in[a,b]} |f(x)|, f \in C\left([a,b],\|.\|_{\infty}\right)$$

Is a normed vector space

Definition 3 *A Convergent sequence*

Let $(f_n)_n$ a sequance of continuous functions and $f : [a, b] \to \mathbb{R}$, such that $\|f_n - f\|_\infty \to 0$ so $f \in C([a, b])$, i.e

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in [a, b], |f_n(x) - f(x)| \leq \varepsilon$$

Definition 4 *A sequence of cauchy*

Let (u_n) a sequence of elements of a normed space $C([a, b], \|.\|_\infty)$, we say that the sequence (u_n) of elements of $C([a, b])$ is a convergent sequence in $C([a, b])$, in other words, we say,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall p, q \geq \mathbb{N}_\varepsilon, \qquad \text{we have} \qquad \|u_p - u_q\| < \varepsilon$$

Definition 5 *the complete space*

A normed vector space $C([a, b], \|.\|_\infty)$, is complet, if all cauchy sequences (u_n) of element of $C([a, b])$ is a convergent sequence in $C([a, b])$, In other words

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall p, q \geq \mathbb{N}_\varepsilon, \qquad \text{we have} \qquad \|u_p - u_q\| < \varepsilon$$

Implies the existence of an element $u \in C([a, b])$ such that $u_n \to u$, Any converge sequencee is a sequence of cauchy but the opposite is not true.

We have the following result :

Theorem 6 *The space $C([a, b], \|.\|_\infty)$ is a complet space of the norme $\|.\|$ so, $C([a, b], \|.\|_\infty)$ is of Banach. For much more details, see[12]*

CHAPTER II

INTEGRAL EQUATIONS

In mathematics, integral equations are equations in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations, and some problems may be formulated either way.

2.1 Notion on operators

Definition 7 *Linear operator*

Let E and F two normed spaces. A an operator defined from E into F is linear if it satisfies the following conditions:

1. **Additive condition:**

$$\forall \varphi_1, \varphi_2 \in E. \text{ we hav. } A(\varphi_1 + \varphi_2) = A(\varphi_1) + A(\varphi_2)$$

2. **Homogeneous condition:**

$$\forall \varphi \in E, \text{ and } \lambda \in \Bbbk = (\mathbb{R} \text{ or } \mathbb{C}), \text{ we have } A(\lambda\varphi) = \lambda A(\varphi)$$

Definition 8 *Bounded linear operator*

A linear operator defined from E into F is said to be bounded if there is a positive constant $c > 0$, such that:

$$\|A(\varphi)\|_F \leq c\,\|\varphi\|_E \;\; \forall \varphi \in E$$

Definition 9 *Integral linear operator*

An integral operator or a linear operator with a kernel defined using a parametric integral on certain functional spaces, the general form of an integral operator is given by:

$$A : C\left([a,b], \|.\|_{\infty}\right) \to C\left([a,b], \|.\|_{\infty}\right)$$

$$\varphi = (A\varphi)(x) = \int_a^b k(x,t)\,\varphi(t)\,dt$$

Or

$$k : C\left([a,b], \|.\|_{\infty}\right) \times C\left([a,b], \|.\|_{\infty}\right) \to \mathbb{R}$$

Is a continuous function called the kernel of the integral operator A. This operator is bounded, with

$$\|A\|_{\infty} = \max_{x\in[a,b]} \int_a^b |k(x,t)\,\varphi(t)|\,dt$$

2.2 Fredholm integral equation

We call a lineaire Fredholm integral equation of the second kind of the form

$$\varphi(x) - \int_a^b k(x,t)\;\varphi(t)\,dt = f(x) \tag{1}$$

With $\varphi(x)$ is the unknown function , $k(x,t)$ and $\mathrm{f}(x)$ is given functions, x and t are two real variables traversing the interval $[a,b]$ and λ a numerical factor.

The function $K(x,t)$ is the kernal of the integral equation (1), supposing the kernal $k(x,t)$ is defined in $\Omega\{\alpha \leq x \leq b, \alpha \leq t \leq b\}$ of plan (x,t) and continuous in Ω, or discontinuous such as the integrale

$$\int_a^b \int_a^b |k(x,t)|^2\,dxdt$$

be finished.

If f $(x) \not\cong 0$, the equation (1) is said to be inhomogeneous , otherwise the integral equation (1) is writting

$$\varphi(x) - \int_a^b k(x,t) \ \varphi(t)\, dt = 0 \tag{2}$$

And we say that it is homogeneous.

An integral equation of the form

$$\int_a^b k(x,t) \ \varphi(t)\, dt = f(x) \tag{3}$$

With the unknown function $\varphi(t)$ only intervenes under the sign of integration, is Fredholm's first kind integral equation. The border α and b in the equations (1), (2) and (3) can be both finished and infinite.

We call solution of integral equations (1), (2) and (3) all the function $\varphi(t)$ after it 's substitution in the equation, it becomes an identity in $x \in [a, b]$.

2.3 The existence and the uniqueness of the solution

We say a Fredholm integral linear equation of the second kind is an equation of the form

$$\varphi(x) - \lambda \int_a^b k(x,t)\, \varphi(t)\, dt = f(x) \tag{4}$$

With $\varphi(x)$ is the unknown function,$k(x,t)$ and $f(x)$ a known functions, x and t are two real variables traversing the interval (a, b) and λ is a real parametre.

The function $k(x,t)$ is the kernel of the function (4), suppose that the kernel $k(x,t)$ is defined in the sequare $[a, b]$ $\{a \leq x \leq b, a \leq t \leq b\}$, of the plan (x,t) and continuous in $[a, b]$ or has a discontinuities such as the integral

$$\int_a^b \int_a^b |k(x,t)|^2\, dx dt < \infty \tag{5}$$

If $f(x) \neq 0$, the equation (4) is not homogenous, on the other hand, the integral equation (4) is written

$$\varphi(x) - \int_a^b k(x,t)\,\varphi(t)\,dt = 0$$

It is said that it's homogeneous. see [10]

The integral equation from a continuous kernel $k(x, y)$

$$\varphi(x) - \int_a^b k(x,t)\,\varphi(t)\,dt = f(x)$$

We can write it also in the form:

$$\varphi - A\varphi = f$$

$$(I - A)\,\varphi = f$$

The existence of the solution for this equation depends on the existence of the inverse of the operator $(I - A)$. Since A is compact, then the inverse operator of $(I - A)$ exist. see [4].

2.4 *Numerical resolution*

There's a lot of methods, in which we can use to solve the Fredholm integral equation:

1. Polynomial interpolation of lagrange.
2. Nyström method.
3. Degenerate kernel method.
4. Projection methods that contains two types :

(a) Galerkin method

(b) collocation method.

But in our search we gonna focus on the Collocation method.

2.4.1 Projection methods

2.4.1.1 General theory

We write the integral equation (1) in the operator form

$$(\lambda - K)\, x = y$$

and the operator K is assumed to be compact on a Banach space χ to χ.

the most popular choices are $C(D)$ and $L^2(D)$. For Galerkin's method and its generalizations, sobolev spaces $H^r(D)$ are also used commonly, with $H^0(D) = L^2(D)$.

In practise, we choose a sequence of finite dimentional subspaces $\chi_n \subset \chi$,$n \geq 1$, with x_n having dimension d_n. Let χ_n have a basis $\{\phi_1, ..., \phi_d\}$, with $d \equiv d_n$ for notational simplicity. We seek a function $x_n \in \chi_n$, and it can be written as

$$x_n(t) = \sum_{j=1}^{d} c_j \ \phi_j \ (t), \qquad t \in d \tag{6}$$

This is substituted into (1), and the coefficients $\{c_1, ..., c_d\}$ are determined by forcing the equation to be almost exact in some sense. for later use, intoduce

$$r_n(t) = \lambda x_n(t) - \int_D K(t,s) x_n(s) ds - y(t)$$

$$= \sum_{j=1}^{d} c_j \left\{ \lambda\phi_j(t) - \int_D \ K(t,s)\phi_j(s) ds \right\} - y(t), \qquad t \in D \tag{7}$$

This is called the residual in the approximation of the equation when using $x \approx x_n$.symbolically.

$$r_n = (\lambda - k)\, x_n - y$$

The coefficients $\{c_1, ...c_d\}$ are chosen by forcing $r_n(t)$ to be approximately zero in some sense. The hope and expectation are that the resulting function $x_n(t)$ will be a good approximation of the true solution $x(t)$.

1. **Galerkin method:** Set to 0 the orthogonal L^2 projection of r into χ_n. $\mathcal{P}_n$ is an orthogonal projection

2. **Collocation method:**

Set to 0 the values of $r(t)$ at a set of points $\{t_1, ..., t_n\} \subseteq [a, b]$. $\mathcal{P}_n$ is an interpolatory projection.

We Pick distinct node points $(t_1, ..., t_d) \in D$, such that

$$r_n(t_i) = 0, \qquad i = 1, ..., d_n$$

This lead to determining $\{c_1, ..., c_d\}$ as the solution of the linear system.

$$\sum_{j=1}^{d} c_j \left\{ \lambda\phi_j(t_i) - \int_D k(t_i, s)\, \phi_j(s)\, ds \right\} = y(t_i), \qquad i = 1, ..., d_n$$

We should have written the node points as $\{t_{1,n}, ..., t_{d,n}\}$; but for national simplicity, the explicit dependence on n has been suppressed, to be understood only implicitly. We introduce a projection operator $\mathcal{P}_n$ that maps $\mathcal{X} = C(D)$ into $\mathcal{X}_n$. Given $x \in C(D)$, define $\mathcal{P}_n x$ to be that element of $\mathcal{X}_n$ that interpolates x at the nodes $\{t_{1,n}, ..., t_{d,n}\}$. This mean writting

$$\mathcal{P}_n x(t) = \sum_{j=1}^{d} \alpha_j\ \phi_j(t)$$

With the coefficients $\{\alpha_j\}$ determined by solving the linear system

$$\sum_{j=1}^{d} \alpha_j\ \phi_j(t) = x(t_i), \qquad i = 1, ..., d_n$$

see [4]. This linear system has a unique solution if

$$\det\left[\phi_j(t_i)\right] \neq 0$$

To see more clearly that $\mathcal{P}_n$ is linear, and to give a more explicit formula, we introduce a new set of basis function. For each i , $1 \leq i \leq d_n$, let $\ell_i \in \mathcal{X}_n$ be that element that satisfies the interpolation conditions

$$\ell_i(t_j) = \delta_{ij}, \qquad i = 1, ..., d_n$$

We can write

$$\mathcal{P}_n x\left(t\right) = \sum_{j=1}^{d} x(t_j)\ell_j\left(t\right), \qquad t \in D$$

Clearly, $\mathcal{P}_n$ is linear and finite rank. In addition, as an operator on $C\left(D\right)$ to $C\left(D\right)$,

$$\|\mathcal{P}_n\| = \max_{t\in D} \sum_{j=1}^{d} x(t_j)\ell_j\left(t\right)$$

CHAPTER III

SPLINE FUNCTIONS

3.1 Numerical interpolation

In the mathematical field of numerical analysis, interpolation is a method of constructing new data points within the range of a discrete set of known data points. In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate, i.e., estimate the value of that function for an intermediate value of the independent variable. A closely related problem is the approximation of a complicated function by a simple function. Suppose the formula for some given function is known, but too complicated to evaluate efficiently. A few data points from the original function can be interpolated to produce a simpler function which is still fairly close to the original. The resulting gain in simplicity may outweigh the loss from interpolation error.

There are multiple families of interpolation functions. Such as:

1. Piecewise constant interpolation

2. Linear interpolation

3. Polynomial interpolation

4. Spline interpolation

Our study is going to be about spline function

Remember that linear interpolation uses a linear function for each of intervals $[x_k, x_{k+1}]$. Spline interpolation uses low-degree polynomials in each of the intervals, and chooses the polynomial pieces such that they fit smoothly together. The resulting function is called a spline. Like polynomial interpolation, spline interpolation incurs a smaller error than linear interpolation and the interpolant is smoother. However, the interpolant is easier to evaluate than the high-degree polynomials used in polynomial interpolation. However, the global nature of the basis functions leads to ill-conditioning. This is completely mitigated by using splines of compact support.

3.2 Cubic spline

A cubic spline is a spline constructed of piecewise third-order polynomials which pass through a set of control points. The second derivative of each polynomial is commonly set to zero at the endpoints, since this provides a boundary condition that completes the system of equations. This produces a so-called "natural" cubic spline and leads to a simple tridiagonal system which can be solved easily to give the coefficients of the polynomials. However, this choice is not the only one possible, and other boundary conditions can be used instead.

3.2.1 Process

The essential idea is to fit a piecewise function of the form

$$S\ (x) = \begin{cases} s_1(x) & \text{if} \quad x_1 < x \leq x_2 \\ s_2(x) & \text{if} \quad x_2 < x \leq x_3 \\ & \vdots \\ s_{n-1}(x) & \text{if} \quad x_{n-1} < x \leq x_n \end{cases} \tag{8}$$

Such that

$$S(x) = \sum_{i=1}^{n-1} s_i(x), i = 1, \ldots.., n$$

Where s_i is a cubic polynomial defined by

$$s_i(x) = a_i (x - x_i)^3 + b_i (x - x_i)^2 + c_i (x - x_i) + d_i \tag{9}$$

for $i = 1, 2, ..., n - 1$.

The first and second derivatives of these $n - 1$ equations are fundamental to this process, and they are

$$s_i'(x) = 3a_i (x - x_i)^2 + 2b_i (x - x_i) + c_i \tag{10}$$

$$s_i''(x) = 6a (x - x_i) + b_i \tag{11}$$

3.2.2 Properties of cubic splines

Our spline will need to confirm to the following stipulations.

1. the piecewise function $S(x)$ will interpolate all data points.

2. $S(x)$ will be continuous on the interval $[x_1, x_n]$.

3. $S'(x)$will be continuous on the interval $[x_1, x_n]$.

4. $S''(x)$will be continuous on the interval $[x_1, x_n]$.

Since the piecewise function $S(x)$ will interpolate all of the data points, see [9]

$$S(x_i) = y_i \tag{12}$$

For $i = 1, 2, ..., n - 1$.Since $x_i \in [x_i, x_{i+1}]$, $S(x_i) = s_i(x_i)$ and we can use (9) to produce

$$y_i = s_i(x_i) \tag{13}$$

$$y_i = a_i (x - x_i)^3 + b_i (x - x_i)^2 + c_i (x - x_i) + d_i$$

$$y_i = d_i$$

For $i = 1, 2, ..., n - 1$.

Since the curve $S(x)$ must be continuous across its entire interval, it can be concluded that each sub-function must join at the data points, so

$$s_i(x_i) = s_{i-1}(x_i) \tag{14}$$

For $i = 1, 2, ..., n - 1$.

From (9),

$$s_i(x_i) = d_i$$

and

$$s_{i-1}(x_i) = a_{i-1}(x - x_i)^3 + b_{i-1}(x - x_i)^2 + c_{i-1}(x - x_i) + d_{i-1} \tag{15}$$

So

$$d_i = a_{i-1}(x - x_i)^3 + b_{i-1}(x - x_i)^2 + c_{i-1}(x - x_i) + d_{i-1} \tag{16}$$

For $i = 1, 2, ..., n - 1$. Letting $h = x_i - x_{i-1}$ in(15), we have

$$d_i = a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1} \tag{17}$$

For $i = 1, 2, ..., n - 1$.

Also, to make the curve smooth acrass the interval, the derivatives must be equal at the data points , that is

$$s_i'(x_i) = s_{i-1}'(x_i) \tag{18}$$

However, by (10)

$$s_i'(x_i) = c_i$$

and

$$s_{i-1}'(x_i) = 3a_{i-1}(x - x_i)^2 + 2b_{i-1}(x - x_i) + c_{i-1}$$

So

$$c_i = 3a_{i-1}(x - x_i)^2 + 2b_{i-1}(x - x_i) + c_{i-1} \tag{19}$$

Again, letting $h = x_i - x_{i-1}$, we arrive at

$$c_i = 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1} \tag{20}$$

For $i = 1, 2, ..., n - 1$.

From (11), $s_i''(x) = 6a_i(x - x_i) + b_i$, so

$$s_i''(x) = 6a_i(x - x_i) + b_i \tag{21}$$

$$s_i''(x_i) = 6a_i(x - x_i) + b_i$$

$$s_i''(x_i) = 2b_i$$

For $i = 1, 2, ..., n - 2$.

Lastly, since $s_i''(x)$ has to be continuous across the interval $[a, b]$, $s_i''(x_i) = s_{i-1}''(x_i)$ for $i = 1, 2, ..., n - 1$. This and (21) lead us to the equation

$$s_i''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \tag{22}$$

$$s_{i+1}''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \tag{23}$$

and, letting $h = x_{i+1} - x_i$ and using the conclusion from (21) and (23) ,

$$s_{i+1}''(x_{i+1}) = 6a_i(x_{i+1} - x_i) + 2b_i \tag{24}$$

$$2b_{i+1} = 6a_ih + 2b_i \tag{25}$$

These equations can be much simplified by substituting M_i and y_i This makes the determination of the weights a_i , b_i , c_i , and d_i a much easier task. Each b_i can be represented by

$$s_i''(x_i) = 2b_i \tag{26}$$

$$M_i = 2b_i$$

$$b_i = \frac{M_i}{2}$$

and d_i has already been determined to be

$$d_i = y_i \tag{27}$$

Similarly, using equation a_i can be-written as

$$2b_{i+1} = 6a_ih + 2b_i \tag{28}$$

$$6a_ih = 2b_{i+1} - 2b_i$$

$$a_i = \frac{2b_{i+1} - 2b_i}{6h}$$

$$a_i = \frac{2\left(\frac{M_{i+1}}{2}\right) - 2\left(\frac{M_i}{2}\right)}{6h}$$

$$a_i = \frac{M_{i+1} - M_i}{6h}$$

and c_i can be-written as

$$d_{i+1} = a_ih^3 + b_ih^2 + c_ih + d_i \tag{29}$$

$$c_ih = -a_ih^3 - b_ih^2 - d_i + d_{i+1}$$

$$c_i = \frac{-a_ih^3 - b_ih^2 - d_i + d_{i+1}}{h}$$

$$c_i = \frac{-a_ih^3 - b_ih^2}{h} + \frac{-d_i + d_{i+1}}{h}$$

$$c_i = \left(-a_ih^2 - b_ih\right) - \frac{d_i - d_{i+1}}{h}$$

$$c_i = -\left(\frac{M_{i+1} - M_i}{6h}h^2 + \frac{M_i}{2}h\right) - \frac{y_i - y_{i+1}}{h}$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - M_i}{6}h + \frac{3M_i}{6}h\right)$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} - M_i + 3M_i}{6}\right)h$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} + 2M_i}{6}\right)h$$

We now have our equations, for determining the weights for our $(n-1)$ equations

$$a_i = \frac{M_{i+1} - M_i}{6h} \tag{30}$$

$$b_i = \frac{M_i}{2}$$

$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} + 2M_i}{6}\right)h$$

$$d_i = y_i$$

This systems can be handled more conveniently, by putting them into matrix form as follows

$$c_{i+1} = 3a_i h^2 + 2b_i h + c_i \tag{31}$$

$$3\left(\tfrac{M_{i+1}-M_i}{6h}\right)h^2 + 2\left(\tfrac{M_i}{2}\right)h + \tfrac{y_{i+1}-y_i}{h} - \left(\tfrac{M_{i+1}+2M_i}{6}\right)h$$
$$= \tfrac{y_{i+2}-y_{i+1}}{h} - \left(\tfrac{M_{i+2}+2M_{i+1}}{6}\right)h$$

$$3\left(\tfrac{M_{i+1}-M_i}{6h}\right)h^2 + 2\left(\tfrac{M_i}{2}\right)h - \left(\tfrac{M_{i+1}+2M_i}{6}\right)h + \left(\tfrac{M_{i+2}+2M_{i+1}}{6}\right)h = \tfrac{y_{i+2}-y_{i+1}}{h} - \tfrac{y_{i+1}-y_i}{h}$$

$$h\ \left(\tfrac{3M_{i+1}-3M_i}{6} + \tfrac{6M_i}{6} - \left(\tfrac{M_{i+1}+2M_i}{6}\right) + \left(\tfrac{M_{i+2}+2M_{i+1}}{6}\right)\right) = \tfrac{y_{i+2}-2y_{i+1}+y_i}{h^2}$$

$$\frac{h}{6}\left(M_i + 4M_{i+1} + M_{i+2}\right) = \tfrac{y_i-2y_{i+1}+y_{i+2}}{h^2}$$

$$M_i + 4M_{i+1} + M_{i+2} = \tfrac{y_i-2y_{i+1}+y_{i+2}}{h^2};\ i = 1, 2, ..., n-1$$

It's a linear system.of $(n-1)$ equations.

Which leads to the matrix equation

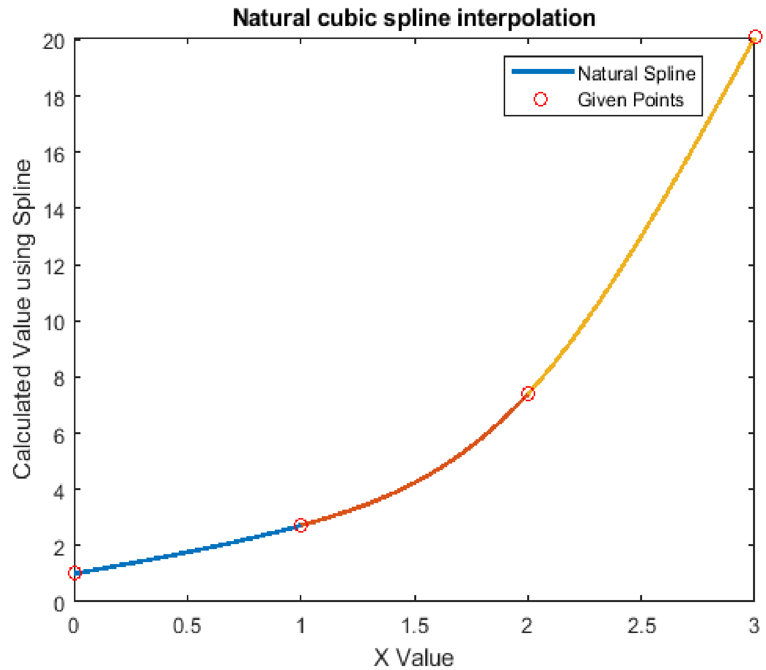
$$\begin{bmatrix} 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \tfrac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Note that this system has$(n-2)$ rows and n columns, and is therefore underdetermined. In order to generate a unique cubic spline, two other conditions must be imposed upon the system. This leads us to our next section.

3.3 Types of splines

3.3.1 Natural cubic splines

Definition 10 *Natural cubic spline is a piece-wise cubic polynomial that is twice continuously differentiable. It is considerably 'stiffer' than a polynomial in the sense that it has less tendency to oscillate between data points.*



Let x_1, x_2, x_3, x_4 be given nodes (strictly increasing) and let (y_1, y_2, y_3, y_4) be given values (arbitrary).Our goal is to produce a function $s(x)$ with the following properties:

1. $s(x_k) = y_k$, $k = 1, .2, 3, 4$

2. $s(x)$ is two times continuously differentiable on $[x_1, x_4]$,

3. $\int_{x_1}^{x_4} s_i''(x)^2 \, dx$ is as small as possible.There is a unique function $s(x)$ that has the required properties .It turns out to also satisfy

4. $s(x)$ restricted to $[x_k , x_{k+1}]$ is a cubic polynomial in the subinterval, $k = 1, 2, 3,$ then we have a particular condition, in order to make the number of unknowns equal to the number of equations.

5. $s''(x_1) = s''(x_4) = 0.$

This function is called the natural cubic spline. It is arbitrarily smooth on every open subinterval $(x_k\ ,x_{k+1})$ and

(a) $s'(x_k)$ exists $(s'(x_k-) = s'(x_k+))$, $k = 2, 3,$

(b) $s''(x_k)$ exists $(s''(x_k-) = s''(x_k+))$, $k = 2, 3,$

Construction:

Introduce the parameters $M_1 = s''(x_1)$, $M_2 = s''(x_2)$, $M_3 = s''(x_3)$, $M_4 = s''(x_4)$. M_2, M_3 are to be specified and M_1, M_4 are zero. write

$$s''(x)\,|_{[x_k\ ,\ x_{k+1}]} = \frac{(x_{k+1}-x)\ M_k + (x - x_k)\ M_{k+1}}{x_{k+1} - x_k}, \qquad k = 1,\ 2,\ 3.$$

so that $s''(x)$ is a piecewise linear continuous function . Integrate $s''(x)$ twice and write the result in the form

$$s(x)\,|_{[x_k\ ,\ x_{k+1}]} = \frac{(x_{k+1}-x)^3 M_k + (x - x_k)^3 M_{k+1}}{6(x_{k+1} - x_k)} + \frac{(x_{k+1}-x)\ y_k + (x - x_k)\ y_{k+1}}{x_{k+1} - x_k}$$

$$-\frac{1}{6}(x_{k+1} - x_k)\left((x_{k+1} - x)\ M_k + (x - x_k)\ M_{k+1}\right), k = 1, 2, 3$$

Thus $s(x)$ is piecewise cubic, $s''(x)$ is piecewise linear and continuous , and it is easy to check that $s(x_k) = y_k$, $k = 1,\ 2,\ 3,\ 4.$

We have:

$$s'(x)\,|_{[x_k\ ,\ x_{k+1}]} = \frac{(x_{k+1}-x)^2 M_k + (x - x_k)^2 M_{k+1}}{2(x_{k+1}-x_k)} + \frac{y_{k+1}-y_k}{x_{k+1}-x_k} - \frac{x_{k+1}-x_k}{6}\left(M_{k+1} - M_k\right)$$

Examine two abutting intervals $[x_{k-1}, x_k]$, $[x_k, x_{k+1}]$ $(k = 2, 3)$ and compute $s'(x_k-)$ and $s'(x_k+)$

$$s'(x_k+) = -\frac{1}{2}(x_{k+1} - x_k)M_k + \frac{y_{k+1}-y_k}{x_{k+1}-x_k} - \frac{x_{k+1}-x_k}{6}(M_{k+1} - M_k)$$

$$= -\frac{x_{k+1}-x_k}{3}M_k - \frac{x_{k+1}-x_k}{6}M_{k+1} + \frac{y_{k+1}-y_k}{x_{k+1}-x_k},$$

$$s'(x_k-) = \frac{x_{k+1}-x_k}{3}M_k + \frac{x_{k+1}-x_k}{6}M_{k-1} + \frac{y_k-y_{k-1}}{x_k-x_{k-1}}.$$

Equate one-sided derivatives:

$$\frac{x_k-x_{k-1}}{6}M_{k-1} + \frac{x_k-x_{k-1}}{3}M_k + \frac{y_k-y_{k-1}}{x_k-x_{k-1}} = -\frac{x_{k+1}-x_k}{3}M_k - \frac{x_{k+1}-x_k}{6}M_{k+1} + \frac{y_{k+1}-y_k}{x_{k+1}-x_k}$$

or

$$\frac{x_k-x_{k-1}}{6}M_{k-1} + \frac{x_{k+1}-x_k}{3}M_k - \frac{x_{k+1}-x_k}{6}M_{k+1} = \frac{y_{k+1}-y_k}{x_{k+1}-x_k} - \frac{y_k-y_{k-1}}{x_k-x_{k-1}}$$

In general , this is a tridiagonal linear system of $(n-2)$ equation in $(n-2)$ unknowns$(M_2, ..., M_{n-1})$. But for $n = 4$ we just have

$$\frac{x_3-x_1}{3}M_2 + \frac{x_3-x_2}{6}M_3 = \frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{x_3-x_2}{6}M_2 + \frac{x_4-x_2}{3}M_3 = \frac{y_4-y_3}{x_4-x_3} - \frac{y_3-y_2}{x_3-x_2}$$

The values of M_2 , M_3 are uniquely determined by this equations .

If the nodes are equispaced, say $h = x_{k+1} - x_k$, we have

$$4M_2 + M_3 = \frac{6}{h^2}(y_1 - 2y_2 - y_3)$$

$$M_2 + 4M_3 = \frac{6}{h^2}(y_2 - 2y_3 - y_4)$$

In this case,

$$M_2 = \frac{2}{5h^2}\left(4y_1 - 9y_2 - 6y_3 - y_4\right), \quad M_3 = \frac{2}{5h^2}\left(-y_1 + 6y_2 - 9y_3 + 4y_4\right)$$

Remark 11 *A direct calculation give*

$$\int_{x_1}^{x_4} s''(x)^2 dx = \frac{1}{3}\left(\begin{array}{c} (x_2 - x_1) M_1^2 \; + (x_2 - x_1) M_1 M_2 \; + (x_3 - x_1) M_2^2 \; + \\ (x_3 - x_2) M_2 M_3 + (x_4 - x_2) M_3^2 \; + (x_4 - x_3) M_3 M_4 \; + (x_4 - x_3) \; M_4^3 \end{array}\right)$$

$$= \frac{1}{3}(\,(x_3 - x_1) M_2^2 + (x_3 - x_2) M_2 M_3 + (x_4 - x_2) M_3^2)$$

3.3.2 Cardinal cubic splines

Hermite cubic spline:

A cubic Hermite spline is a spline with each polynomial in Hermit form. Basically, each interval will use two control points and two tangents. these tangent values can be defined by the user, or defined automatically by more surrounding control points, there three types of Hermite cubic splines.

1. Catmull-Rom spline
2. Kochanek-Bartels spline
3. Cardinal spline

Our study is about cardinal splines. Also, we can call it a B-spline.

A cardinal spline is a sequence of individual curves joined to form a larger curve. The spline is specified by an array of points and a tension parameter. It's given by

$$B_{i,0}(t) = \begin{cases} 1 & \text{if} \quad t_i < t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{i,x}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(x) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t) \tag{32}$$

A cardinal spline passes smoothly through each point in the array, there are no sharp corners and no abrupt changes in the tightness of the curve.

A shape parameter called *tension* causes a spline to bend more sharply, it increase the magnitude of the tangent vector of the knots. The cubic cardinal spline generalization ot the Catmull-Rom spline adds tension with the parameter labeled α in the following defining matrix.Given the control points y_0, y_1, y_2 , and we can use a Cardinal spline to interpolate between $y_1$and y_2. Assuming uniformly spaced control points.

This belongs to a class of spline that is called deffective spline functions. The name indicates that such a spline function does not satisfy the definition of a standard spline function. Acubic cardinal hermite spline interpolates the given data, whose first derative is continuous at nodes, but the second derivative might be discontinuous.

Consider the data set (t_k, y_k) for $0 \leq k \leq n$, as well as the derivatives y'_k. On each interval $[t_k, t_{k+1}]$, we seek a cubic polynomial $S_k(t_{k+1})$, such that:

$$S_k(t_k) = y_k,\ S_k(t_{k+1}) = y_{k+1},\ S'_k(t_k) = y'_k,\ S'_k(t_{k+1}) = y'_{k+1}.$$

Recall the cubic Hermite basis functions for the interval $\overline{x} \in [0, 1]$

$$\begin{aligned}
\overline{\phi}_0(\overline{x}) &= 2\overline{x}^3 - 3\overline{x}^2 + 1 \quad , \quad \overline{\phi}_1(\overline{x}) = -2\overline{x}^3 - 3\overline{x}^2 \\
\overline{\psi}_0(\overline{x}) &= \overline{x}^3 - 2\overline{x}^2 + \overline{x} \quad , \quad \overline{\psi}_1(\overline{x}) = \overline{x}^3 - \overline{x}^2
\end{aligned}$$

On interval $[t_k, t_{k+1}]$, the basis functions are:

$$\begin{aligned}
\phi_0(x) &= 2\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^3 - 3\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^2 + 1 \\
\phi_1(x) &= -2\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^3 + 3\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^2 \\
\psi_0(x) &= (t_{k+1} - t_k)\left[\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^3 - 2\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^2 + \left(\frac{x - t_k}{t_{k+1} - t_k}\right)\right] \\
\psi_1(x) &= (t_{k+1} - t_k)\left[\left(\frac{x - t_k}{t_{k+1} - t_k}\right)^3 - \left(\frac{x - t_k}{t_{k+1} - t_k}\right)^2\right]
\end{aligned}$$

One can easily verify that the basis functions, have the following properties

$$\begin{array}{llll}\phi_0(t_k) = 1 & , \quad \phi_0(t_{k+1}) = 0 & , \quad \phi_0'(t_k) = 0 & , \quad \phi_0'(t_{k+1}) = 0 \\ \phi_0(t_k) = 0 & , \quad \phi_0(t_{k+1}) = 1 & , \quad \phi_0'(t_k) = 0 & , \quad \phi_0'(t_{k+1}) = 0 \\ \psi_0(t_k) = 0 & , \quad \psi_0(t_{k+1}) = 0 & , \quad \psi_0'(t_k) = 1 & , \quad \psi_0'(t_{k+1}) = 0 \\ \psi_0(t_k) = 0 & , \quad \psi_0(t_{k+1}) = 0 & , \quad \psi_0'(t_k) = 0 & , \quad \psi_0'(t_{k+1}) = 1\end{array}$$

We can now assemble the piecewise polynomial function as

$$S_k(x) = y_k\ \phi_0(x) + y_{k+1}\ \phi_1(x) + y_k'\psi_0(x) + y_{k+1}'\psi_1(x) \qquad , \qquad t_k \leq x \leq t_{k+1}$$

Very often, one only has the data set (t_k, y_k), without the derivatives. There are various approximations. For exemple, one can approximate the first derivative at t_k as

$$y_k' = \frac{y_{k+1} - y_{k-1}}{t_{k+1} - t_k} \qquad , \qquad 1 \leq k \leq n-1$$

and

$$y_0' = \frac{y_1 - y_0}{t_1 - t_0} \qquad , \qquad y_n' = \frac{y_n - y_{n-1}}{t_n - t_{n-1}}$$

for moe details, see [11]

3.4 The cubic spline collocation method

3.4.1 Natural cubic spline

Let $\Pi_n = \{t_i = a + ih, i = 0, 1, ..., n\}$ be a uniform partition of the interval I, where the stepsize $h = \frac{b-a}{n}$,and let $t_{-3} = t_{-2} = t_{-1} = t_0$, $t_{n+3} = t_{n+2} = t_{n+1} = t_{n.}$. We denote by $S \in S_3^2(I, \Pi_n)$ the space of C^2 cubic splines with knot set Π_n and multiple set at the endpoints. It has dimension $(n+3)$ and the cubic B-splines $\{B_j, i = 0, 1, ..., n+2\}$ with support $[t_{i-3}, t_{i+1}]$ form a basis of this space. Let $S \in S_3^2(I, \Pi_n)$ be a cubic spline generated by boundary conditions interpolating the $(n+1)$ values $x_i = x(t_i)$. $i = 0, 1, ..., n$ i.e. $S(t_i) = y_i$, With natural boundary conditions $S''(a) = S''(b) = 0$.

Then. The restrictions of S to the intervals $\sigma_i = [t_i, t_{i+1}]\, , i = 0, \ldots.., n-1$. Can be written in the form:

$$S_i\,(t) = \tfrac{M_{i+1}}{6h}\,(t-t_i)^3 + \tfrac{M_i}{6h}\,(t_{i+1}-t)^3 + \left(\tfrac{y_{i+1}}{h} - \tfrac{h}{6}M_{i+1}\right)(t-t_i) + \left(\tfrac{y_i}{h} - \tfrac{h}{6}M_i\right)(t_{i+1}-t)$$

Where $M_i = S''\,(t_i)\, , i = 0, \ldots, n.$

The values z_i are determined as the solutions of the following linear system:

$$M_{i-1} + 4M_i + M_{i+1} = \tfrac{6}{h^2}\,(y_{i-1} - 2y_i + y_{i+1})\, , i = 1, \ldots\ldots., n-1,$$

With $M_0 = S''\,(a) = M_n = S''\,(b) = 0.$

In the matrix notation . The above system has the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{pmatrix} = \tfrac{6}{h^2} \begin{pmatrix} 0 \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ \vdots \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + xy_n \\ 0 \end{pmatrix}$$

We obtain for all $i = 0, \ldots\ldots, n$.. Also, for more details See [3]

$$x_i = f_i + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} k\,(t_i, x)\,\varphi_j\,(x)\,dx + O\left(h^4\right)$$

$$= f_i + \sum_{j=0}^{n-1} \tfrac{M_{j+1}}{6h} \underbrace{\int_{t_j}^{t_{j+1}} k\,(t_i, x)\,(x - t_j)^3\,dx}_{a_{i,j+1}} + \sum_{j=0}^{n-1} \tfrac{M_{j+1}}{6h} \underbrace{\int_{t_j}^{t_{j+1}} k\,(t_i, x)\,(t_{j+1} - x)^3\,dx}_{b_{i,j+1}}$$

$$+\sum_{j=0}^{n-1}\left(\tfrac{y_{j+1}}{h}-\tfrac{h}{6}M_{j+1}\right)\underbrace{\int_{t_j}^{t_{j+1}}k\left(t_i,x\right)\left(x-t_j\right)dx}_{c_{i,j+1}}+\sum_{j=0}^{n-1}\left(\tfrac{y_j}{h}-\tfrac{h}{6}M_j\right)\underbrace{\int_{t_j}^{t_{j+1}}k\left(t_i,x\right)\left(t_{j+1}-x\right)dx}_{b_{i,j}}+O\left(h^4\right)$$

$$=f_i+\tfrac{1}{6h}\sum_{j=0}^{n}M_ja_{i,j}+\frac{1}{6h}\sum_{j=0}^{n}M_jb_{i,j}-\frac{1}{6h}\sum_{j=0}^{n}M_jc_{i,j}+\tfrac{1}{6h}y_jc_{i,j}+\tfrac{1}{6h}\sum_{j=0}^{n}y_jd_{i,j}-\tfrac{h}{6}\sum_{j=0}^{n}M_jd_{i,j}+O\left(h^4\right)$$

Such that $a_{i,0}=b_{i,n}=c_{i,0}=d_{i,n}=0$ for $i=0,....,n$

Now we approximate$y_i$by $\widehat{y_i}$ andM_i by $\widehat{M_i}$ such that $\widehat{y_i}$ and $\widehat{M_i}$, and for all $i=0,....,n$, we have

$$\widehat{Y_i}=f_i+\tfrac{1}{6h}\sum_{j=0}^{n}\widehat{M_j}a_{i,j}+\tfrac{1}{6h}\sum_{j=0}^{n}\widehat{M_j}b_{i,j}-\tfrac{1}{6h}\sum_{j=0}^{n}\widehat{M_j}c_{i,j}+\tfrac{1}{6h}\sum_{j=0}^{n}\widehat{y_j}c_{i,j}+\tfrac{1}{6h}\sum_{j=0}^{n}\widehat{y_j}d_{i,j}-\tfrac{h}{6}\sum_{j=0}^{n}\widehat{M_j}d_{i,j}$$

In the matrix notation. we get

$$\widehat{Y}=F+\tfrac{1}{6h}A\widehat{M}+\tfrac{1}{6h}B\widehat{M}-\tfrac{h}{6}C\widehat{M}+\tfrac{1}{h}C\widehat{y}+\tfrac{1}{h}D\widehat{y}-\tfrac{h}{6}D\widehat{M}$$

Where $\widehat{M}=\left(\widehat{M_0},.....,\widehat{M_n}\right)^T,\widehat{y}=\left(\widehat{y_0},.....,\widehat{y_n}\right)^T,F=\left(f_0,.....,f_n\right)^T,A=a_{i,j},B=b_{i,j},C=c_{i,j}$ ana $D=d_{i,j}$. It follows that

$$\left(I-\tfrac{1}{h}\left(C+D\right)\right)\widehat{x}=F+\tfrac{1}{6}\underbrace{\left(\frac{1}{h}A+\frac{1}{h}B-hC-hD\right)}_{E_h}\widehat{z}=F+\tfrac{1}{h^2}E_h\left(M_1^{-1}-2M_2^{-1}+M_3^{-1}\right)\widehat{y}$$

Such that $M^{-1}=\left(u_{i,j}\right)_{1\leq i,j\leq n+1}$,

$$M_1^{-1}=\begin{pmatrix} u_{1,2} & \cdots & u_{1,n} & 0 & 0\\ u_{2,2} & \cdots & u_{2,n} & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ u_{(n+1),2} & \cdots & u_{(n+1),n} & 0 & 0 \end{pmatrix}$$

and

$$M_2^{-1} = \begin{pmatrix} 0 & u_{1,2} & \dots & u_{1,n} & 0 \\ 0 & u_{2,2} & \dots & u_{2,n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & u_{(n+1),2} & \dots & u_{(n+1),n} & 0 \end{pmatrix},$$

$$M_3^{-1} = \begin{pmatrix} 0 & 0 & u_{1,2} & \dots & u_{1,n} \\ 0 & 0 & u_{2,2} & \dots & u_{2,n} \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & u_{(n+1),2} & \dots & u_{(n+1),n} \end{pmatrix}$$

Hence

$$\left(I - \frac{1}{h}(C+D) - \frac{1}{h^2}E_h\left(M_1^{-1} - 2M_2^{-1} + M_3^{-1}\right)\right)\widehat{y} = F.$$

Finally. We approximate the solution x by the natural spline function $\widehat{S}$ such that $\widehat{S} = \widehat{S}_i$ on $\sigma_i, i = 0, ...n-1$, where

$$\widehat{S}_i(t) = \frac{\widehat{M_{i+1}}}{6h}(t-t_i)^3 + \frac{\widehat{M_i}}{6h}(t_{i+1}-t)^3 + \left(\frac{1}{h}\widehat{y}_{i+1} - \frac{h}{6}\widehat{M}_{i+1}\right)(t-t_i) + \left(\frac{1}{h}\widehat{y}_i - \frac{h}{6}\widehat{M}_i\right)(t_{i+1}-t_i).$$

3.4.2 Cardinal cubic spline

To develop the collocation methods Based on a cardinal cubic spline for the solution of Fredholm integral equation of the second kind, let π be an uniform partition in the interval $[a,b]$ $\pi : \{a = t_0 < t_1 < ... < t_{n+2} = b\}$,where $h = \frac{b-a}{n+2}, t_i = a + ih, t_i = x_i, i = 0, ..., n+2$.Where P_3 is the class of cubic polynomials. By introducing knots $t_{-2} < t_{-1} < t_0$ and $t_{n+2} < t_{n+3} < t_{n+4}$ and the function $B_i(t)$. defined by

$$B_i(t) = \frac{1}{h^3}\begin{cases} (t-t_{i-2})^3, & \text{if } t \in [t_{i-2}, t_{i-1}] \\ h^3 + 3h^2(t-t_{i-1}) + 3h(t-t_{i-1})^2 - 3(t-t_{i-1})^3, & \text{if } t \in [t_{i-1}, t_i] \\ h^3 + 3h^2(t_{i+1}-t) + 3h(t_{i+1}-t)^2 - 3(t_{i+1}-t)^3, & \text{if } t \in [t_i, t_{i+1}] \\ (t_{i+2}-t)^3, & \text{if } t \in [t_{i+1}, t_{i+2}] \\ 0, \text{otherwise} & \end{cases}$$

and

$$S\left(t\right)=\sum_{i=-2}^{n+2}c_{i}B_{i}\left(t\right) \tag{33}$$

See the formula (32)

By using cardinal cubic spline (33) we can approximated the solution and also we can approximate the Fredholm integral equation by Newton-cotes type methods , when n is even then we have $h=\frac{b-a}{n+2}$ the Simpson rule can be used

$$\int_a^b f(x)dx = \frac{\Delta x}{3}\left[f\ (x_0)+4f\ (x_1)+2f\ (x_2)+4f\ (x_3)+\ldots+2f\ (x_{n-2})+4f\ (x_{n-1})+f\ (x_n)\right]$$

and when n is odd we have to use the three-eighth rule. see [6]

$$\sum_{i=-2}^{n+2}c_iB_j\left(x_j\right)-h\sum_{i=-2}^{n+2}c_i\begin{bmatrix}\omega_{j,-1}k\left(x_j,t_{-1}\right)B_i\left(t_{-1}\right)+\omega_{j,0}k\left(x_j,t_0\right)B_i\left(t_0\right)+\\ \ldots+\omega_{j,n+1}k\left(x_j,t_{n+1}\right)\times B_i\left(t_{n+1}\right)\end{bmatrix}=y\left(x_i\right) \tag{34}$$

Where $z_j=a+jh, h=\frac{b-a}{n+2}, j=0,\ldots,n+2.t_j=x_j=z_{j+1}, j=-1,\ldots,n+1.$

By solving the system (34) we obtain the vector c_i and also we set $c_{-2}=c_{n+2}=0$ in order to have the cardinal spline relation ,then with substituting c_j in(33) we obtain the approximation solution.

CHAPTER IV

COMPARISON BETWEEN THE TWO CUBIC SPLINES METHODS

We compared the result of the two methods on those examples. with The absolute error is given by

$$|E| = |exact - approch|$$

Example 12

$$\varphi(x) - \int_0^1 \frac{1}{12}\frac{sx-1}{1+x^2}\varphi(s)\,ds = \sin(x), \qquad 0 \le x \le 1$$

With the exact solution $\varphi(x) = \sin(t) + 1$.

Comparison of the two methods.

n=10	$e = \lvert\varphi(x_i) - ns(x_i)\rvert$	$e = \lvert\varphi(x_i) - cs(x_i)\rvert$
t=0	0.87×10^{-3}	0.18×10^{-8}
t=0.1	0.93×10^{-3}	0.75×10^{-8}
t=0.2	0.96×10^{-3}	0.24×10^{-8}
t=0.3	0.95×10^{-3}	0.63×10^{-8}
t=0.4	0.93×10^{-3}	0.86×10^{-8}
t=0.5	0.88×10^{-3}	0.26×10^{-8}
t=0.6	0.82×10^{-3}	0.47×10^{-8}
t=0.7	0.75×10^{-3}	0.43×10^{-8}
t=0.8	0.68×10^{-3}	0.58×10^{-8}
t=0.9	0.60×10^{-3}	0.51×10^{-8}
t=1.0	0.53×10^{-3}	0.42×10^{-8}

For $n = 20$, we have the following result

n=20	$e = \lvert\varphi(x_i) - ns(x_i)\rvert$	$e = \lvert\varphi(x_i) - cs\,(x_i)\rvert$
t=0	0.46×10^{-3}	0.19×10^{-11}
t=0.1	0.55×10^{-3}	0.26×10^{-11}
t=0.2	0.61×10^{-3}	0.65×10^{-11}
t=0.3	0.65×10^{-3}	0.33×10^{-11}
t=0.4	0.68×10^{-3}	0.24×10^{-11}
t=0.5	0.68×10^{-3}	0.12×10^{-11}
t=0.6	0.66×10^{-3}	0.10×10^{-11}
t=0.7	0.64×10^{-3}	0.21×10^{-11}
t=0.8	0.60×10^{-3}	0.45×10^{-11}
t=0.9	0.56×10^{-3}	0.17×10^{-11}
t=1.0	0.51×10^{-3}	0.22×10^{-11}

Example 13

$$\varphi(x) - \frac{1}{11}\int_0^1 \sin(sx + 3s)\,\varphi(s)\,ds = \cos(t+1), \qquad 0 \le x \le 1.$$

With the exact solution $\varphi(x) = 2\cos(t) + 1$.

Comparison of the two methods.

n=10	$e = \lvert\varphi(x_i) - cs\,(x_i)\rvert$	$e = \lvert\varphi(x_i) - cs\,(x_i)\rvert$
t=0	0.36×10^{-3}	0.65×10^{-8}
t=0.1	0.33×10^{-3}	0.25×10^{-8}
t=0.2	0.25×10^{-3}	0.19×10^{-8}
t=0.3	0.13×10^{-3}	0.69×10^{-8}
t=0.4	0.10×10^{-4}	0.41×10^{-8}
t=0.5	0.15×10^{-3}	0.16×10^{-8}
t=0.6	0.27×10^{-3}	0.18×10^{-8}
t=0.7	0.34×10^{-3}	0.72×10^{-8}
t=0.8	0.36×10^{-3}	0.63×10^{-8}
t=0.9	0.33×10^{-3}	0.27×10^{-8}
t=1.0	0.24×10^{-3}	0.19×10^{-8}

For $n = 20$, we have the following result

n=20	$e = \lvert\varphi(x_i) - cs\,(x_i)\rvert$	$e = \lvert\varphi(x_i) - cs\,(x_i)\rvert$
t=0	0.24×10^{-3}	0.13×10^{-11}
t=0.1	0.22×10^{-3}	0.18×10^{-11}
t=0.2	0.17×10^{-4}	0.24×10^{-11}
t=0.3	0.89×10^{-3}	0.54×10^{-11}
t=0.4	0.72×10^{-3}	0.27×10^{-11}
t=0.5	0.10×10^{-3}	0.36×10^{-11}
t=0.6	0.18×10^{-3}	0.18×10^{-11}
t=0.7	0.23×10^{-3}	0.98×10^{-11}
t=0.8	0.24×10^{-3}	0.44×10^{-11}
t=0.9	0.22×10^{-3}	0.78×10^{-11}
t=1.0	0.16×10^{-3}	0.43×10^{-11}

Conclusion 14 *The numerical results related to the above examples are illustrated in Tables1-2 where, for different value of n, and $h = \frac{1}{10}$. From this tables , it can be seen that the result obtained by the method based on the cardinal spline are better than those obtained by the natural spline, due to the fact that the free-boundary conditions affect negatively the rate of convergence in the whole interval.*

Conclusion

We have proposed two main methods, for approximating the solution of Fredholm linear integral equations of the second kind,based on cubic splines because of their simplicity,stabilityand their high order of convergence. In spite of its positive sounding name. the natural spline interpolation has little to recommend it from an approximation theoretic point of view. Due to its global character and the fact that the free-end conditions $S^{''}(a) = S^{''}(b) = 0.$ produce an $O(h^2)$ error.The natural spline interpolation method is complicated and requires many computations. However, the simulair results given for the method based on the cardinal spline are simple and more closer to the exacte solution. Also , we verified that the presented method can be applied with large number of and remains stable.

Perspective

In the future, for a further research we hope to have a better comprehension, for splines and it's diverting types, that gives as a lot of doors to open in mathematics world.

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