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Particular intuitionstic fuzzy subsets on a lattice

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Introduction

In 1983, Atanassov [2] introduced the notion of intuitionistic fuzzy set (or Atanassov's intuitionistic fuzzy set) as a generalization of Zadeh's fuzzy set [29] by considering two degrees, the membership degree and the non-membership degree of an element on a subset of a given universe. In fuzzy set theory, the non-membership degree of an element x can be viewed as $\nu_A(x) = 1 - \mu_A(x)$ (using the standard strong negation on the real interval $[0, 1]$), which is fixed, while in intuitionistic fuzzy setting, the non-membership degree has no fixed formula, it satisfies only the condition: $\nu_A(x) \leq 1 - \mu_A(x)$, for any $x \in A$. Certainly, fuzzy sets are Atanassov's intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$. Based on Atanassov's intuitionistic fuzzy set, Burillo and Bustince [7] introduced the concept of intuitionistic fuzzy relation, in particular, they introduced the intuitionistic fuzzy order (or intuitionistic fuzzy ordered set) as a natural generalization of fuzzy order relation previously introduced by Zadeh [30]. Intuitionistic fuzzy relations theory has been applied in many different fields, such as decision making, mathematical modeling, medical diagnosis, control systems, machine learning, market prediction.

The notion of an ideal or its dual (a filter) is recognized as one of the most important concepts in the lattices theory and the theory of other algebraic structures used in formal fuzzy logic. These notions are mainly used to translate connections between properties on algebraic structures and to define congruence relations and quotient algebras . They played a central role in Stone representation theorem for Boolean lattice and distributive lattice. In topology and approaches to its analysis, ideals and filters appeared to provide very general contexts to unify the various notions of sequences convergence and limit in arbitrary topological spaces, and to express completeness and compactness in metric spaces. In fuzzy setting, for the same purposes, several authors introduced and investigated the concepts of some kinds of fuzzy ideals and fuzzy filters in different ways and on different structures. The first approach considered fuzzy ideal and fuzzy filter as fuzzy sets on crisp structures, like on lattices or on residuated lattices

The objective of this work is to study some notions and properties of intuitionistic fuzzy ideals and filters on lattice. Also, we provide some characterizations of these notions in terms of the intuitionistic fuzzy ordered lattice operations, as well as, in terms of their level sets.

This memoire is structured as follows.

- In Chapter 1, we provide notions and properties related to sets, order relations, ideals, fillters, fuzzy sets, intuitionistic fuzzy sets and intuitionistic fuzzy relations.

- In Chapter 2, we study the notions of intuitionistic fuzzy ideal and filter on a lattice and investigate their fundamental properties. Further, we present interesting characterizations of these notions in terms of the lattice meet and join operations, and in terms of their (α, β)-level sets.

Chapter 1

Generalities on fuzzy sets and fuzzy relations

The purpose of this first chapter is to provide a basic introduction to the binary relations, posets, lattices, t-norm. Next, we recall some basic notions of fuzzy sets, intuitionistic fuzzy sets and intuitionistic fuzzy relations. Many of the properties of these concepts will be used in the next chapters.

1.1 Sets, order relations, posets, ideals and fillters

This section contains the basic definitions and properties of binary relations, posets, lattices.

1.1.1 Crisp sets

This section contains the basic definitions of crisp sets with several operations.

Definition 1.1. *A set of reference X is a collection of objects, this set con be defined by*

(i) Writing of all its elements, whose elements are $a_1, a_2, \cdots, a_n$, and we write, $X = \{a_1, a_2, \cdots, a_n\}$.

(ii) *A property or properties are satisfied by its elements, and we write,* $A = \{x \mid P(x)\}$. *Where the symbol "$|$" denotes the sentence "such that" and* $P(x)$ *a proposition of the form "x" has a property.*

(iii) *A function called characteristic function* $\mathcal{X}_A(x)$ *which takes the value 0 for the elements that do not belong to* A *and the value 1 for those that belong to* A*:*

$$\chi_A : X \longrightarrow \{0, 1\}$$
$$x \longmapsto \begin{cases} 0 \;\; if \;\; x \notin A; \\ 1 \;\; if \;\; x \in A. \end{cases}$$

Definition 1.2. ***(Operations on crisp sets)*** *Let* X *be a set, let* A *and* B *be two subsets on* X*.*

(i) *Inclusion:* $A \subset B$ *if* $(x \in A) \Rightarrow (x \in B)$,*i.e.,*$\mathcal{X}_A(x) \leq \mathcal{X}_B(x)$, *for any* $x \in X$;

(ii) *Equality:* $A = B$ *if* $A \subseteq B$ *and* $B \subseteq A$ *i.e.,* $(\mathcal{X}_A(x) = \mathcal{X}_B(x))$, *for any* $x \in X$;

(iii) *Complement:* $A^c = \{x \in X \mid x \notin A\}$ *i.e.,* $\mathcal{X}_{A^c}(x) = 1 - \mathcal{X}_A(x)$;

(iv) *Intersection:* $A \cap B = \{x \in X \mid x \in A$ *and* $x \in B\}$*i.e.,*$\mathcal{X}_{A \cap B}(x) = min(\mathcal{X}_A(x), \mathcal{X}_B(x))$;

(v) *Union:* $A \cup B = \{x \in X \mid x \in A$ *or* $x \in B\}$*i.e.,*$\mathcal{X}_{A \cup B}(x) = max(\mathcal{X}_A(x), \mathcal{X}_B(x))$;

(vi) *Relative complement:* $A \backslash B = A - B = A \cap B^c = \{x \in X \mid x \in A$ *and* $x \notin B\}$ *i.e.,* $\mathcal{X}_{A-B}(x) = \mathcal{X}_{A \cap B^c}(x) = min(\mathcal{X}_A(x), \mathcal{X}_{B^c}(x))$.

Example 1.1. *Let* $X = \{x, y, z, t, w\}$ *be a set, let* A *and* B *be two subsets of* X *such that* $A = \{x, y, w\}$ *and* $B = \{x, y, z\}$*. Then,*

$$A^c = \{z, t\};$$

$$B^c = \{t, w\};$$

$$A \cap B = \{x, y\};$$

$$A \cup B = \{x, y, z, w\};$$

$$A \backslash B = \{w\};$$

$$B \backslash A = \{z\}.$$

Example 1.2. *Let* $X = [36, 42]$ *the universe of speech which expresses the degree of temperature of a human body, a person with hepatitis generally presents the following symptoms:*

1. *The person has a high fever;*
2. *His skin is yellow in color;*
3. *He has nausea.*

We will now study the first symptom or the property having a high fever which is indicated by the degree of temperature in the classic case. we define of temperature in the classic set A *of* X *associated with the property (having a high fever) by:*
$$\mathcal{X}_A(x) = \begin{cases} 1, & if \quad x \geq 39 \\ 0, & otherwise. \end{cases}$$

1.1.2 Order relations

A binary relation on a set X is a subset of X^2, i.e., it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements, denoted by $x \nparallel y$, if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel_R y$, or simply $x \parallel y$ when no confusion can occur. We denote by R^c the complement of the relation R on X, i.e., for any $x, y \in X$, xR^cy denotes the fact that $(x, y) \notin R$. We denote by R^t the transpose of the relation R on X, i.e., for any $x, y \in X$, xR^ty denotes the fact that yRx. We denote by R^d the dual of the relation R on X, i.e., for any $x, y \in X$, xR^dy denotes the fact that yR^cx. A relation R on a set X is said to be included in a relation S on the same set X, denoted by $R \subseteq S$, if, for any $x, y \in X$, xRy implies that xSy. The union of two relations R and S on a set X is the relation $R \cup S$ on X defined as $R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}$. Similarly, the intersection of two relations R and S on a set X is the relation $R \cap S$ on X defined as $R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}$. If $R \cap S = \emptyset$, then R and S are called disjoint relations. The composition of two relations R and S on a set X is the relation $R \circ S$ on X defined as $R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}$. For any $n \in \mathbb{N}^*$, the $n-$th power relation R^n of R is recursively defined as follows:

$$(R^1 = R) \wedge (\forall n \geqslant 1)(R^{n+1} = R^n \circ R)\,.$$

A binary relation R on a set X is called:

(i) reflexive, if, for any $x \in X$, it holds that xRx;

(ii) irreflexive, if, for any $x \in X$, it holds hat xR^cx;

(iii) symmetric, if, for any $x, y \in X$, it holds that xRy implies that yRx;

(iv) antisymmetric, if, for any $x, y \in X$, it holds that xRy and yRx imply that $x = y$;

(v) asymmetric, if, for any $x, y \in X$, it holds that xRy implies that yR^cx;

(vi) transitive, if, for any $x, y, z \in X$, it holds that xRy and yRz imply that xRz;

(vii) complete, if, for any $x, y \in X$, either xRy or yRx holds.

A binary relation R on a set X is called:

(i) a pseudo-order relation, if it is reflexive and antisymmetric;

(ii) a strict order, if it is irreflexive and transitive;

(iii) an order relation, if it is reflexive, antisymmetric and transitive;

(iv) a total order relation, if it is reflexive, antisymmetric, transitive and complete;

1.1.3 Posets

A partial order (order, for short) is a binary relation $\leqslant$ over a set X which is reflexive ($a \leqslant a$, for any $a \in X$), antisymmetric ($a \leqslant b$ and $b \leqslant a$ implies $a = b$, for any $a, b \in X$) and transitive ($a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$, for any $a, b, c \in X$). A set with an order relation is called an ordered set (also called a poset). Further, $\{x, y\}^u$ denotes the set of all upper bounds of x and y, while $\{x, y\}^l$ denotes the set of all lower bounds of x and y, i.e., $\{x, y\}^u = \{z \in X \mid x \leqslant z \ \wedge \ y \leqslant z\}$ and $\{x, y\}^l = \{z \in X \mid z \leqslant x \ \wedge \ z \leqslant y\}$.

A strict order is a binary relation $<$ on a set X that is irreflexive ($a < a$ does not hold for any $a \in X$), asymmetric (if $a < b$, thus $b < a$ does not hold for any $a, b \in X$) and transitive. A given binary relation $\sim$ on a set X is said to be an equivalence relation if it is reflexive, symmetric ($a \sim b$ implies $b \sim a$, for any $a, b \in X$) and transitive. If $\leqslant$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \ \text{ if } \ a \leqslant b \ \text{ and } \ a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leqslant$ is the reflexive closure given by

$$a \leqslant b \ \text{ if } \ a < b \ \text{ or } \ a = b.$$

Two elements x and y of X are called comparable if $x \leqslant y$ or $y \leqslant x$; otherwise they are called incomparable, and we write $x \parallel y$. Using the strict order $<$, the relation (x is covered by y) denoted as $x \lll y$, if $x < y$ and there exists no $z \in X$ such that $x < z < y$. A poset can be conveniently represented by a Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y.

Definition 1.3. *[15] Let $(X, \leqslant_1)$ and $(Y, \leqslant_2)$ two posets, then the mapping $f : X \to Y$ is monotone (or order-preserving) if $x \leqslant_1 y \Rightarrow f(x) \leqslant_2 f(y)$, for any $(x, y \in X)$*

Definition 1.4. *[15] Let P be an ordered set. Then P is a chain if for any $x, y \in P$, either $x \leq y$ or $y \leq x$ (that is, if any two elements of P are comparable). Alternative* *names for a chain are linearly ordered set and totally ordered set.*

Zorn's lemma is a result in set theory that appears in proofs of some non-constructive existence theorems throughout mathematics.

Theorem 1.1. *[15](Zorn's lemma) Let P be a non-empty ordered set in which every nonempty chain has an upper bound. Then P has a maximal element.*

1.1.4 Ideals and filters on a crisp lattice

Ideals are of fundamental importance in algebra. Filters, the order duals of lattice ideals, have a variety of applications in logic and topology.

Definition 1.5. *[15] A nonempty subset I on a lattice L is called an ideal of L if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in I$ and $x \leq y$, then $x \in I$,*

2. *if $x, y \in I$ implies $x \vee y \in I$.*

The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.

A dual ideal is called a filter. Specifically, a non-empty subset of L determined by the following definition.

Definition 1.6. *[15] A nonempty subset F on a lattice L is called a filter if, for any $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in F$ and $y \leq x$, then $x \in F$,*

2. *if $x, y \in F$ implies $x \wedge y \in F$.*

The set of all ideals (resp. filters) of L is denoted by $\mathcal{I}(L)$ (resp. $\mathcal{F}(L)$), and carries the usual inclusion order.

More precisely, an ideal or filter is called proper if it does not coincide with L. More precisely, an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$. For any $a \in L$, the set $\downarrow a$ is an ideal (also known as the principal ideal generated by a). Dually, $\uparrow a$ is a principal filter.

Definition 1.7. *[15] An ideal I on a lattice L is called a prime ideal if, $x \wedge y \in L$, then $x \in I$ or $y \in I$, for any $x, y \in L$.*

Definition 1.8. *[15] A filter F on a lattice L is called a prime filter if, $x \vee y \in F$, then $x \in F$ or $y \in F$, for any $x, y \in L$.*

Definition 1.9. *[15] Let L be a lattice. A proper ideal (resp. filter) A is said to be a maximal ideal (resp. maximal filter or more usually known as an ultrafilter) if the only ideal (resp. filter) properly containing A is L.*

Example 1.3. *(i) The following are ideals in $\mathcal{P}(X)$*

(a) all subsets not containing a fixed element of X,

(b) all finite subsets (this ideal is non-principal if X is infinite).

(ii) Let $(X, \mathfrak{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in \mathfrak{T}) x \in U \subseteq V\}$ is a filter in $(X, \mathfrak{T})$.

1.2 Fuzzy sets and intuitionistic fuzzy sets

In this section, we provide some definition on fuzzy sets and intuitionistic fuzzy sets and operations of fuzzy sets,characteristics of fuzzy sets .

1.2.1 Fuzzy sets

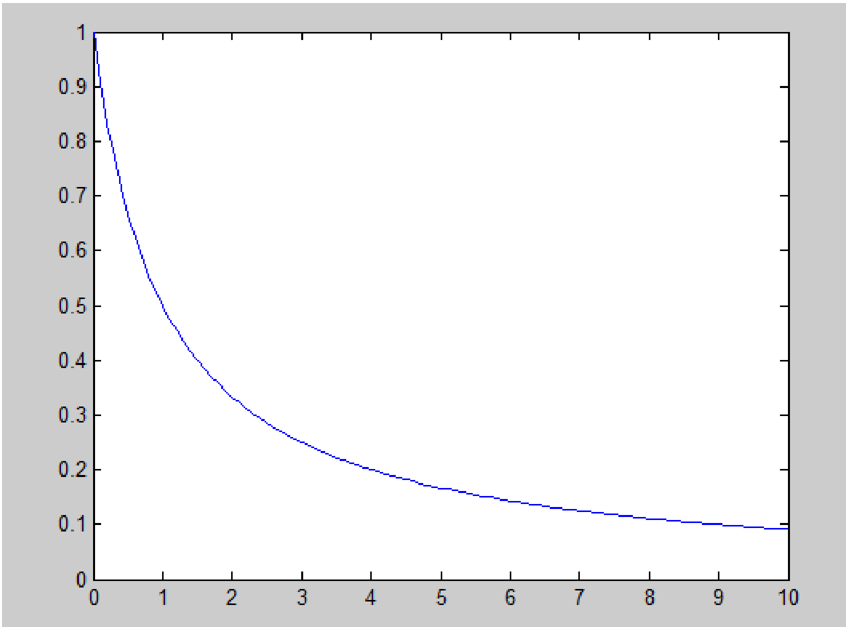
The notion of fuzzy sets was first introduced by Zadeh [29].

Definition 1.10. *[15] Let X be a non empty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.*

Example 1.4. *(1) Let $X = \{a, b, c\}$ be a universal set. $A = \{(a, 0.2), (b, 0.8), (c, 1)\}$ is an fuzzy subset on X ;*

(2) Let $X = [0, 10]$, and A be a fuzzy subset on X defined by :

$$\mu_A(x) = \frac{1}{1 + x}$$



graph of μ_A

Operations of fuzzy sets

In this section, we will give definitions for operations of fuzzy sets : equality, inclusion, intersection, union, sum and product of two fuzzy subsets, and complement of a fuzzy set, and we will give an example.

Definition 1.11 (Equality)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, we say that $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.*

Definition 1.12 (Inclusion)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, we say that $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for all x in X.*

Definition 1.13 (Intersection)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, the intersection defined by for all $x \in X$*

$$\mu_{A\cap B}(x) = min\,\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$$

Definition 1.14 (Union)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, the union defined by for all $x \in X$*

$$\mu_{A\cup B}(x) = max\,\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$$

Definition 1.15 (Complement)**.** [29] *The complement of a fuzzy set A is de noted by $C(A)$ and is defined by : for all $x \in X$*

$$\mu_{C(A)}(x) = 1 - \mu_A(x)$$

Definition 1.16 (Sum)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, the sum defined by for all $x \in X$*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

Definition 1.17 (Product)**.** [29] *Let X be a non empty set and let A and B two fuzzy subsets, the product defined by for all $x \in X$*

$$\mu_{A\times B}(x) = \mu_A(x)\mu_B(x)$$

Example 1.5. *Let $X = \{a, b, c\}$,and let $A = \{(a, 0.2), (b, 0.6), (c, 0.5)\}$, and $B = \{(a, 0.7), (b, 0.1), (c, 1)\}$ we have :*

1. $A \cap B = \{(a, 0.2), (b, 0.1), (c, 0.5)\}$
2. $A \cup B = \{(a, 0.7), (b, 0.6), (c, 1)\}$
3. $A \times B = \{(a, 0.14), (b, 0.06), (c, 0.5)\}$
4. $A + B = \{(a, 0.76), (b, 0.74), (c, 1)\}$
5. $C(A) = \{(a, 0.8), (b, 0.4), (c, 0.5)\}$

Example 1.6. *If we consider the fuzzy sets* $A_1(x) = \begin{cases} 1, & if \quad 40 \le x < 50, \\ 1 - \frac{x-50}{10}, & if \quad 50 \le x < 60, \\ 0, & if \quad 60 \le x \le 100. \end{cases}$

$$A_2(x) = \begin{cases} 0, & if \quad 40 \le x < 50, \\ \frac{x-50}{10}, & if \quad 50 \le x < 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \le x < 70, \\ 0, & if \quad 70 \le x \le 100. \end{cases}$$

Then their union is $(A_1 \cup A_2)(x) = \begin{cases} 1, & if \quad 40 \le x < 50, \\ 1 - \frac{x-50}{10}, & if \quad 50 \le x < 55, \\ \frac{x-50}{10}, & if \quad 55 \le x \le 60, \\ 1 - \frac{x-60}{10}, & if \quad 60 \le x \le 70, \\ 0, & if \quad 70 \le x \le 100. \end{cases}$

The intersection can be expressed as $(A_1 \cap A_2)(x) = \begin{cases} 0, & if \quad 40 \le x < 50, \\ \frac{x-50}{10}, & if \quad 50 \le x < 55, \\ 1 - \frac{x-50}{10}, & if \quad 55 \le x < 60, \\ 0, & if \quad 60 < x \le 100. \end{cases}$

The complement of A_1 *can be written* $\overline{A_1}(x) = \begin{cases} 0, & if \quad 40 \le x < 50, \\ \frac{x-50}{10}, & if \quad 40 \le x < 60, \\ 1, & if \quad 60 \le x \le 100. \end{cases}$

Example 1.7. *Let* $X = \mathbb{R}$ *and let* A *be the set of reals greater than* 10 *and* B *the set of reals close to* 1 *are characterized respectively by its membership functions*

$$\mu_A(x) = \begin{cases} 0, & if \quad x \le 10, \\ (1 + (x - 10)^{-2})^{-1}, & if \quad x > 10, \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0, & if \quad x \le 10, \\ (1 + (x - 10)^{4})^{-1}, & if \quad x > 10. \end{cases}$$

So, we get $A \cap B$ *set of reals greater than* 10 *and close to* 11 *given by its membership* *function*

$$\mu_{A\cap B}(x) = \begin{cases} 0, \quad if \quad x \leq 10, \\ min[(1+(x-10)^{-2})^{-1},(1+(x-10)^{4})^{-1}], \quad if \quad x > 10. \end{cases}$$

And $A \cup B$ the set of real numbers greater than 10 or close to 11 given by its membership function

$$\mu_{A\cup B}(x) = max[(1+(x-10)^{-2})^{-1},(1+(x-10)^{4})^{-1}], x \in X.$$

Characteristics of fuzzy sets.

In this section, we remember definitions of characteristics of fuzzy sets : support, kernel, height and cardinality of a fuzzy set, and we will give an example.

Definition 1.18 (α-cuts)**.** [31] Let A be a fuzzy set in X and let $\alpha \in]0,1]$, The α-cut of A, denoted A_α. we mean all elements of X that belong to A to a degree of at least α. That is A_α is a classical set defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

Definition 1.19 (Support)**.** *[31] The support of a fuzzy set A, denoted by $S(A)$, we mean all elements of X that belong to a nonzero degree. That is $S(A)$is a calssical set defined by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

Definition 1.20 (Kernel)**.** *[31] The ker of a fuzzy set A, denoted by ker(A), we mean all elements of X that belong to a equal one. That is $ker(A)$is a calssical set defined by*

$$ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

Definition 1.21 (Height)**.** *[31] The height of a fuzzy set A is the largest membership grade of any element in A.*

$$H(A) = Max\mu_A(x).$$

Definition 1.22 (Cardinality)**.** *[31] Cardinality of a finite fuzzy set* A*, denoted* $\mid A \mid$ *is defined as*

$$\mid A \mid = \sum_{x \in X} \mu_A(x).$$

Example 1.8. *Let* $X = \{a, b, c, d\}$*, and* $A = \{(a, 0.6), (b, 1), (c, 0.3), (d, 0)\}$

$A_{0.5} = \{a, b\}$

$S(A) = \{a, b, c\}$

$ker(A) = \{b\}$

$H(A) = 1$

$|A| = 1.9$

Example 1.9. *Let* $X = \{a, b, c\}$ *be a set.* $A_1 = \{(a, 0.3), (b, 1.0), (c, 0.7)\}$ *and* $A_2 = \{(a, 0.0), (b, 0.9),$
$(c, 1.0)\}$ *are two fuzzy subsets on* X*.*
Then, $Supp(A_1) = \{a, b, c\}$ *and* $Supp(A_2) = \{b, c\}$*.*
$Ker(A_1) = \{b\}$ *and* $Ker(A_2) = \{c\}$*.* $H(A_1) = 1$ *and* $H(A_2) = 1$*.* $\mid A_1 \mid = 2$ *and* $\mid A_2 \mid = 1.9$*.*

Example 1.10. *Let* $X = [0, 1]$ *with* $\alpha, \beta \in \mathbb{R}$ *and let* $a, b \in \mathbb{R}$*. We define the fuzzy set* A *on* X *by*

$$\mu_A(x) = \begin{cases} 0, & if \quad x < a - \alpha \quad or \quad b + \beta < x, \\ 1, & if \quad a < x < b, \\ 1 + (\frac{x-a}{\alpha}), & if \quad a - \alpha < x < a, \\ 1 - (\frac{b-x}{\beta}), & if \quad b < x < b + \beta. \end{cases}$$

Then, $Ker(A) = [0, 1]$*,* $Supp(A) = [a - \alpha, b + \beta]$ *and* $H(A) = 1$*.*

Example 1.11. *Let* B *the fuzzy subset given on the set* $X = [36, 42]$*. Then,* $Supp(A) =]37, 42]$*,* $H(A) = 1$*,* $Ker(A) = [41, 42]$*,* $\mid A \mid$ *is infinite.*

Example 1.12. *Let* $X = \{1, 2, 3, ..., 10\}$*, and* A *be a fuzzy subset of* X *given by* $A = \{< 1; 0.2 >, < 2; 0.5 >, < 3; 0.8 >, < 4; 1 >, < 5; 0.7 >, < 6; 0.3 >, < 7; 0 >, <$

$8;0>$,

$<9;0>,<10;0>\}$. *Then, the α-cut of A is given by:*

$A_0 = \{x \in X, A(x) > 0\} = X$;

$A_{0.1} = \{x \in X, A(x) \geq 0.1\} = \{1,2,3,4,5,6\}$;

$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1,2,3,4,5,6\}$;

$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2,3,4,5,6\}$;

$A_{0.4} = \{x \in X, A(x) \geq 0.4\} = \{2,3,4,5\}$;

$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2,3,4,5\}$;

$A_{0.6} = \{x \in X, A(x) \geq 0.6\} = \{3,4,5\}$;

$A_{0.7} = \{x \in X, A(x) \geq 0.7\} = \{3,4,5\}$;

$A_{0.8} = \{x \in X, A(x) \geq 0.8\} = \{3,4\}$;

$A_{0.9} = \{x \in X, A(x) \geq 0.9\} = \{4\}$;

$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}$.

Example 1.13.

let $X = [0,35]$ *(set of ages)* , $\alpha \in [0,1]$

and let A a fuzzy set of X . The ages young defined by :

$$\mu_A(x) = \begin{cases} 1, & if \quad x \in [20,30] \\ 0, & if \quad x \geq 35 \;\; and \;\; x \leqslant 152 \\ \alpha, & if \quad x \in]15,35[\;\; and \;\; x \in]30,35[\end{cases}$$

The kernel of A is $Ker(A) = [20,30]$ and the support of A is $Supp(A) =]15,35[$ and the height of A is $H(A) = 1$.

T-norms and T-conorms.

The history of triangular-norms (*t-norms*) started with Menger . His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar , provided the axioms of *t-norms*, as they are used today.

Tringular norm.

Definition 1.23. *Tiangular norm is a binary operation T on the unit interval* $[0, 1]$, *i.e., it is a function* $T : [0, 1]^2 \longrightarrow [0, 1]$ *: the following four axioms are satisfied :*

(T1) Commutativity i.e., $T(x, y) = T(y, x)$ *;*

(T2) Associativity i.e., $T(x, T(y, z)) = T(T(x, y), z)$ *;*

(T3) Monotonicity i.e., $T(x, y) \leq T(x, z)$ *whenever* $y \leq z$ *;*

(T4) Boundary condition i.e., $T(x, 1) = x$*.*

Example 1.14. *The following four operations are the most common t-norms: The dual t-conorms w.r.t.* T_M, T_P, T_L *and* T_D *are given by:*

(T5) *Minimum:* $T_M(x, y) = \min\{x, y\}$

(T6) *Product:* $T_P(x, y) = x.y$

(T7) *Lukasiewicz:* $T_L(x, y) = \max\{x + y - 1, 0\}$

(T8) *Drastic product:*

$$T_D(x, y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{if } x, y < 1. \end{cases}$$

Triangular conorm.

Definition 1.24. *A triangular conorm is a binary operation S on the unit interval* $[0, 1]$*, i.e., it is a function* $S : [0, 1]^2 \longrightarrow [0, 1]$ *: the following four axioms are satisfied :*

(S1) Commutativity : $S(x, y) = S(y, x)$ *;*

(S2) Associativity : $S(x, S(y, z)) = S(S(x, y), z)$ *;*

(S3) Monotonicity : $S(x, y) \leq S(x, z)$ *whenever* $y \leq z$ *;*

(S4) Boundary condition : $S(x,0) = x$.

Remarque 1.1. *Given a t-norm T, we find the associated dual t-conorm S by* $S(x,y) = 1 - T(1-x, 1-y)$.

The dual t-conorms w.r.t. T_M, T_P, T_L and T_D are given by:

(S1) Maximum: $S_M(x,y) = \max\{x,y\}$

(S2) Probabilistic sum: $S_P(x,y) = x + y - s.y$

(T7) Lukasiewicz: $S_L(x,y) = \min\{x+y, 1\}$

(T8) Drastic sum:

$$S_D(x,y) = \begin{cases} 1, & \text{if } \ (x,y) \in [0,1)^2 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

Cartesian product and projection on fuzzy set.

Cartesian product on fuzzy set.

The cartesian product of the fuzzy subsets is the minimum of these degrees of belonging.

Definition 1.25. *The cartesian product applied to n fuzzy sets can be defined as follows : Let $\mu_{A_1}, \mu_{A_2}, ...\mu_{A_n}$, be membership functions of $A_1, A_2, ..., A_n$. Then, the membership degree of $(x_1, x_2, ...x_n) \in X_1 \times X_2 \times ... \times X_n$ on the fuzzy set $A_1 \times A_2 \times ... \times A_n$ is ,*

$$\mu_{A_1 \times A_{2 \times ... \times A_n}}(x_1, x_2, ..., x_n) = min\left\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)\right\}.$$

Example 1.15. *Lets $X_1 = \{a, b, c, \}$, $X_2 = \{\alpha, \beta\}$ and lets A_1, A_2 two fuzzy subset respectively defined on X_1 and X_2 given by:*
$A_1 = \{(a, 0.1), (b, 0.4), (c, 0.8)\}$;
$A_2 = \{(\alpha, 0.2), (\beta, 0.6)\}$.
So, we get:

$A_1 \times A_2 = \{((a,\alpha), 0.1), ((a,\beta), 0.1), ((b,\alpha), 0.2), ((b,\beta), 0.4), ((c,\alpha), 0.2), ((c,\beta), 0.6)\}$

Let X_1 *be a set of animals* $X_1 = \{cat, cheetah, tiger\}$ *and* X_2 *be a set of country choices by temperature* $X_2 = \{hot, cold\}$. *The fuzzy subset* A_1 *represents the choices of an individual that the animals would like to own and the fuzzy subset* A_2 *represents its choices to the type of country in which the animal would like to live such as*
$A_1 = \{\langle cat, 0.5\rangle, \langle cheetah, 0.8\rangle, \langle tiger, 0.3\rangle\}, A_2 = \{\langle hot, 0.9\rangle, \langle cold, 0.1\rangle\}$. *We get*
$A_1 \times A_2 = \{\langle (cat, hot), 0.5\rangle, \langle (cat, cold), 0.1\rangle, \langle (cheetah, hot), 0.8\rangle, \langle (cheetah, cold), 0.1\rangle,$
$\langle (cheetah, hot), 0.8\rangle,$
$\langle (tiger, hot), 0.3\rangle, \langle (tiger, cold), 0.1\rangle\}$.
Let be a fuzzy subset A *defined on a universe* $X_1 \times X_2$ *cartesian product of two reference sets* X_1 *and* X_2.

Projection on fuzzy set.
The projections is the maximum of these cartesian products.

Definition 1.26. *The projection on* X_1 *of the fuzzy set* A *of* $X_1 \times X_2 \times ... \times X_n$ *is the fuzzy set* $Proj_{X_1}(A)$ *of* X_1*, whose membership function is defined by: for any* $x_1 \in X_1$,

$$\mu_{ProjX_{1(A)}}(x_1) = sup_{x_2 \in X_2, x_3 \in X_3, ..., x_n \in X_n}(\mu_A(x_1, x_2, ..., x_n)).$$

Example 1.16. *Let* $X = X_1 \times X_2$ *the set of reference such that* X_1 *and* X_2 *two sets, we consider* $A_1 \times A_2 = A$ *given by:*
$A = \{((a, \alpha), 0.1), ((a, \beta), 0.1), ((b, \alpha), 0.2), ((b, \beta), 0.4), ((c, \alpha), 0.2), ((c, \beta), 0.6)\}$

So, we get:

$$\begin{aligned} Proj_{X_1}(A) &= \{(a, max(0.1, 0.1)), (b, max(0.2, 0.4)), (c, max(0.2, 0.6))\}; \\ &= \{(a, 0.1), (b, 0.4), (c, 0.6)\}. \end{aligned}$$

1.2.2 Intuitionistic fuzzy sets

In 1983, Atanassov [2] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 1.27. [2] *Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfy the condition:*

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

For any $x \in X$ the number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree or the intuitionistic index of x to A.

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Example 1.17. *Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate that have voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number corresponds to the part of electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define $\nu(x)$ (degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be $\pi(x) = 1 - \mu(x) - \nu(x)$ (degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set $\{\langle x, \mu(x), \nu(x)\rangle \mid x \in X\}$ and obviously,*

$$0 \leq \mu(x) + \nu(x) \leq 1.$$

For two intuitionistic fuzzy sets A and B on a set X, several operations are defined as follows (see, e.g., Atansssov [3, 4, 5], Biswas [6] and Gy [12]). Here we will present only those which are related to the present work.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for any $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for any $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X\}$;

(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X\}$;

(vi) $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in X\}$;

(vii) $\langle A \rangle = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle \mid x \in X\}$;

(viii) $Ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}$;

(ix) $Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}$.

In the sequel, we need the following definition of level sets (which is also often called (α, β)-cuts) of an intuitionistic fuzzy set.

Definition 1.28. *[15] Let* A *be an intuitionistic fuzzy set on a set* X*. The* (α, β)*-cut of* A *is a crisp subset*

$$A_{\alpha,\beta} = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$.

Example 1.18. *Let* $X = a,b,c,d,e$ *let IfS* A *and* B *have the froms :*

$A = \{< a, 0.01, 0.4 >, < b, 0.02, 0.8 >, < c, 0.9, 0.0 >, < d, 0.1, 0.03 >, < e, 0.0, 1.0 >$
$\}$

$B = \{< a, 0.5, 0.3 >, < b, 0.1, 0.4 >, < c, 0.8, 0.1 >, < d, 0.2, 0.4 >, < e, 0.2, 0.6 >\}$

Then

$A^c = \{< a, 0.4, 0.01 >, < b, 0.8, 0.02 >, < c, 0.0, 0.9 >, d, 0.03, 0.1 >, < e, 1.0, 0.0 >\}$

$A \cap B = \{< a, 0.01, 0.4 >, < b, 0.1, 0.8 >, < c, 0.8, 0.1 >, < d, 0.1, 0.4 >, < e, 0.0, 0.6 >$
$\}$

$A \cup B = \{< a, 0.5, 0.3 >, < b, 0.1, 0.4 >, < c, 0.9, 0.0 >, < d, 0.2, 0.03 >, < e, 0.2, 0.6 >$
$\}$

Definition 1.29. *(The support) [15]*
The support of an intuitionistic fuzzy set $A = \{x, \mu_A(x), \upsilon_A(x)\}$ *is defined as :*

$$Supp(A) = \{x \in X : \mu_A(x) \neq 0 \text{ and } \nu_A(x) \neq 1\}$$

Example 1.19. *Let* $X = \{a, b, c, d\}$ *and* $A = \{< a, 0.1, 0.7 >, < b, 0.9, 0 >, < c, 0.3, 0.01 >, < d, 0, 0.4 >\}$
Then, $Supp(A) = \{a, b, c\}$

Definition 1.30. *(the kernel) [15]*
Let A *be an intuitionistic fuzzy set on universe* X
The kernel of A *is the crisp subset of* X *given by:*

$$Ker(A) = \{x \in X / \mu_A(x) = 1 \text{ and } \upsilon_A(x) = 0 \}.$$

Example 1.20. *Let* $X = \{a, b, c, d\}$ *and*
$A = \{< a, 0.2, 0.5 >, < b, 1, 0 >, < c, 0.01, 0.2 >, < d, 0.05, 0.5 >\}$ *then,*
$ker(A) = b$.

Example 1.21. *Let's consider an example where* X *is the set of integers from 1 to 10, and* A *is an intuitionistic fuzzy set defined as:*

$$A = \{(1, \frac{1}{2}, \frac{1}{3}), (2, \frac{1}{3}, \frac{1}{2}), (3, 1, 0), (4, \frac{2}{3}, \frac{1}{4}), (5, 1, 0), (6, \frac{1}{4}, \frac{1}{3}), (7, \frac{1}{2}, \frac{1}{2}), (8, \frac{1}{4}, \frac{1}{2}), (9, \frac{2}{3}, \frac{1}{3}), (10, \frac{1}{3}, \frac{1}{3})\}$$

The kernel set K_A *consists of the elements with* $\mu_A(x) = 1$*, which are 3 and 5.*

The support set S_A *includes the kernel set elements and the boundary elements with* $\mu_A(x) + \nu_A(x) = 1$*, which are 1, 2, 4, 6, 7, 8, 9, and 10.*

1.3 Fuzzy relations and intuitionistic fuzzy relations

This section contains the basic definitions and properties of intuitionistic fuzzy relations which introduced by Burillo and Bustince, as a natural generalization of fuzzy relation.

1.3.1 Fuzzy relations

In this section, we recall the basic definitions and properties of fuzzy relations and several operations on fuzzy relations. The notion of fuzzy set was introduced in 1971 by Lotfi A.Zadeh in the paper [30].

Definition 1.31. [30] *Let X and Y be two nonempty sets. A binary fuzzy relation R from X to Y, is a fuzzy subset of $X \times Y$ characterized by a membership function R which associates with each pair (x, y) its grade of membership $R(x, y)$ in the interval $[0, 1]$.*

Definition 1.32. ***(Operations on fuzzy relations)***[29, 30]
Let R, P be two fuzzy relations. R is said to be contained in P (or P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it holds that $R(x, y) \leq P(x, y)$. The transpose(or the inverse) R^t of R is the fuzzy relation from Y to X defined by
$R^t = \{\langle (x, y), R^t(x, y)\rangle \mid (x, y) \in X \times Y\}$, where $R^t(x, y) = R(y, x)$ for any $(x, y) \in X \times Y$.
The intersection of two fuzzy relations R and P is defined as
$R \cap P = \{\langle (x, y), min(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}$.
The union of two fuzzy relations R and P is defined as
$R \cup P = \{\langle (x, y), max(R(x, y), P(x, y))\rangle \mid (x, y) \in X \times Y\}$.
The complement R^c of R is the fuzzy relation defined as
$R^c = \{\langle (x, y), 1 - R(x, y)\rangle \mid (x, y) \in X \times Y\}$.

Example 1.22. *Let R and S be two fuzzy relations on $X \times X$ such that $x = \{x, y, z\}$, represented by the following tables*

R	x	y	z
x	1.0	0.9	0.8
y	0.9	1.0	0.8
z	0.8	0.8	1.0

S	x	y	z
x	0.6	0.2	0.7
y	0.9	0.0	1.0
z	0.1	0.7	0.6

The union and intersection relations defined by

$R \cup S$	x	y	z
x	1.0	0.9	0.8
y	0.9	1.0	1.0
z	0.8	0.8	1.0

$R \cap S$	x	y	z
x	0.6	0.2	0.7
y	0.9	0.0	0.8
z	0.1	0.7	0.6

The transpose relation is given by the following table

S^t	x	y	z
x	0.6	0.9	0.1
y	0.2	0.0	0.7
z	0.7	1.0	0.6

The complementary relation is given by the following table

S^c	x	y	z
x	0.4	0.8	0.3
y	0.1	1.0	0.0
z	0.9	0.3	0.4

Definition 1.33. [30] *Let R, P and Q be three fuzzy relations from a universe X to a universe Y.*

(i) *if* $R \subseteq P$, *then* $R^t \subseteq P^t$;

(ii) $(R \cup P)^t = R^t \cup P^t$;

(iii) $(R \cap P)^t = R^t \cap P^t$;

(iv) $(R^t)^t = R$;

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ *and* $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(vi) $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;

(vii) *if* $R \supseteq P$ *and* $R \supseteq Q$, *then,* $R \supseteq P \cup Q$;

(viii) *if* $R \subseteq P$ *and* $R \subseteq Q$, *then,* $R \subseteq P \cap Q$.

Definition 1.34. [30] *Let* R *be an fuzzy relation on* X*, for short. The following properties are crucial :*

(i) *Reflexive: if* $R(x,x) = 1$*, for any* $x \in X$;

(ii) *Irreflexive: if* $R(x,x) = 0$*, for any* $x \in X$;

(iii) *Symmetrical: if* $R(x,y) = R(y,x)$*, for all* $x, y \in X$;

(iv) *Asymmetrical: if* $R(x,y) \wedge R(y,x) = 0$*, with* $x \neq y$*, for all* $x, y \in X$;

(v) *Antisymmetry: if* $(R(x,y) > 0) \wedge (R(y,x) > 0)$ *then,* $x = y$ *, for all* $x, y \in X$;

(vi) *Transitive: if* $R(x,z) \geq max_{y \in X}(min(R(x,y), R(y,z)))$*, for all* $x, y, z \in X$.

Definition 1.35. [30] *Let* X *be a nonempty crisp set and* R *be a fuzzy relation on* X*.* R *is called a fuzzy order or a partial fuzzy order if the following condition are satisfies:*

(i) *Reflexive: if* $R(x,x) = 1$*, for any* $x \in X$;

(ii) *Antisymmetry: if* $\begin{cases} R(x,y) > 0 \\ R(y,x) > 0 \end{cases} \Rightarrow x = y$;

(iii) *Transitivity: if* $R(x,z) \geq max_{y \in X}(min(R(x,y), R(y,z)))$*, for all* $x, y, z \in X$.

A nonempty set X *with a fuzzy order* R *defined on it is called a fuzzy ordered set and is denoted by* $(X; R)$*. It easily follows that each partially ordered set* $(X; \leq)$ *and each fuzzy ordered set* $(X; R)$ *can be viewed as fuzzy ordered sets.*

Example 1.23. *Let* $X = \{a, b, c, d, e\}$*. Then, the fuzzy relation* R *defined on* X *by* $R = \{\langle (x, y), R(x, y)\rangle \mid x, y \in X\}$.
where R *is given by the following table:*

$R(.,.)$	a	b	c	d	e
a	1	0	0	0.65	0.50
b	0	1	0	0.35	0.45
c	0	0	1	0	0.80
d	0	0	0	1	0
e	0	0	0	0	1

is an the fuzzy order on X.

Example 1.24. *Let* $m, n \in \mathbb{N}$*. Then, the following fuzzy relation* R *on* $\mathbb{N}$ *is an fuzzy order, where*

$$R(m, n) = \begin{cases} 1, & if \quad m = n, \\ 1 - \frac{m}{n}, & if \quad m < n, \\ 0, & if \quad m > n. \end{cases}$$

On the basis of the above definition of antisymmetry we define a complete (or total) fuzzy order as follows.

Definition 1.36. [30] *A fuzzy order* R *on a universe* X *is called complete (or total) if for all* $x, y \in X$ *it holds that* $[R(x;y) > 0 \quad or \quad (R(x;y) = 0)] \quad or \quad [R(y;x) > 0 \quad or \quad (R(y;x) = 0)]$.

Example 1.25. *Let* R *be a fuzzy relation on* $X = \{x, y, z\}$ *given by*

R	x	y	z
x	1	0.6	0.6
y	0	1	0.4
z	0	0	1

R *is a total fuzzy order.*

1.3.2 Intuitionistic fuzzy relations

Burillo and Bustince [7] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.

Definition 1.37. [7] *Let X and Y be two nonempty sets. An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from X to Y is an intuitionistic fuzzy subset of $X \times Y$, i.e., is an expression R given by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0, 1], \text{ and } \nu_A : X \times Y \to [0, 1]$$

satisfy the condition

$$0 \le \mu_R(x, y) + \nu_R(x, y) \le 1, \tag{1.1}$$

for any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of membership of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R.

Several operations on intuitionistic fuzzy relations are defined (see, e.g., Atansssov [3, 4, 5], Biswas [6] and Gy [12]). Here we will present only those which are related to the present work.

Let R, P be two intuitionistic fuzzy relations. R is said to be contained in P (or we say that P contains R), denoted by $R \subseteq P$, if for any $(x, y) \in X \times Y$ it holds that $\mu_R(x, y) \le \mu_P(x, y)$ and $\nu_R(x, y) \ge \nu_P(x, y)$. The *transpose* (or the *inverse*) R^t of R is the intuitionistic fuzzy relation from Y to X defined by $R^t = \{\langle (x, y), \mu_{R^t}(x, y), \nu_{R^t}(x, y)\rangle \mid (x, y) \in X \times Y\}$, where $\mu_{R^t}(x, y) = \mu_R(y, x)$ and $\nu_{R^t}(x, y) = \nu_R(y, x)$ for any $(x, y) \in X \times Y$.

The *intersection* of two intuitionistic fuzzy relations R and P is defined as

$$R \cap P = \{\langle (x, y), \min(\mu_R(x, y), \mu_P(x, y)), \max(\nu_R(x, y), \nu_P(x, y))\rangle \mid (x, y) \in X \times Y\}$$

The *union* of two intuitionistic fuzzy relations R and P is defined as

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}.$$

Let R, P and Q be three intuitionistic fuzzy relations from a universe X to a universe Y.

(i) if $R \subseteq P$, then $R^t \subseteq P^t$;

(ii) $(R \cup P)^t = R^t \cup P^t$;

(iii) $(R \cap P)^t = R^t \cap P^t$;

(iv) $(R^t)^t = R$;

(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;

(vi) $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;

(vii) if $R \supseteq P$ and $R \supseteq Q$, then $R \supseteq P \cup Q$;

(viii) if $R \subseteq P$ and $R \subseteq Q$, then $R \subseteq P \cap Q$.

Let R be an intuitionistic fuzzy relation (intuitionistic fuzzy relation on X, for short). The following properties are crucial in (see e.g., [7, 28, 32]):

(i) *Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.

(ii) *Antisymmetry*: for any $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases}, \tag{1.2}$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) *Perfect antisymmetry*: for any $x, y \in X$ with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0 \\ \text{ or} \\ \mu_R(x,y) = 0 \;\text{ and } \nu_R(x,y) < 1\,, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \text{ and} \\ \nu_R(y,x) = 1\,. \end{cases} \tag{1.3}$$

(iv) *Transitivity*:

$$R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R. \tag{1.4}$$

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that $R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\}\rangle \mid x, z \in X\}$, where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\} \leq 1\,,$$

for any $x, z \in X$. The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in

Note that Bustince and Burillo in [8], mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh [30] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order.

Definition 1.38. [7] *Let X be a nonempty crisp set and R be an intuitionistic fuzzy relation on X. R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

A nonempty set X with an intuitionistic fuzzy order R defined on it is called an intuitionistic fuzzy ordered set and is denoted by (X, μ_R, ν_R). It easily follows that each partially ordered set $(X, \leq)$ and each fuzzy ordered set (X, R) can be viewed as intuitionistic fuzzy ordered sets.

Example 1.26. *Let $X = \{a, b, c, d, e\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \mid x, y \in X\},$$

where μ_R and ν_R are given by the following tables:

$\mu_R(.,.)$	*a*	*b*	*c*	*d*	*e*
a	1	0	0	0.55	0.40
b	0	1	0	0.35	0.45
c	0	0	1	0	0.70
d	0	0	0	1	0
e	0	0	0	0	1

$\nu_R(.,.)$	*a*	*b*	*c*	*d*	*e*
a	0	1	0.40	0.45	0.25
b	0.30	0	0.20	0.35	0.10
c	1	1	0	0.85	0.15
d	1	1	1	0	1
e	1	1	1	0.90	0

is an intuitionistic fuzzy order on X.

Example 1.27. *Let $m, n \in \mathbb{N}$. Then the following intuitionistic fuzzy relation R on $\mathbb{N}$ is an intuitionistic fuzzy order, where*

$$\mu_R(m,n) = \begin{cases} 1\ ,\ if\ m = n \\ 1 - \frac{m}{n}\ ,\ \ if\ \ m < n \\ 0\ ,\ \ if\ \ m > n\ , \end{cases} \quad and\ \nu_R(m,n) = \begin{cases} 0\ ,\ if\ \ m = n \\ \frac{m}{2n}\ ,\ \ if\ \ m < n \\ 1\ ,\ \ if\ \ m > n \end{cases}$$

On the basis of the above definition of perfect antisymmetry we define a complete (or total) intuitionistic fuzzy order as follows

Definition 1.39. [32] *An intuitionistic fuzzy order R on a universe X is called complete (or total) if for any $x, y \in X$ it holds that*

$$[\mu_R(x,y) > 0 \text{ or } (\mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1)]$$

or

$$[\mu_R(y,x) > 0 \text{ or } (\mu_R(y,x) = 0 \text{ and } \nu_R(y,x) < 1)].$$

Definition 1.40. [32] *An intuitionistic fuzzy ordered set (X, μ_R, ν_R) in which R is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

Chapter 2

Particular intuitionstic fuzzy subsets on a lattice

In this chapter, we introduce the notions of intuitionistic fuzzy ideal and filter on an intuitionistic fuzzy ordered lattice and we present interesting characterizations of these notions in terms of the intuitionistic fuzzy ordered lattice operations and in terms of their (α, β)-level sets.

2.1 Intuitionistic fuzzy lattices

In this section, we recall the basic definitions and properties of intuitionistic fuzzy lattices and some related notions that will be needed throughout the next chapters. The concept of an intuitionistic fuzzy lattice was introduced by Thomas and Nair [27] as an intuitionistic fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations $\sqcap$ and $\sqcup$.

To avoid any confusion or misunderstanding in some formulas, we use the notation $(\leqslant, \sqcap, \sqcup)$ to refer the (order, min, max) on the lattice L and $(\leq, \wedge, \vee)$ to refer the (usual order, min, max) on the real interval $[0, 1]$.

Definition 2.1. [27] *Let L be a lattice and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFS on L. Then A is called an intuitionistic fuzzy sub-lattice (fuzzy lattice, for* *short) if for any $x, y \in L$, the following conditions are satisfied:*

(i) $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$;

(iii) $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv) $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$.

Example 2.1. *Figure 1.1 shows the Hasse diagram of a lattice* $L = \{0, a, b, 1\}$. *The intuitionistic fuzzy set* A *on* L *given by* $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$ *is an intuitionistic fuzzy lattice.*

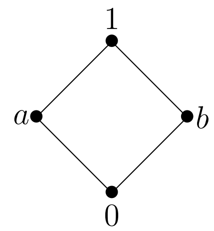


Figure 2.1: Hasse diagram of a lattice $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

For further details on intuitionistic fuzzy lattices, we refer to [14, 26, 27].

2.2 Intuitionistic fuzzy ideals and filters on a lattice

The notion of intuitionistic fuzzy ideal (resp. filter) on a lattice $(L, \leqslant, \sqcap, \sqcup)$ was first introduced by Thomas and Nair [27].

Definition 2.2. [27] *Let* L *be a lattice and* $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ *be an IFS on* L. *Then* I *is called an intuitionistic fuzzy ideal on* L *(IF-ideal, for short) if for all* $x, y \in L$ *the following conditions are satisfied :*

(i) $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

(ii) $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

(iii) $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$;

(iv) $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

Example 2.2. *Let L be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set I on L defined by:* $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$ *is an IF-ideal.*

Definition 2.3. [27] *Let L be a lattice and* $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ *be an IFS on L. Then F is called an intuitionistic fuzzy filter on L (IF-filter, for short) if for any* $x, y \in L$*, the following conditions are satisfied :*

(i) $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

(ii) $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

(iii) $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$;

(iv) $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

Example 2.3. *Let L be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set F on L defined by:* $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$ *is an IF-filter.*

Remarque 2.1. *Notice that every IF-ideal on L is an L-IF-lattice, but the converse is not true in general. Indeed, let L be the lattice given by the Hasse diagram in Figure 2.1 and* $A \in IFS(L)$ *defined by* $A = \{< 0, 0.3, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$*. Then A is an L-IF-lattice, but since* $\mu_A(a) = \mu_A(a \sqcap 1) = 0.4 \not\geq \max\{0.4; 0.7\}$*, then it holds that A is not an IF-ideal on L. As well since* $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \not\geq \min\{0.4; 0.4\}$*, then it holds that A is not an IF-filter on L.*

The following results will be needed throughout this chapter.

Proposition 2.1. *Let L be a lattice, L^d be its order-dual lattice and $A \in IFS(L)$. Then it holds that A is an IF-ideal on L if and only if A is an IF-filter on L^d and conversely.*

Proof

Given L is lattice and $(F, A) \in IFSs(L)$ and (F, A) is an IF-ideal on L .

To prove: (F,A) is an IF-filter on L^d .

We have L is a lattice. and $I_{(F,A)}$ *is an* $IFSs$ on L then

$I_{(F,A)}$

is called on $IF - ideal$ on L. if for all $x, y \in L$

(i)

$\mu_{I_{(F,A)}}(x \vee y) \geq \mu_{I_{(F,A)}}(x) \wedge \mu_{I_{(F,A)}}(y)$

Then is a $IFSs$ on L then is called an IF-filter on L.

$\mu_{I_{(F,A)}}(x \vee y) \geq \mu_{I_{(F,A)}}(x) \wedge \mu_{I_{(F,A)}}(y)$, we show that IF-filter on L^d

$\mu_{I_{(F,A)}}(x \vee y) \geq \mu_{I_{(F,A)}}(x) \vee \mu_{I_{(F,A)}}(y)$ on L

(ii)

$\mu_{F_{(F,A)}}(x \vee y) \leqslant \mu_{F_{(F,A)}}(x) \wedge \mu_{F_{(F,A)}}(y)$ on L^d

From equation (i) and (ii)

$\mu_{I_{(F,A)}}(x \vee y) \geq \mu_{I_{(F,A)}}(x) \wedge \mu_{I_{(F,A)}}(y)$ then $\mu_{F_{(F,A)}}(x \vee y) \leqslant \mu_{F_{(F,A)}}(x) \wedge \mu_{F_{(F,A)}}(y)$

$\mu_{I_{(F,A)}}(x \vee y) = \mu_{F_{(F,A)}}(x \vee y)$

Proposition 2.2. *[14]*

Let L be a lattice, A and B are two intuitionistic fuzzy sets on L. Then it holds that

(i) if A and B are two IF-ideals on L, then $A \cap B$ is an IF-ideal on L;

(ii) if A and B are two IF-filters on L, then $A \cap B$ is an IF-filter on L.

proof

If (F, A) and(F, B) are two IF-ideals on L then, To prove: $(F, A) \cap (F, B)$ is an IF-ideal on L

Let L be a lattice is a $IFSs$ on L then $I_{(F,A)\cap(F,B)}$ is called IF-ideal on L. if for all $x, y \in L$.

(i) $I_{((F,A)\cap(F,B))}(x \vee y) \geq I_{((F,A)\cap(F,B))}(x) \wedge I_{((F,A)\cap(F,B))}(y)$

(ii) $I_{((F,A)\cap(F,B))}(x \wedge y) \geq I_{((F,A)\cap(F,B))}(x) \vee I_{((F,A)\cap(F,B))}(y)$

(iii) $I_{((F,A)\cap(F,B))}(x \vee y) \leqslant I_{((F,A)\cap(F,B))}(x) \vee I_{((F,A)\cap(F,B))}(y)$

(iv) $I_{((F,A)\cap(F,B))}(x \wedge y) \leqslant I_{((F,A)\cap(F,B))}(x) \wedge I_{((F,A)\cap(F,B))}(y)$

Hence $(F, A) \cap (F, B)$ is an $IF - ideal$ on L.

2.3 Characterizations of intuitionistic fuzzy ideals and filters in terms of their level sets

In this section, we characterize the notion of IF-ideals and IF-filters on IF-lattice in terms of the IF-ordered lattice operations. First, we need the following key results.

Theorem 2.1. [22] *(definition)*
Let L be a lattice and $A \in IFS(L)$. Then for any $x, y \in L$ the following four statements hold

(i) $(\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$;

(ii) $(\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$;

(iii) $(\nu_A(x \sqcap y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \leq \nu_A(y))$;

(iv) $(\nu_A(x \sqcup y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \geq \nu_A(y))$.

Proof. Let $x, y \in L$.

(i) Suppose that $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$ then $x \sqcap y = x$. Since $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(x) = \mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \geq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$. Then it follows that $\mu_A(x \sqcap y) \geq \mu_A(x)$ and $\mu_A(x \sqcap y) \geq \mu_A(y)$. Hence, $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$.

(ii) Suppose that $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$ then $x \sqcup y = y$. Since $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(y) = \mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \leq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$. Then it follows that $\mu_A(x) \leq \mu_A(x \sqcup y)$ and $\mu_A(y) \leq \mu_A(x \sqcup y)$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$.

(iii) The proof is similar to (i).

(iv) The proof is similar to (ii).

□

As corollaries, we obtain the following interesting properties of IF-ideals and IF-filters.

Corollaire 2.1. *Let L be a lattice and I be an IF-ideal on L. Then for any $x, y \in L$ it holds that*

(i) If $x \leqslant y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : L \to [0,1]$ is antitone);

(ii) If $x \leqslant y$, then $\nu_I(x) \leq \nu_I(y)$, (i.e., the map $\nu_I : L \to [0,1]$ is monotone).

Remarque 2.2. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and I the intuitionistic fuzzy ideal on L given by $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$. It is easy to verify that $\mu_I(a) = 0.4 \geq \mu_I(b) = 0.1$, but a, b are incomparable elements.*

Corollaire 2.2. *Let L be a lattice and F be an IF-filter on L. Then for any $x, y \in L$ it holds that*

(i) If $x \leqslant y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : L \to [0,1]$ is monotone);

(ii) If $x \leqslant y$, then $\nu_F(x) \geq \nu_F(y)$, (i.e., the map $\nu_F : L \to [0,1]$ is antitone).

Remarque 2.3. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and F the intuitionistic fuzzy ideal on L given by $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$. It is easy to verify that $\mu_F(b) = 0.1 \leq \mu_F(a) = 0.2$, but a, b are incomparable elements.*

Corollaire 2.3. *Let L be a lattice has smallest element $\perp$ or greatest element $\top$ and I be an IF-ideal on L. Then it holds that*

(i) $\mu_I(\perp) = \max \mu_I(L)$ and $\mu_I(\top) = \min \mu_I(L)$, where $\mu_I(L) = \{\mu_I(x) \mid x \in L\}$;

(ii) $\nu_I(\perp) = \min \nu_I(L)$ and $\nu_I(\top) = \max \nu_I(L)$, where $\nu_I(L) = \{\nu_I(x) \mid x \in L\}$.

Corollaire 2.4. *[15] Let L be a lattice has smallest element $\perp$ or greatest element $\top$ and F be an IF-filter on L. Then it holds that*

(i) $\mu_F(\perp) = \min \mu_F(L)$ and $\mu_F(\top) = \max \mu_F(L)$, where $\mu_F(L) = \{\mu_F(x) \mid x \in L\}$;

(ii) $\nu_F(\perp) = \max \nu_F(L)$ and $\nu_F(\top) = \min \nu_F(L)$, where $\nu_F(L) = \{\nu_F(x) \mid x \in L\}$.

In the following theorem, Milles et al . provided a basic characterization of IF-ideals on a lattice.

Theorem 2.2. *[15] Let L be a lattice and $I \in IFS(L)$. Then it holds that I is an IF-ideal on L if and only if the following two conditions are satisfied:*

(i) $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ for $x, y \in L$;

(ii) $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$ for $x, y \in L$.

Proof. Suppose that I is an IF-ideal on L. Then for any $x, y \in L$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$. Since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, from Corollary 2.1 it follows that

$$\mu_I(x) \geq \mu_I(x \sqcup y)$$

and

$$\mu_I(y) \geq \mu_I(x \sqcup y)\,.$$

Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.

In addition, since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$ we obtain from Corollary 2.1 that

$$\nu_I(x) \leq \nu_I(x \sqcup y)$$

and

$$\nu_I(y) \leq \nu_I(x \sqcup y)\,.$$

Hence, $\nu_I(x) \vee \nu_I(y) \leq \nu_I(x \sqcup y)$. Thus, $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$.

Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ and $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in L$. Then it is easy to see that

$$\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$$

and

$$\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y).$$

Next, we show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$ and $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$ for $x, y \in L$. Let $x, y \in L$. Since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$. From Definition 2.2 (hypothesis (i) and (ii)) it follows that

$$\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$$

and

$$\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y)\,.$$

Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$. In the same way, we obtain that $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$, for any $x, y \in L$. Therefore, I is an IF-ideal on L. □

In the same manner, the following theorem, Milles et al . provided a basic characterization of IF-filters on a lattice.

Theorem 2.3. [22] *Let L be a lattice and $F \in IFS(L)$. Then it holds that F is an intuitionistic fuzzy filter on L if and only if the following conditions are satisfied:*

(i) $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$ *for* $x, y \in L$;

(ii) $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$ *for* $x, y \in L$.

Proof. The proof is a direct application of Proposition 2.1 and Theorem 2.2. □

2.4 Characterizations of intuitionistic fuzzy ideals and filters in terms of their level sets

In this section, we provide some interesting characterizations and properties of IF-ideals and IF-filters in terms of their level sets.

Proposition 2.3. [22] *Let L be a lattice and $A \in IFS(L)$. The following statements hold*

(i) if A is an IF-ideal, then its support $Supp(A)$ is an ideal on L;

(ii) if A is an IF-filter, then its support $Supp(A)$ is a filter on L.

Proof. Let $A \in IFS(L)$.

(i) Suppose that $A \in IFS(L)$ is an IF-ideal. We show that $Supp(A)$ is an ideal on L.

(a) Let $x \in Supp(A)$ and $y \leqslant x$, then it holds that

$$\mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\,.$$

There are two cases to consider : ($y \leqslant x$ and $\mu_A(x) > 0$) and ($y \leqslant x$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$)).

First case: suppose that $y \leqslant x$ and $\mu_A(x) > 0$. Since $y \leqslant x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) > 0$. From Theorem 2.2 (i), it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $\mu_A(y) > 0$. Thus, $y \in Supp(A)$.

Second case: $y \leqslant x$ and ($\mu_A(x) = 0$ and $\nu_A(x) < 1$). Since $y \leqslant x$, then it holds that $x \sqcup y = x$. This implies that $\mu_A(x) = \mu_A(x \sqcup y) = 0$ and $\nu_A(x) = \nu_A(x \sqcup y) < 1$. From Theorem 2.2, it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) = 0$ and $\nu_A(x \sqcup y) = \nu_I(x) \vee \nu_I(y) < 1$. Hence, $\mu_A(y) > 0$ or ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$. We show now that $x \sqcup y \in Supp(A)$. There are four cases to consider.

First case: let $\mu_A(x) > 0$ and $\mu_A(y) > 0$. Since A is an IF-ideal, then from Theorem 2.2 (i) it follows that $\mu_A(x \sqcup y) = \mu_I(x) \wedge \mu_I(y) > 0$. Hence, $x \sqcup y \in Supp(A)$.

Second case: let ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

Third case: $\mu_A(x) > 0$ and ($\mu_A(y) = 0$ and $\nu_A(y) < 1$). From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = 0$ and $\nu_A(x \sqcup y) < 1$. Hence, $x \sqcup y \in Supp(A)$.

The last case ($\mu_A(x) = 0$ and $\nu_A(x) < 1$) and $\mu_A(y) > 0$ is analogous to the third case. Thus, $Supp(A)$ is an ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Remarque 2.4. *The converse of the above implications are not necessarily hold. Indeed, let us consider the lattice L given by the Hasse diagram in Figure 2.1 and* $A \in IFS(L)$ *given by* $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 > \}$. *It is easy to verify that* $Supp(A) = L$ *is an ideal and a filter on* L*, but* A *is neither an IF-ideal nor an IF-filter on* L*.*

The following theorem, Milles et al . provided a characterization of IF-ideals (resp. IF-filters) in terms of their level sets.

Theorem 2.4. [22] *Let* L *be a lattice and* $A \in IFS(L)$*. The following statements hold*

(i) *A is an IF-ideal if and only if their level sets are ideals on* L*;*

(ii) *A is an IF-filter if and only if their level sets are filters on* L*.*

Proof. Let $A \in IFS(L)$ and $A_{\alpha,\beta}$ their level set, where $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

(i) Suppose that A is an IF-ideal on L. We show that $A_{\alpha,\beta}$ is an ideal on L for $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

  (a) Let $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$, $x \in A_{\alpha,\beta}$ and $y \in L$ such that $y \leqslant x$. Since $x \in A_{\alpha,\beta}$, then it holds that $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Since $y \leqslant x$, from Corollary 2.1 it follows that $\mu_A(y) \geq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$. This implies that $\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$. Hence, $y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

  (b) Let $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Then it holds that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). From Theorem 2.2 it follows that $\mu_A(x \sqcup y) = \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\nu_A(x \sqcup y) = \nu_A(x) \vee \nu_A(y) \leq \beta$. Hence, $x \sqcup y \in A_{\alpha,\beta}$ for $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$. Thus, $A_{\alpha,\beta}$ is an ideal on L for $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$.

Conversely, suppose that all level sets of A are ideals on L. We show that A is an IF-ideal on L. Let $x, y \in L$, $\alpha = \mu_A(x) \wedge \mu_A(y)$ and $\beta = \nu_A(x) \vee \nu_A(y)$. Then it follows that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). The case $\alpha = 0$ or $\beta = 0$ is obvious. Let $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$ and $x, y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on L, then it holds that $x \sqcup y \in A_{\alpha,\beta}$ $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta \leq 1$. This implies that $\mu_A(x \sqcup y) \geq \alpha$ and $\nu_A(x \sqcup y) \leq \beta$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$.

On the other hand, let $\alpha = \mu_A(x \sqcup y)$ and $\beta = \nu_A(x \sqcup y)$. The case $\alpha = 0$ or $\beta = 0$ is also obvious. Otherwise $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$ and $x \sqcup y \in A_{\alpha,\beta}$. Since $A_{\alpha,\beta}$ is an ideal on L, $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, it follows that $x, y \in A_{\alpha,\beta}$, for any $\alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \leq 1$. This implies that ($\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$) and ($\mu_A(y) \geq \alpha$ and $\nu_A(y) \leq \beta$). Hence, $\mu_A(x) \wedge \mu_A(y) \geq \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) \leq \nu_A(x \sqcup y)$. Thus, $\mu_A(x) \wedge \mu_A(y) = \mu_A(x \sqcup y)$ and $\nu_A(x) \vee \nu_A(y) = \nu_A(x \sqcup y)$. Therefore, Theorem 2.2 guarantees that A is an IF-ideal on L.

(ii) Follows from Proposition 2.1 and (i).

□

Conclusion

In this memoire, we have studied the notions of intuitionistic fuzzy ideals and intuitionistic fuzzy filters on a lattice and provided some basic properties. Furthermore, we have characterized these notions in terms of the lattice meet and join operations, and in terms of their level sets as associated crisp sets.

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ملخص

في هذه المذكرة قمنا بدراسة انواع خاصة من مجموعات ضبابية حدسية داخل شبكة. أكثر تفصيلا، المثاليات الضبابية الحدسية والمرشحات الضبابية الحدسية. كما قمنا بدراسة مميزات هذه المفاهيم وفق العمليات الداخلية للشبكة، وكذلك وفق مجموعاتهم المستوياتية.

كلمات مفتاحية

مجموعة ضبابية حدسية، مثالي ضبابي حدسي، مرشح ضبابي حدسي .

Abstract

In this memoire, we study particular intuitionistic fuzzy subsets on a lattice and provide some examples. More precisely, the intuitionistic fuzzy ideal and the intuitionistic fuzzy filter on a given lattice. Further, we give some characterizations of these notions in terms of the lattice meet and join operations and in terms of their level sets .

Key words

Intuitionistic fuzzy set, Intuitionistic fuzzy ideal, Intuitionistic fuzzy filter .

Résumé

Dans ce mémoire, nous étudions des types particulières d'ensembles flous intuitionnistes. Plus précisément, l'idéal flou intuitionniste et le filtre flou intuitionniste. De plus, nous donnons quelques caractérisations de ces notions en termes de lattice min max opérations, ainsi que, en termes de ses coupes sets.

Mot-clés

Ensemble flou intuitionniste, Idéal flou intuitionniste, Filter flou intuitionniste .