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THEME

On the Schechter essential spectrum and application to singular transport equation

Rédige Par:

KHALILI Zineb

Dirigé par:

Toufik HERAIZ

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A ma grand-père

A ma grand-mère

que dieu nous la grand Et a mes aimes

Notations

$\mathcal{L}(X)$	The algebra of all bounded linear operators from X into itself
$N(A)$	The null space of A
$R(A)$	The range space of A
$\mathcal{K}(X)$	The ideal of all compact operators on X
$\mathcal{C}(X)$	The set of all closed densly defined linear operators on X
$\mathcal{W}(X,Y)$	The family of weakly compact operators from X to Y
$\mathcal{S}(X,Y)$	The set of strictly singular operators from X to Y
$C\mathcal{S}(X,Y)$	The set of strictly cosingular operators from X to Y
$\mathcal{I}(X)$	Anarbitrary non zero two sided ideal of $\mathcal{L}(X)$
$\sigma(A)$	The spectrum of A
$\rho(A)$	The resolventset of A
$\Phi(X)$	The set of Fredholm operators on X
$\alpha(A)$	The nullity of A is defined as the dimension of $N(A)$
$\beta(A)$	The deficiency of A is defined as the codimension of $R(A)$
I	Operator of identity
$\mathcal{F}(X)$	The set of fredholm perturbation on X
$\Phi_{+}(X)$	The set of upper semi-Fredholm operators on X
$\Phi_{-}(X)$	The set of lower semi-Fredholm operators on X

Table of contents

Introduction

The theory of the essential spectra of linear operators in Banach space is a modern section of the spectral analysis widely used in the mathematical and physical sense when resolving a number of applications that can be formulated in terms of linear operators.

In first chapter we recal the basic properities of the spectrun of bounded and unbounded linear operator in a Banach space, introducing the basic notaions necessary to study linear adjoint operators for norme and Hilbert spaces and we present some classical quantities associated with an operators, this quantities such as the closed and closable of an operators are defined in the first section and are the basic bricks in the construction of one of the most important branches of spectral theory, the theory of fredholm operators and essential spectrum

In the second chapter we gives a survey of results concerning various types and fredholm perturbations, the ideal of strictly singular and strictly cosingular operators which occur in Fredholm theory.

The third chapter provides of Invariance of Schechter Essential spectrum by their bounded prturbations and extention to unbounded perturbations

The last chapter deal with the schechter essential spectrum of singular transport equations

Chapitre 1

Basic Properties

1.1 Generalities about Closed operators

Let X and Y be two Banach spaces. A mapping A which assigns to each element x of a set $D(A) \subset X$ a unique element $y \subset Y$ is called an operator (or transformation), the set $D(A)$ on which A acts is called the domain of A.

Definition 1.1.1
The operator A is called linear, if $D(A)$ is a subspace of X and if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all scalars α, β and all elements x, y in $D(A)$.

We note by a pair $(A, D(A))$ the operator A with domain $D(A)$

Definition 1.1.2
The operator A is called bounded, if there is a constant M such that $\| Ax \| \leq M \| x \|$. The norm of such an operator is dfined by:

$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}$$

Example 1.1.1
The diferential operator of first order $A = i\partial_x$ on $\mathcal{L}^2([0;1])$. It readily seen to be unbounded since one can find a sequence of functions $\varphi_n \in \mathcal{L}^2([0;1])$, given by $\varphi_n = e^{inx}$ for $n \in \mathbb{N}$, satisfying $\| \varphi_n \|_{\mathcal{L}^2([0,1])} = 1$ and $\| A\varphi_n \|_{\mathcal{L}^2([0,1])} = n \to \infty$ as $n \to \infty$

Definition 1.1.3

The graph $G(A)$ *of a linear operator* A *on* $D(A) \subset X$ *into* Y *is the set* $\{(x, Ax) \text{ such that } x \in D(A)\}$ *in the product space* $X \times Y$ *then* A *is called a closed linear operator when its graph* $G(A)$ *constitutes a closed linear subspace of* $X \times Y$. *Hence, the notion of a closed linear operator is an extension of the notion of a bounded linear operator.*

In other word, we say that A *is closed if for every sequence* (x_n) *in* $D(A)$ *such that* $x_n \to x$ *and* $Ax_n \to y$, *we have* $x \in D(A)$ *and* $Ax = y$

- We denote by $\mathcal{C}(X, Y)$ the set of all closed,densely defined linear operator if $X = Y$ then $\mathcal{C}(X, Y) = \mathcal{C}(X)$

1.1.1 Closable Operators

When A and B are operators from X to Y and $D(B) \subset D(A)$ with $Bx = Ax$ for $x \in D(B)$, we say that A is an extension of B and B is a restriction of A, and we write $B \subset A$. Equivalently, $B \subset A$ if and only if $G(B) \subset G(A)$. One often wants to know whether a given operator A has a closed extension, If A is bounded, this always holds, since we can simply take the operator $\overline{A}$ with graph $\overline{G(A)}$, here $\overline{G(A)}$ is a graph since $x_n \to 0 \in X$ implies $Ax_n \to 0$ but when A is unbounded, one cannot be certain that it has a closed extension. But if A has a closed extension A_1, then $\overline{G(A_1)}$ is a closed subspace of $X \times Y$ containing $G(A)$, hence also containing $\overline{G(A)}$, in that case $\overline{G(A)}$ is a graph, it is in fact the graph of the smallest closed extension of A, we call it the closure of A and denote it $\overline{A}$ (observe that when A is unbounded, then $D\left(\overline{A}\right)$ is a proper subset of $\overline{D(A)}$).

Definition 1.1.4

An operator A *is called closable if it has a closed extension, the smallest closed extension of* A *whose graph equals* $\overline{G(A)}$ *is denoted by* $\overline{A}$ *and called the closure of* A, *every closable operator has a closure.*

Proposition 1.1.1

Let $A : X \to Y$ *be an operator, the following conditions are equivalent:*

1. A is closable.

2. $\overline{G(A)}$ is a graph of an operator.

3. If $(0, y) \in \overline{G(A)}$ then $y = 0$.

4. If for any sequence $(x_n)_n \subset D(A)$ such that $x_n \to 0 \in X$ and $Ax_n \to y \in Y$ implies $y = 0$

Theorem 1.1.1 [1]
Let A be closed (resp. closable) operator from X to Y and $B \in \mathcal{L}(Z, X)$, then the operator AB is closed (resp. closable) with $D(AB) = \{x \in D(B) : Ax \in D(A)\}$.

Proof.
Let $z_n \in D(AB), n \in \mathbb{N}$, and $z \in Z, y \in Y$ such that $z_n \to z$ in Z and $ABz_n \to y$ in Y as $n \to \infty$. Since B is bounded, then $x_n = Bz_n$ converges to Bz and so $Ax_n \to y$ as $n \to \infty$ Since A is closed, we deduce that $Bz \in D(A)$ and $ABz = y$. Now if A is closable then AB has a closed extension $\overline{A}B$.

Note that the product of two closed operator need not be closed operator. ■

Example 1.1.2
Let $X = C([0; 1]), Af = f\,'$ with $D(A) = C^1([0; 1])$ and $\varphi \in C([0, 1])$ such that $\varphi = 0$ on $[0, 1/2]$, define $B \in \mathcal{L}(X)$ by $Bf = \varphi f$ for all $f \in X$, then the operator BA with $D(BA) = D(A)$ is not closed.see this, take functions $f_n \in D(A)$ such that $f_n = 1$ on $\left[\frac{1}{2}, 1\right]$ and $f_n \to f$ in X with $f \notin C^1([0, 1])$, then, $BAf_n = \varphi f_n = 0$ converges to 0, but $f \notin D(A)$

1.2 Relatively Boundedness and Relatively Compactness

Definition 1.2.1
Let X, Y and Z be Banach spaces, and let A and S be two linear operators from X into Y and from X into Z, respectively.

(i) S is called relatively bounded with respect to A, or A-bounded, if $D(A) \subset D(S)$ and there exist two constants $a_s \geq 0$ and $b_s \geq 0$ such that

$$\| Sx \| \leq a_s \| x \| + b_s \| Ax \| \,,\ x \in D(A) \tag{1.2.1}$$

The infimum δ of all b_s that holds for some $a_s \geq 0$ is called relative bound of S with respect to A (or A-bound of S).

(ii) S is called relatively compact with respect to A (or A-compact), if $D(A) \subset D(S)$ and for every bounded sequence $(X_n)_n \in D(A)$ such that, $(AX_n)_n \subset Y$ is bounded, the sequence $(SX_n)_n \subset Z$ contains a convergent subsequence.

Remark 1.2.1

The inequality (1.2.1) is equivalent to

$$\| SX \|^2 \leq a^2 \| x \|^2 + b^2 \| Ax \|^2 \text{ for all } x \in D(A)$$

where $a = \sqrt{a_s^2 + a_s b_s}$ *and* $b = \sqrt{b_s^2 + a_s b_s}$

Lemma 1.2.1

If S is A-bounded with an $A-$bound $\delta < 1$, then S is $(A+S)-$bounded with an $A-$bound $\leq \dfrac{\delta}{1-\delta}$.

Proof.

First of all, it should be mentioned that $A+S$ is will defined as

$$D(A+S) = D(S) \cap D(A) = D(A) \subset D(S)$$

The fact that S is A-bounded, there exist $a_s \geq 0$ and $\delta \leq b < 1$ such that, for all $x \in D(A)$ we have

$$\| Sx \| \leq a_s \| x \| + b_s \| Ax \|$$

$$= a_s \| x \| + b_s \| Ax + Sx - Sx \|$$

$$= a_s \| x \| + b_s \| Ax + Sx \| + b_s \| Sx \|$$

Since $b_s < 1$ it follows that

$$\| Sx \| \leq \frac{a_s}{1-b_s} \| x \| + \frac{b_s}{1-b_s} \| (A+S)\, x \|, x \in D$$

Q.E.D.

Let $A \in C(X,Y)$, the graph norm of A is defined by $\| x \|_A = \| x \| + \| Ax \|, x \in D(A)$ from the closedness on A it follows that $D(A)$ endowed with the norm $\| \cdot \|_A$ is a Banach space, Let us denote by X_A, the space $D(A)$ equipped with the norm $\| \cdot \|_A$ clearly, the operator A satisfies $\| Ax \| \leq \| x \|_A$ and consequently $A \in$L(X)

Let $J : X \to Y$ be a linear operator on X if $D(A) \subset D(J)$ then J will be called A-defined. If J is A-defined, we will denote by $\hat{J}$ its restriction to $D(A)$ Moreover, if $\hat{J} \in$ L(X) we say that J is A-bounded. We can easily check that, if J is closed (or closable), then J is A-bounded. ■

Proposition 1.2.1

If A is closed and B is closable, then $D(A) \subset D(B)$ implies that B is A-bounded.

Proof.

If A is a closed operator, then $D(A)$ equipped with the graph norm is a Banach space, if we suppose that $D(A) \subset D(B)$ and $(B, D(B))$ is closable, then $D(A) \subset D\left(\bar{B}\right)$ because the graph norm on $D(A)$ is stronger than the norm induced from X, the operator $\bar{B}$, considered as an operator from $D(A)$ into X is every where defined and closed. On the other hand, $\overline{B}_{|D(A)} = B$ hence $B : D(A) \to X$ is bounded by the closed graph and thus B is A-bounded.

Q.E.D. ■

Definition 1.2.2

We say that an operator J is A-closed, if $x_n \to x, Ax_n \to y$, $Jx_n \to z$ for $(x_n)_n \subseteq D(A)$ implies that $x \in D(J)$ and $Jx = z$. An operator J will be called $A-$closable, if $x_n \to 0, Ax_n \to 0$, $Jx_n \to z$ implies $z = 0$

Remark 1.2.2

(i) If J is bounded, then J is A-bounded.

(ii) If J is closed, then J is A-closed.

(iii) If J is closable, then J is A-closable.

1.3 Compact Operators

Definition 1.3.1 ***(Compact linear operators)***
A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if for every bounded subset G of E; the image $A(G)$ is relatively compact in F. In other words, the closure $\overline{A(G)}$ is compact

Theorem 1.3.1 [5] ***(Compactness criterion)***
A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence $\varphi_n$in E, the sequence $A\varphi_n$ in F has a convergent subsequence.

Proof.
Let φ_n be a bounded sequence in E since the operator A is compact, then the set $\{A\varphi_n\}$ is relatively compact in F where this property shows that $(A\varphi_n)$ contains a convergent subsequence. Conversely, let us consider any bounded subset G in E and let ψ_n be any sequence in $A(G)$, then there exists a bounded sequence φ_n in G; such that $\psi_n = A\varphi_n$, by assumption, $A\varphi_n = \psi_n$ contains a convergent subsequence ψ_{nk} in F, thus $A(G)$ is relatively compact, because for any bounded sequence ψ_n in $A(G)$ there exits a convergent subsequence ψ_{nk} in F, in other words, for all bounded set $G \subset E$ the set $A(G)$ is relatively compact in F, hence A is compact. ■

Theorem 1.3.2 [5]
The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β

Proof.
Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F, then $A\varphi_n(x) = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x)$ with $\varphi_n \in E$, $n \in \mathbb{N}$. The operators A_1 and A_2 are compact,

one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences which give by their sum a convergent subsequence of $A\varphi_n$, hence A is compact. ■

Theorem 1.3.3 [5]
The product AB of two bounded operators A and B is compact if either of operators A or B is compact.

Proof.
Let φ_n be a bounded sequence in E then if we consider B as a bounded operator the sequence $B\varphi_n(x)$ is bounded, and from the compactness of the operator A gives a convergent subsequence $A(B\varphi_{nk}(x))$ of $A(B\varphi_n(x))$, then the operator AB is compact. On the other hand, if we consider B as a compact, one can extract from $B\varphi_n(x)$ a convergent subsequence $B\varphi_n(x)$ and from the boundedness of the operator A gives the convergence of the sequence $A(B\varphi_{nk}(x))$, hence the operator AB is compact ■

Theorem 1.3.4 [5]
A sequence A_n of compact operators defined from a normed space E into a Banach space F converges uniformly to an operator A, say $\lim_{n\to\infty} \| A_n - A \| = 0$, then the limit operator A is compact.

Proof.
Let φ_n be a bounded sequence in E, the operator A_1 is compact, then one can extract from the sequence $A_1\varphi_n$ a convergent subsequence, say φ_n^1 a subsequence from φ_n such that $A_1\varphi_n^1$ converges. In the same way, we can extract from the sequence $A_2\varphi_n^1$ a convergent subsequence, say φ_n^2 a subsequence from φ_n^1 such that $A_2\varphi_n^2$ converges. Noting that, we obtain from the bounded sequence φ_n a subsequence φ_n^2 such that $A_1\varphi_n^2$ and $A_2\varphi_n^2$ both converge, Continuing in this way, we see that, for the compact operators $A_1, A_2, ..., A_p$, there exists a nested subsequences $\varphi_n^p \subset ....\varphi_n^2 \subset \varphi_n^1 \subset \varphi_n$such that the sequences $A_k\varphi_n^p$ converge for all $k = 1, 2, ..., p$: In order to show the compactness of the operator limit A we must use the completeness of the space F and showing that the sequence $A\varphi_n^p$ is Cauchy sequence. Noting that the sequence φ_n is bounded,

say $\| \varphi_n \| \leq M$ for all n hence $\| \varphi_n^p \| \leq M$ for each n and p hoose $n = p$ so that $\| A_n - A \| < \frac{\varepsilon}{3M}$

Since the sequence $A_n \varphi_n^p$ is Cauchy, because it converges there is N such that, for all $p > N$ and $q > N$;

we get

$$\| A_n \varphi_n^p - A_n \varphi_n^q \| < \frac{\varepsilon}{3}$$

Hence we obtain

$$\| A\varphi_n^p - A\varphi_n^q \| = \| A\varphi_n^p - A\varphi_n^p + A\varphi_n^p - A\varphi_n^q + A\varphi_n^q - A\varphi_n^q \|$$

$$\leq \| A_n - A \| \| \varphi_n^p \| + \| A_n\varphi_n^p - A_n\varphi_n^q \| \| A_n\varphi_n^q - A\varphi_n^q \|$$

$$\leq \frac{\varepsilon}{3M} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} = \varepsilon$$

Remembering that, due to the completeness of the space F the Cauchy sequence $A\varphi_n^p$ converges as a subsequence of $A\varphi_n$ where φ_n^p is a subsequence of an arbitrary bounded sequence φ_n hence the compactness of the operator A. ■

1.4 Adjoint operator.

Recall that when X is a Banach space, the dual space $X^* := \mathcal{L}(X; \mathbb{C})$, consists of the bounded linear functionals x^* on X it is a Banach space with the norm

$$\| x \|_{X^*} = \inf \{ | x^*(x) | : x \in X, \| x \| = 1 \}$$

When $A : X \to Y$ is densely defned, we can define the adjoint operator $A^* : Y^* \to X^*$ as follows:

The domain $D(A^*)$ consists of the $y \in Y^*$, for which the linear functional

$$x \mapsto y^*(Ax), x \in D(A)$$

is continuous (from X to $\mathbb{C}$), this means that there is a constant c (depending on y^*) such that

$$| y^*(Ax) | \leq c \| x \|_X, \text{for all } x \in D(A)$$

Since $D(A)$ is dense in X, the mapping extends by continuity to X, so there is a uniquely determined $x^* \in X^*$ so that

$$y^*(Ax) = x^*(x), \text{for } x \in D(A)$$

Since x^* is determined from y^*, we can define the operator A^* from Y^* to X^* by:

$$A^* y^* = x^*, \text{for } y^* \in D(A^*)$$

Theorem 1.4.1 [6]
Let $A \in C(X, Y)$, then there is an adjoint operator $A^* : Y^* \to X^*$, moreover, A^* is closed. If A is a bounded operator then A^* is also a bounded operator from Y^* into X^* and moreover, $\| A^* \| = \| A \|$, for nonempty sets $M \subseteq X$ and $N \subseteq X^*$ we define the annihilators

$$M^{\perp} = \{f \in X^* : f(x) = 0 \textit{ for all } x \in M\}$$

$$N^{\perp} = \{x \in X : f(x) = 0 \textit{ for all } x \in N\}$$

Even if M and N are not subspaces, and $M^{\perp}$ and $N^{\perp}$ are closed subspaces of X^* and X respectively, we have $M^{\perp} = X^* \left(resp. N^{\perp} = X\right)$ if and only if $M = \{0\}\,(resp. N = \{0\})$

Proposition 1.4.1
Let $A \in C(X, Y)$ then we have

$$N(A) = R(A^*)^{\perp},\; N(A^*) = R(A)^{\perp},\; \overline{R(A)} = N(A^*)^{\perp} \textit{ and } \overline{R(A^*)} \subset N(A)^{\perp}$$

Recall that if X and Y are Hilbert spaces with scalar product $\langle ., . \rangle$ we can defne the adjoint of $A \in C(X, Y)$ as follows

$$D(A^*) = \{y \in Y : \exists z \in Y \;\forall x \in D(A) : \langle Ax, y \rangle = \langle x, z \rangle\}, A^* y = z$$

Moreover, if A is a bounded, we have

$$\textit{for all } x \in X, y \in Y : \langle Ax, y \rangle = \langle x, A^* y \rangle.$$

Theorem 1.4.2 [6]
Suppose that H is a Hilbert space and A is a densely defined operator from H to itself then A is closable if and only if A^* is densely defined, in this case $\overline{A} = A^{**}$

Definition 1.4.1

Let H is a Hilbert space and A is a densely defined operator from H to itself.

- *If A^* is an extension of A, that is $A \subset A^*$, then A is called a symmetric operator.*
- *If $A = A^*$, then A is called a self-adjoint operator.*
- *If $\overline{A} = A^*$, then A is called essentially self-adjoint.*
- if A is a symmetric operator, then for any $x, y \in D(A)$

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

- If A is a symmetric bounded operator, then A is self-adjoint.
- If A is a symmetric operator, then $D(A) \subset D(A^*)$, So A^* is densely defined. and A is closable and $\overline{A} = A^{**} \subset A^*$, thus if A is essentially self-adjoint, then $\overline{A} = A^{**} = A^*$
- Let A be a self-adjoint operator and B be a symmetric operator such that $A \subset B$, then $A = B$, this is because $A \subset B \subset B^* \subset A^* = A$. We see that a symmetric operator can have diferent self-adjoint extension.

1.5 Spectrum

Definition 1.5.1

Let A be a closable linear operator in a Banach space X. The resolvent set and the spectrum of A are, respectively, defined as

$$\rho(A) = \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is injective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\right\}, \sigma(A) = \mathbb{C} \backslash \rho(A)$$

and the point spectrum, continuous, and the residual spectrum are defined as

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is not injective}\},$$

$$\sigma_c(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} = X, R(\lambda - A) \neq X\right\}$$

$$\sigma_r(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} \neq X, \right\}$$

Remark 1.5.1

Note that, if $\rho(A) \neq \emptyset$ then A is closed. In fact, if $\lambda \in \rho(A)$ then $(\lambda - A)^{-1}$is closed, which is also valid for $\lambda - A$ then, according to the closed graph theorem we deduce that

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ sucht hat } \lambda - A \text{ is bijective } \}$$

and hence

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Proposition 1.5.1

Let $(A, D(A))$ be a closed, densely defined, and linear operator with a nonempty resolvent set $.\rho(A)$, for each $\lambda_0 \in \rho(A)$ we have

$$\sigma\left((\lambda_0 - A)^{-1}\right) = (\lambda_0 - \sigma(A))^{-1}$$

1.6 Essential Fredholm spectra

1.6.1 Fredholm Operators

Definition 1.6.1

A closed, densely defined linear operator A on a Banach space X is said to be a Fredholm operator if it satisfies

(i) $\alpha(A) = \dim(N(A)) < \infty$,

(ii) $R(A)$ *is closed*

(iii) $\beta(A) = co\dim(R(A)) < \infty$

Remark 1.6.1

The number $i(A) = \alpha(A) - \beta(A)$ is called the index of A densely defined, closed operators on X satisfying (i) and (ii) are said to be upper semi-Fredholm operators, and those satisfying (ii) and (iii) are called lower semi-Fredholm operators.

- *Let $\Phi(X)$ denote the set of Fredholm operators, $\Phi_+(X)$ denote the set of upper semi-Fredholm operators, and $\Phi_-(X)$ denote the set of lower semi-Fredholm operators.*

Then we have

$\Phi_{+}(X) := \{A \in \mathcal{C}(X)$ such that $\alpha(A) < \infty$ and $R(A)$ is closed in $X\}$,

$\Phi_{-}(X) := \{A \in \mathcal{C}(X)$ such that $\beta(A) < \infty$ and $R(A)$ is closed in $X\}$,

$\Phi(X) := \Phi_{+}(X) \cap \Phi_{-}(X)$,

and $\Phi_{\overset{-}{+}}(X) := \Phi_{+}(X) \cup \Phi_{-}(X)$ the set of semi-Fredholm operators from X into itself

Proposition 1.6.1

If $A \in L(X)$ we consider the quantities $r(A) = \lim\limits_{n\to+\infty} \alpha(A^n)$ and $r'(A) = \lim\limits_{n\to+\infty} \beta(A^n)$ It is well known that if $A = I - K$ with $K \in \mathcal{K}(X)$ then both $r(A)$ and $r'$are finite, we next denote by $\Phi^*(X)$ the set $\Phi^*(X) := \{A \in \Phi(X)$ such that $r(A) < \infty, r'(A) < \infty\}$

1.6.2 Schechter essential spectrum

Let A be a self-adjoint operator on a Hilbert space. A spectral singularity is said to be in the essential spectrum if it is a limit point of the spectrum of A (where eigenvalues are counted according to their multi- plicities and hence infinite-dimensional eigenvalues are included.), i.e, all points of the spectrum except isolated eigenvalues of finite multiplicity

- When dealing with non-self-adjoint closed, densely defined linear operators on Banach spaces, there are many ways to define the essential spectrum, densely defined linear operators

Definition 1.6.2 *Let X and Y be two Banach spaces and let $A \in \mathcal{C}(X,Y)$. We define the Schechter essential spectrum of the operator A by*

$$\sigma_{ess}(A) = \bigcap_{k\in\mathcal{K}(X)} \sigma(A+K)$$

The following proposition gives a characterization of the Schechter essential spectrum by means of Fredholm operators

Proposition 1.6.2 *Let X and Y be two Banach spaces and let $A \in \mathcal{C}(X,Y)$, then*

$$\lambda \notin \sigma_{ess} \text{ if and only if } \lambda \in \Phi_A^0$$

where $\Phi_A^0 = \{\lambda \in \mathbb{C}$, such that $\lambda - A \in \Phi(X)$ and $i(\lambda - A) = 0\}$

Remark 1.6.2 *Let $\mathbb{C}$ be the set of complex numbers. If $A \in \mathcal{C}(X)$ we define the sets*

$$\Phi_A := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi(X)\}$$

$$\Phi_{+A} := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_+(X)\}$$

$$\Phi_{-A} := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_-(X)\}$$

As is well known, Φ_A, Φ_{+A} and Φ_{-A} are open subsets of the complex plane

Proposition 1.6.3

Let $A \in \mathcal{C}(X)$ and define the sets

$$\sigma_{e1}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\} := \mathbb{C} \backslash \Phi_{+A}$$

$$\sigma_{e2}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X)\} := \mathbb{C} \backslash \Phi_{-A}$$

$$\sigma_{e3}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X) \cup \Phi_+(X)\} := \mathbb{C} \backslash \{\Phi_{-A} \cup \Phi_{+A}\}$$

$$\sigma_{e4}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi(X)\} := \mathbb{C} \backslash \Phi$$

$$\sigma_{e5}(A) := \mathbb{C} \backslash \rho_5(A)$$

$$\sigma_{e6}(A) := \mathbb{C} \backslash \rho_6(A)$$

σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra , σ_{e3} is the Kato essential spectrum, σ_{e4} is the Wolf essential spectrum , σ_{e5} is the Schechter essential spectrum , and σ_{e6} denotes the Browder essential spectrum.

- Note that, in general, we have $\sigma_{e3}(A) \subset \sigma_{e4}(A) \subset \sigma_{e5}(A) \subset \sigma_{e6}(A)$, but if A is a self-adjoint operator on a Hilbert space, then $\sigma_{e1}(A) = \sigma_{e2}(A) = \sigma_{e3}(A) = \sigma_{e4}(A) = \sigma_{e5}(A) = \sigma_{e6}(A)$,

Chapitre 2

classes of perturbations operators

In this chapter we concentrate our attention on some perturbation classes of operators which occur in Fredholm theory, in this theory we find two fundamentally different classes of operators, such as the class of Fredholm operators, the classes of upper and lower semi-Fredholm operators, and ideals of operators, such as the classes of finite-dimensional, compact operators.

2.1 weakly compact

Definition 2.1.1
An operator $A \in L(X)$ *is said to be weakly compact, if* $A(B)$ *is relatively weakly compact in* Y*, for every bounded subset* $B \subset X$

Definition 2.1.2
Let X *and* Y *be Banach spaces, and let* S *and* A *be linear operators from* X *into* Y*.* S *is called relatively weakly compact with respect to A.or (A-weakly compact)* $D(A) \subset D(S)$ *and, for every bounded sequence* $x_n \in D(A)$ *such that .*$(Ax_n)_n$ *is bounded, the sequence* $(Sx_n)_n$ *contains a weakly convergent subsequence.*

Definition 2.1.3
Let $A \in L(X)$*,* A *is called a power-compact operator, if there exists* $m \in \mathbb{N}^*$ *satisfying* $A^m \in \mathcal{K}(X)$

Remark 2.1.1

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$ if $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X,X) = \mathcal{W}(X)$ is a closed two-sided ideal of $L(X)$ containing $\mathcal{K}(X)$.

2.2 strictly singular and strictly cosingular operator

Definition 2.2.1

Let X and Y be two Banach spaces. An operator $A \in L(X)$ is called strictly singular if, for every infinite-dimensional subspace M the restriction of A to M is not a homeomorphism, Let $\mathrm{S}(X,Y)$ denote the set of strictly singular operators from X into Y

- *The concept of strictly singular operators as a generalization of the notion of compact operators.*

Remark 2.2.1

The set $\mathcal{S}(X,Y)$ is a closed subspace of $L(X)$ if $X = Y$ then $\mathcal{S}(X) := \mathcal{S}(X,X)$ is a closed two-sided ideal of $L(X)$ containing $\mathcal{K}(X)$ if X is a Hilbert space then $\mathcal{K}(X) = \mathcal{S}(X)$ and the class of weakly compact operators on $\mathcal{L}_1$-spaces (respectively $\mathcal{C}(K)$-spaces with K a compact Hausdorff space) is nothing else but the family of strictly singular operators on $\mathcal{L}_1$-spaces (respectively $\mathcal{C}(K)$-spaces)

Remark 2.2.2

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N^X the quotient map $X \to X/N$ the codimension of N, $co\dim(N)$ is defined to be the dimension of the vector space X/N

Definition 2.2.2

Let X and Y be two Banach spaces and $S \in L(X)$. S is said to be strictly cosingular from X into Y if there exists no closed subspace N of Y with $co\dim = \infty$ such that $\pi_N^Y S : X \to Y/N$ is surjective.

Let $CS(X,Y)$ denote the set of strictly cosingular operators from X into Y

Definition 2.2.3

A normed linear space X is called subprojective if for every closed infinite-dimensional subspace W of X there exists a closed infinite-dimensional subspace W_1 of W and a projection from W into W_1

Proposition 2.2.1 [2]

The L_P spaces $2 \leq p < \infty$ are subprojective and the spaces $L_p\,(1 < p < 2)$ are not subprojective

Proposition 2.2.2

Suppose X is reflexive and X' is subprojective, then if $S \in L(X)$ is strictly singular, So is S'

Proposition 2.2.3

Let X, Y and Z be Banach spaces and let A be a closed, densely defined linear operator from X to Y . Suppose $B \in L(Y, Z)$ with $\alpha\,(B) < \infty$ and $BA \in \Phi\,(X, Z)$ then $A \in \Phi\,(X, Y)$

Proposition 2.2.4

Let X, Y and Z be Banach spaces and let A be an operator in $\Phi\,(X, Y)$ Suppose B is a closed, densely defined linear operator from Y to Z such that $BA \in \Phi\,(X, Z)$ Then $B \in \Phi\,(Y, Z)$

- *Let X_p denote the space $L_p\,(\Omega, d\mu)$ $(1 \leq p \leq \infty)$, where (Ω, Σ, μ) stands for a positive measure space*

Proposition 2.2.5 [2]

Let $A \in \mathcal{C}\,(X_p)$ and let $S \in L(X)$then we have:

(i) If $A \in \Phi_+\,(X_p)$ then $A + S \in \Phi_+\,(X_p)$ for each $1 \leq p \leq \infty$

(ii) if $p \in [1, 2]$ and $A \in \Phi_-\,(X_p)$ then $A + S \in \Phi_-\,(X_p)$

(iii) if $p \in [1, 2]$ and $A \in \Phi_+\,(X_p) \cup \Phi_-\,(X_p)$ then $A + S \in \Phi_+\,(X_p) \cup \Phi_-\,(X_p)$.

(iv) if $A \in \Phi\,(X_p)$ then $A + S \in \Phi\,(X_p)$ and $i\,(A + S) = i\,(A)$ for each $1 \leq p \leq \infty$.

Proof.

(i) This consequence of [1]

(ii) If $p \in [1, 2]$, then X_p is reflexive. Hence, by (Proposition 2.2.1), X'_p is subprojective. Therefore (Proposition 2.2.2) shows that $S' \in$L(X) , it suffices to apply (i) to $A' + S'$

(iii) This immediately follows from (i) and (ii)

(iv) If $A \in \Phi(X_p)$, there exist $A_1 \in$L(X) and $k \in \mathcal{K}(X_p)$ such that

$$AA_1 = I - K \text{ on } X_p \tag{2.2.1}$$

This implies

$$(A+S)A_1 = I - K + SA_1 = I - F \tag{2.2.2}$$

Since $A + S \in \mathcal{C}(X_p)$, we can make $D(A+S) = D(A)$ into a Banach space by introducing the norm

$$|||\, x \,||| = \|\, x \,\| + \|\, Ax \,\|$$

Let Z_p denote this Banach space and $\hat{S}$ denote the restriction of S to $D(A)$ Clearly $A + \hat{S}$ and $\hat{S}$ are bounded operators from Z_p into X_p Clearly Eq (2.2.1) shows that AA_1 is a Fredholm operator satisfying $i(AA_1) = 0$.

Moreover, applying (Proposition 2.2.3) to AA_1 we get $A_1 \in \Phi(X_p, Z_p)$ and

$$i(A) = -i(A_1) \tag{2.2.3}$$

On the other hand, since $F \in P(X_p)$, we see that $F^2 \in K(X_p)$ therefore Eq.(2.2.2) imply that

$$(A+S)A_1 \in \Phi(X_p) \text{ and } i[(A+S)A_1] = 0$$

Now using the fact that $A_1 \in \Phi(X_p, Z_p)$ and (Proposition 2.2.4), we obtain

$$A + \hat{S} \in \Phi(Z_p, X_p), A + S \in \Phi(X_p)$$

and

$$i(A+S) = -i(A_1) \tag{2.2.4}$$

Finally, Eqs (2.2.3) and (2.2.4) imply $i(A+S) = i(A)$, which achieves the proof. Q.E.D.

■

2.3 Fredholm perturbations

Definition 2.3.1

Let X and Y be two Banach spaces and let $F \in$L(X). F is called a Fredholm perturbation if $U + F \in \Phi^b(X,Y)$ whenever $U \in \Phi^b(X,Y)$

- The set of Fredholm perturbations is denoted by $\mathcal{F}(X,Y)$ this class of operators is shown that $\mathcal{F}^b(X,Y)$ is closed subset of $\mathcal{I}(X,Y)$ and if $X = Y$ then $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is closed two-sided ideal of $\mathrm{L}(X)$

Proposition 2.3.1

Let X, Y and Z be three Banach spaces. If at least one of the sets $\Phi^b(X,Y)$ and $\Phi^b(Y,Z)$ is not empty, then

(i) $F \in \mathcal{F}^b(X,Y), A \in L(Y,Z)$ imply $AF \in \mathcal{F}^b(X,Z)$

(ii) $F \in \mathcal{F}^b(Y,Z), A \in L(X)$ imply $FA \in \mathcal{F}^b(X,Z)$

Definition 2.3.2

Let X be a Banach space and $R \in L(X), R$ is said to be a Riesz operator if $\Phi_R = \mathbb{C}\backslash\{0\}$

Remark 2.3.1

(a) The family of Riesz operators is not an ideal of $L(X)$

(b) it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $L(X)$ contained in the family of Riesz operators.

Remark 2.3.2

Let X and Y be two Banach spaces, If in (Definition 2.3.1) we replace $\Phi^b(X,Y)$ by $\Phi(X,X)$ we obtain the set $\mathcal{F}(X,Y)$, let X be an arbitrary Banach space.

$$\mathcal{F}(X) = \mathcal{F}^b(X)$$

Lemma 2.3.1

An immediate consequence of this result is that $\mathcal{F}(X)$ is a closed two-sided ideal of $L(X)$

Definition 2.3.3

Let X and Y be two Banach spaces and let $F \in L(X)$, F is called a upper (respectively lower) Fredholm perturbation if $U+F \in \Phi^b_+(X,Y)$ (respectively $\Phi^b_-(X,Y)$ whenever $U \in \Phi^b_+(X,Y)$ (respectively $\Phi^b_-(X,Y)$).

Remark 2.3.3

The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by

$\mathcal{F}_{+}^{b}(X,Y)$*and* $\mathcal{F}_{-}^{b}(X,Y)$ *respectively. it is shown that* $\mathcal{F}_{+}^{b}(X,Y)$ *and* $\mathcal{F}_{-}^{b}(X,Y)$*are closed subset of* $L(X)$ *and if* $X = Y$,*then* $\mathcal{F}_{+}^{b}$:=$\mathcal{F}_{+}^{b}(X,Y)$ *is a closed two-sided ideal of* $L(X)$ *in general, we have the following inclusions*

$$\begin{aligned}\mathcal{K}(X) &\subset \mathcal{S}(X) \subset \mathcal{F}_{+}^{b} \subset \mathcal{J}(X)\\ \mathcal{K}(X) &\subset C\mathcal{S}(X) \subset \mathcal{F}_{-}^{b} \subset \mathcal{J}(X)\end{aligned}$$

where $\mathcal{F}_{-}^{b}(X) := \mathcal{F}_{-}^{b}(X,X)$ *and* $\mathcal{J}(X)$ *denotes the set*

$$\mathcal{J}(X) = \{F \in \mathcal{L}(X) \text{ such that } I - F \in \Phi(X) \text{ and } i(\lambda - F) = 0\}$$

Lemma 2.3.2 *Let* $A \in \mathcal{C}(X)$ *and let* $\mathcal{I}(X)$ *be any nonzero ideal of* $L(X)$ *satisfying* $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X)$, *where* $\mathcal{F}_0(X)$ *stands for the ideal of finite rank operators, if* $A \in \Phi(X)$ *and if* $J \in \mathcal{I}(X)$ *then* $A + J \in \Phi(X)$ *and* $i(A+J) = i(A)$

Chapitre 3

Invariance of Schechter Essential Spectrum

Let X be a fixed Banach space and let $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of $\mathrm{L}(X)$ satisfying the condition

$$\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X) \tag{3.0.1}$$

3.1 Bounded perturbations

The first result in this section is given on the following theorem

Theorem 3.1.1 [7]
Let $A \in \mathcal{C}(X_p)$ Then

$$\sigma_{ess}(A) = \underset{S \in \mathcal{S}(X)}{\cap} \sigma(A+S)$$

Proof. *see* [3] ■

3.1.1 D-P property

Definition 3.1.1
*A Banach space X is said to have the **Dunford-Pettis** property (for short property **DP**) if for each Banach space Y every weakly compact operator $T : X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y.*

Proposition 3.1.1
*It is well known that any L_1 space has the property **DP**, also if Ω is a compact Hausdorff space then $C(\Omega)$ has the property **DP**, note that the property **DP** is not preserved under conjugation.*

*However, if X is a Banach space whose dual has the property **DP** then X has the property **DP***

Theorem 3.1.2
Let $A \in C(X)$ if X has the Dunford–Pettis property, then

$$\sigma_{ess}(A) = \bigcap_{F \in \mathcal{W}(X)} \sigma(A+F)$$

Theorem 3.1.3 [2]
Let X be a Banach space and let $A \in C(X)$. if X has the Dunford–Pettis property then

$$\sigma_{ess}(A) = \bigcap_{C \in \mathcal{G}_A^F(X)} \sigma(A+C)$$

where $\mathcal{G}_A^F(X) = \left\{K \in \mathcal{L}(X) \text{ such that } (\lambda - A)^{-1} K \in \mathcal{W}(X) \text{ for some } \lambda \in \rho(A)\right\}$

- Let X be a fixed Banach space and let $\mathcal{I}(X)$ an arbitrary nonzero two-sided ideal of L(X) satisfying the condition

$$\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X) \tag{3.1.1}$$

Remark 3.1.1
It should be observed that if $\mathcal{I}(X)$ is a nonzero two-sided ideal of $L(X)$ satisfying the condition $\mathcal{I}(X) \subset \mathcal{F}(X)$ then $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{F}(X)$, where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators.

Definition 3.1.2
If $A \in \mathcal{C}(X)$ we define the sets:

$$\mathcal{G}_A(X) = \left\{K \in \mathcal{L}(X) \text{ such that } (\lambda - A)^{-1} K \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\right\}$$

$$\mathcal{D}_A(X) = \left\{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A)^{-1} \in \mathcal{I}(X) \text{ for some } \lambda \in \rho(A)\right\}$$

$$\mathcal{O}_A(X) = \left\{K \in \mathcal{L}(X) \text{ such that } (\lambda - A - K)^{-1} K \in \mathcal{J}(X) \text{ for all } \lambda \in \rho(A+K)\right\}$$

$$\mathcal{V}_A(X) = \left\{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A - K)^{-1} \in \mathcal{J}(X) \text{ for all } \lambda \in \rho(A+K)\right\}$$

Remark 3.1.2

(i) Observe that, in the definition of the sets $\mathcal{G}_A(X)$ *and* $\mathcal{D}_A(X)$ *if an operator satisfies the required condition for a fixed* $\lambda \in \rho(A)$ *then it satisfies it for every* $\lambda \in \rho(A)$

(ii) We have the following inclusion:

$$\mathcal{K}(X) \subset \mathcal{G}_A(X) \subset \mathcal{O}_A(X), \mathcal{K}(X) \subset \mathcal{D}_A(X) \subset \mathcal{V}_A(X)$$

In fact, let $K \in \mathcal{G}_A(X)$ *(resp.*$K \in \mathcal{D}_A(X)$ *) then* $(\lambda - A)^{-1} K \in \mathcal{I}(X)$ *(resp.* $K(\lambda - A)^{-1} \in \mathcal{I}(X)$*) where* $\mathcal{I}(X)$ *is an arbitrary nonzero two-sided ideal of* $L(X)$ *satisfying the condition* (3.0.1)

Let $\mu \in \rho(A)$, *we have*

$$(\lambda - A - K)^{-1} K = \left[I + (\lambda - A - K)^{-1} (\mu - \lambda + K)\right] (\mu - A)^{-1} K \tag{3.1.2}$$

and

$$K(\lambda - A - K)^{-1} = K(\mu - A)^{-1} \left[I + (\mu - \lambda + K)(\lambda - A - K)^{-1}\right] \tag{3.1.3}$$

Using (3.1.2) (resp. (3.1.3)), and the fact that $\mathcal{I}(A)$ *is a two-sided ideal of* $L(X)$ *we infer that* $(\lambda - A - K)^{-1} K \in \mathcal{I}(X)$*(resp.*$K(\lambda - A - K)^{-1} \in \mathcal{I}(X)$ *) So,* $K \in \mathcal{O}_A(X)$ *(resp. .*$K \in \mathcal{V}_A(X)$

(iii) If we take $A := I$ *and* $K := I$ *then* $(\lambda - A)^{-1} K$ *is not in an ideal* $\mathcal{I}(X)$ *But* $(\lambda - A - K)^{-1} K$ *is in* $\mathcal{J}(X)$ *So, the set* $\mathcal{G}_A(X)$ *(resp.* $\mathcal{D}_A(X)$*) is strictly included in* $\mathcal{O}_A(X)$ *(resp.* $\mathcal{V}_A(X)$*).*

We define the right spectrum of A by

$$\sigma_{rg}(A) = \bigcap_{K \in \mathcal{O}_A(X)} \sigma(A + K)$$

Similarly, we define the left spectrum of A by

$$\sigma_l(A) = \bigcap_{K \in \mathcal{V}_A(X)} \sigma(A + K)$$

Theorem 3.1.4 [2]

Let X be a Banach space and let $A \in \mathcal{C}(X)$, *then*

$$\sigma_{ess}(A) = \sigma_{rg}(A) = \sigma_l(A).$$

Remark 3.1.3

a) Note that the sets $\mathcal{O}_A(X)$ *and* $\mathcal{V}_A(X)$ *may characterize the Schechter essential spectrum. Since* $\mathcal{K}(X) \subset \mathcal{O}_A(X)$ *and* $\mathcal{K}(X) \subset \mathcal{V}_A(X)$ *so* $\mathcal{K}(X)$ *is then the minimal subset of* $L(X)$ *(in the sense of inclusion) for which Theorem 3.1.4 holds true. Hence Theorem 3.1.4 provides an improvement of the definition of* $\sigma_{ess}(.)$ *valid for a somewhat large variety of subsets of* $L(X)$

b) Note that in applications (transport operators, operators arising in dynamic populations, etc), the operator B *is, in general, a bounded perturbation of* $A \in \mathcal{C}(\mathcal{L}_p)$ *by an integral operator on* $\mathcal{L}_p$*-spaces* $p > 1$*. The integral operator* $K := B - A$ *is not compact. For some physical conditions on* K*, the operator* $(\lambda - A)^{-1} K$ *or* $K(\lambda - A)^{-1}$ *are compact on* $\mathcal{L}_p$*-spaces* $p > 1$*, This implies that* $(\lambda - A - k)^{-1} K$ *or* $K(\lambda - A - k)^{-1}$ *are compact on* $\mathcal{L}_p$*-spaces* $p > 1$

So $\mathcal{K}(X) \subset\neq \mathcal{O}_A(X)$ *and* $\mathcal{K}(X) \subset\neq \mathcal{V}_A(X)$

c) For all $K \in \mathcal{O}_A(X), \sigma_{ess}(A+K) = \sigma_{ess}(A)$

d) For all $K \in \mathcal{V}_A(X), \sigma_{ess}(A+K) = \sigma_{ess}(A)$

Proof. **(of Theorem 3.1.4)**

We first claim that $\sigma_{ess}(A) \subset \sigma_r(A)$ (resp.$\sigma_{ess}(A) \subset \sigma_{\mathcal{L}}(A)$). Indeed, if $\lambda \notin \sigma_r(A)$ (resp.$\lambda \notin \sigma_{\mathcal{L}}(A)$) then there exists $K \in \mathcal{O}_A(X)$ (resp.$K \in \mathcal{V}_A(X)$) such that $\lambda \in \rho(A+K)$, hence $\lambda \in \Phi_{(A+K)}$ and $i(\lambda - A - K) = 0$.

Using the equality

$$\lambda - A = (\lambda - A - K)\left(I + (\lambda - A - K)^{-1} K\right)$$

$$(\text{resp.}\lambda - A = \left(I + (\lambda - A - K)^{-1} K\right)(\lambda - A - K))$$

together with Atkinson's theorem one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally,shows that $\lambda \notin \sigma_{ess}(A)$ which proves the claim. On the other hand, since $\mathcal{K}(X) \subset \mathcal{O}_A(X)$ (resp. $\mathcal{K}(X) \subset \mathcal{V}_A(X)$) we infer that $\sigma_r(A) \subset \sigma_{ess}(A)$ (resp.$\sigma_{\mathcal{L}}(A) \subset \sigma_{ess}(A)$) which completes the proof of theorem. ■

Corollary 3.1.1

Let X *be a Banach space,* $A \in \mathcal{C}(X)$ *and let* $\mathcal{T}(X)$ *and* $\mathcal{U}(X)$ *be any subset of* $L(X)$ *(not* *necessarily an ideal) satifying the condition*

$$\mathcal{K}(X) \subset \mathcal{T}(X) \subset \mathcal{O}_A(X), \tag{3.1.4}$$

$$\mathcal{K}(X) \subset \mathcal{U}(X) \subset \mathcal{V}_A(X) \tag{3.1.5}$$

Then

$$\sigma_{ess}(A) = \bigcap_{K\in\mathcal{T}_A(X)} \sigma(A+K) = \bigcap_{K\in\mathcal{U}_A(X)} \sigma(A+K)$$

Remark 3.1.4

a) Note that any subset $\mathcal{T}(X)$ and $\mathcal{U}(X)$ of $L(X)$ (not necessarily an ideal) satisfying the condition (3.1.4) and (3.1.5) may characterize the Schechter essential spectrum.

b) For all $K \in \mathcal{T}_A(X), \sigma_{ess}(A+K) = \sigma_{ess}(A)$.

c)For all $K \in \mathcal{U}_A(X), \sigma_{ess}(A+K) = \sigma_{ess}(A)$

Proof. (**of Corollary 3.1.1**)

Set

$$\mathcal{Z} := \bigcap_{K\in\mathcal{T}_A(X)} \sigma(A+K) \text{ and } \mathcal{Z}' := \bigcap_{K\in\mathcal{U}_A(X)} \sigma(A+K)$$

We already know from (3.1.5) and (3.1.4) that $\mathcal{K}(X) \subset \mathcal{T}_A(X) \subset \mathcal{O}_A(X)$, and $\mathcal{K}(X) \subset \mathcal{U}_A(X) \subset \mathcal{V}_A(X)$ From this we infer that $\sigma_r(A) \subset \mathcal{Z} \subset \sigma_{ess}(A)$ and $\sigma_{\mathcal{L}}(A) \subset \mathcal{Z}' \subset \sigma_{ess}(A)$.

■

3.2 Extention to Unbounded Perturbations

Let X be a fixed Banach space.Unless otherwise stated in all that follows $\mathcal{I}(X)$ will denote an arbitrary nonzero two-sided ideal of $L(X)$ satisfying the condition

$$\mathcal{K}(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X) \tag{3.2.1}$$

Definition 3.2.1

Let X be a Banach space, $A \in C(X)$ and let F be an A-defined linear operator on X. We say that F is an A-Fredholm perturbation if $\hat{F} \in \mathcal{F}^b(X_A, X)$, F is called a upper (respectively lower) $A-$semi-Fredholm perturbation if $\hat{F} \in \mathcal{F}_+^b(X_A, X)$ (respectively $\hat{F} \in \mathcal{F}_-^b(X_A, X)$).

- *The sets of A−Fredholm, upper A−semi-Fredholm and lower A−semi-Fredholm perturbations are denoted by* $AF(X)$, $AF_+(X)$ *and* $AF_-(X)$ *, respectively.*

Definition 3.2.2

Let X *be a Banach space,* $A \in C(X)$ *and let* J *be an A-defined linear operator on* X*. We say that* J *is A-compact (respectively A-weakly compact, A-strictly singular, A-strictly cosingular) if* $\hat{J} \in \mathcal{K}(X_A, X)$ *(respectively* $\hat{J} \in \mathcal{W}(X_A, X)$, $\hat{J} \in \mathcal{S}(X_A, X)$, $\hat{J} \in C\mathcal{S}(X_A, X)$*).*

Remark 3.2.1

Let $A\mathcal{K}(X)$, $A\mathcal{W}(X)$, $A\mathcal{S}(X)$ *,and* $AC\mathcal{S}(X)$ *denote, respectively, the sets of A−compact, A−weakly compact, A−strictly singular and A−strictly cosingular on* X

Remark 3.2.2

If J *is A−defined and compact (respectively weakly compact, strictly singular, strictly cosingular) then* J *is A-compact (respectively A-weakly compact, A-strictly singular, A-strictly cosingular).*

Definition 3.2.3

Let X *be a Banach space,* $A \in C(X), \rho(A) \neq \emptyset$ *and let* F *be an A−defined linear operator on* X

We say that F *is an A−resolvent Fredholm perturbation if*

$$\left(\lambda - \hat{A}\right)^{-1} \hat{F} \in \mathcal{F}^b(X_A) \ \ , for\ some\ \lambda \in \rho(A).$$

Let $A\mathcal{RF}(X)$ *designate the set of A−resolvent Fredholm perturbation.*

Remark 3.2.3

Let $A \in \mathcal{C}(X)$ *and let* J *be an A-bounded operator on* X*, Let* $\lambda \in \rho(A)$ *Since* J *is A-bounded ,*$J(\lambda - A)^{-1}$*is a closed linear operator defined on all of* X *and therefore bounded, by the closed graph theorem, We define the set* $\sigma_a(A)$ *by*

$$\sigma_a(A) = \bigcap_{K \in \mathcal{N}_A(X)} \sigma(A + K)$$

where

$$\mathcal{N}_A(X) = \left\{ J \in \mathcal{C}(X) \ \text{such that } J \text{ is } A-\text{bounded and } J(\lambda - A - J)^{-1} \in \mathcal{J}(X) \ \text{for all } \lambda \in \rho(A+J) \right\}$$

Remark 3.2.4

Note that any arbitrary nonzero two-sided ideal $\mathcal{I}(X)$ *of* $L(X)$ *realizes the condition* $\mathcal{I}(A) \subset \mathcal{N}_A(X)$.

Theorem 3.2.1 [2]

Let X *be a Banach space and let* $A \in \mathcal{C}(X)$. *Then*

$$\sigma_{ess}(A) = \sigma_a(A)$$

Remark 3.2.5

a) Note that the set $\mathcal{N}_A(X)$ *may characterize the Schechter essential spectrum. Since* $\mathcal{K}(X) \subset \mathcal{N}_A(X)$, $\mathcal{K}(X)$ *is the minimal subset of* $\mathcal{C}(X)$ *(in the sense of inclusion) for which Theorem 3.2.1 holds true. Hence Theorem 3.2.1 provides an improvement of the definition of* $\sigma_{ess}(.)$ *valid for a some what large variety of subsets of* $\mathcal{C}(X)$ *to unbounded linear operators.*

b) For all $K \in \mathcal{N}_A(X)$, $\sigma_{ess}(A+K) = \sigma_{ess}(A)$

Proof. (Proof of Theorem 3.2.1)

By hypothesis (3.2.1) and Remark (3.2.4) we have $\mathcal{K}(X) \subset \mathcal{N}_A(X)$. So, $.\sigma_a(A) \subset \sigma_{ess}(A)$

Conversely, let $\lambda \notin \sigma_a(A)$ then there exists $J \in \mathcal{N}_A(X)$ such that $\lambda \in \rho(A+J)$

hence $\lambda \in \Phi_{(A+J)}$ and $i(\lambda - A - J) = 0$

Since $J \in \mathcal{N}_A(X)$ we infer that $I + J(\lambda - A - J)^{-1}$ is a Fredholm operator and $i(I + J(\lambda - A - J)^{-1}) = 0$

Using the equality $\lambda - A = (I + J(\lambda - A - J)^{-1})(\lambda - A - J)$ together with Atkinson's theorem one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, shows that $\lambda \notin \sigma_{ess}(A)$ which completes the proof of theorem. ■

Corollary 3.2.1

Let X *be a Banach space,* $A \in \mathcal{C}(X)$ *and let* $\mathcal{M}(X)$ *be any subset of* $\mathcal{J}(X)$ *(not necessarily an ideal) satisfying the condition*

$$\mathcal{K}(X) \subset \mathcal{M}(X) \subset \mathcal{J}(X) \tag{3.2.2}$$

Then

$$\sigma_{ess}(A) = \bigcap_{K \in \mathcal{H}_A(X)} \sigma(A+K)$$

where $\mathcal{H}_A(X) = \{J \in \mathcal{C}(X)$ *such that* J *is* $A-$ *bounded and* $J(\lambda - A - J)^{-1} \in \mathcal{M}(X)$ *for all* $\lambda \in \rho$

Remark 3.2.6

a) Note that any subset $\mathcal{M}(X)$ *of* $L(X)$ *(not necessarily an ideal) satisfying the condition (3.2.2) may characterize the Schechter essential spectrum.*

b) For all $K \in \mathcal{H}_A(X), \sigma_{ess}(A+K) = \sigma_{ess}(A)$

Proof. (**Proof of Corollary 3.2.1**).

Set $\mathcal{Z} := \underset{K \in \mathcal{H}_A(X)}{\cap} \sigma(A+K)$ We already know from (3.2.2) that $\mathcal{K}(X) \subset \mathcal{H}_A(X) \subset \mathcal{N}_A(X)$. From this we infer that $\sigma_a(A) \subset \mathcal{Z} \subset \sigma_{ess}$. Now, the result follows from Theorem 3.2.1. ■

Proposition 3.2.1

Let J *be an arbitrary A-bounded operator, hence we can regard* A *and* J *as operators from* X_A *into* X *they will be denoted by* $\hat{A}$ *and* $\hat{J}$ *respectively, these belong to* $L(X)$ *furthermore, we have the obvious relations*

$$\begin{cases} \alpha\left(\hat{A}\right) = \alpha(A), \beta\left(\hat{A}\right) = \beta(A), R\left(\hat{A}\right) = R(A), \\ \alpha\left(\hat{A}+\hat{J}\right) = \alpha(A+J), \\ \beta\left(\hat{A}+\hat{J}\right) = \beta(A+J) \text{ and } R\left(\hat{A}+\hat{J}\right) = R(A+J) \end{cases} \qquad (**)$$

Remark 3.2.7

(i) For all $\lambda \in \rho(A)$, *the operator* $\left(\lambda - \hat{A}\right)^{-1} \in L(X)$ *in fact, let* $x \in X$ *and put* $Y = (\lambda - A)^{-1} x$ *it follows from the estimate*

$$\begin{aligned} \| y \|_A &= \| y \| + \| \hat{A} y \| = \| y \| + \| \lambda y - x \| \\ &= \| \left(\lambda - \hat{A}\right)^{-1} x \| + \| \left(\lambda - \hat{A}\right)^{-1} x - x \| \\ &\leq \left(1 + (1 + |\lambda|) \| \left(\lambda - \hat{A}\right)^{-1} \|\right) \| x \| \end{aligned}$$

that $\left(\lambda - \hat{A}\right)^{-1} \in L(X)$

(ii) If F *is an A-Fredholm perturbation then* F *is an A-resolvent Fredholm perturbation, In fact, if* $F \in AF(X)$, *then* $\hat{F} \in F^b(X_A, X)$ *we have* $\left(\lambda - \hat{A}\right)^{-1} \in L(X)$ *and we have* $\left(\lambda - \hat{A}\right)^{-1} \hat{F} \in F^b(X_A)$ *This proves that* $F \in ARF(X)$.

Chapitre 4

Application to Transport Equations

In this chapter we are concerned with the Schechter essential spectrum of singular transport operators

$$A\Psi(x,v) = -v.\nabla_x \Psi(x,v) - \sigma(v)\Psi(x,v) + \int_{\mathbb{R}^n} k(v,v')\Psi(x,v')\,d\mu(v')$$

with vacuum boundary conditions, i.e, $\Psi\Gamma_- = 0$ with

$$\Gamma_- = \{(x,v) \in \partial D \times \mathbb{R}^n \text{ such that } u \cdot v_x < 0\}$$

where v_x stands for the outer unit normal vector at $x \in \partial D$.

Here $x \in D$ and $v \in \mathbb{R}^n$ where D is an open bounded subset of $\mathbb{R}^n, d\mu(.)$ is a bounded positive Radon measure on $\mathbb{R}^n$. The functions $\sigma(.)$ and $k(.,.)$ represent, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded

We introduce the different notions and notations which we shall need in the sequel Let us first make precise the functional setting of the problem:

Let $X_p := \mathcal{L}_p(D \times \mathbb{R}^n, dxd\mu(v)), 1 \leq p < \infty,$

$$X_p^\sigma := \mathcal{L}_p(D \times \mathbb{R}^n, \sigma(v)\,dxd\mu(v)) \text{ and } \mathcal{L}_p^\sigma(\mathbb{R}^n) := \mathcal{L}_p(\mathbb{R}^n, \sigma(v)\,d\mu(v))$$

We define the partial Sobolev space W_p by

$$W_p = \{\Psi \in X_p \text{ such that } v.\nabla_x \in X_p\}$$

Next we introduce the following subspace of W_p by

$$W_p = \left\{\Psi \in W_p \text{ such that } \Psi_{|\Gamma_-} = 0\right\}$$

Now, we define the streaming operator T by

$$T\Psi\left(x,v\right) = -v.\nabla_x \Psi\left(x,v\right) - \sigma\left(v\right)\Psi\left(x,v\right), \Psi \in D\left(T\right)$$

$$D\left(T\right) = W_p^0 \cap X_p^\sigma$$

The transport operator A can be formulated as follows $A = T + K$, where K is the following collision operator

$$K : \Psi \rightarrow K\Psi\left(v\right) := \int_{\mathbb{R}^n} k\left(v,v'\right)\Psi\left(x,v'\right)d\mu\left(v'\right) \in \mathcal{L}_p\left(\mathbb{R}^n\right)$$

with $\mathcal{L}_p\left(\mathbb{R}^n\right) := \mathcal{L}_p\left(\mathbb{R}^n, d\mu\left(v\right)\right)$ that $K \in \mathrm{L}\big(\mathcal{L}_p^\sigma\left(\mathbb{R}^n\right), \mathcal{L}_p\left(\mathbb{R}^n\right)\big)$ and

$$\parallel K \parallel_{\mathcal{L}\left(\mathcal{L}_p^\sigma(\mathbb{R}^n),\mathcal{L}_p(\mathbb{R}^n)\right)} \leq \parallel \left[\int \left(\frac{k\left(.,v'\right)}{\sigma\left(v'\right)^{1/p}}\right)^q d\mu\left(v'\right)\right]^{1/q} \parallel_{\mathcal{L}_p(\mathbb{R}^n)}$$

Moreover, using the boundedness of D we find that $K \in \mathrm{L}(X)$, with

$$\parallel K \parallel_{\mathcal{L}(X)} \leq \parallel \left[\int \left(\frac{k\left(.,v'\right)}{\sigma\left(v'\right)^{1/p}}\right)^q d\mu\left(v'\right)\right]^{1/q} \parallel_{\mathcal{L}_p(\mathbb{R}^n)}$$

shows that X_p^σ is a subset of X_p and the embedding $X_p^\sigma \hookrightarrow X_p$ is continuous.

Let $\varphi \in X_p$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \sigma_0$. We seek Ψ in $D\left(T\right)$ satisfying

$$\left(\lambda - T\right)\Psi = \varphi \tag{4.0.1}$$

The solution of (4.0.1) reads as follows

$$\Psi\left(x,v\right) = \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s}\varphi\left(x - sv, v\right)ds,$$

where $t^-\left(x,v\right) = \sup\left\{t > 0, x - sv \in D, 0 < s < t\right\}$ So, we have $\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda > -\sigma_0\right\} \subset \rho\left(T\right)$, where $\rho\left(T\right)$ denotes the resolvent set of T.

Since $\sigma\left(.\right)$ is bounded below by σ_0, shows that

$$\sigma\left(T\right) = \left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\sigma_0\right\}$$

Remark 4.0.8 *The functions $\sigma(.)$ and $k(.,.)$ are called, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded. more precisely, we will assume that there exist a closed subset $\mathcal{O} \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that*

$$\sigma(.) \in L^{\infty}_{loc}(\mathbb{R}^n \backslash \mathcal{O}), \ \sigma(v) > \sigma_0 \tag{4.0.2}$$

$$\left[\int_{\mathbb{R}^n} \left(\frac{k(., v')}{\sigma(v')^{1/p}} \right)^q d\mu(v') \right]^{1/q} \in L_P(\mathbb{R}^n) \tag{4.0.3}$$

where q denotes the conjugate exponent of p.

Lemma 4.0.1

Let D be a bounded subset of $\mathbb{R}^n$ and $1 < p < \infty$ If, the hypotheses 4.0.2 and 4.0.3 are satisfied, the measure $d\mu$ satisfies

$$\begin{cases} \text{the hyperplanes have zero } d\mu \text{ measure,ie} \\ \text{for each } e \in S^{n-1}, d\mu\{v \in \mathbb{R}^n, v.e = 0\} \end{cases}$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ and the collision operator $K := \mathcal{L}^{\sigma}_p(\mathbb{R}^n) \to \mathcal{L}_p(\mathbb{R}^n)$ is compact, then for any λ satisfying $\operatorname{Re}\lambda > -\sigma_0$, the operator $K(\lambda - T)^{-1}$ is compact on X_p

Theorem 4.0.2 [2]

Assume that the hypotheses of (Lemma 4.0.1) are satisfied. Then

(i) For all $\lambda \in \rho(T + K)$ we have $K(\lambda - T - K)^{-1}$ is compact on X_p

(ii) $\sigma_{ess}(A) = \sigma_{ess}(T) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\sigma_0\}$

Proof.

(i) Let $\lambda \in \rho(T + K)$ and $\mu \in \rho(T)$, we have

$$K(\lambda - T - K)^{-1} = K(\mu - T)^{-1}\left[I + (\mu - \lambda + K)(\lambda - T - K)^{-1}\right] \tag{4.0.4}$$

Using (Lemma 4.0.1) and (Eq. (4.0.4)) we have $K(\lambda - T - K)^{-1}$ is compact on X_p

(ii) The hypothesis on K together with (Theorem 4.0.2) (i) implies that $K \in \mathcal{N}_T(X_p)$, now the result follows from (see [7, theorem 1.3]) and (see [2, remark 3.3] ■

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