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Fuzzy Lattices

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بسم الله الرحمن الرحيم

الشكر و التقدير:

الحمدلله نحمده و نستعينه و نثني عليه الخير كله ، ونصلي ونسلم على الرحمة المهداة حبيبنا ورسولنا محمد صلى الله عليه وسلم.

الحمدلله الذي وهبنا التوفيق و السداد ومنحنا الثبات و أعاننا على إتمام هذا العمل .

إن أصبنا فمن الله وإن أخطأنا فمن أنفسنا ومن الشيطان.

نتقدم بأرقى و أسمى عبارات الشكر و التقدير و التوقير ،

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إلى والداي رمز العطاء بلا حدود و التضحية دون انتظار والبذل من غير مقابل ،إلى الملاذ الآمن و الحصن المنيع ، إلى قدوتي في الحياة إلى بابي جنتي

(والدي مبروك و والدتي زنوبة).

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(سَنَشُدُّ عَضُدَكَ بِأَخِيكَ) القصص٥٣

إخوتي السند العظيم و الضلعَ القَويم الثابت الذي لا يميل :

(عبد اللطيف - محمد الامين - سارة - عماد) وفقهم الله ورعاهم.

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Contents

Introduction

Fuzzy Lattices are one of the abstract structures that have a prominent role and extensively studied and used in mathematics with wide ranging applications in many disciplines within which the information are incomplete or imprecise, like computer sciences, chemistry, economics, social sciences ... etc. This provides sufficient motivation to researchers to devote themselves to studying it by reviewing various concepts and results from the classical and fuzzy logic.

The fuzzy mathematics is the branch of mathematics including fuzzy set theory and fuzzy logic which refer to a logical system that generalizes the classical two-value logic for reasoning under uncertainty.

In 1965, Lotfi Zadeh [16] presented the theory of fuzzy sets as an extension of the classical notion of a set (within classical mathematics). Such that in fuzzy set theory, the set is characterized by an application associates with each element a real number in unite interval $[0, 1]$ is interpreted as the degree of membership of t he element in a fuzzy set. While in classical set theory, the indicator functions of classical sets are special cases of the membership functions of fuzzy sets if the latter only take the values (0 or 1) i,e: it is evaluated in binary terms according to a bivalent condition: an element either belongs or does not belong to the set.

In 1971, the Iranian Zadeh [17]generalized the notion of reflexivity, antisymmetry and transitivity and defined fuzzy orderings, which are transitive fuzzy relations. In particular, a fuzzy partial ordering is a fuzzy ordering which is also reflexive and antisymmetric.

Chon Inheung in [5] , proposed a new notion for fuzzy lattices considering Zadeh's fuzzy orders as and studied the level sets of such structures, he also provided some results for distributive and modular fuzzy lattices.

Amroune.A, Oumhani and Davvaz [2] characterized fuzzy filters and fuzzy t-filters using their α-cuts .

Mezzomo [8] characterized the fuzzy lattice throughout a fuzzy partial order relation, he defined ideals, filters of fuzzy lattices.

In 1983, Atanassov [3] introduced a new notion called an intuitionistic fuzzy set as an extension of concept of fuzzy sets .The Bulgarian Krassimir added the non-membership of an element to the fuzzy set, such that the intuitionistic fuzzy set are sets whose elements have degrees of membership and non-membership. In the intuitionistic fuzzy environment, the degree of non-membership is a more or less independent degree: the only condition is that $\nu(x) \leq 1 - \mu(x)$.Certainly, fuzzy sets are intuitionistic fuzzy sets where $\nu(x) = 1 - \mu(x)$.

The main objective of this memory is to improve our knowledge about the notions of lattices in classical and fuzzy logic and in the concept of intuitionistic.

This memory is organized as follows:
In first chapter, we recall some concepts and well-known results on classical ordered relation and classical lattices, then cite their types.
In second chapter, we introduce concepts of fuzzy sets and fuzzy relations then, we study and characterize diverse types of fuzzy lattices and we define filters and ideals on fuzzy lattice.
In third chapter, we present concept of intuitionistic sets and study intuitionistic lattices and intuitionistic fuzzy ideals (filters).
In last chapter, we offer a comparaison between fuzzy structures and intuitionistic structures.

Chapter 1

Generalities on Classical Orders and Lattices.

The main of this chapter is to recall some definitions and properties of lattices in classical mathematics, and to mention their types, then to give definitions of ideals and filters in lattice.
Basic reference for this chapter is :[[1][6],[4],[7]].

1.1 Partial Order

1.1.1 Definitions and Properties

Definition 1.1.1.1 *(Binary relation)*
Let E *be a non-empty set. A binary relation* $\mathcal{R}$ *from a set* E *is a subset of the cartesian product* $\mathrm{E} \times \mathrm{E}$*. The relation between* x *and* y *written as :* $x\mathcal{R}y$ *or* $\mathcal{R}(x, y)$*.*

Example 1.1.1.2 *- The relation* $\mathcal{R}$ *is "greater than :* $>$*" , on the set* $\mathrm{E} = \{2, 3, 4\}$ *, we have :*

$$\mathrm{E}^2 = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}.$$

Then: all pairs $(x, y) \in \mathrm{E}$ *where :* $x > y$ *,are:*

$$\mathcal{R} = \{(3,2), (4,2), (4,3)\}.$$

Propertie 1.1.1.3
There are many properties that binary relations may satisfy, such as:

1) $\mathcal{R}$ *is reflexive if:*

$$\forall x \in \mathrm{E} : x\mathcal{R}xi, e : ((x, x) \in \mathcal{R}).$$

2) $\mathcal{R}$ *is irreflexive if:*

$$\forall x \in \mathrm{E} : (x, x) \notin \mathcal{R}.$$

3) $\mathcal{R}$ *is symmetric if:*

$$\forall x, y \in \mathrm{E} : x\mathcal{R}y \Rightarrow y\mathcal{R}x.$$

4) $\mathcal{R}$ *is anti-symmetric if:*

$$\forall x, y \in \mathrm{E} : \begin{cases} x\mathcal{R}y \\ \textit{and} \quad \Rightarrow x = y. \\ y\mathcal{R}x. \end{cases}$$

5) $\mathcal{R}$ *is transitive if:*

$$\forall x, y, z \in \mathrm{E} : \begin{cases} x\mathcal{R}y \\ \textit{and} \quad \Rightarrow x\mathcal{R}z. \\ y\mathcal{R}z. \end{cases}$$

Definition 1.1.1.4 *(Partial order)*
Let E *be a non-empty set. A binary relation* $\mathcal{R}$ *on* E *is called an* ***order relation*** *(or partial order, partial ordering) if and only if satisfies the following three properties : reflexivity, anti-symmetry and transitivity.*

Notation 1 - *Most order relations are denoted* $\leq$ *or* $\preceq$.

Example 1.1.1.5 *The relation* $\mathcal{R}$ *is "greater than or equal :* $\geq$*" is a partial order on the set of integers. For the reason that :*

- ➢ *Reflexivity :* $\forall x \in \mathrm{E} : x \geq x$.
- ➢ *Anti-symmetry:* $\forall x, y \in \mathrm{E}$ *: if* $x \geq y$ *and* $y \geq x$*, then:* $x = y$.
- ➢ *Transitivity :* $\forall x, y, z \in \mathrm{E}$ *: if* $x \geq y$ *and* $y \geq z$*, then:* $x \geq z$.

Definition 1.1.1.6 *(Partially ordered set)*
A set E *equipped with an order relation* $\mathcal{R}$ *is called a partially ordered set (or Poset) and denoted by the pair* $(\mathrm{E}, \mathcal{R})$.

Remark 1.1.1.7
1. *We say that elements* x, y *of an ordered set* $(\mathrm{E}, \leq)$ *are comparable if: either* $x \leq y$ *or* $y \leq x$.
2. *A poset* $(\mathrm{E}, \leq)$ *is totally ordered if every* $x, y \in \mathrm{E}$ *: are comparable.*

   In this case, the order $\leq$ *is said to be total or linear.*

Definition 1.1.1.8 *(Chains)*
Let $(\mathrm{E}, \leq)$ *be a partially ordered set.* E *forms a chain if all pairs of elements of* E *are comparable. The chain is also called : totally ordered set and linearly ordered set.*

Example 1.1.1.9 - *The set of natural numbers under usual order ,* $(\mathbb{N}, \leq)$ *forms a chain.*

Definition 1.1.1.10 *(Upper and Lower Bound)*
Let E *be an ordered set and* $\mathrm{X} \subseteq \mathrm{E}$.

1) *An element* $u \in \mathrm{E}$ *is an **upper bound** of* X *if* $\forall x \in \mathrm{X} : x \leq u$.

   - *The set of all upper bounds of* X *denoted by:* X^u *and defined by:*

$$\mathrm{X}^u = \{x \in \mathrm{E} : y \leq x, \forall y \in \mathrm{X}\}$$

2) *Dually, An element* $l \in \mathrm{E}$ *is an **lower bound** of* X *if* $\forall x \in \mathrm{X} : l \leq x$.

   - *The set of all lower bounds of* X *denoted by:*

$$\mathrm{X}^l = \{x \in \mathrm{E} : x \leq y, \forall y \in \mathrm{X}\}$$

☆ *The **greatest** lower bound of* X *whenever it exists, is called the **infimum** of* X *and is denoted by* $\inf \mathrm{X}$.

☆ *The **least** upper bound of* X *, whenever it exists, is called the **supremum** of* X *and is denoted by* $\sup X$.

Definition 1.1.1.11 *(Greatest and Least element)*
Let $(E, \leq)$ *be an ordered set.*
E *has a greatest element if there exists* $t \in E$ *with* $\forall x \in E : x \leq t$*, and it has a least element if there exists* $b \in E$ *with* $\forall x \in E : b \leq x$.

Notation 1 *1. The greatest element (Top)* t *noted by :* $\mathbf{1}$ *and least element (Bottom)* b *noted by :* $\mathbf{0}$.

Remark 1.1.1.12 *The top and bottom are unique when they exist.*

Proof 1.1.1.13 *We show the uniqueness of top element* $\mathbf{1}$ *by the absurd:*
Suppose that exists another top element $\mathbf{1}^*$*, i,e:* $\forall x \in E : \begin{cases} x \leq \mathbf{1} \\ \\ x \leq \mathbf{1}^* \end{cases}$
and we have:

$$\begin{cases} \mathbf{1}^* \in E \\ \\ \mathbf{1} \in E \end{cases} \Rightarrow \begin{cases} \mathbf{1}^* \leq \mathbf{1} \\ \\ \mathbf{1} \leq \mathbf{1}^* \end{cases} \Rightarrow \mathbf{1} = \mathbf{1}^*$$

(The antisymmetry of $\leq$ *). Then:* $\mathbf{1}$ *is unique.*

✓ *The same method with* $\mathbf{0}$.

1.2 Lattices

1.2.1 Definitions and properties

Definition 1.2.1.1 *(Order Lattice)*
A lattice consists of a partially ordered set $(E, \mathcal{R})$ *in which every two elements* (x, y) *have a unique least upper bound (supremum), denoted by:* $x \vee y$ *and a unique greatest lower bound(infimum), denoted by:* $x \wedge y$*. i,e:*

$$\forall x, y \in \mathbf{E} : \exists \begin{cases} \sup\{x, y\} \\ \text{and} \\ \inf\{x, y\} \end{cases}$$

Examples 1.2.1.2 ❑ *The non-zero natural numbers equipped with divisibility :* $(\mathbb{N}^*, |)$ *is a lattice, when:*

$$\forall a, b \in \mathbb{N}^* : \begin{cases} a \wedge b = gcd(a, b) \\ a \vee b = lcm(a, b) \end{cases}$$

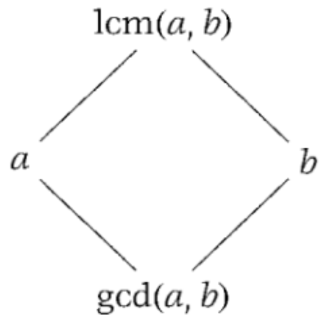

Figure 1.1: The Hasse diagram of $(\mathbb{N}^*, |)$

❑ *For all set* X $,(\mathcal{P}(\mathrm{X}), \subseteq)$ *is a lattice , when:*

$$\forall A, B \in (\mathcal{P}(\mathrm{X})) : \begin{cases} A \wedge B = A \cap B \\ A \vee B = A \cup B \end{cases}$$

1. *For example: The poset consisting of all the subsets of* $\{1, 2, 3\}$ *under the relation* $\subseteq$ *forms* $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$ *a lattice.*

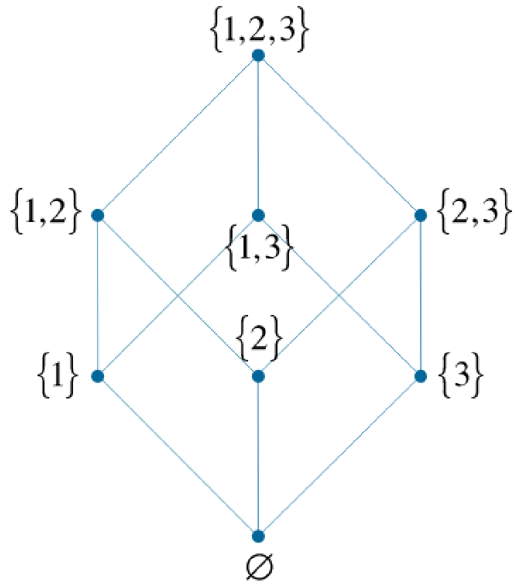

Figure 1.2: This is an example for a lattice $((\mathcal{P}(\{1, 2, 3\}), \subseteq))$

Definition 1.2.1.3 *(Algebraic Lattice)*
An algebraic structure $(\mathrm{E}, \vee, \wedge)$*, consisting of a set* E *and two binary operations* $\wedge$ *(**meet**) and* $\vee$ *(**join**) on* E *is a lattice if the following axiomatic identities hold for all elements* a, b, c *of* E*:*

1) *Commutative laws,i.e:*

$$\begin{aligned} a \vee b &= b \vee a \\ a \wedge b &= b \wedge a. \end{aligned}$$

2) Associative laws,i.e:

$$\begin{aligned} a \vee (b \vee c) &= (a \vee b) \vee c \\ a \wedge (b \wedge c) &= (a \wedge b) \wedge c. \end{aligned}$$

3) Absorption laws,i.e:

$$\begin{aligned} a \vee (a \wedge b) &= a \\ a \wedge (a \vee b) &= a. \end{aligned}$$

4) Idempotent laws,i.e:

$$\begin{aligned} a \vee a &= a \\ a \wedge a &= a. \end{aligned}$$

Examples 1.2.1.4 *As in the preceding example 1.2.1.2 : For all set* X *,*$(\mathcal{P}(\mathrm{X}), \subseteq)$*, we have:*

$$\forall A, B \in (\mathcal{P}(\mathrm{X})) : \begin{cases} A \wedge B = A \cap B \\ A \vee B = A \cup B \end{cases}$$

Thus: $\forall A, B, C \in (\mathcal{P}(\mathrm{X}))$

i) Commutative laws,i.e:

$$\begin{aligned} A \vee B &= B \vee A \\ A \wedge B &= B \wedge A. \end{aligned}$$

ii) Associative laws,i.e:

$$\begin{aligned} A \vee (B \vee C) &= (A \vee B) \vee C \\ A \wedge (B \wedge C) &= (A \wedge B) \wedge C. \end{aligned}$$

iii) Absorption laws,i.e:

$$\begin{aligned} A \vee (A \wedge B) &= (A \vee A) \wedge (A \vee B) \\ &= A \wedge (A \vee B) \\ &= A. \\ A \wedge (A \vee B) &= (A \wedge A) \vee (A \wedge B) \\ &= A \vee (A \wedge B) \\ &= A. \end{aligned}$$

iv) Idempotent laws, i.e:

$$\begin{aligned} A \vee A &= A \\ A \wedge A &= A. \end{aligned}$$

Then: For all set X *,* $(\mathcal{P}(\mathrm{X}), \subseteq)$ *is a lattice.*

1.2.2 Types of Lattices

Definition 1.2.2.1 *(Bounded Lattice)*
Let E *be a lattice. If it contains a greatest element* **1** *and a least element* **0** *(with respect of order or binary operations) is called bounded lattice. i,e:*
$(\forall x \in \mathrm{E} : \mathbf{0} \leq x \leq \mathbf{1})$.

- $(\mathrm{E}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *is bounded lattice ,* **0** *and* **1** *are called universal bounds.*

Example 1.2.2.2 *1. The set of divisors of the integer 6,* $\mathrm{D}(6) = \{1, 2, 3, 6\}$ *equipped with the relation divide* $(\mathrm{D}(6), |)$ *is a bounded lattice.*
because: the least element is $(1 \in \mathrm{D}(6))$ *and the greatest element is* $(6 \in \mathrm{D}(6))$.

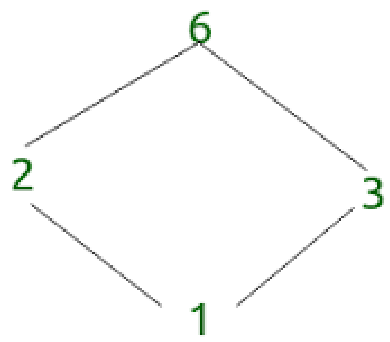

Figure 1.3: Hass diagram of D(6)

Definition 1.2.2.3 *(Complete Lattice)*
Let E *be a lattice. If all its subsets have both a supremum and infimum becomes a complete lattice.*

Remark 1.2.2.4 *Every complete lattice is a bounded lattice.*

Definition 1.2.2.5 *(Complemented Lattice)*
Let $(\mathrm{E}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ *be a lattice.*
We say b is a complement of a if : $a \wedge b = \mathbf{0}$ *and* $a \vee b = \mathbf{1}$*, if a has a unique complement it denotes by:* $a^{'}$.

- E *is called a complemented lattice if it is a bounded lattice and every element* $x \in \mathrm{E}$ *has a complement* $x^{'}$.

Proposition 1.2.2.6 *1. Let* $(\mathrm{E}, \leq)$ *the following statements are equivalent for all x and y in* E *:*

i) $x \leq y$.

ii) $\sup\{x, y\} = x \vee y = y$.

iii) $\inf\{x, y\} = x \wedge y = x$.

Lemma 1.2.2.7 *Let* $(\mathrm{E}, \leq)$ *be a lattice, then:* $\forall x, y, z, t \in \mathrm{E}$ *:*

✸ $x \leq y$ *implies* $\begin{cases} x \wedge y \leq y \wedge z \\ \text{and} \\ x \vee z \leq y \vee z \end{cases}$

✸ $x \leq y$ *and* $z \leq t$ *imply* $\begin{cases} x \wedge z \leq y \wedge t \\ \text{and} \\ x \vee z \leq y \vee t \end{cases}$

Proof 1.2.2.8 *Demonstrate that :*

1)- *For all* x, y, z *in* E : $x \leq y \Rightarrow x \wedge y \leq y \wedge z$ *: we have:* $y \leq w \Rightarrow y \wedge z = y$ *according to 1.2.2.7,*

$$\begin{array}{rcl} x \wedge y & = & (x \wedge x) \wedge (y \wedge z) \\ & = & (x \wedge y) \wedge (x \wedge z) \end{array} \Rightarrow x \wedge y \leq x \wedge z.$$

✓ *The same method for :* $x \vee z \leq y \vee z$.

2)- *For all* $x, y, z, t \in \mathrm{E}$: $x \leq y$ *and* $z \leq t \quad \Rightarrow x \wedge z \leq y \wedge t$ *:*

$$\begin{array}{rcl} (x \wedge z) \wedge (y \wedge t) & = & ((x \wedge z) \wedge y) \wedge ((x \wedge z) \wedge t) \\ & = & (x \wedge z \wedge y) \wedge (x \wedge z \wedge t) \\ & = & (x \wedge z) \wedge (y \wedge t) \wedge (x \wedge z) \\ & = & (x \wedge z) \wedge (y \wedge t). \end{array} \Rightarrow x \wedge z \leq y \wedge t.$$

✓ *The same method for :* $x \vee z \leq y \vee t$.

Definition 1.2.2.9 *(Distributive Lattice)*
Let $(\mathrm{E}, \wedge, \vee)$ *be a lattice. If it satisfies the following two distribute properties:*

$$\begin{array}{rcl} x \vee (y \wedge z) & = & (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) & = & (x \wedge y) \vee (x \wedge z). \end{array}$$

It is called distributive lattice.

Theorem 1.2.2.10 *A lattice* E *be distributive, it is necessary and sufficient that it satisfies the condition :* $\forall x, y, z \in \mathrm{E}$:

$$\begin{cases} x \wedge z = y \wedge z \\ \\ x \vee z = y \vee z \end{cases} \Rightarrow x = y$$

Proof 1.2.2.11 *We show that : for all* x, y, z *in* E , $\begin{cases} x \wedge z = y \wedge z \\ \\ x \vee z = y \vee z \end{cases} \Rightarrow x = y.$

we have :

$$\begin{aligned} x &= x \wedge (x \vee z) \text{ "from absorption"} \\ &= x \wedge (y \vee z) \\ &= (x \wedge y) \vee (x \wedge z) \\ &= (x \wedge y) \vee (y \wedge z) \\ &= y \wedge (x \vee z) \\ &= y \wedge (y \vee z) \text{ "from absorption"} \\ &= y. \end{aligned}$$

Example 1.2.2.12 ◆ $(\mathcal{P}(\mathrm{X}))$ *is a distributive lattice.*

◆ *The* ***"pentagon lattice"*** *and* ***"diamond lattice"*** *are not divisibulite lattice, because for pentagon lattice in the Hasse diagram we have :*
$a \vee (b \wedge c) = (a \wedge 1) = a \neq 1 = (1 \wedge 1) = (a \vee b) \wedge (a \vee c).$

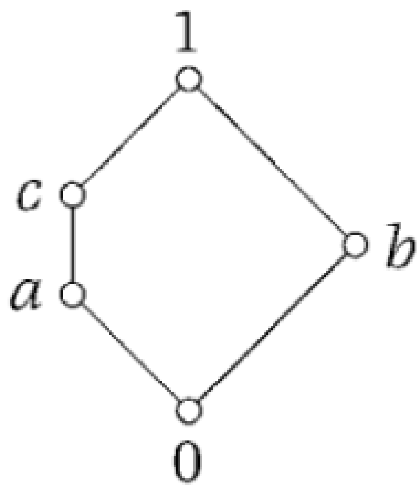

Figure 1.4: Pentagon Lattice

and for diamond lattice :
$a \wedge (b \vee c) = (a \wedge 1) = a \neq 0 = (0 \vee 0) = (a \wedge b) \vee (a \wedge c).$

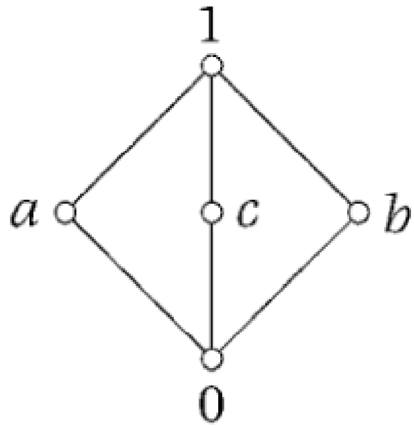

Figure 1.5: Diamond Lattice

Definition 1.2.2.13 *(Sublattice)*
Let $(\mathrm{E}, \wedge, \vee)$ *be a lattice. A non-empty subset* S *of* E *is a sublattice if* $\forall x, y \in \mathrm{S}$*:*

$$\begin{cases} x \wedge y \in \mathrm{S} \\ \quad and \\ x \vee y \in \mathrm{S} \end{cases}$$

Theorem 1.2.2.14 *A lattice is distributive if and only if it does not contain a sublattice isomorphic to the diamond or the pentagon.*

Example 1.2.2.15 *The lattice of the following Hasse diagram*

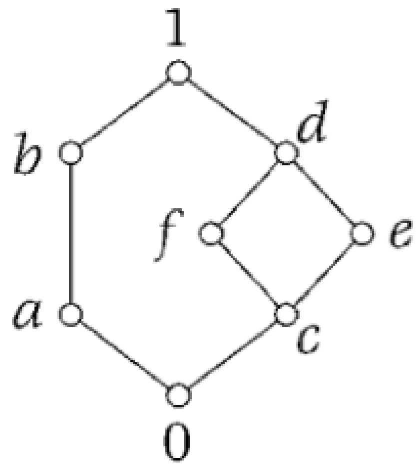

Figure 1.6:

contains the pentagon lattice,therefore is not distributive.

Definition 1.2.2.16 *(Modular Lattice)*
Let $(\mathrm{E}, \wedge, \vee)$ *be a lattice , if it satisfies the modular laws:*

$$\forall x, y, z \in \mathrm{E} : z \leq x \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z.$$

It is a modular lattice.

Remark 1.2.2.17 *Every distributive lattice is modular.*

1.2.3 Filters and Ideals in a Lattice

Ideals are of fundamental importance in algebra. Filters, the order duals of lattice ideals, have a variety of applications in logic,we recall some dfinitions and notions of filters and ideals in a lattice.

Definition 1.2.3.1 *(Filter)*
Let $(\mathrm{E}, \leq)$ *be a lattice. A non-empty subset* F *of* E *is called a filter, if it satisfies the following conditions:*

i) If $x \in \mathrm{F}$ *and* $x \leq y$*, then* $y \in \mathrm{F}$*.*

ii) If $x \in \mathrm{F}$ *and* $y \in \mathrm{F}$*, then* $(x \wedge y) \in \mathrm{F}$*.*

Definition 1.2.3.2 *(Ideal)*
Let $(\mathrm{E}, \leq)$ *be a lattice. A non-empty subset* I *of* E *is called a ideal, if it satisfies the following conditions:*

i) If $y \in \mathrm{I}$ *and* $x \leq y$*, then* $x \in \mathrm{I}$*.*

ii) If $x, y \in \mathrm{I}$*, then* $(x \vee y) \in \mathrm{I}$*.*

Remark 1.2.3.3

- ❖ E *is a filter of* E*, it will be called the improper filter. In addition, it is an ideal of* E *will be called the improper ideal.*
- ❖ *The greatest element belongs to all filters. Dually, the least element belongs to all ideals (from condition 1.2.3.1).*
- ❖ *The intersection of any family of filters is also a filter.*

Definition 1.2.3.4 *(Prime ideal)*
Let $\mathbf{E}$ *be a lattice. A proper ideal* I *of* E *is prime if and only if:*

$$\forall x, y \in \mathrm{E} : x \wedge y \in \mathrm{F} \Rightarrow \begin{cases} x \in \mathrm{I} \\ or \\ y \in \mathrm{I} \end{cases}.$$

Definition 1.2.3.5 *(Proper filter and ideal)*
A filter (or an ideal) is called ***proper*** *if it does not* ***coincide*** *with* E*.*
In bounded lattice $\mathbf{E}$*, a filter* F *of* E *is proper if and only if:*
$0 \notin \mathrm{F}$ *,and an ideal* I *of* E *is* ***proper*** *if and only if* $\mathbf{1} \notin \mathrm{I}$*.*

Definition 1.2.3.6 *(Prime Filter)*
Let **E** *be a lattice. A proper filter* F *of* E *is prime if and only if* $\forall x, y \in \mathrm{E} : x \vee y \in \mathrm{F} \Rightarrow \begin{cases} x \in \mathrm{F} \\ or \\ y \in \mathrm{F} \end{cases}$.

Definition 1.2.3.7 *(Generated Filter)*

- ➢ ***"By an element"***:
  Let α *be a fixed element of* E*, we define the set* $\mathrm{F}_\alpha = \{x \in \mathrm{E} : \alpha \leq x\}$*, if* F_α *is a filter ,says a filter* ***generated*** *by* α*.*

- ➢ ***"By a subset"***:
  Let G *be an arbitrary subset* $\mathrm{G} \subseteq \mathrm{E}$*. The filter* ***generated*** *by subset* G *is the smallest filter that contains* G *denoted* F_G*, such as:*
  $\mathrm{F}_\mathrm{G} = \{x \in \mathrm{E} : a_1, a_2, \dots, a_n \in \mathrm{G} | (a_1 \wedge a_2 \wedge \cdots \wedge a_n) \leq x\}$.

Remark 1.2.3.8 ➪ *A filter* F *of* E *be a* ***principal*** *filter if:* $\exists \alpha \in \mathrm{E}$ *such as* F *generated by* α*.*
i,e: the filter $\uparrow \alpha = \{y \in \mathrm{E} : \alpha \leq y\}$ *is principal .*
Dually,the ideal $\downarrow \alpha = \{y \in \mathrm{E} : y \leq \alpha\}$ *is principal.*

Definition 1.2.3.9 *(Maximal Filter)*
A proper filter F *of* E *be an* ***ultra-filter*** *(or maximal filter) if for every filter* X *of* E *:*

$$\mathrm{F} \subseteq \mathrm{X} \subseteq \mathrm{E} \Rightarrow \begin{cases} \mathrm{X} = \mathrm{E} \\ or \\ \mathrm{X} = \mathrm{F} \end{cases}$$

Example 1.2.3.1 1. *The set of divisors of the integer 30* $\mathrm{D}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ *equipped with the relation divide* $(\mathrm{D}(30), |)$*:*

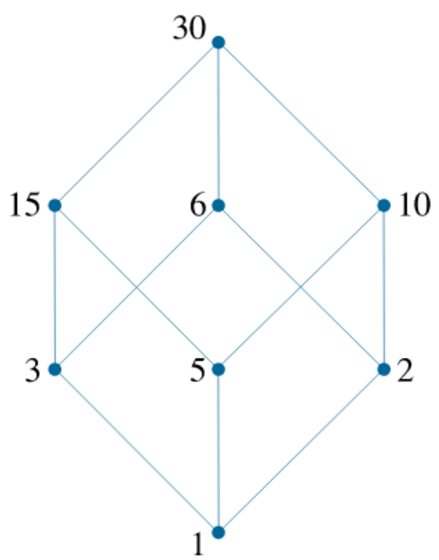

Figure 1.7: Hasse diagram of $(\mathrm{D}(30), |)$

1) *The proper filters are :* $\{2, 6, 10, 30\}$, $\{5, 10, 15, 30\}$, $\{3, 6, 15, 30\}$, $\{6, 30\}$, $\{10, 30\}$, $\{15, 30\}$, $\{30\}$.

2) *The maximal filters are :* $\{2, 6, 10, 30\}$, $\{3, 6, 15, 30\}$, $\{5, 10, 15, 30\}$.
   We can observe that all the proper filters are included in maximal filters.

Chapter 2

Fuzzy Ordered Relations and Lattices

In this chapter, we study and characterize diverse types of lattices. Also, we give definitions of ideals and filters in fuzzy lattice , fuzzy ideals and fuzzy filters in fuzzy lattice.
Basic references for this chapter are :[[16],[17],[2],[5],[8],[9]].

2.1 Fuzzy sets

2.1.1 Basic definition

Definition 2.1.1.1 *(Fuzzy Set) [16]*

➢ *The membership is a real number with a range* $[0,1]$ *. An application associates with each point* x *in* X *a real number in unite interval is called membership function (degree of membership or degree of truth). i,e:*

$$\mu : \ \mathbf{X} \ \longrightarrow \ [0,1] \ .$$

A fuzzy set $A \subseteq \mathbf{X}$ *be represented as a set of ordered pairs :*$A = \{\langle x, \mu_A(x)\rangle \, | x \in \mathrm{X}\}$.

Notation 1 *Let* X *be a non-empty set. The set of all fuzzy subset of* X *denoted by* $\mathcal{F}(\mathrm{X})$.

2.1.2 Operations on Fuzzy Sets

Zadeh in first paper [16] defined the following operations for fuzzy sets:

Definition 2.1.2.1 *(Equality and inclusion of a fuzzy set)*
Let X *be a non-empty set.* A *and* B *in* $\mathcal{F}(\mathrm{X})$

- *We say that* A *equal to* B*. i,e:* $A = B \Leftrightarrow (\forall x \in \mathrm{X} : \mu_A(x) = \mu_B(x))$.

- *We say that* A *is included in* B*. i,e:* $A \subseteq B \Leftrightarrow (\forall x \in \mathrm{X} : \mu_A(x) \leq \mu_B(x))$.
  Thus :A *be a fuzzy subset of* B.

Definition 2.1.2.2 *(Complement and difference of a fuzzy set)*

- *The complement of a fuzzy set* A *is defined as:*
  $A^c = \{\langle x, \mu_{A^c}\rangle \, | x \in \mathrm{X}\} = \{\langle x, 1 - \mu_A(x)\rangle \, | x \in \mathrm{X}\}$.
- *The difference of two fuzzy sets* A *and* B *(where* $A - B = A \cap B^c$*) is defined as:*
  $A - B = A \cap B^c = \{\langle x, \mu_{A\cap B^c}(x)\rangle \, | x \in \mathrm{X}\} = \{\langle x, \min(\mu_A, 1 - \mu_B(x))\rangle \, | x \in \mathrm{X}\}$.

Proposition 2.1.2.3 *Let* X *be a non-empty set.* A, B*in* $\mathcal{F}(\mathrm{X})$*, the following properties is satisfied by the complement of fuzzy sets:*

❑ $(A^c)^c = A$.

❑ $(A \cup B)^c = A^c \cap B^c$.

❑ $(A \cap B)^c = A^c \cup B^c$.

Definition 2.1.2.4 *(Union and intersection of a fuzzy set)*

- *The union of two fuzzy sets A and B in $\mathcal{F}(\mathrm{X})$ is defined as:*
  $A \cup B = \{\langle x, \mu_{A\cup B}(x)\rangle \,|x \in \mathrm{X}\} = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}\rangle \,|x \in \mathrm{X}\}$.
  Thus: $\mu_{A\cup B} = \mu_A \vee \mu_B$.

- *The intersection of two fuzzy sets A and B in $\mathcal{F}(\mathrm{X})$ is defined as:*
  $A \cap B = \{\langle x, \mu_{A\cap B}(x)\rangle \,|x \in \mathrm{X}\} = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}\rangle \,|x \in \mathrm{X}\}$.
  Thus: $\mu_{A\cap B} = \mu_A \wedge \mu_B$.

Properties 2.1.2.5 *Let X be a non-empty set. A, B in $\mathcal{F}(\mathrm{X})$, the following properties satisfie :*

- *Identity:*

$$\begin{cases} A \cap \mathrm{X} = A & A \cup \mathrm{X} = \mathrm{X} \\ A \cap \emptyset = \emptyset & A \cup \emptyset = A. \end{cases}$$

- *Commutativity:*

$$\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$$

- *Associativity:*

$$\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{cases}$$

- *Distributivity:*

$$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$$

- *Involution:*

$$(A^c)^c = A.$$

- *Idempotency:*

$$\begin{cases} A \cup A = A \\ A \cap A = A \end{cases}$$

- *Absorption:*

$$\begin{cases} A \cup (A \cap B) = A \\ A \cap (A \cup B) = A \end{cases}$$

2.1.3 Characteristics of fuzzy sets

Definition 2.1.3.1 *(Support)*
Let A be a fuzzy set on a non-empty X*. The support of A is the crisp (classic) set of all $x \in \mathrm{X}$ such as : $S(A) = \{x|\mu_A(x) > 0\}$.*

Definition 2.1.3.2 *(α-cuts)*
Let A be a fuzzy set on X*. The α-cuts of a fuzzy set A is the crisp set of all $x \in \mathrm{X}$ and for every $\alpha \in [0, 1]$, denoted by A_α (All elements of* X *that have a membership grades in A of least equal to α) such as: $A_\alpha = \{x|\mu_A(x) \geq \alpha\}$.*

- *The strong α-cut is defined as:*
  $A'_\alpha = \{x|\mu_A(x) > \alpha\}$.

Example 2.1.3.3 *Let $\mathrm{X} = \{a, b, c, d\}$ and $A = \{(a, 0.2), (b, 0.5), (c, 1), (d, 0.9)\}$.*
$Supp(A) = \{x|\mu_A(x) > 0\} = \{a, b, c, d\}$.
$A_{0.5} = \{x|\mu_A(x) \geq 0.5\} = \{b, c, d\}$.
$A_1 = \{x|\mu_A(x) \geq 1\} = \{c\}$.

Proposition 2.1.3.4 *Let A, B be two fuzzy sets. For every $\alpha, \beta \in [0, 1]$:*

(i) *If $\alpha \leq \beta$, then: $A_\beta \subset A_\alpha$.*

(ii) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.

(iii) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.

2.2 Fuzzy ordered relation

Definition 2.2.1 *(Fuzzy binary relation)[17]*
Let X *and* Y *be a non-empty sets. A fuzzy binary relation $\mathcal{R}$ from* X *to* Y *is a fuzzy subset of the cartesian space $\mathrm{X} \times \mathrm{Y}$, defined as:*

$$\begin{aligned} \mathcal{R} &= \{\langle (x, y), \mu_{\mathcal{R}}(x, y)\rangle \,|(x, y) \in \mathrm{X} \times \mathbf{Y}\} \\ & \quad or \\ &= \{\langle (x, y), \mathcal{R}(x, y)\rangle \,|(x, y) \in \mathrm{X} \times \mathrm{Y}\} \end{aligned}$$

Where: $\mu_{\mathcal{R}} : \mathrm{X} \times \mathrm{Y} \longrightarrow [0, 1]$. the degree of relation between x and y under $\mathcal{R}$.

Properties 2.2.2 *Let $\mathcal{R}$ be a fuzzy relation (fuzzy relation on* X *) which may satisfies the following properties :*

1) *$\mathcal{R}$ is reflexive if:*

$$\forall x \in \mathrm{X} : \mathcal{R}(x, x) = 1.$$

2) *$\mathcal{R}$ is symmetric if:*

$$\forall x, y \in \mathrm{X} : \mathcal{R}(x, y) = \mathcal{R}(y, x).$$

3) $\mathcal{R}$ is anti-symmetric if:

$$\forall x, y \in \mathrm{X} \begin{cases} (\mathcal{R}(x,y) > 0 \quad \textit{and} \quad \mathcal{R}(y,x) > 0) \Rightarrow x = y \\ \\ \textit{or} \\ \\ x \neq y : \min\{\mathcal{R}(x,y), \mathcal{R}(y,x)\} = 0. \end{cases}$$

4) $\mathcal{R}$ is transitive if:

$$\forall x, y, z \in \mathrm{X} : \sup_{y \in \mathrm{X}} \quad (\min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\}) \leq \mathcal{R}(x,z).$$

Definition 2.2.3 *(Fuzzy partial ordering relation)[17]*
A relation $\mathcal{R}$ on X be a partial fuzzy order (fuzzy ordered relation) if and only if: it is reflexive, anti-symmetric and transitive.

Remark 2.2.4 ➢ *In this case: the pair $(\mathrm{X}, \mathcal{R})$ is called a partially ordered fuzzy set (Fuzzy Poset).*

Definition 2.2.5 *(Fuzzy chain)[5]*
*The pair $(\mathrm{X}, \mathcal{R})$ is called a fuzzy totally ordered set (or a fuzzy **chain**) if:*
$\mathcal{R}$ is a fuzzy total order relation in a set X.(i,e: $\forall x, y \in \mathrm{X} : \mathcal{R}(x,y) > 0$ or $\mathcal{R}(y,x)0$).

Proposition 2.2.6 *[5]*
Let $(\mathrm{X}, \mathcal{R})$ be a fuzzy poset (or fuzzy chain) and $\mathrm{Y} \subseteq \mathrm{X}$.
If $\mathcal{R}'$ is a fuzzy relation on Y such that :

$$\forall x, y \in \mathrm{Y} : \mathcal{R}'(x,y) = \mathcal{R}(x,y)$$

Then $(\mathrm{Y}, \mathcal{R}')$ is a fuzzy poset (or fuzzy chain).

Definition 2.2.7 *(α-cut of fuzzy relation)[2]*
Let $\mathcal{R}$ be a fuzzy relation. The α-cut of $\mathcal{R}$ is the subset $\mathcal{R}_\alpha$, defined by :

$$\mathcal{R}_\alpha = \{(x,y) \in \mathrm{X} \times \mathrm{X} : \mathcal{R}(x,y) \geq \alpha\}.$$

Definition 2.2.8 *(Support of fuzzy relation)[8]*
Let $\mathcal{R}$ be a fuzzy relation. The support of $\mathcal{R}$ defined by:

$$S(\mathcal{R}) = \{(x,y) \in \mathrm{X} \times \mathrm{X} : \mathcal{R}(x,y) > 0\}.$$

Proposition 2.2.9 *[5]*
Let $\mathcal{R}$ be a fuzzy relation and $\mathcal{R}_\alpha$ the α-cut of $\mathcal{R}$.
$\mathcal{R}$ is a fuzzy order relation if and only if for every $\alpha \in]0,1]$: $\mathcal{R}_\alpha$ is an order relation in X.

Proof 2.2.10 *Demonstrate that :*
$\mathcal{R}$ *is a fuzzy partial order relation* $\Leftrightarrow \mathcal{R}_\alpha$ *is a partial order relation in* X.
$\Rightarrow$)

i) *Reflexivity of* $\mathcal{R}_\alpha$ *:*
We have: $\mathcal{R}$ *is fuzzy reflexive (because* $\mathcal{R}$ *is a fuzzy partial order relation).*
i,e: $\forall x \in \mathrm{X} : \mathcal{R}(x,x) = 1 \geq \alpha \Rightarrow (x,x) \in \mathcal{R}_\alpha$.
Thus : $\mathcal{R}_\alpha$ *is reflexive.*

ii) *Antisymmetry of* $\mathcal{R}_\alpha$ *:*
We suppose that : $\begin{cases} (x,y) \in \mathcal{R}_\alpha \\ and \\ (y,x) \in \mathcal{R}_\alpha \end{cases}$
$\Rightarrow \begin{cases} \mathcal{R}(x,y) \geq \alpha > 0 \\ and \\ \mathcal{R}(y,x) \geq \alpha > 0 \end{cases} \Rightarrow x = y$ *(because* $\mathcal{R}$ *is fuzzy antisymmetric).*
Thus : $\mathcal{R}_\alpha$ *is antisymmetric.*

iii) *Transitivity of* $\mathcal{R}_\alpha$ *:*
We suppose that : $\begin{cases} (x,y) \in \mathcal{R}_\alpha \\ and \\ (y,z) \in \mathcal{R}_\alpha \end{cases}$

$$\begin{array}{ll} \Rightarrow & \mathcal{R}(x,z) \geq \sup_{y\in \mathrm{Y}}(\min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\}) \\ \Rightarrow & \mathcal{R}(x,z) \geq \min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\} \\ \Rightarrow & \mathcal{R}(x,z) \geq \min\{\alpha,\alpha\} \geq \alpha \\ \Rightarrow & (x,z) \in \mathcal{R}_\alpha \end{array}.$$

Thus : $\mathcal{R}_\alpha$ *is transitive.*
$\Leftarrow$)

i) *Reflexivity of* $\mathcal{R}$ *:*
We have: $\mathcal{R}_\alpha$ *is reflexive (because* $\mathcal{R}_\alpha$ *is a partial order relation). i,e:* $\forall x \in \mathrm{X} : (x,x) \in \mathcal{R}_\alpha \Rightarrow \mathcal{R}(x,x) \geq \alpha$, *for* $\alpha = 1$ *:* $\mathcal{R}(x,x) = 1$.
Thus : $\mathcal{R}$ *is fuzzy reflexive.*

ii) *Antisymmetry of* $\mathcal{R}$ *:*
We suppose that : $\begin{cases} (x,y) \in \mathcal{R} \\ and \\ (y,x) \in \mathcal{R} \end{cases} \Rightarrow \begin{cases} \mathcal{R}(x,y) > 0 \\ and \\ \mathcal{R}(y,x) > 0 \end{cases} \Rightarrow \begin{cases} \mathcal{R}(x,y) > a > 0 \\ and \\ \mathcal{R}(y,x) > b > 0 \end{cases}$
$\Rightarrow \begin{cases} \mathcal{R}(x,y) > a \wedge b \\ and \\ \mathcal{R}(y,x) > a \wedge b \end{cases}$ *, and then, we pose:* $a \wedge b = c \Rightarrow (x,y),(y,x) \in \mathcal{R}_c \Rightarrow$
$x = y$ *(because* $\mathcal{R}_\alpha$ *is antisymmetric).*
Thus : $\mathcal{R}$ *is fuzzy antisymmetric.*

iii) *Transitivity of* $\mathcal{R}$ *:*

$$\textit{We have :}\begin{cases}(x,y)\in\mathcal{R}_\alpha\\ \textit{and}\\ (y,z)\in\mathcal{R}_\alpha\end{cases}\Rightarrow\begin{cases}\mathcal{R}(x,y)\geq\alpha\\ \textit{and}\\ \mathcal{R}(y,z)\geq\alpha\end{cases}\Rightarrow\mathcal{R}(x,z)\geq\alpha$$

$$\begin{aligned}\Rightarrow\mathcal{R}(x,z) &\geq \min\{\mathcal{R}(x,y),\mathcal{R}(y,z)\}\\ &\geq \alpha\geq\min\{\alpha,\alpha\}\end{aligned}.$$

Thus : $\mathcal{R}$ *is fuzzy transitive.*

2.3 Fuzzy Lattice

Definition 2.3.1 *(Upper and Lower Bound on fuzzy poset) [5]*
Let $(\mathrm{X},\mathcal{R})$ *be a partially ordered fuzzy set and* $A\subseteq\mathrm{X}$*.*

1) *An element* $u\in\mathrm{X}$ *is an* ***upper bound*** *of the subset* A *if* $\forall a\in A:\mathcal{R}(a,u)>0$*.*
   ▪ *The least upper bound* u_0 *of* A *satisfies :* $\mathcal{R}(u_0,u)>0$*, for all* u *upper bounds of* A*.*

2) *An element* $l\in\mathrm{X}$ *is an* ***lower bound*** *of the subset* A *if* $\forall a\in A:\mathcal{R}(l,a)>0$*.*
   ▪ *The greatest lower bound* l_0 *of* A *satisfies :* $\mathcal{R}(l,l_0)>0$*, for all* l *lower bounds of* A*.*

☆ *The* ***least*** *upper bounds (supremum) of* $\{x,y\}$ *whenever it exists, is denoted by :* $x\sqcup y$*.*

☆ *The* ***greatest*** *lower bounds (infimum) of* $\{x,y\}$ *whenever it exists, is denoted by :* $x\sqcap y$*.*

Definition 2.3.2 *A fuzzy poset* $(\mathrm{X},\mathcal{R})$ *is called fuzzy inf-lattice if any pair of element has infimum on* X *then, it is called fuzzy sup-lattice if any pair of element has supremum on* X*.*

Definition 2.3.3 *(Fuzzy Lattice)*
A fuzzy poset $(\mathrm{X},\mathcal{R})$ *be a fuzzy lattice if and only if for every two elements* $\{x,y\}$ *have a* ***least*** *upper bounds* $(x\sqcup y)$ *and a* ***greatest*** *lower bounds* $(x\sqcap y)$*. i,e:*

$$\forall x,y\in\mathrm{X}:\exists\begin{cases}x\sqcup y\\ \\ \textit{and}\\ \\ x\sqcap y\end{cases}$$

➢ *In another way:* $(\mathrm{X},\mathcal{R})$ *be a fuzzy lattice if and only if* $(\mathrm{X},\mathcal{R})$ *is fuzzy sup-lattice and fuzzy inf-lattice.*

Example 2.3.4 *Let* $\mathrm{A}=\{a,b,c\}$ *and* $\mathcal{R}\ :\ \mathrm{A}\times\mathrm{A}\longrightarrow[0,1]$ *be a fuzzy ordered relation ,such that :* $\mathcal{R}(a,a)=\mathcal{R}(b,b)=\mathcal{R}(c,c)=1$
$\mathcal{R}(a,c)=\mathcal{R}(b,a)=\mathcal{R}(b,c)=0$

$\mathcal{R}(a,b) = 0.5 \quad \mathcal{R}(c,a) = 0.3 \quad \mathcal{R}(c,b) = 0.7$

$\mathcal{R}$	a	b	c
a	1	0.5	0
b	0	1	0
c	0.3	0.7	1

In addition, we have : $\begin{cases} a \sqcup b = b \\ a \sqcap b = a \end{cases}$, $\begin{cases} a \sqcup c = a \\ a \sqcap c = c \end{cases}$ *and* $\begin{cases} b \sqcup c = b \\ b \sqcap c = c. \end{cases}$

then, their tables of cayley are written as follows:

$\sqcup$	a	b	c
a	a	b	a
b	b	b	b
c	a	b	c

$\sqcap$	a	b	c
a	a	a	c
b	a	b	c
c	c	c	c

Thus : $(\mathrm{A}, \mathcal{R})$ *is a fuzzy lattice.*

Proposition 2.3.5 *[5]*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. For all* $x, y, z \in \mathrm{X}$*:*

1)- $\begin{cases} \mathcal{R}(x \sqcap y, x) > 0 \\ \mathcal{R}(x \sqcap y, y) > 0. \end{cases}$ *and* $\begin{cases} \mathcal{R}(x, x \sqcup y) > 0 \\ \mathcal{R}(y, x \sqcup y) > 0. \end{cases}$

2)- $\begin{cases} \mathcal{R}(x,y) > 0 \\ \text{and} \\ \mathcal{R}(x,z) > 0 \end{cases} \Rightarrow \mathcal{R}(x, y \sqcap z) > 0.$

3)- $\begin{cases} \mathcal{R}(x,z) > 0 \\ \text{and} \\ \mathcal{R}(y,z) > 0 \end{cases} \Rightarrow \mathcal{R}(x \sqcup y, z) > 0.$

4)- $\begin{cases} x \sqcap y = x \Leftrightarrow \mathcal{R}(x,y) > 0 \\ \text{and} \\ x \sqcup y = x \Leftrightarrow \mathcal{R}(y,x) > 0. \end{cases}$

5)- If $\mathcal{R}(x,y) > 0 \Rightarrow \begin{cases} \mathcal{R}(z \sqcap x, z \sqcap y) > 0 \\ \text{and} \\ \mathcal{R}(z \sqcup x, z \sqcup y) > 0. \end{cases}$

Proof 2.3.6 *Let* $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice.* $\forall x, y, z \in \mathrm{X}$ *:*

1)- *We have :* $x \sqcap y$ *is the greatest lower bound of* $\{x, y\}$ *, then :* $x \sqcap y$ *is a lower bound of* x *and* y *.*

$$\text{So: } \begin{cases} \mathcal{R}(x \sqcap y, x) > 0 \\ \text{and} \\ \mathcal{R}(x \sqcap y, y) > 0. \end{cases}$$

In addition, we have : $x \sqcup y$ *is the least upper bound of* $\{x, y\}$ *, then :* $x \sqcup y$ *is an upper bound of* x *and* y *.*

$$\text{So: } \begin{cases} \mathcal{R}(x, x \sqcup y) > 0 \\ \text{and} \\ \mathcal{R}(y, x \sqcup y) > 0. \end{cases}$$

2)- *Suppose :* $\begin{cases} \mathcal{R}(x, y) > 0 \\ \text{and} \\ \mathcal{R}(x, z) > 0 \end{cases}$ *(that is mean :* x *is a lower bound of* y *and* z*).*

Then: $\mathcal{R}(x, y \sqcap z) > 0$*. (because* $y \sqcap z$ *the greatest lower bound).*

3)- *Suppose :* $\begin{cases} \mathcal{R}(x, z) > 0 \\ \text{and} \\ \mathcal{R}(y, z) > 0 \end{cases}$ *(that is mean :* z *is an upper bound of* x *and* y*).*

Then: $\mathcal{R}(x \sqcup y, z) > 0$*. (because* $x \sqcup y$ *the least upper bound).*

4)- $(\Leftarrow)$ *Suppose :* $\mathcal{R}(x, y) > 0$

Since $\mathcal{R}(y, y) = 1 > 0$ *, and we have (From 1)-):* $\begin{cases} \mathcal{R}(x \sqcup y, y) > 0 \\ \mathcal{R}(y, x \sqcup y) > 0 \end{cases} \Rightarrow$ $x \sqcup y = y$ *(From the antisymmetry of* $\mathcal{R}$*).*

$(\Rightarrow)$ *Suppose :* $x \sqcup y = y$ *then,* $\mathcal{R}(x, y) = \mathcal{R}(y, x \sqcup y) > 0$ *(From 1)-).*

5)- *Suppose :* $\mathcal{R}(x, y) > 0$*, then :*

$$\begin{aligned} \mathcal{R}(z \sqcap x, y) &\geq \sup_{r \in \mathrm{X}} \min\left[\mathcal{R}(z \sqcap x, r), \mathcal{R}(r, y)\right] \\ &\geq \min\left[\mathcal{R}(z \sqcap x, x), \mathcal{R}(x, y)\right] > 0 \end{aligned}$$

Thus, we have : $\mathcal{R}(z \sqcap x, y) > 0$ *that is mean :* $z \sqcap x$ *is a lower bound of* $\{y, z\}$ *and* $y \sqcap z$ *is the greatest lower bound of* $\{y, z\}$*.*

Then : $\mathcal{R}(z \sqcap x, z \sqcap y) > 0$*.*

In addition :

$$\begin{aligned} \mathcal{R}(x, z \sqcup y) &\geq \sup_{r\in \mathrm{X}} \min\left[\mathcal{R}(x,r), \mathcal{R}(r, z\sqcup y)\right] \\ &\geq \min\left[\mathcal{R}(x,y), \mathcal{R}(y, z\sqcup y)\right] > 0 \end{aligned}$$

Thus, we have : $\mathcal{R}(x, z \sqcup y) > 0$ *that is mean :* $z \sqcup y$ *is an upper bound of* $\{x, z\}$ *and* $z \sqcup y$ *is the least upper bound of* $\{y, z\}$.
Then : $\mathcal{R}(z \sqcup x, z \sqcup y) > 0$.

2.3.1 Types of fuzzy Lattices

Proposition 2.3.1.1 *(Distributive inequalities) [5]*

Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice and let* $x, y, z \in \mathrm{X}$. *Then:*
$$\begin{cases} \mathcal{R}((x \sqcap y) \sqcup (x \sqcap z), (x \sqcap (y \sqcup z))) > 0 \\ \text{and} \\ \mathcal{R}(x \sqcup (y \sqcap z), (x \sqcup y) \sqcap (x \sqcup z)) > 0. \end{cases}$$

Definition 2.3.1.2 *(Distributive Lattice) [5]*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. It is distributive if and only if :*

- $\forall x, y, z \in \mathrm{X}:$ $$\begin{cases} x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \\ \text{and} \\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z). \end{cases}$$

➢ *In another way (from distributive inequalities):*
$(\mathrm{X}, \mathcal{R})$ *is called fuzzy distributive lattice if and only if:*

- $$\begin{cases} \mathcal{R}((x \sqcup y) \sqcap (x \sqcup z), x \sqcup (y \sqcap z)) > 0. \\ \text{and} \\ \mathcal{R}(x \sqcap (y \sqcup z), (x \sqcap y) \sqcup (x \sqcap z)) > 0. \end{cases}$$

Example 2.3.1.3 *Let* $(\mathrm{A}, \mathcal{R})$ *be a fuzzy totally ordered set. such that :* $\mathrm{A} = \{a, b, c\}$ *and* $\mathcal{R}$ *be a fuzzy order relation which is defined by:*
$\mathcal{R}(a,a) = \mathcal{R}(b,b) = \mathcal{R}(c,c) = 1$
$\mathcal{R}(b,a) = \mathcal{R}(c,a) = \mathcal{R}(c,b) = 0$
$\mathcal{R}(a,b) = 0.8 \quad \mathcal{R}(b,c) = 0.5 \quad \mathcal{R}(a,c) = 0.4.$

$\mathcal{R}$	a	b	c
a	1	0.8	0.4
b	0	1	0.5
c	0	0	1

We see that : $\begin{cases} a \sqcup b = b \\ \\ a \sqcap b = a \end{cases}$, $\begin{cases} a \sqcup c = c \\ \\ a \sqcap c = a \end{cases}$ *and* $\begin{cases} b \sqcup c = c \\ \\ b \sqcap c = b. \end{cases}$

i,e:$(\mathrm{A}, \mathcal{R})$ *be a fuzzy lattice.*
In addition, for all elements of A *satisfie distribitive conditions's. Then :*$(\mathrm{A}, \mathcal{R})$ *be a distribitive fuzzy lattice.*

Proposition 2.3.1.4 *(Modular inequality) [5]*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice and let* $x, y, z \in \mathrm{X}$. *Then:*
$\mathcal{R}(x, z) > 0 \Rightarrow \mathcal{R}(x \sqcup (y \sqcap z), (x \sqcup y) \sqcap z)$

Definition 2.3.1.5 *(Modular Lattice) [5]*
A fuzzy lattice $(\mathrm{X}, \mathcal{R})$ *is modular if and only if :*

$$\forall x, y, z \in \mathrm{X} : \mathcal{R}(x, z) > 0 \Rightarrow \mathcal{R}((x \sqcup y) \sqcap z, x \sqcup (y \sqcap z)) > 0.$$

Example 2.3.1.6 *In the precedent example (2.3.1.3) , we take:* $\mathcal{R}(a, c) > 0$ *then:*
$\mathcal{R}((a \sqcup b) \sqcap c, a \sqcup (b \sqcap c)) = \mathcal{R}(b, b) = 1 > 0$.
Thus: $(\mathrm{X}, \mathcal{R})$ *is a fuzzy modular lattice.*

Definition 2.3.1.7 *(Bounded Lattice)*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. If it contains a greatest element* **1** *and a least element* **0***(with respect of binary operations) is called bounded fuzzy lattice . i,e:*

$$\forall x \in \mathrm{X} : \begin{cases} \mathcal{R}(\mathbf{0}, x) > 0 \\ \\ \textit{and} \\ \\ \mathcal{R}(x, \mathbf{1}) > 0. \end{cases}$$

Definition 2.3.1.8 *(Complete Lattice)*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. If for every non-empty fuzzy set* A *has supremum and infimum, then it is called complete fuzzy lattice.*

2.3.2 Properties of fuzzy Lattices

Proposition 2.3.2.1 *[8]*
Let $\mathcal{R} : \mathrm{X} \times \mathrm{X} \longrightarrow [0, 1]$ *be a fuzzy relation.* $\mathcal{R}$ *is a fuzzy ordering relation on* X *if and only if :* $\alpha \in]0, 1]$, *the* α*-level set* $\mathcal{R}_\alpha$ *is an order relation in* X.

Lemma 2.3.2.2 *[8]*
Let $\mathcal{R}$ *be a fuzzy relation. If* $\mathcal{R}$ *is a fuzzy ordering relation on* X, *then* $S(\mathcal{R})$ *is a partial order relation on* X.

Proof 2.3.2.3 *Let* $\mathcal{R}$ *be a fuzzy ordering relation.*

✓ *Since* $\mathcal{R}$ *is reflexive* $(\mathcal{R}(x, x) = 1)$, $\forall x \in \mathrm{X} : (x, x) \in S(\mathcal{R})$. *i,e:* $S(\mathcal{R})$ *is reflexive.*

✓ *Suppose that:*

$$\begin{cases} (x,y) \in S(\mathcal{R}) \\ and \\ (y,x) \in S(\mathcal{R}) \end{cases} \Rightarrow \begin{cases} \mathcal{R}(x,y) > 0 \\ and \\ \mathcal{R}(y,x) > 0 \end{cases} \Rightarrow x = y \quad (\mathcal{R} \quad anti-symmetric).$$

Thus: $S(\mathcal{R})$ is anti-symmetric .

✓ *Suppose that :*

$$\begin{cases} (x,y) \in S(\mathcal{R}) \\ and \\ (y,z) \in S(\mathcal{R}) \end{cases} \Rightarrow \begin{cases} \mathcal{R}(x,y) > 0 \\ and \\ \mathcal{R}(y,z) > 0 \end{cases}.$$

$$(\mathcal{R} \quad transitive) \Rightarrow \mathcal{R}(x,z) \begin{array}{l} \geq \sup_{y\in \mathrm{X}}(\min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\}) \\ \geq \min\{\mathcal{R}(x,y), \mathcal{R}(y,z)\} > 0 \end{array}$$

Thus: $S(\mathcal{R})$ is transitive .

Hence: $S(\mathcal{R})$ is a partial order relation on X.

Proposition 2.3.2.4 *[5]*
Let $\mathcal{R}$ be a fuzzy relation and let $\mathcal{R}_\alpha = \{(x,y) \in \mathrm{X}\times\mathrm{X} : \mathcal{R}(x,y) \geq \alpha\}$. If $(\mathrm{X}, \mathcal{R}_\alpha)$ is a lattice for every $\alpha \in]0,1]$, then $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice.

Proof 2.3.2.5 *Let $(\mathrm{X}, \mathcal{R}_\alpha)$ is a lattice for every $\alpha \in]0,1]$, that is mean:*
For $x, y \in \mathrm{X}$ there exists a greatest lower bound l_0, such that:

$$\begin{cases} (l_0,x) \in \mathcal{R}_\alpha \\ and \\ (l_0,y) \in \mathcal{R}_\alpha \end{cases} \Rightarrow \begin{cases} \mathcal{R}(l_0,x) \geq \alpha > 0 \\ and \\ \mathcal{R}(l_0,y) \geq \alpha > 0 \end{cases}$$

and for every lower bound l for $\{x,y\}$ exists $l_0 \in \mathrm{X}$,such that:

$$(l, l_0) \in \mathcal{R}_\alpha \Rightarrow \mathcal{R}(l, l_0) \geq \alpha > 0$$

Then: there exists a greatest lower bound (infimum).
In addition : for $x, y \in \mathrm{X}$ there exists a least upper bound u_0, such that:

$$\begin{cases} (x,u_0) \in \mathcal{R}_\alpha \\ and \\ (y,u_0) \in \mathcal{R}_\alpha \end{cases} \Rightarrow \begin{cases} \mathcal{R}(x,u_0) \geq \alpha > 0 \\ and \\ \mathcal{R}(y,u_0) \geq \alpha > 0 \end{cases}$$

and for every upper bound u for $\{x,y\}$ exists $u_0 \in \mathrm{X}$,such that:

$$(u_0, u) \in \mathcal{R}_\alpha \Rightarrow \mathcal{R}(u_0, u) \geq \alpha > 0.$$

Then: there exists a least upper bound (supremum).
Thus: $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice.

Proposition 2.3.2.6 *[5]*
Let $\mathcal{R}$ be a fuzzy relation. If $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice, then: $(\mathrm{X}, \mathcal{R}_\alpha)$ is a lattice for some $\alpha \in]0,1]$.

Proof 2.3.2.7 *(Counter example)*
Let $\mathrm{X} = \{a, b, c, d\}$ and $\mathcal{R} \ : \ \mathrm{X} \times \mathrm{X} \ \longrightarrow \ [0,1]$ be a fuzzy ordered relation ,such that : $\mathcal{R}(a,a) = \mathcal{R}(b,b) = \mathcal{R}(c,c) = \mathcal{R}(d,d) = 1$
$\mathcal{R}(b,a) = \mathcal{R}(c,a) = \mathcal{R}(d,a) = \mathcal{R}(c,b) = \mathcal{R}(d,b) = \mathcal{R}(d,c) = \mathcal{R}(b,c) = 0$
$\mathcal{R}(a,b) = 0.3 \quad \mathcal{R}(a,c) = 0.5 \quad \mathcal{R}(a,d) = 0.7 \quad \mathcal{R}(b,d) = 0.9 \quad \mathcal{R}(c,d) = 0.2$

We define this relation by the following table:

$\mathcal{R}$	a	b	c	d
a	1	0.3	0.5	0.7
b	0	1	0	0.9
c	0	0	1	0.2
d	0	0	0	1

The poset $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice.

Consider the relation $\mathcal{R}_{0.7}$:

$\mathcal{R}$	a	b	c	d
a	1	0	0	1
b	0	1	0	1
c	0	0	1	0
d	0	0	0	1

We observe that : $(\mathrm{X}, \mathcal{R})$ is not a lattice.

Proposition 2.3.2.8 *[5]*
Let $\mathcal{R}$ be a fuzzy relation and let $\mathcal{R}_\alpha = \{(x,y) \in \mathrm{X} \times \mathrm{X} : \mathcal{R}(x,y) \geq \alpha\}$. If $(\mathrm{X}, \mathcal{R}_\alpha)$ is a lattice for every $\alpha \in]0,1]$, then $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice.

Theorem 2.3.2.9 *Let $(\mathrm{X}, \mathcal{R})$ be a fuzzy totally ordered set. Then it is a distributive fuzzy lattice.*

Proof 2.3.2.10 *Demonstrate that :*
$(\mathrm{X}, \mathcal{R})$ is a fuzzy chain $\Rightarrow$ $(\mathrm{X}, \mathcal{R})$ is a fuzzy distributive lattice.

Suppose $x, y \in \mathrm{X}$, then: $\begin{cases} \mathcal{R}(x,y) > 0 \\ \text{or} \\ \mathcal{R}(y,x) > 0. \end{cases}$

✓ *Demonstrate that $(\mathrm{X}, \mathcal{R})$ is a fuzzy lattice:*
Let $\mathcal{R}(x,y) > 0$ and we have: $\mathcal{R}(y,y) = 1 > 0$, then: y is an upper bound of $\{x, y\}$.

Let u another upper bound of $\{x,y\}$, that is mean: $\begin{cases} \mathcal{R}(x,u) > 0 \\ \text{and} \\ \mathcal{R}(y,u) > 0. \end{cases}$

Thus: $x \sqcup y = y$.

★ *Moreover:* $\mathcal{R}(x,y) > 0$ *and* $\mathcal{R}(x,x) = 1 > 0$*, then:* x *is a lower bound of* $\{x,y\}$.

Let l *another lower bound of* $\{x,y\}$*, that is mean:* $\begin{cases} \mathcal{R}(l,x) > 0 \\ and \\ \mathcal{R}(l,y) > 0. \end{cases}$

Thus: $x \sqcap y = x$.
We can see that :$(\mathrm{X}, \mathcal{R})$ *is a fuzzy lattice.*

✓ *Demonstrate that lattice is distributive:*
We have:

$$\begin{aligned} \mathcal{R}(x,y) > 0 \ &\Rightarrow \ x \sqcap y = x \\ &\Rightarrow \ \mathcal{R}(x \sqcap (y \sqcup z), x) > 0 \\ &\Rightarrow \ \mathcal{R}(x \sqcap (y \sqcup z), x \sqcap y) > 0. \end{aligned}$$

In addition:

$$\begin{aligned} \mathcal{R}(x,y) > 0 \ &\Rightarrow \ x \sqcup y = y \\ &\Rightarrow \ \mathcal{R}(x \sqcap y, (x \sqcap y) \sqcup (x \sqcap z)) > 0 \\ &\Rightarrow \ \mathcal{R}(x \sqcap (y \sqcup z), (x \sqcap y) \sqcup (x \sqcap z) \\ &> \ 0 \\ &\geq \ \min \{\mathcal{R}(x \sqcap (y \sqcup z), x \sqcap y), \mathcal{R}(x \sqcap y, (x \sqcap y) \sqcup (x \sqcap z)\} \\ &\Rightarrow \begin{cases} \mathcal{R}((x \sqcup y) \sqcap (x \sqcup z), x \sqcup (y \sqcap z)) > 0 \\ \\ and \\ \\ \mathcal{R}(x \sqcap (y \sqcup z), (x \sqcap y) \sqcup (x \sqcap z)) > 0. \end{cases} . \end{aligned}$$

That is means : $(\mathrm{X}, \mathcal{R})$ *is a fuzzy distributive lattice.*

Proposition 2.3.2.11 *[9] Let* $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice and let* $x, y \in \mathrm{X}$*. Then* $x \sqcup y$ *and* $x \sqcap y$ *in terms of* $(\mathrm{X}, \mathcal{R})$ *coincide with* $x \vee y$ *and* $x \wedge y$ *in terms of the bounded lattice* $(\mathrm{X}, S(\mathcal{R}))$ *.*

2.4 Fuzzy ideal and fuzzy filter of fuzzy Lattice

Definition 2.4.1 *(Fuzzy ideal)[9]*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. A fuzzy set* I *on* X *is a* ***fuzzy ideal*** *in* $(\mathrm{X}, \mathcal{R})$ *when it verifies the following conditions :* $\forall x, y, z \in \mathrm{X}$

❑ *If* $\begin{cases} \mu_{\mathrm{I}}(y) > 0 \\ \\ and \\ \\ \mathcal{R}(x,y) > 0 \end{cases} \Rightarrow \mu_{\mathrm{I}}(x) > 0.$

❑ *If* $\begin{cases} \mu_{\mathrm{I}}(x) > 0 \\ \text{and} \\ \mu_{\mathrm{I}}(y) > 0 \end{cases} \Rightarrow \mu_{\mathrm{I}}(x \sqcup y) > 0.$

Example 2.4.2 *Let* $(\mathrm{X}, \mathcal{R})$ *be a lattice of the example 2.3.2.7 defined by the following Hass diagram :*

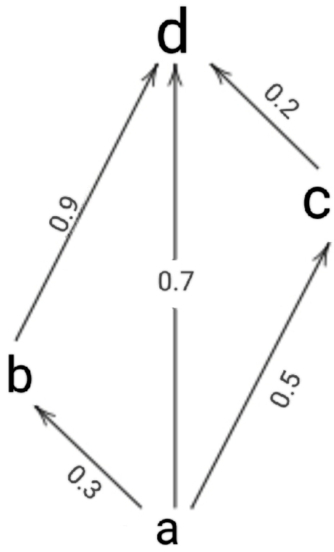

Figure 2.1: Hass diagram of a fuzzy lattice.

The fuzzy set $\mathrm{I} = \{\langle x, \mu_{\mathrm{I}}(x)\rangle \,|x \in \mathrm{X}\} = \{\langle a, 0.3\rangle, \langle b, 0.2\rangle, \langle c, 0\rangle\}$ *is a fuzzy ideal in* $(\mathrm{X}, \mathcal{R})$ *.*

Definition 2.4.3 *(Fuzzy filter)[9]*
Let $(\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice. A fuzzy set* F *on* X *is a* ***fuzzy filter*** *in* $(\mathrm{X}, \mathcal{R})$ *if it verifies the following conditions :* $\forall x, y, z \in \mathrm{X}$

❑ *If* $\begin{cases} \mu_{\mathrm{F}}(x) > 0 \\ \text{and} \\ \mathcal{R}(x, y) > 0 \end{cases} \Rightarrow \mu_{\mathrm{F}}(y) > 0.$

❑ *If* $\begin{cases} \mu_{\mathrm{F}}(x) > 0 \\ \text{and} \\ \mu_{\mathrm{F}}(y) > 0 \end{cases} \Rightarrow \mu_{\mathrm{F}}(x \sqcap y) > 0.$

Example 2.4.4 *Let* $(\mathrm{X}, \mathcal{R})$ *be a lattice of the example 2.3.2.7 and*
$\mathrm{F} = \{\langle x, \mu_{\mathrm{F}}(x)\rangle \,| x \in \mathrm{X}\}$ *be a fuzzy set.*
Since the conditions precedents, $\mathrm{F} = \{\langle a, 0.1\rangle, \langle b, 0.3\rangle, \langle c, 0.6\rangle, \langle d, 0.8\rangle\}$ *is a fuzzy filter of* $(\mathrm{X}, \mathcal{R})$ *.*

Proposition 2.4.5 *Let* A *and* B *be two fuzzy sets.* F *be a fuzzy filter.*

$$\text{If} \begin{cases} \mathrm{A} \subseteq \mathrm{F} \\ \\ \text{and} \qquad \Rightarrow (\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{F}. \\ \\ \mathrm{B} \subseteq \mathrm{F} \end{cases}$$

Proposition 2.4.6 *i)- Let* I *be a fuzzy set on* X*.* I *is a fuzzy ideal of* $(\mathrm{X}, \mathcal{R})$ *if and only if* $S(\mathrm{I})$ *is an ideal of* $(\mathrm{X}, S(\mathcal{R}))$ *.*

ii)- Let F *be a fuzzy set on* X*.* F *is a fuzzy filter of* $(\mathrm{X}, \mathcal{R})$ *if and only if* $S(\mathrm{F})$ *is an ideal of* $(\mathrm{X}, S(\mathcal{R}))$ *.*

Proposition 2.4.7 *Let* $\mathcal{L} = (\mathrm{X}, \mathcal{R})$ *be a fuzzy lattice.*
If $\mathrm{Y} \in \mathrm{X}$ *is an ideal (or filter) of* $\mathcal{L}$ *, then* $(\mathrm{X}, \mathcal{R}')$ *is a fuzzy sublattice of* $\mathcal{L}$*.*

Let $\mathcal{A}$ be a fuzzy relation, $\mathcal{A}_\alpha$ be the α-level set and let $\mathrm{Y}_\alpha = \{x \in \mathrm{Y} : \mathcal{A}(x, y) \geq \alpha, \forall y \in \mathrm{Y}\}$ be an ideal of the lattice $(\mathrm{X}, \mathcal{A}_\alpha)$.

Theorem 2.4.8 *Let* $(\mathrm{X}, \mathcal{A})$ *be a fuzzy lattice. For every* $\alpha \in]0, 1]$ *the fuzzy poset* $(\mathrm{X}, \mathcal{A}_\alpha)$ *is a lattice .*

- $\mathrm{Y} \subseteq \mathrm{X}$ *is an ideal of fuzzy lattice* $(\mathrm{X}, \mathcal{A})$ *if and only if* $\forall \alpha \in]0, 1]: \quad \mathrm{Y}_\alpha$ *is an ideal of* $(\mathrm{X}, \mathcal{A}_\alpha)$ *.*
- $\mathrm{Y} \subseteq \mathrm{X}$ *is a filter of fuzzy lattice* $(\mathrm{X}, \mathcal{A})$ *if and only if* $\forall \alpha \in]0, 1]: \quad \mathrm{Y}_\alpha$*is a filter of* $(\mathrm{X}, \mathcal{A}_\alpha)$ *.*

Definition 2.4.9 *[18]*
Let L *be a lattice and* $\mathrm{I} = \{\langle x, \mu_{\mathrm{I}}\rangle \,| x \in \mathrm{L}\}$ *be an fuzzy set.* I *is called a* ***fuzzy ideal*** *of* L *if* $\forall x, y \in \mathrm{L}$ *the following conditions are satisfied:*

- $\mu_{\mathrm{I}}(x \sqcup y) \geq \mu_{\mathrm{I}}(x) \wedge \mu_{\mathrm{I}}(y)$.
- $\mu_{\mathrm{I}}(x \sqcap y) \geq \mu_{\mathrm{I}}(x) \vee \mu_{\mathrm{I}}(y)$.

Definition 2.4.10 *Let* L *be a lattice and* $\mathrm{F} = \{\langle x, \mu_{\mathrm{F}}\rangle \,| x \in \mathrm{L}\}$ *be an fuzzy set.* F *is called a* ***fuzzy filter*** *of* L *if* $\forall x, y \in \mathrm{L}$ *the following conditions are satisfied:*

- $\mu_{\mathrm{F}}(x \sqcup y) \geq \mu_{\mathrm{F}}(x) \vee \mu_{\mathrm{F}}(y)$.
- $\mu_{\mathrm{F}}(x \sqcap y) \geq \mu_{\mathrm{F}}(x) \wedge \mu_{\mathrm{F}}(y)$.

Example 2.4.11 *Let* L *be a lattice of the example 2.3.2.7 defined by the following Hass diagram :*

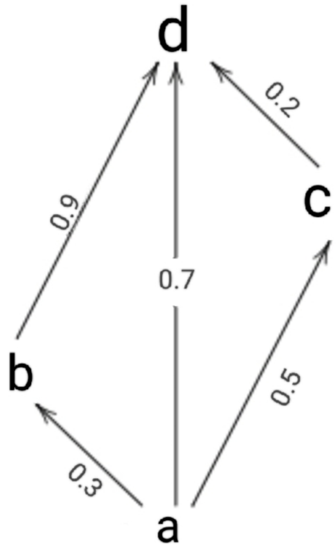

Figure 2.2: Hass diagram of a fuzzy lattice.

The fuzzy set:

- $\mathrm{I} = \{\langle x, \mu_{\mathrm{I}}(x)\rangle \,|x \in \mathrm{X}\} = \{\langle a, 0.3\rangle, \langle b, 0.2\rangle, \langle c, 0\rangle\}$ *is a fuzzy ideal of* L .
- $\mathrm{F} = \{\langle a, 0.1\rangle, \langle b, 0.3\rangle, \langle c, 0.6\rangle, \langle d, 0.8\rangle\}$ *is a fuzzy filter of* L .

Chapter 3

Intuitionistic ordered relations and Lattices

In this chapter, we study some types of intuitionistic fuzzy lattices. Also, we mention definitions of intuitionistic fuzzy ideals and intuitionistic fuzzy filters in intuitionistic fuzzy lattice. Basic references for this chapter are :[[3],[13],[14],[18]]

3.1 Intuitionistic fuzzy sets

3.1.1 Basic definition

Definition 3.1.1.0.1 *(Intuitionistic fuzzy set)[3]*
Let X *be a set and* $\mathrm{A} \subset \mathrm{X}$ *. An intuitionistic fuzzy set* A *in* X *is a fuzzy object presented as follows:*

$$\mathrm{A} = \{\langle x, \mu_{\mathrm{A}}(x), \nu_{\mathrm{A}}(x)\rangle \,|x \in \mathrm{X}\}$$

Where: $\mu_{\mathrm{A}} : \mathrm{X} \longrightarrow [0,1]$ *is the degree membership function of* $x \in \mathrm{X}$ *to* A *and* $\nu_{\mathrm{A}} : \mathrm{X} \longrightarrow [0,1]$ *is the degree non-membership function of* x *to* A.
With the following condition:

$$0 \leq \mu_{\mathrm{A}}(x) + \nu_{\mathrm{A}}(x) \leq 1.$$

Notation 3.1.1.0.2 *The set of all intuitionistic fuzzy sets of* X *will be denoted by* $IFS(\mathrm{X})$.

Remark 3.1.1.0.3 *For all* $x \in \mathrm{X}$*, when* $\nu_{\mathrm{A}}(x) = 1 - \mu_{\mathrm{A}}(x)$ *: the set* A *is fuzzy set.*

3.1.2 Operations over intuitionistic fuzzy sets

Definition 3.1.2.1 *[3]*
Let X *be a non-empty set,* A *and* B *be two intuitionistic fuzzy sets such that :*
$\mathrm{A} = \{\langle x, \mu_{\mathrm{A}}(x), \nu_{\mathrm{A}}(x)\rangle \,|x \in \mathrm{X}\}$ *and* $\mathrm{B} = \{\langle x, \mu_{\mathrm{B}}(x), \nu_{\mathrm{B}}(x)\rangle \,|x \in \mathrm{X}\}$ *. Then:*

- ❍ $\mathrm{A} \subseteq \mathrm{B}$ *if and only if :* $\forall x \in \mathrm{X} : \begin{cases} \mu_{\mathrm{A}}(x) \leq \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}}(x) \geq \nu_{\mathrm{B}}(x) \end{cases}$

- ❍ $\mathrm{A} = \mathrm{B}$ *if and only if :* $\forall x \in \mathrm{X} : \begin{cases} \mu_{\mathrm{A}}(x) = \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}}(x) = \nu_{\mathrm{B}}(x) \end{cases}$

- ❍ $\mathrm{A} \cap \mathrm{B} = \{\langle x, \mu_{\mathrm{A}}(x) \wedge \mu_{\mathrm{B}}(x), \nu_{\mathrm{A}}(x) \vee \nu_{\mathrm{B}}(x)\rangle \,|\forall x \in \mathrm{X}\}$.

- ❍ $\mathrm{A} \cup \mathrm{B} = \{\langle x, \mu_{\mathrm{A}}(x) \vee \mu_{\mathrm{B}}(x), \nu_{\mathrm{A}}(x) \wedge \nu_{\mathrm{B}}(x)\rangle \,|\forall x \in \mathrm{X}\}$

- ❍ $\begin{aligned} \mathrm{A}^c &= \{\langle x, \mu_{\mathrm{A}^c}(x), \nu_{\mathrm{A}^c}(x)\rangle \,|\forall x \in \mathrm{X}\} \\ &= \{\langle x, \nu_{\mathrm{A}}(x), \mu_{\mathrm{A}}(x)\rangle \,|\forall x \in \mathrm{X}\} . \end{aligned}$

3.1.3 Characteristics of intuitionistic fuzzy sets

Definition 3.1.3.1 *(Support)[14]*
Let A *be an intuitionistic fuzzy set on a non-empty* X*. The support of* A *is the crisp (classic) subset defined by :*

$$\begin{aligned} S(\mathrm{A}) &= \{x | x \in \mathrm{X} : \mu_{\mathrm{A}}(x) > 0.\} \\ & \quad Or \\ &= \{x | x \in \mathrm{X} : \mu_{\mathrm{A}}(x) = 0 \quad and \quad \nu_{\mathrm{A}}(x) < 1.\} \end{aligned} .$$

Definition 3.1.3.2 *((α, β)-cuts)[14]*
Let A *be an intuitionistic fuzzy set on a non-empty* X*. The* (α, β)*-cuts* ((α, β)*-level) of* A *is the crisp (classic) subset defined by :*
$A_{(\alpha,\beta)} = \{x | x \in X : \mu_A(x) \geq \alpha \quad and \quad \nu_A(x) \leq \beta\}$.
Where : $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta \leq 1$.

3.2 Intuitionistic fuzzy orderd relation

Definition 3.2.1 *(Intuitionistic fuzzy relation)[13]*
Let X, Y *be a non-empty sets. An intuitionistic fuzzy binary relation* $\mathcal{R}$ *from* X *to* Y *is an intuitionistic fuzzy subset of the cartesian space* X × Y*, defined as:*

$$\mathcal{R} = \{\langle (x, y), \mu_{\mathcal{R}}(x, y), \nu_{\mathcal{R}}(x, y)\rangle \, | (x, y) \in X \times \mathbf{Y}\}$$

Where: $\mu_{\mathcal{R}} : X \times Y \longrightarrow [0, 1]$. *the membership of relation between* x *and* y *under* $\mathcal{R}$ *and* $\nu_{\mathcal{R}} : X \times Y \longrightarrow [0, 1]$. *the non-membership of relation between* x *and* y *under* $\mathcal{R}$.
satisfies the following condition: $\forall (x, y) \in X \times \mathbf{Y} : 0 \leq \mu_{\mathcal{R}}(x, y) + \nu_{\mathcal{R}}(x, y) \leq 1$.

Definition 3.2.2 *(Composition of intuitionistic fuzzy relation)[18]*
Let $\mathcal{R}_1, \mathcal{R}_2$ *be two intuitionistic fuzzy relation. The composition* "$\circ$" *of* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *defined by:*

$$\begin{cases} \mu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) &= \max_{y \in Y} [\min \{\mu_{\mathcal{R}_1}(x, y), \mu_{\mathcal{R}_2}(y, z)\}] \\ \nu_{\mathcal{R}_1 \circ \mathcal{R}_2}(x, z) &= \min_{y \in Y} [\max \{\nu_{\mathcal{R}_1}(x, y), \nu_{\mathcal{R}_2}(y, z)\}] \end{cases}$$

Properties 3.2.3 *[10]*
Let $\mathcal{R}$ *be an intuitionistic fuzzy relation on* X *which may satisfies the following properties :*

1) $\mathcal{R}$ *is reflexive if:*

$$\forall x \in X : \begin{cases} \mu_{\mathcal{R}}(x, x) = 1. \\ \\ or \\ \\ \nu_{\mathcal{R}}(x, x) = 0. \end{cases}$$

2) $\mathcal{R}$ *is anti-symmetric if:*

$$\forall x, y \in X, x \neq y \quad then : \begin{cases} \mu_{\mathcal{R}}(x, y) \neq \mu_{\mathcal{R}}(y, x) \\ \\ and \\ \\ \nu_{\mathcal{R}}(x, y) \neq \nu_{\mathcal{R}}(y, x) \\ \\ and \\ \\ \Pi_{\mathcal{R}}(x, y) = \Pi_{\mathcal{R}}(y, x). \end{cases}$$

Where $\Pi_{\mathcal{R}}(x,y)$ *is the degree of uncertainty of the membership of* $(x,y) \in \mathrm{X}$ *under* $\mathcal{R}$*. It is defined by:*
$\Pi_{\mathcal{R}}(x,y) = 1 - \mu_{\mathcal{R}}(x,y) - \nu_{\mathcal{R}}(x,y)$

3) $\mathcal{R}$ *is transitive if:* $\mathcal{R}^2 = \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$

Definition 3.2.4 *(Intuitionistic fuzzy order relation)*
A relation $\mathcal{R}$ *on* X *be an intuitionistic partial fuzzy order (intuitionistic fuzzy ordered relation) if and only if :it is reflexive, anti-symmetric and transitive.*

Definition 3.2.5 *(Intuitionistic fuzzy chain)[13]*
$(\mathrm{X}, \mathcal{R})$ *is called an intuitionistic fuzzy chain, if :*

$$\forall x, y \in \mathrm{X} : \begin{cases} \mu_{\mathcal{R}}(x,y) > 0 \quad or \quad \{\mu_{\mathcal{R}}(x,y) = 0 \quad and \quad \nu_{\mathcal{R}}(x,y) < 1\} \\ \\ Or \\ \\ \mu_{\mathcal{R}}(y,x) > 0 \quad or \quad \{\mu_{\mathcal{R}}(y,x) = 0 \quad and \quad \nu_{\mathcal{R}}(y,x) < 1\} \end{cases}$$

3.3 Intuitionistic fuzzy Lattice

Definition 3.3.1 *(Intuitionistic fuzzy lattice)[18]*
Let L *a lattice and* $\mathrm{A} = \{\langle x, \mu_{\mathrm{A}}, \nu_{\mathrm{A}} \rangle \, | x \in \mathrm{L}\}$ *be an intuitionistic fuzzy set on* L*. Then* A *is called an intuitionistic fuzzy lattice if* $\forall x, y \in \mathrm{L}$ *the following conditions are satisfied :*

- ✗ $\mu_{\mathrm{A}}(x \sqcup y) \geq \mu_{\mathrm{A}}(x) \wedge \mu_{\mathrm{A}}(y)$.
- ✗ $\mu_{\mathrm{A}}(x \sqcap y) \geq \mu_{\mathrm{A}}(x) \wedge \mu_{\mathrm{A}}(y)$.
- ✗ $\nu_{\mathrm{A}}(x \sqcup y) \leq \nu_{\mathrm{A}}(x) \vee \nu_{\mathrm{A}}(y)$.
- ✗ $\nu_{\mathrm{A}}(x \sqcap y) \leq \nu_{\mathrm{A}}(x) \vee \nu_{\mathrm{A}}(y)$.

Example 3.3.2 *Let the lattice* $\mathrm{L} = \{0, a, b, c, 1\}$ *definded by the Hasse diagram:*

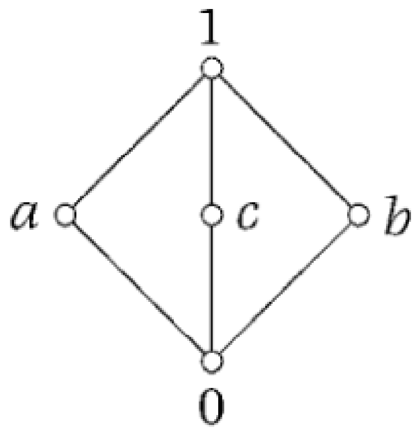

Figure 3.1: Hasse diagram of L

The intuitionistic fuzzy set A *on* L *giveen by:*
$\mathrm{A} = \{\langle 0, 0.4, 0.1\rangle, \langle a, 0.3, 0.2\rangle, \langle b, 0.6, 0.1\rangle, \langle c, 0.4, 0.3\rangle, \langle 1, 0.6, 0.2\rangle\}$ *is an intuitionistic fuzzy lattice.*

Definition 3.3.3 *(Upper and lower bound on intuitionistic fuzzy poset)[10]*
Let $(\mathrm{X}, \mathcal{R})$ *be an intuitionistic fuzzy ordered set and* A *be a subset of* X.

i- The set of all upper bounds of A *with respect to* $\mathcal{R}$ *is the intuitionistic fuzzy subset of* X *defined by:*

$$\forall y \in \mathrm{X} : U(\mathcal{R}, \mathrm{A})(y) = \cap_{x \in \mathrm{A}} \mathcal{R}_{\geq [x]}(y).$$

When:

$$\begin{aligned} \mathcal{R}_{\geq [x]}(y) &= \left\{\left\langle y, \mu_{\mathcal{R}_{\geq [x]}}(y), \nu_{\mathcal{R}_{\geq [x]}}(y)\right\rangle | y \in \mathrm{X}\right\}. \\ &= \{\langle y, \mu_{\mathcal{R}}(x, y), \nu_{\mathcal{R}}(x, y)\rangle | y \in \mathrm{X}\}. \end{aligned}$$

ii- The set of all lower bounds of A *with respect to* $\mathcal{R}$ *is the intuitionistic fuzzy subset of* X *defined by:*

$$\forall y \in \mathrm{X} : L(\mathcal{R}, \mathrm{A})(y) = \cap_{x \in \mathrm{A}} \mathcal{R}_{\leq [x]}(y).$$

When:

$$\begin{aligned} \mathcal{R}_{\leq [x]}(y) &= \left\{\left\langle y, \mu_{\mathcal{R}_{\leq [x]}}(y), \nu_{\mathcal{R}_{\leq [x]}}(y)\right\rangle | y \in \mathrm{X}\right\}. \\ &= \{\langle y, \mu_{\mathcal{R}}(y, x), \nu_{\mathcal{R}}(y, x)\rangle | y \in \mathrm{X}\}. \end{aligned}$$

- *The least upper bound (or a supremum) of* A *with respect to* $\mathcal{R}$ *if:*

$$\begin{cases} x \in S(U(\mathcal{R}, \mathrm{A})). \\ \\ \textit{and} \\ \\ \forall y \in S(U(\mathcal{R}, \mathrm{A})) : \mu_{\mathcal{R}}(x, y) > 0 \quad \textit{or} \quad (\mu_{\mathcal{R}}(x, y) = 0 \quad \textit{and} \quad \nu_{\mathcal{R}}(x, y) < 1). \end{cases}$$

- *The greatest lower bound (or a infimum) of* A *with respect to* $\mathcal{R}$ *if:*

$$\begin{cases} x \in S(L(\mathcal{R}, \mathrm{A})). \\ \\ \textit{and} \\ \\ \forall y \in S(L(\mathcal{R}, \mathrm{A})) : \mu_{\mathcal{R}}(y, x) > 0 \quad \textit{or} \quad (\mu_{\mathcal{R}}(y, x) = 0 \quad \textit{and} \quad \nu_{\mathcal{R}}(y, x) < 1). \end{cases}$$

3.3.1 Types of intuitionistic fuzzy Lattice

Definition 3.3.1.1 *(Distrubitive Lattice)[13]*
Let $(\mathrm{X}, \mathcal{R})$ *be an intuitionistic fuzzy lattice. It is distributive if and only if :*

$$\forall x, y, z \in \mathrm{X} : \quad \begin{cases} x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \\ \\ \textit{and} \\ \\ x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \end{cases}$$

Definition 3.3.1.2 *(Modular Lattice)[13]*
An intuitionistic fuzzy lattice $(\mathrm{X}, \mathcal{R})$ *is modular if and only if :*
$\forall x, y, z \in \mathrm{X} : (\mu_{\mathcal{R}}(x, z) > 0 \quad or \, \{\mu_{\mathcal{R}}(x, z) = 0 \quad and \quad \nu_{\mathcal{R}}(x, z) < 1\})$.
Then : $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z$.

Theorem 3.3.1.3 *Every distributive intuitionistic fuzzy lattice is modular .*

Definition 3.3.1.4 *(Bounded Lattice)[13]*
An intuitionistic fuzzy lattice $(\mathrm{X}, \mathcal{R})$ *is bounded if :*

$$\forall x \in \mathrm{X} : \exists \mathbf{0}, \mathbf{1} \in \mathrm{X} : \begin{cases} \mu_{\mathcal{R}}(\mathbf{0}, x) > 0 \quad Or \quad \{\mu_{\mathcal{R}}(\mathbf{0}, x) = 0 \quad and \quad \nu_{\mathcal{R}}(\mathbf{0}, x) < 1\} \\ \\ and \\ \\ \mu_{\mathcal{R}}(x, \mathbf{1}) > 0 \quad Or \quad \{\mu_{\mathcal{R}}(x, \mathbf{1}) = 0 \quad and \quad \nu_{\mathcal{R}}(x, \mathbf{1}) < 1\} \end{cases}$$

Definition 3.3.1.5 *(Complete Lattice)[10]*
Let $\mathrm{L} = (\mathrm{X}, \mathcal{R})$ *be a complete lattice and* $\mathrm{A} = \{\langle x, \mu_{\mathrm{A}}, \nu_{\mathrm{A}}\rangle \, | x \in \mathrm{L}\}$ *be an intuitionistic fuzzy set.* A *is called an intuitionistic fuzzy complete lattice if for any* $\mathrm{B} \in IFS(\mathrm{L})$ *the following conditions are satisfied:*

1. $\mu_{\mathrm{A}}(\sqcup S(\mathrm{B})) \geq \inf \mu_{\mathrm{A}}(S(\mathrm{B})) = \inf_{x \in S(\mathrm{B})} \mu_{\mathrm{A}}(x)$.
2. $\mu_{\mathrm{A}}(\sqcap S(\mathrm{B})) \geq \inf \mu_{\mathrm{A}}(S(\mathrm{B}))$.
3. $\nu_{\mathrm{A}}(\sqcup S(\mathrm{B})) \leq \sup \nu_{\mathrm{A}}(S(\mathrm{B})) = \sup_{x \in S(\mathrm{B})} \nu_{\mathrm{A}}(x)$.
4. $\nu_{\mathrm{A}}(\sqcup S(\mathrm{B})) \leq \sup \nu_{\mathrm{A}}(S(\mathrm{B}))$

3.4 Intuitionistic fuzzy ideals and filters of Lattice

Definition 3.4.1 *(Intuitionistic fuzzy ideal)[12]*
Let L *be a lattice and* $\mathrm{I} = \{\langle x, \mu_{\mathrm{I}}, \nu_{\mathrm{I}}\rangle \, | x \in \mathrm{L}\}$ *be an intuitionistic fuzzy set.* I *is called an intuitionistic fuzzy ideal of* L *if* $\forall x, y \in \mathrm{L}$ *the following conditions are satisfied:*

- ❖ $\mu_{\mathrm{I}}(x \sqcup y) \geq \mu_{\mathrm{I}}(x) \wedge \mu_{\mathrm{I}}(y)$.
- ❖ $\mu_{\mathrm{I}}(x \sqcap y) \geq \mu_{\mathrm{I}}(x) \vee \mu_{\mathrm{I}}(y)$.
- ❖ $\nu_{\mathrm{I}}(x \sqcup y) \leq \nu_{\mathrm{I}}(x) \vee \nu_{\mathrm{I}}(y)$.
- ❖ $\nu_{\mathrm{I}}(x \sqcap y) \leq \nu_{\mathrm{I}}(x) \wedge \nu_{\mathrm{I}}(y)$.

Example 3.4.2 *Let* L *be a lattice defined in 3.3.2. The intuitionistic fuzzy set* $\mathrm{I} = \{\langle 0, 0.4, 0.2\rangle, \langle a, 0.3, 0.2\rangle, \langle b, 0.3, 0.3\rangle, \langle c, 0.4, 0.3\rangle, \langle 1, 0.3, 0.3\rangle\}$ *is an intuitionistic fuzzy ideal.*

Definition 3.4.3 *(Intuitionistic fuzzy filter)[12]*
Let L *be a lattice and* F *be an intuitionistic fuzzy set.* F *is called an intuitionistic fuzzy ideal of* L *if* $\forall x, y \in \mathrm{L}$ *the following conditions are satisfied:*

- ❖ $\mu_{\mathrm{F}}(x \sqcup y) \geq \mu_{\mathrm{F}}(x) \vee \mu_{\mathrm{F}}(y)$.
- ❖ $\mu_{\mathrm{F}}(x \sqcap y) \geq \mu_{\mathrm{F}}(x) \wedge \mu_{\mathrm{F}}(y)$.
- ❖ $\nu_{\mathrm{F}}(x \sqcup y) \leq \nu_{\mathrm{F}}(x) \wedge \nu_{\mathrm{F}}(y)$.
- ❖ $\nu_{\mathrm{F}}(x \sqcap y) \leq \nu_{\mathrm{F}}(x) \vee \nu_{\mathrm{F}}(y)$.

Example 3.4.4 *Let* L *be a lattice defined in 3.3.2. The intuitionistic fuzzy set* $\mathrm{F} = \{\langle 0, 0.2, 0.5\rangle, \langle a, 0.3, 0.5\rangle, \langle b, 0.2, 0.5\rangle, \langle c, 0.2, 0.3\rangle, \langle 1, 0.5, 0.2\rangle\}$ *is an intuitionistic fuzzy filter.*

Proposition 3.4.5 *Let* A *and* B *be two intuitionistic fuzzy sets,* F *be an intuitionistic fuzzy filter.*

$$\textit{If} \begin{cases} \mathrm{A} \subseteq \mathrm{F} \\ \textit{and} \\ \mathrm{B} \subseteq \mathrm{F} \end{cases} \Rightarrow (\mathrm{A} \cap \mathrm{B}) \subseteq \mathrm{F}.$$

Remark 3.4.6 *Every intuitionistic fuzzy ideal on* L *is an intuitionistic fuzzy lattice, but the duality is not true.*

Proposition 3.4.7 *Let* L *be a lattice ,* A *and* B *are two intuitionistic fuzzy sets on* L*:*

1)- *If* A *and* B *are two intuitionistic fuzzy ideals of* L*, then :* $\mathrm{A} \cap \mathrm{B}$ *is an intuitionistic fuzzy ideal of* L.

2)- *If* A *and* B *are two intuitionistic fuzzy filters of* L*, then :* $\mathrm{A} \cap \mathrm{B}$ *is an intuitionistic fuzzy filter of* L.

Chapter 4

Comparison between some intuitionistic fuzzy structures and fuzzy structures

The main of this chapter is to compare between fuzzy sets and intuitionistic fuzzy sets, then compare between fuzzy lattices and intuitionistic fuzzy lattices. Also, we compare between intuitionistic fuzzy ideals of intuitionistic fuzzy lattice and fuzzy ideals of fuzzy lattice (intuitionistic fuzzy filters of intuitionistic fuzzy lattice and fuzzy filters of fuzzy lattice).

4.1 Comparison between fuzzy sets (fuzzy relation) and intuitionistic fuzzy sets (intuitionistic fuzzy relations)

Fuzzy sets (Fuzzy relation) and intuitionistic fuzzy sets (intuitionistic fuzzy relations) are mathematical models used to defined uncertainty and vaguness in set theory. Such that, they are represented extensions of classical set theory.
However, in the following table we compare between the fuzzy mathematical models and fuzzy intuitionistic mathematical models:

Property	Fuzzy set	Intuitionistic fuzzy set
The membership functions:	Fuzzy sets represent uncertainty by using membership function. Such that: an element x belongs to fuzzy set has degree of membership ranging between 0 and 1.	Intuitionistic fuzzy set represent uncertainty by using membership function and non-membership function. Such that: an element x belongs to fuzzy set has degree of membership and non-membership ranging between 0 and 1 with: $0 \leq \mu(x)+\nu(x) \leq 1$ where the non-membership degree expresses lack of confidence or uncertainty.
Operations:	The mathematical operations are defined by the membership degrees of elements. Such that: **Equality**: $\mathrm{A} = \mathrm{B} \Leftrightarrow \mu_{\mathrm{A}}(x) = \nu_{\mathrm{B}}(x)$ **Inclusion**: $\mathrm{A} \subseteq \mathrm{B} \Leftrightarrow \mu_{\mathrm{A}}(x) \leq \nu_{\mathrm{B}}(x)$ **Intersection**: $\mathrm{A} \cap \mathrm{B} \Leftrightarrow \mu_{\mathrm{A}}(x) \wedge \mu_{\mathrm{B}}(x)$. **Union**: $\mathrm{A} \cup \mathrm{B} \Leftrightarrow \mu_{\mathrm{A}}(x) \vee \mu_{\mathrm{B}}(x)$. **Complement**: $\mu_{\mathrm{A}^c} = 1 - \mu_{\mathrm{A}}(x)$.	The mathematical operations are defined by both membership and non-membership degrees of elements. Such that: **Equality**: $\mathrm{A} = \mathrm{B} \Leftrightarrow \begin{cases} \mu_{\mathrm{A}}(x) = \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}}(x) = \nu_{\mathrm{B}}(x) \end{cases}$ **Inclusion**: $\mathrm{A} \subseteq \mathrm{B} \Leftrightarrow \begin{cases} \mu_{\mathrm{A}}(x) \leq \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}}(x) \geq \nu_{\mathrm{B}}(x) \end{cases}.$ **Intersection**: $\begin{cases} \mu_{\mathrm{A}\cap\mathrm{B}}(x) = \mu_{\mathrm{A}}(x) \wedge \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}\cap\mathrm{B}}(x) = \nu_{\mathrm{A}}(x) \vee \nu_{\mathrm{B}}(x) \end{cases}$ **Union**: $\begin{cases} \mu_{\mathrm{A}\cup\mathrm{B}}(x) = \mu_{\mathrm{A}}(x) \vee \mu_{\mathrm{B}}(x) \\ and \\ \nu_{\mathrm{A}\cap\mathrm{B}} = \nu_{\mathrm{A}}(x) \wedge \nu_{\mathrm{B}}(x) \end{cases}$ **Complement**: $\begin{cases} \mu_{\mathrm{A}^c}(x) = \nu_{\mathrm{A}}(x) \\ and \\ \nu_{\mathrm{A}^c}(x) = \mu_{\mathrm{A}}(x) \end{cases}$
Support:	The support of fuzzy sets defined by membership function. Such that: $S(\mathrm{A}) = \{x \mid \mu_{\mathrm{A}}(x) > 0\}$.	The support of intuitionistic fuzzy sets defined by membership and non-membership function. Such that: $S(\mathrm{A}) = \{x \mid x \in \mathrm{X} : \mu_{\mathrm{A}}(x) > 0.\}$ Or $= \{x \mid x \in \mathrm{X} : \mu_{\mathrm{A}}(x) = 0 \;\; and \;\; \nu_{\mathrm{A}}(x) < 1.\}$
Cuts or levels	In fuzzy sets, defined the α-cuts $\mathrm{A}_\alpha$by: $\{x \mid \mu_{\mathrm{A}}(x) \geq \alpha\}$.	In intuitionistic fuzzy sets, defined the α, β-cut by: $\mathrm{A}_{\alpha,\beta} = \{x \mid \mu_{\mathrm{A}}(x) \geq \alpha \;\; and \;\; \nu_{\mathrm{A}}(x) \leq \beta\}$.

Relation:	Fuzzy relations represent uncertainty by membership function that range between 0 and 1. Such that, $\mathcal{R}$ defined by: $\{\langle (x,y), \mu_{\mathcal{R}}(x,y)\rangle \,\vert (x,y) \in \mathrm{X} \times \mathrm{Y}\}$	Intuitionistic fuzzy relations represent uncertainty by both membership and non-membership functions that range between 0 and 1. Such that, $\mathcal{R}$ defined by: $\{\langle (x,y), \mu_{\mathcal{R}}(x,y), \nu_{\mathcal{R}}(x,y)\rangle \,\vert (x,y) \in \mathrm{X} \times \mathrm{Y}\}$
Order relation:	The order relation is defined by the membership function. Such that: Let $x, y \in \mathrm{X} : (\mu(x) \leq \mu(y)) \Rightarrow (x \leq y)$.	The order relation is defined by both membership and non-membership functions. Such that: Let $x, y \in \mathrm{X} : \begin{cases} \mu(x) \leq \mu(y) \\ and \\ \nu(x) \geq \nu(y) \end{cases} \Rightarrow (x \leq y).$
Cuts:	The α-cuts of fuzzy relation $\mathcal{R}_\alpha$ is defined by the membership, such that: $\forall (x,y) \in \mathrm{X} \times \mathrm{X} :$ $\mu_{\mathcal{R}}(x,y) \geq \alpha$.	**The** α, β**-cuts** of intuitionistic fuzzy relation $\mathcal{R}_{\alpha,\beta}$ is defined by both membership and non-membership, such that: $\forall (x,y) \in \mathrm{X} \times \mathrm{X} :$ $\mu_{\mathcal{R}}(x,y) \geq \alpha \quad and \quad \mu_{\mathcal{R}}(x,y) \leq \beta$

4.2 Comparison between fuzzy lattices and intuitionistic fuzzy lattices

Fuzzy lattice and intuitionistic fuzzy lattice are mathematical structures used in fuzzy set theory and intuitionistic fuzzy set theory. They model for uncertainty and imprecision.

In addition, there are another structures used to define subsets of a universe called the fuzzy filters (ideals) and intuitionistic fuzzy filters (ideals).

However, in the following table we represent some differences between fuzzy lattice and intuitionistic fuzzy lattice :

<table>
<tr><th></th><th>Fuzzy Lattice</th><th>Intuitionistic fuzzy Lattice</th></tr>
<tr><td>Least upper bound and greatest lower bound:</td><td>The infimum l_0 of A satisfies : $\mu_{\mathcal{R}}(l, l_0) > 0$, for all l lower bounds of A.
The supermum u_0 of A satisfies: $\mu_{\mathcal{R}}(u_0, u) > 0$ for all u upper bounds of A.</td><td>The infimum of A is defined by:
$$\begin{cases} l_0 \in S(L(\mathcal{R}, \mathrm{A})). \\ and \\ \forall l \in S(L(\mathcal{R}, \mathrm{A})) : \begin{cases} \mu_{\mathcal{R}}(l, l_0) > 0 \\ or \\ (\mu_{\mathcal{R}}(l, l_0) = 0 \\ and \\ \nu_{\mathcal{R}}(l, l_0) < 1). \end{cases} \end{cases}$$
The supermum of A is defined by:
$$\begin{cases} u_0 \in S(U(\mathcal{R}, \mathrm{A})). \\ and \\ \forall u \in S(U(\mathcal{R}, \mathrm{A})) : \begin{cases} \mu_{\mathcal{R}}(u_0, u) > 0 \\ or \\ (\mu_{\mathcal{R}}(u_0, u) = 0 \\ and \\ \nu_{\mathcal{R}}(u_0, u) < 1) \end{cases} \end{cases}.$$</td></tr>
<tr><td>Definition:</td><td>For every pair elements in poset have a least upper bound and a greatest lower bound. That is mean:
$\forall x, y$ in the poset: $\exists \begin{cases} x \sqcup y \\ and \\ x \sqcap y \end{cases}$
which are defined by membership function ranging between 0 and 1.
$$\begin{cases} \begin{cases} \mu_{\mathcal{R}}(x, x \sqcup y) > 0 \\ and \\ \mu_{\mathcal{R}}(y, x \sqcup y) > 0 \end{cases} \\ and \\ \begin{cases} \mu_{\mathcal{R}}(x \sqcap y, x) > 0 \\ and \\ \mu_{\mathcal{R}}(x \sqcap y, y) > 0 \end{cases} \end{cases}$$</td><td>For every pair elements have an intuitionistic least upper bound and an intuitionistic greatest lower bound, which are defined by both membership and non-membership functions ranging between 0 and 1.</td></tr>
<tr><td>Modularity</td><td>Modular fuzzy Lattice $(\mathrm{X}, \mathcal{R})$ is defined by the membership function, such that: $\forall x, y, z \in \mathrm{X}$:
If $\mu_{\mathcal{R}}(x, z) > 0$
$\Rightarrow x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z.$</td><td>Modular intuitionistic fuzzy Lattice $(\mathrm{X}, \mathcal{R})$ is defined by both membership function and non-membership function, such that:
$\forall x, y, z \in \mathrm{X}$:
If $\mu_{\mathcal{R}}(x, z) > 0$
$or \{\mu_{\mathcal{R}}(x, z) = 0 \quad and \quad \nu_{\mathcal{R}}(x, z) < 1\}$
$\Rightarrow x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z.$</td></tr>
</table>

Completion	Bounded fuzzy Lattice $(\mathrm{X}, \mathcal{R})$ is defined by membership function, such that: $\forall x \in \mathrm{X} : \begin{cases} \mu_{\mathcal{R}}(\mathbf{0}, x) > 0 \\ and \\ \mu_{\mathcal{R}}(x, \mathbf{1}) > 0. \end{cases}$	Bounded intuitionistic fuzzy Lattice (Complete) $(\mathrm{X}, \mathcal{R})$ is defined by both membership function and non-membership function, such that: $\forall x \in \mathrm{X} : \exists \mathbf{0}, \mathbf{1} \in \mathrm{X} :$ $\begin{cases} \mu_{\mathcal{R}}(\mathbf{0}, x) > 0 \\ Or \\ \{\mu_{\mathcal{R}}(\mathbf{0}, x) = 0 \quad and \quad \nu_{\mathcal{R}}(\mathbf{0}, x) < 1\} \\ and \\ \mu_{\mathcal{R}}(x, \mathbf{1}) > 0 \\ Or \\ \{\mu_{\mathcal{R}}(x, \mathbf{1}) = 0 \quad and \quad \nu_{\mathcal{R}}(x, \mathbf{1}) < 1\} \end{cases}$

4.3 Comparison between fuzzy ideals (fuzzy filters) and intuitionistic fuzzy ideals (intuitionistic fuzzy filters)

Structurs	**Fuzzy**	**Intuitionistic fuzzy**
Ideal	In a fuzzy lattice L, a fuzzy ideal I verifies the following conditions: $\forall x, y \in \mathrm{L}:$ • $\mu_{\mathrm{I}}(x \sqcup y) \geq \mu_{\mathrm{I}}(x) \wedge \mu_{\mathrm{I}}(y)$. • $\mu_{\mathrm{I}}(x \sqcap y) \geq \mu_{\mathrm{I}}(x) \vee \mu_{\mathrm{I}}(y)$.	In an intuitionistic fuzzy lattice L, an intuitionistic fuzzy ideal I verifies the following conditions: $\forall x, y \in \mathrm{L}:$ • $\mu_{\mathrm{I}}(x \sqcup y) \geq \mu_{\mathrm{I}}(x) \wedge \mu_{\mathrm{I}}(y)$. • $\mu_{\mathrm{I}}(x \sqcap y) \geq \mu_{\mathrm{I}}(x) \vee \mu_{\mathrm{I}}(y)$. • $\nu_{\mathrm{I}}(x \sqcup y) \leq \nu_{\mathrm{I}}(x) \vee \nu_{\mathrm{I}}(y)$. • $\nu_{\mathrm{I}}(x \sqcap y) \leq \nu_{\mathrm{I}}(x) \wedge \nu_{\mathrm{I}}(y)$.
Filter	In a fuzzy lattice L, a fuzzy filter F verifies the following conditions: $\forall x, y \in \mathrm{L}:$ • $\mu_{\mathrm{F}}(x \sqcup y) \geq \mu_{\mathrm{F}}(x) \vee \mu_{\mathrm{F}}(y)$. • $\mu_{\mathrm{F}}(x \sqcap y) \geq \mu_{\mathrm{F}}(x) \wedge \mu_{\mathrm{F}}(y)$. .	In an intuitionistic fuzzy lattice L, an intuitionistic fuzzy filter F verifies the following conditions: $\forall x, y \in \mathrm{L}:$ • $\mu_{\mathrm{F}}(x \sqcup y) \geq \mu_{\mathrm{F}}(x) \vee \mu_{\mathrm{F}}(y)$. • $\mu_{\mathrm{F}}(x \sqcap y) \geq \mu_{\mathrm{F}}(x) \wedge \mu_{\mathrm{F}}(y)$. • $\nu_{\mathrm{F}}(x \sqcup y) \leq \nu_{\mathrm{F}}(x) \wedge \nu_{\mathrm{F}}(y)$. • $\nu_{\mathrm{F}}(x \sqcap y) \leq \nu_{\mathrm{F}}(x) \vee \nu_{\mathrm{F}}(y)$.

4.4 The benefits and drawbacks of Fuzzy Lattices

4.4.1 Benefits of Fuzzy Lattices:

- Fuzzy lattices present complex phenomena in simple and concise way.
- Fuzzy lattices may be used for various contexts either to present uncertain information or situation or be tailored to fit certain structured need (Flexibility and adaptable).
- They present effective reasoning capacities with their strong combination of set theoretic and littice-based operation s which analyse data effectively (Reasoning effectively).

- Fuzzy lattices present robust presentation of items and their meanings and make the best use of lattice based operations for more flexibility and scalability (Robest Representation).

4.4.2 Drawbacks of Fuzzy Lattices:

- Fuzzy lattices are based on uncertain data sets, hence they are difficult to contract and waste time.
- They not appropriate for large datasets since they lack scalability. It is not easy to add items since the littice will be complected (Limited scalability).
- Fuzzy lattices need huge memory for data storage (Memory overhead).
- Difficult for prediction making they are hard to interpret or to get a clear result thus it non-experts couldn't understand them.

4.5 The benefits and drawbacks of Intuitionistic Fuzzy Lattices

4.5.1 Benefits of Intuitionistic Fuzzy Lattices:

- Intuitionistic fuzzy lattice provides more flexible approach to problem solving by using multiple criteria.
- It allows combining variables from various fields in away that variables with values also can be included in analysis.
- It can offers more refined effects than the traditional one
- Intuitionistic fuzzy lattices offer the chance to reflect on how various criteria can impact the result.

4.5.2 Drawbacks of Intuitionistic Fuzzy Lattices:

- Intuitionistic fuzzy lattice is proved that intuitionistic fuzzy lattices are computationally .
- The risk of complexity is higher in decision-making process owing to the greater number of variables.
- The quality of fuzzy lattices membership function used affect accuracy.
- Intuitionistic fuzzy lattices need huge amount of data to generate meaningful consequences.

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Abstract:

In this memory, We did study the concept of fuzzy lattice and their properties and intuitionistic fuzzy lattice. In addition, we did mentioned some characterizations of ideals and filters. After that, we did compare between fuzzy lattices and intuitionistic fuzzy lattices. Then we did cited the benefits and drawbacks of fuzzy lattices and intuitionistic fuzzy lattices.

Keywords: Fuzzy ordering relation, fuzzy lattice , fuzzy ideal , fuzzy filter, intuitionistic fuzzy ordering relation, intuitionistic fuzzy lattice , intuitionistic fuzzy ideal , intuitionistic fuzzy filter.

Résumé:

Dans ce mémoire, nous avons étudié le concept de treillis flou et leurs propriétés ainsi que le treillis flou intuitionniste. De plus, nous avons mentionné quelques caractérisations des idéaux et des filtres. Ensuite, nous avons comparé les treillis flous et les treillis flous intuitionnistes. Nous avons ensuite cité les avantages et les inconvénients des treillis flous et des treillis flous intuitionnistes.

Mots clés: Relation d'ordre floue, treillis flou, idéal flou, filtre flou, relation d'ordre floue intuitionniste, treillis flou intuitionniste, idéal flou intuitionniste, filtre flou intuitionniste.

خلاصة:

في هذه المذكرة ، قمنا بدراسة مفهوم الشبكة الضبابية وخصائصها والشبكة الضبابية الحدسية. بالإضافة إلى ذلك ، ذكرنا بعض خصائص المثل العليا والمرشحات. بعد ذلك ، قارنا بين المشابك الضبابية والمشابك الضبابية الحدسية. ثم ذكرنا مزايا وعيوب الضبابية المشابك والمشابك الضبابية الحدسية.

الكلمات المفتاحية: علاقة ترتيب ضبابية ، شبكة ضبابية ، مثالية ضبابية ، مرشح ضبابي ، علاقة ترتيب ضبابية حدسية ، شبكة ضبابية حدسية ، مثالي ضبابي حدسي ، مرشح ضبابي حدسي.


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