ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2019, **11** (2), 3–11
doi:10.15330/cmp.11.2.3-11

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.3–11

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WEIGHTS OF THE $\mathbb{F}_q$-FORMS OF 2-STEP SPLITTING TRIVECTORS OF RANK 8 OVER A FINITE FIELD

Grassmann codes are linear codes associated with the Grassmann variety $G(\ell, m)$ of ℓ-dimensional subspaces of an m dimensional vector space $\mathbb{F}_q^m$. They were studied by Nogin for general q. These codes are conveniently described using the correspondence between non-degenerate $[n, k]_q$ linear codes on one hand and non-degenerate $[n, k]$ projective systems on the other hand. A non-degenerate $[n, k]$ projective system is simply a collection of n points in projective space $\mathbb{P}^{k-1}$ satisfying the condition that no hyperplane of $\mathbb{P}^{k-1}$ contains all the n points under consideration. In this paper we will determine the weight of linear codes $C(3, 8)$ associated with Grassmann varieties $G(3, 8)$ over an arbitrary finite field $\mathbb{F}_q$. We use a formula for the weight of a codeword of $C(3, 8)$, in terms of the cardinalities certain varieties associated with alternating trilinear forms on $\mathbb{F}_q^8$. For $m = 6$ and 7, the weight spectrum of $C(3, m)$ associated with $G(3, m)$, have been fully determined by Kaipa K.V, Pillai H.K and Nogin Y. A classification of trivectors depends essentially on the dimension n of the base space. For $n \leq 8$ there exist only finitely many trivector classes under the action of the general linear group $GL(n)$. The methods of Galois cohomology can be used to determine the classes of nondegenerate trivectors which split into multiple classes when going from $\bar{\mathbb{F}}$ to $\mathbb{F}$. This program is partially determined by Noui L and Midoune N and the classification of trilinear alternating forms on a vector space of dimension 8 over a finite field $\mathbb{F}_q$ of characteristic other than 2 and 3 was solved by Noui L and Midoune N. We describe the $\mathbb{F}_q$-forms of 2-step splitting trivectors of rank 8, where char $\mathbb{F}_q \neq 3$. This fact we use to determine the weight of the $\mathbb{F}_q$-forms.

Key words and phrases: trivector, Grassmannian, weight.

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Introduction

Let V be an 8-dimensional vector space over a field K and let $\wedge^3 V$ denote the exterior power of degree 3 over V, the classification of trivectors is the study of the action of general linear group $GL(V)$ on the space $\wedge^3 V$ defined by $f.\omega = (\wedge^3 f)(\omega)$. The equivalence classes are the $GL(V)$-orbits under this action. As $\wedge^3 V^* \simeq (\wedge^3 V)^*$, there is no difference between trilinear alternating forms and trivectors. The support of the trivector ω is the least subspace F of V such that $\omega \in \wedge^3 F$, its dimension is the rank of ω. Let ω be a trilinear alternating form on V. The set $\{u \in V, \ \ \omega(u, \cdot, \cdot) = 0\}$ is called the radical of ω and will be denoted by $Rad\omega$. If $Rad\omega = \{0\}$, then ω is called nondegenerate (full rank).

This classification is motivated by many important applications, especially in the theory of codes. See [2,4,5,7]. Let $C(3, 8)$ be a grassman code (linear code) associated with the Grassmann

УДК ???
2010 *Mathematics Subject Classification:* 15A69, 20B40.

variety $G(3,8)$ of 3-dimensional subspaces of an 8-dimensional vector space $\mathbb{F}_q^8$, where $\mathbb{F}_q$ is a finite field with q elements. The parameters n and k of the code $C(3,8)$ are

$$n = |C(3,8)| = \frac{(q^8-1)(q^{8-1}-1)(q^{8-3+1}-1)}{(q^3-1)(q^{3-1}-1)(q-1)},$$
$$k = \binom{8}{3}.$$

The minimum distance of grassmann codes $C(3,8)$ equals $d = q^{3(8-3)} = q^{15}$. The weight of $C(3,7)$, $C(3,6)$ and $C(2,m)$ is determined by [2], [5] and [4] respectively. In this paper,we are interested in the classification of $\mathbb{F}_q$-forms of the 2-step splitting trivectors of rank 8, and in determining the weights of $\mathbb{F}_q$-forms where $\mathbb{F}_q$ is a finite field of characteristic other than 3. Some undefined terms can be found in references [2, 3, 6] and [5].

1 $\mathbb{F}_q$-FORMS OF 2-STEP SPLITTING TRIVECTORS OF RANK ≤ 8

If ω is a trivector defined over the field K, a K-form of ω is another trivector of the same type as that of ω, defined over K which is isomorphic to ω over $\overline{K}$, the algebraic closure of K. The element ω of $\wedge^3 V$ is called splitting if there exists a decomposition $V = V_1 \oplus V_2$ such that $\omega \in V_1 \otimes \wedge^2 V_2$. If $dim V_1 = 2$, ω is called 2-step splitting.

Preliminary result

1.1 Degenerate forms

Theorem 1 ([1]). *Let V be a vector space of dimension 7 over a finite field $\mathbb{F}_q$. Then any trivector of rank ≤ 7 in $\wedge^3 V$ is equivalent to one of the trivectors in Table 1.*

Table1. Trivectors of rank ≤ 7 over $\mathbb{F}_q$ (degenerate forms).

Name	*Trivector*
ω_3	$e_1e_2e_3$
ω_5	$e_1(e_2e_3 + e_4e_5)$
$\omega_{6,1}$	$e_1e_2e_3 + e_4e_5e_6$
$\omega_{6,1,d_1}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_6 - d_1e_4e_5)$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{6,1,d_2}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_4 - d_2e_3e_5 + e_4e_5)$ *if char* $\mathbb{F}_q = 2$
$\omega_{6,2}$	$e_1e_2e_4 + e_2e_3e_5 + e_1e_3e_6$
$\omega_{7,1}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7)$
$\omega_{7,2}$	$\omega_{7.1} + e_2e_4e_6$
$\omega_{7,3}$	$e_1e_2e_3 + e_3e_4e_5 + e_5e_6e_7$
$\omega_{7,3,d_1}$	$e_1(e_2e_5 + e_3e_7) + e_4(e_2e_3 + d_1e_5e_7) + e_6e_5e_3$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{7,3,d_2}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_4 - d_2e_3e_5 + e_4e_5) + e_1e_6e_7$ *if char* $\mathbb{F}_q = 2$
$\omega_{7,4}$	$e_1(e_2e_3 + e_4e_5) + e_2e_4e_6 + e_3e_5e_7$
$\omega_{7,5}$	$\omega_{7.2} + e_3e_5e_7$

where $d_1 \notin (\mathbb{F}_{q^*})^2$, $d_2 \in (\mathbb{F}_{q^*})^2$.

Main results

1.2 Nondegenerate forms (full rank)

Theorem 2. *Let V be a vector space of dimension 8 over a finite field $\mathbb{F}_q$. Then any $\mathbb{F}_q$-form of 2-step splitting trivector of rank 8 in $\wedge^3 V$ is equivalent to one of the Table 2.*

Table2. Trivectors of rank 8 over $\mathbb{F}_q$ (nondegenerate forms).

Name	Trivector
$\omega_{8,1}$	$e_1(e_2e_3 + e_4e_5) + e_6e_7e_8$
$\omega_{8,2}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_5e_6e_8$
$\omega_{8,3}$	$e_1(e_3e_4 + e_5e_6) + e_2(e_3e_5 + e_7e_8)$
$\omega_{8,4}$	$e_1(e_2e_5 + e_3e_6) + e_4(e_7e_2 + e_8e_3)$
$\omega_{8,4,d_1}$	$e_5(e_1e_2 + e_3e_4) + e_6(e_1e_3 + d_1e_2e_4) + e_7(e_1e_4) + e_8(e_2e_3)$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{8,4,d_2}$	$e_8(e_1e_4 + e_3e_2) + e_7(e_1e_4 + e_4e_2 + d_2e_1e_3) + e_6e_1e_2 + e_5e_3e_4$ *if char* $\mathbb{F}_q = 2$
$\omega_{8,5}$	$e_1(e_2e_3 + e_4e_5) + e_6(e_2e_3 + e_7e_8)$
$\omega_{8,5,d_1}$	$e_7(e_1e_2 + e_3e_4 + e_5e_6) + e_8[e_1(e_4 + d_1e_5) + e_2e_6 + \frac{1}{d_1}e_3e_5]$ *if char* $\mathbb{F}_q \neq 2$
$\omega_{8,5,d_2}$	$e_3(e_1e_2 + e_4e_7 + e_6e_8) + e_5(e_1e_4 + e_8e_2 + d_2e_6e_7)$ *if char* $\mathbb{F}_q = 2$
$\omega_{8,5,d_3}$	$e_1(d_3e_3e_4 + d_3e_5e_6 + e_7e_8) + e_2(e_3e_5 + e_4e_7 + e_6e_8)$ *if char* $\mathbb{F}_q \neq 3$
$\omega_{8,6}$	$e_1(e_2e_3 + e_4e_5 + e_6e_7) + e_8(e_4e_3 + e_5e_6)$

where $d_1 \notin (\mathbb{F}_{q^*})^2, d_2 \in (\mathbb{F}_{q^*})^2, d_3 \notin (\mathbb{F}_{q^*})^3$.

Proof. The $\mathbb{F}_q$-forms of 2-step splitting trivectors of rank 8 where $\mathbb{F}_q$ is a field of characteristic other than 2 and 3 has been done in [6], hence, in characteristic 2, it is sufficient to study the case of orbits of type $\omega_{8,i}$, for $i = 4, 5$.

In characteristic 2, the trivectors of type $\omega_{8,i}$, for $i = 4, 5$ are written
$\omega_{8,4,d_2} = e_8(e_1e_4 + e_3e_2) + e_7(e_1e_4 + e_4e_2 + d_2e_1e_3) + e_6e_1e_2 + e_5e_3e_4$
$\omega_{8,5,d_2} = e_3(e_1e_2 + e_4e_7 + e_6e_8) + e_5(e_1e_4 + e_8e_2 + d_2e_6e_7)$.

If L is the quadratic extension of K, there exists a trivector $\omega_L \in \wedge^3 V$ such that $\omega_L \not\simeq \omega_{8,4}$ and $\omega_L \otimes L \in \wedge^3(V \otimes_K L)$ is L-isomorphic to $\omega_{8,4}$. We construct ω_L as follows: $\omega_{8,4} = e_1(e_2e_5 + e_3e_6) + e_4(e_7e_2 + e_8e_3)$ is a 4-step splitting because $\omega_{8,4} = e_5u_1 + e_6u_2 + e_7u_3 + e_8u_4$ where $u_1 = e_1e_2$, $u_2 = e_1e_3$, $u_3 = e_2e_4$ and $u_4 = e_3e_4$, thus $E = vect\{u_1, u_2, u_3, u_4\}$ is a subspace of dimension 4 of $\wedge^4 K^4$. We put $\omega_L = \omega_{8,4,d_2} = e_5v_1 + e_6v_2 + e_7v_3 + e_8v_4$, with $v_1 = e_3e_4$, $v_2 = e_1e_2$, $v_3 = e_1e_4 + e_4e_2 + d_2e_1e_3$, and $v_4 = e_1e_4 + e_3e_2$, where $K' = K(\alpha)$, $\alpha^2 + \alpha = d_2$, $\alpha \in K$. To each of the forms $\omega_{8,4}$, $\omega_{8,4,d_2}$, we associate a quadratic form on E [6]: $\gamma_2(xu_1 + yu_2 + zu_3 + tu_4)$, then we get $\gamma_2(xu_1 + yu_2 + zu_3 + tu_4) = (xt - yz)$, $\gamma_2(xv_1 + yv_2 + zv_3 + tv_4) = (y^2d_2 - x^2 + zt)$ respectively. The two forms are not equivalent over K but they may become equivalent over the algebraic closure $\overline{K}$. We can also prove that $\omega_{8,4}$ is not equivalent to $\omega_{8,4,d_2}$ by using the arithmetical invariant $d_1(\omega)$ [6].

Similar arguments apply to the case for $\omega_{8,5}$. □

2 FORMULA FOR THE WEIGHT OF A TRIVECTOR

The correspondence between equivalence classes of nondegenerate forms and equivalence classes of nondegenerate linear $[n, k]$-codes, is one-to-one. In what follows, we speak by abuse of language not only of a weight of a codeword, but also of a weight of hyperplane and a weight of a form

$\omega \in \wedge^3 V$. Therefore, the problem on the spectrum of a Grassmann code (at least, on the weights of the codewords) is closely related to that on the classification of the elements of $\wedge^3 V$.

The cardinality of the general linear group $GL(8, \mathbb{F}_q)$ will be denoted by $[8]_q$

$$[8]_q = q^{8(8-1)/2}(q^8 - 1)(q^{8-1} - 1) \cdots (q - 1).$$

Given a codeword of $C(3, 8)$, let ω be the corresponding trivector on $\mathbb{F}_q^8$, and let $\mathcal{H}$ be the corresponding hyperplane of $\mathbb{P}(\wedge^3 \mathbb{F}_q^8)$. The weight of the codeword ω

$$\mathrm{wt}(\omega) = |\{P_i \ : \ 1 \leq i \leq n, \quad P_i \notin \mathcal{H}\}|.$$

We have

$$[3]_q \cdot \mathrm{wt}(\omega) = |\{[v_1, v_2, v_3] \ : \ \langle \omega, v_1 \wedge v_2 \wedge v_3 \rangle \neq 0\}|.$$

2.1 Weight of a degenerate trivector

If ω is degenerate, let $Rad\omega$ be r-dimensional. We pick a basis $\{e_1, \ldots, e_8\}$ of V such that $\{e_{8-r+1}, \ldots, e_8\}$ is a basis for $Rad\omega$. Let W denote the span of $\{e_1, \ldots, e_{8-r}\}$.

Let $\tilde{\omega}$ denote the restriction of the form ω to W. Since $W \cap Rad\omega = \{0\}$, it is clear that $\tilde{\omega}$ is a nondegenerate trivector on W. Thus, $\tilde{\omega}$ can be thought of as codeword in $C(3, 8 - r)$.

Proposition 1 ([2]). *The weight of a degenerate trivector ω in $\mathbb{F}_q^8$ is given by*

$$\mathrm{wt}(\omega) = q^{3r}\mathrm{wt}(\tilde{\omega}).$$

The proposition shows that in order to calculate the weights of codewords of $C(3, 8)$, it is enough to know only the weights of nondegenerate codewords of $C(3, m)$ for $m \leq 8$.

Lemma 1. *The weights of degenerate trivectors are*

$$\begin{aligned}
\mathrm{wt}(\omega_3) &= q^{15} \\
\mathrm{wt}(\omega_5) &= q^{15} + q^{13} \\
\mathrm{wt}(\omega_{6,1}) &= q^{15} + q^{13} + q^{12} - q^{10} \\
\mathrm{wt}(\omega_{6,1,d}) &= q^{15} + q^{13} + q^{12} + q^{10} \\
\mathrm{wt}(\omega_{6,2}) &= q^{15} + q^{13} + q^{12} \\
\mathrm{wt}(\omega_{7,1}) &= q^{15} + q^{13} + q^{11} \\
\mathrm{wt}(\omega_{7,2}) &= q^{15} + q^{13} + q^{12} + q^{11} \\
\mathrm{wt}(\omega_{7,3}) &= q^{15} + q^{13} + q^{12} + q^{11} - q^{10} \\
\mathrm{wt}(\omega_{7,3,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} \\
\mathrm{wt}(\omega_{7,4}) &= q^{15} + q^{13} + q^{12} + q^{11} \\
\mathrm{wt}(\omega_{7,5}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{9}.
\end{aligned}$$

Proof. According to Proposition 1, the weight of a degenerate form ω is q^3 times the weight of ω viewed as a trivector on $\mathbb{F}_q^7$ the span of $\{e_1, \ldots, e_7\}$. The latter weights were determined in [2]. We multiply them by q^3; we get the weights of ω. □

2.2 Weight varieties of a nondegenerate trivector

Let V be an 8-dimensional vector space over an arbitrary field F.

We consider the map $\varphi_w : V \longrightarrow \wedge^2 V^*$ sending $v \longmapsto \iota_v\omega$ where ι_v is the operation of the interior multiplication defined by

$$\langle \iota_v\omega, \beta\rangle = \langle \omega, v \wedge \beta\rangle, \quad \text{for all } \beta \in \wedge^2 V.$$

Here, $\langle , \rangle$ is the pairing between $\wedge^j V^*$ and $\wedge^j V$ for each j.

Given a two-from $\lambda \in \wedge^2 V^*$, we define certain quantities $\mathrm{Pf}_k(\lambda) \in \wedge^{2k} V^*$, for each $k \geq 1$ which we call the k-th Pfaffian of λ. Let $\mathrm{Pf}_0(\lambda) = 1$. We define $\mathrm{Pf}_k(\lambda) \in \wedge^{2k} V^*$ inductively by requiring

$$\iota_v\lambda \wedge \mathrm{Pf}_{k-1}(\lambda) = \iota_v \mathrm{Pf}_k(\lambda), \quad \text{for all } v \in V.$$

This $\mathrm{Pf}_k(\lambda)$ generalizes the forms $\frac{\lambda^k}{k!} = \frac{1}{k!}(\lambda \wedge \cdots \wedge \lambda)$.

Definition 1 ([2]). *Given a nondegenerate trivector ω on $\mathbb{F}_q^8$, the k-th weight variety of ω is the subvariety of $\mathbb{P}^7$ given by*

$$X_k(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_{k+1}(\iota_x\omega) = 0\}.$$

We have

$$\varnothing = X_0(\omega) \subset X_1(\omega) \subset X_2(\omega) \subset X_{\lfloor \frac{8-1}{2} \rfloor = 3}(\omega) = \mathbb{P}^7.$$

Lemma 2. *Given a nondegenerate trivector ω on $\mathbb{F}_q^8$.*

Let

$$n_i := |X_i(\omega)| - |X_{i-1}(\omega)|.$$

The weight $\mathrm{wt}(\omega)$ is given by

$$\mathrm{wt}(\omega) = q^6 \left[(q^9 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + 1) - \frac{n_2 + n_1(1 + q^2)}{1 + q + q^2} \right]. \tag{1}$$

Proof. We use Theorem 7 in [2], we get

$$\mathrm{wt}(\omega) = \frac{q^{2m-4}}{(q^2-1)(1+q+q^2)} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} n_i(1 - q^{-2i}),$$

for the case $m = 8$, we use $n_1 + n_2 + n_3 = |\mathbb{P}^7|$, we get this result in (1). □

3 WEIGHT CLASSIFICATION OF TRIVECTORS ON $\mathbb{F}_q^8$

The weights of the nondegenerate forms $\omega_{8,i}$, $1 \leq i \leq 6$ can be determined from formula (1) once the cardinalities of the varieties $X_1(\omega_{8,i})$ and $X_2(\omega_{8,i})$ are known. We recall that

$$X_1(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_2(\iota_x\omega) = 0\}$$

$$X_2(\omega) = \mathbb{P}\{x \in \mathbb{F}_q^8 \backslash \{0\} \mid \mathrm{Pf}_3(\iota_x\omega) = 0\}.$$

Proposition 2. *The varieties $X_1(\omega_{8,i})$ and their cardinalities for $1 \le i \le 6$ are*

$\omega_{8,i}$	$X_1(\omega_{8,i})$	$n_1(\omega_{8,i})$
$\omega_{8,1}$	$\mathbb{P}^2 \cup \mathbb{P}^3$	$q^3 + 2q^2 + 2q + 2$
$\omega_{8,2}$	$\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^3$	$q^3 + 2q^2 + q + 1$
$\omega_{8,3}$	$\mathbb{P}^1 \cup \mathbb{P}^1$	$2q + 2$
$\omega_{8,4}$	$\mathbb{P}^1 \times \mathbb{P}^1$	$q^2 + 2q + 1$
$\omega_{8,4,d}$	$\mathbb{P}^1(\mathbb{F}_{q^2})$	$q^2 + 1$
$\omega_{8,5}$	$\mathbb{P}^1 \cup_{\mathbb{P}^0} \mathbb{P}^1$	$2q + 1$
$\omega_{8,5,d}$	$\varnothing$	0
$\omega_{8,6}$	$\mathbb{P}^1$	$q + 1$

Proof. Let $x = \sum_{j=1}^{8} x_j e_j$. We have

$$\mathrm{Pf}_2(\iota_x \omega) = \sum_{j=1}^{8} x_j^2 \mathrm{Pf}_2(\iota_{e_j}\omega) + \sum_{i<j} x_i x_j (\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega). \tag{2}$$

We calculate $\mathrm{Pf}_2(\iota_x\omega_i)$ using the above formula (2) and set it equal to zero to determine the varieties $X_1(\omega_i)$. We begin with $\omega_{8,1}$. The forms $\iota_{e_j}\omega_{8,1}$ for $j = 1 \cdots 8$ are $e_2e_3 + e_4e_5$, e_3e_1, e_1e_2, e_5e_1, e_1e_4, e_7e_8, e_8e_6, e_6e_7, respectively. For $j \ge 2$; the forms $\iota_{e_j}\omega_{8,1}$ are decomposable and hence $\mathrm{Pf}_2(\iota_{e_j}\omega_{8,1}) = 0$, whereas $\mathrm{Pf}_2(\iota_{e_1}\omega_{8,1}) = e_2e_3e_4e_5$.

We also note that $\iota_{e_2}\omega_{8,1} \wedge \iota_{e_j}\omega_{8,1} = 0$ for all $j = 3, 4, 5$, and $\iota_{e_3}\omega_{8,1} \wedge \iota_{e_4}\omega_{8,1} = \iota_{e_3}\omega_{8,1} \wedge \iota_{e_5}\omega_{8,1} = \iota_{e_4}\omega_{8,1} \wedge \iota_{e_5}\omega_{8,1} = \iota_{e_6}\omega_{8,1} \wedge \iota_{e_7}\omega_{8,1} = \iota_{e_6}\omega_{8,1} \wedge \iota_{e_8}\omega_{8,1} = \iota_{e_7}\omega_{8,1} \wedge \iota_{e_8}\omega_{8,1} = 0$. Using these relations, we get

$$\begin{aligned} \mathrm{Pf}_2(\iota_x\omega_{8,1}) &= x_1^2 e_2e_3e_4e_5 + x_1[x_2e_4e_5e_3e_1 + x_3e_4e_5e_1e_2 + x_4e_2e_3e_5e_1 + x_5e_2e_3e_1e_4 \\ &+ x_6(e_2e_3e_7e_8 + e_4e_5e_7e_8) + x_7(e_2e_3e_8e_6 + e_4e_5e_8e_6 + x_8(e_2e_3e_6e_7 + e_4e_5e_6e_7)] \\ &+ x_2(x_6e_3e_1e_7e_8 + x_7e_3e_1e_8e_6 + x_8e_3e_1e_6e_7) + x_3(x_6e_1e_2e_7e_8 + x_7e_1e_2e_8e_6 + x_8e_1e_2e_6e_7) \\ &+ x_4(x_6e_5e_1e_7e_8 + x_7e_5e_1e_8e_6 + x_8e_5e_1e_6e_7) + x_5(x_6e_1e_4e_7e_8 + x_7e_1e_4e_8e_6 + x_8e_1e_4e_6e_7) = 0 \end{aligned}$$

Since the coefficient of $e_2e_3e_4e_5$ above is x_1^2, $x_1 = 0$ is necessary for $\mathrm{Pf}_2(\iota_x\omega_{8,1}) = 0$. Setting $x_1 = 0$ in the above equation, we get

$$\begin{aligned} \mathrm{Pf}_2(\iota_x\omega_{8,1})_{x_1=0} &= x_2(x_6e_3e_1e_7e_8 + x_7e_3e_1e_8e_6 + x_8e_3e_1e_6e_7) + x_3(x_6e_1e_2e_7e_8 \\ &+ x_7e_1e_2e_8e_6 + x_8e_1e_2e_6e_7) + x_4(x_6e_5e_1e_7e_8 + x_7e_5e_1e_8e_6 + x_8e_5e_1e_6e_7) \\ &+ x_5(x_6e_1e_4e_7e_8 + x_7e_1e_4e_8e_6 + x_8e_1e_4e_6e_7) = e_1 \wedge (x_3e_2 - x_2e_3 \\ &+ x_5e_4 - x_4e_5) \wedge (x_6e_7e_8 + x_7e_8e_6 + x_8e_6e_7). \end{aligned}$$

Therefore,

$$\begin{aligned} X_1(\omega_{8,1}) &= \{x_1 = 0\} \cap [\{x_2 = x_3 = x_4 = x_5 = 0\} \cup \{x_6 = x_7 = x_8 = 0\}] \\ &= \mathbb{P}\{e_6, e_7, e_8\} \cup \mathbb{P}\{e_2, e_3, e_4, e_5\} \simeq \mathbb{P}^2 \cup \mathbb{P}^3. \end{aligned}$$

Next, we consider $\mathrm{Pf}_2(\iota_x\omega_{8,2})$. The coefficients of $e_2e_3e_4e_5 + e_2e_3e_6e_7 + e_4e_5e_6e_7$, $e_1e_4e_6e_8$, $e_7e_1e_8e_5$ are x_1^2 and x_5^2 and x_6^2 respectively, $x_1 = x_5 = x_6 = 0$ is necessary for $\mathrm{Pf}_2(\iota_x\omega_{8,2}) = 0$. By Setting x_1, x_5 and x_6 to zero, in the equation, we get

$\mathrm{Pf}_2(\iota_x\omega_{8,2})_{x_1=x_5=x_6=0} = e_1 \wedge (x_3e_2 - x_2e_3) \wedge x_8e_5e_6.$

Therefore,

$$\begin{aligned} X_1(\omega_{8,2}) &= \{x_1 = x_5 = x_6 = 0\} \cap [\{x_2 = x_3 = 0\} \cup \{x_8 = 0\}] \\ &= \mathbb{P}\{e_4, e_7, e_8\} \cup_{\mathbb{P}\{e_4,e_7\}} \mathbb{P}\{e_2, e_3, e_4, e_7\} \simeq \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^3. \end{aligned}$$

In $\mathrm{Pf}_2(\iota_x\omega_{8,3})$, the coefficients of $e_3e_4e_5e_6$, $e_3e_5e_7e_8$, $e_4e_1e_5e_2$ and $e_6e_1e_2e_3$ are x_1^2, x_2^2, x_3^2 and x_5^2, respectively. By setting x_1, x_2, x_3 and x_5 to zero, $\mathrm{Pf}_2(\iota_x\omega_{8,3})$ is reduced to $e_1e_2 \wedge (x_4e_3 + x_6e_5) \wedge (x_8e_7 - x_7e_8)$.

Therefore, $X_1(\omega_{8,3}) = \{x_1 = x_2 = x_3 = x_5 = 0\} \cap [\{x_4 = x_6 = 0\} \cup \{x_7 = x_8 = 0\}] \simeq \mathbb{P}^1 \cup \mathbb{P}^1$. Similar arguments apply to the case for $X_1(\omega_{8,i})$ for $i = 4, \ldots, 6$. □

We now compute the varieties $X_2(\omega)$ and their cardinalities.

Proposition 3. *The varieties $X_2(\omega_{8,i})$ and their cardinalities for $1 \leq i \leq 6$ are*

$\omega_{8,i}$	$X_2(\omega_{8,i})$	$\lvert X_2(\omega_{8,i})\rvert$
$\omega_{8,1}$	$\mathbb{P}^6 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^6\rvert + \lvert\mathbb{P}^4\rvert - \lvert\mathbb{P}^3\rvert$
$\omega_{8,2}$	$\mathbb{P}^6$	$\lvert\mathbb{P}^6\rvert$
$\omega_{8,3}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^5\rvert + 2\lvert\mathbb{P}^4\rvert - 2\lvert\mathbb{P}^3\rvert$
$\omega_{8,4}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^5$	$2\lvert\mathbb{P}^5\rvert - \lvert\mathbb{P}^3\rvert$
$\omega_{8,4,d}$	$\mathbb{P}^3$	$\lvert\mathbb{P}^3\rvert$
$\omega_{8,5}$	$(\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4) \cup (\mathbb{F}_q)^2$	$\lvert\mathbb{P}^5\rvert + 2\lvert\mathbb{P}^4\rvert - 2\lvert\mathbb{P}^3\rvert + q^2$
$\omega_{8,5,d}$	$\mathbb{P}^5$	$\lvert\mathbb{P}^5\rvert$
$\omega_{8,6}$	$\mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4$	$\lvert\mathbb{P}^5\rvert + \lvert\mathbb{P}^4\rvert - \lvert\mathbb{P}^3\rvert$

Proof. Let $x = \sum_{j=1}^{8} x_je_j$. We have

$$\mathrm{Pf}_3(\iota_x\omega) = \sum_{j=1}^{8} x_j^3\mathrm{Pf}_3(\iota_{e_j}\omega) + \sum_{i<j}[x_i^2x_j\mathrm{Pf}_2(\iota_{e_i}\omega) \wedge (\iota_{e_j}\omega) + x_ix_j^2(\iota_{e_i}\omega) \wedge \mathrm{Pf}_2(\iota_{e_j}\omega)]. \tag{3}$$

We calculate $\mathrm{Pf}_3(\iota_x\omega_{8,i})$ using the above formula (3), and set it equal to zero to determine the varieties $X_2(\omega_{8,i})$. We begin with $\omega_{8,1}$.

For $j \geq 1$, $\mathrm{Pf}_3(\iota_{e_j}\omega_{8,1}) = 0$ and $\mathrm{Pf}_2(\iota_{e_1}\omega_{8,1}) = e_2e_3e_4e_5$, we get

$$\mathrm{Pf}_3(\iota_x\omega_{8,1}) = x_1^2x_6e_2e_3e_4e_5e_7e_8 + x_1^2x_7e_2e_3e_4e_5e_8e_6 + x_1^2x_8e_2e_3e_4e_5e_6e_7.$$

Since the coefficients of $e_2e_3e_4e_5e_7e_8$ and $e_2e_3e_4e_5e_8e_6$ and $e_2e_3e_4e_5e_6e_7$ above are $x_1^2x_6$ and $x_1^2x_7$ and $x_1^2x_8$, respectively, $x_1 = 0$ or $x_6 = x_7 = x_8 = 0$ is nececessary for $\mathrm{Pf}_3(\iota_x\omega_{8,1}) = 0$.

Therefore,

$$\begin{aligned} X_2(\omega_{8,1}) &= \{x_1 = 0\} \cup \{x_6 = x_7 = x_8 = 0\} \\ &= \mathbb{P}\{e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \cup_{\mathbb{P}\{e_2,e_3,e_4,e_5\}} \mathbb{P}\{e_1, e_2, e_3, e_4, e_5\} \simeq \mathbb{P}^6 \cup_{\mathbb{P}^3} \mathbb{P}^4. \end{aligned}$$

Next, we consider $\mathrm{Pf}_3(\iota_x\omega_{8,2})$. The coefficient of $e_2e_3e_4e_5e_6e_7$ is x_1^3; moreover, x_1 divides $\mathrm{Pf}_3(\iota_x\omega_{8,2})$. Therefore,

$$X_2(\omega_{8,2}) = \{x_1 = 0\} \simeq \mathbb{P}^6.$$

For $\mathrm{Pf}_3(\iota_x\omega_{8,3})$, the coefficients of $e_3e_4e_5e_6e_7e_8$, $e_3e_4e_5e_6e_8e_2$, $e_3e_4e_5e_6e_2e_7$, $e_3e_5e_7e_8e_4e_1$, $e_3e_5e_7e_8e_6e_1$, $e_7e_8e_4e_1e_5e_2$ and $e_7e_8e_6e_1e_2e_3$ are $x_1^2x_2$, $x_1^2x_7$, $x_1^2x_8$, $x_2^2x_3$, $x_2^2x_5$, $x_2x_3^2$ and $x_2x_5^2$ respectively. Reducing x_1x_2, x_1x_7, x_1x_8, x_2x_3 and x_2x_5 to zero is necessary for $\mathrm{Pf}_3(\iota_x\omega_{8,3}) = 0$.

Therefore,

$$\begin{aligned} X_2(\omega_{8,3}) &= \{x_1 = x_2 = 0\} \cup \{x_2 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_5 = 0\} \\ &= \mathbb{P}\{e_3, e_4, e_5, e_6, e_7, e_8\} \cup_{\mathbb{P}\{e_3,e_4,e_5,e_6\}} \mathbb{P}\{e_1, e_3, e_4, e_5, e_6\} \cup_{\mathbb{P}\{e_4,e_6,e_7,e_8\}} \mathbb{P}\{e_2, e_4, e_6, e_7, e_8\} \\ &\simeq \mathbb{P}^5 \cup_{\mathbb{P}^3} \mathbb{P}^4 \cup_{\mathbb{P}^3} \mathbb{P}^4. \end{aligned}$$

Similar arguments apply to the case for $X_2(\omega_{8,i})$ for $i = 4, \ldots, 6$. □

Theorem 3. *The weights of the nondegenerate forms* $\omega_{8,1}, \ldots, \omega_{8,6}$ *are*

$$\begin{aligned} \mathrm{wt}(\omega_{8,1}) &= q^{15} + q^{13} + q^{12} + q^{11} - q^8 \\ \mathrm{wt}(\omega_{8,2}) &= q^{15} + q^{13} + q^{12} + q^{11} \\ \mathrm{wt}(\omega_{8,3}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^8 \\ \mathrm{wt}(\omega_{8,4}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^9 \\ \mathrm{wt}(\omega_{8,4,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 \\ \mathrm{wt}(\omega_{8,5}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} - q^8 \\ \mathrm{wt}(\omega_{8,5,d}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^8 \\ \mathrm{wt}(\omega_{8,6}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10}. \end{aligned}$$

Proof. We use the formula (1) with $n_2(\omega) + n_1(\omega) = |X_2(\omega)|$, we get

$$\mathrm{wt}(\omega_{8,i}) = q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^6 - q^6 \left(\frac{|X_2(\omega_{8,i})| + q^2 |X_1(\omega_{8,i})|}{1 + q + q^2} \right),$$

the quantities $|X_1(\omega_{8,i})|$ and $|X_2(\omega_{8,i})|$ have been computed in Proposition 2 and 3.

For $\mathrm{wt}(\omega_{8,1})$,

we have $|X_1(\omega_{8,1})| = q^3 + 2q^2 + 2q + 2$ and $|X_2(\omega_{8,1})| = |\mathbb{P}^6| + |\mathbb{P}^4| - |\mathbb{P}^3| = q^6 + q^5 + 2q^4 + q^3 + q^2 + q + 1$, substituting these in the above equation we find

$$\begin{aligned} \mathrm{wt}(\omega_{8,1}) &= q^{15} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^6 \\ &- q^6 \left(\frac{|\mathbb{P}^6| + |\mathbb{P}^4| - |\mathbb{P}^3| + q^2(q^3 + 2q^2 + 2q + 2)}{1 + q + q^2} \right) = q^{15} + q^{13} + q^{12} + q^{11} - q^8. \end{aligned}$$

Similarly for the weights $\mathrm{wt}(\omega_{8,2}), \ldots, \mathrm{wt}(\omega_{8,6})$.

□

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Received 10.02.2019

Ракді М.А., Мідоун Н. *Ваги $\mathbb{F}_q$-форм 2-ступінчастих тривекторів розщеплення рангу 8 над скінченним полем* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 3–11.

Коди Грассмана - це лінійні коди, пов'язані з многовидом Грассмана $G(\ell, m)$ ℓ-вимірного підпростору у m-вимірному векторному просторі $\mathbb{F}_q^m$. Їх вивчав Ногін для загальних q. Ці коди зручно описати за допомогою відповідності між невиродженими $[n,k]_q$ лінійними кодами з одного боку, і невиродженими $[n,k]$ проективними системами з іншого боку. Невироджена $[n,k]$ проективна система - це просто набір n точок у проективному просторі $\mathbb{P}^{k-1}$, який задовольняє умови, що жодна гіперплощина $\mathbb{P}^{k-1}$ не містить n точок, що розглядаються. У цій роботі ми визначимо вагу лінійних кодів $C(3,8)$ асоційованих із многовидом Грассмана $G(3,8)$ над довільним скінченним полем $\mathbb{F}_q$. Ми використовуємо формулу для ваги кодового слова $C(3,8)$, у сенсі потужності певних многовидів, пов'язаних з чергуванням трилінійних форм на $\mathbb{F}_q^8$. Для $m=6$ і 7, звужений спектр $C(3,m)$ асоційований з $G(3,m)$, був повністю визначений в роботах Кайпа К.В., Пілаі Х.К. і Ногіна Й. Класифікація тривекторів істотно залежить від розмірності n базового простору. Для $n \leq 8$ існує тільки скінченна кількість класів тривекторів під дією загальної лінійної групи $GL(n)$. Методи когомології Галуа можуть використовуються для визначення класів невироджених тривекторів, які поділяються на кілька класів при переході від $\bar{\mathbb{F}}$ до $\mathbb{F}$. Ця програма частково визначена Ноуі Л. і Мідуне Н. та класифікація трилінійних змінних форм на векторному просторі розмірності 8 над скінченним полем $\mathbb{F}_q$ характеристик, відмінних від 2 і 3 була зроблена у роботах Ноуі Л. і Мідуне Н. Ми описали $\mathbb{F}_q$-форми 2-ступінчастих тривекторів розщеплення рангу 8, де $\operatorname{char} \mathbb{F}_q \neq 3$. Цей факт ми використовуємо для визначення ваги $\mathbb{F}_q$-форм.

Ключові слова і фрази: тривектор, грасманіан, вага.