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ADAPTED SOLUTION WITH NEWTON-KANTOROVICH METHOD FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. Find an approximate solution is one of the most important problems in our days, in this paper we look for an approximate solution for Volterra nonlinear integral equation using a combination between Newton-Kantorovich method and adapted trapezoidal method. Then we do a comparaison between the numerical results obtained by this method against ones obtained by another authors.

Keywords and phrases. Nonlinear Volterra integral equation, Newton-Kantorovich method, adapted trapezoidal method

1. Introduction

Integral equations is a very useful mathematical tools in both pure and applied mathematics, appears in various fields of science and numerous applications such that elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc.

An integral equation is defined as an equation in which the unknown function $\varphi(x)$ to be determined appear under the integral sign. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations.

A general form of an integral equation in $\varphi(x)$ is of the form

$$\varphi(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t,\varphi(t))dt$$

where $K(x,y)$ is called the kernel of the integral equation , and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It is to be noted here that both the kernel $K(x,y)$ and the function $f(x)$ in the integral equation are given functions; and λ is a constant parameter.

If the lower limit of integration is constant and the upper one is variable we are in the case of Volterra integral equations which is our subject. So we will study the equations in the form

$$\varphi(x) = f(x) + \lambda \int_{a}^{x} K(x,t,\varphi(t))dt, \quad x \in [a,b] \tag{1}$$

This kind of equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by Dubois Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908.

Many authors try to approximate the solution of the nonlinear Volterra integral equations using quadrature methods, in 2010 Jafar Saberi-Nadjafi and Mahdi Heidari used Newton–Kantorovich method combined with a quadrature method such that Simpson method. In our work we will use as a quadrature method the adaptive trapezoidal method. The advantage of this method is to reduce the error in order to obtain a better approximation.

In this method we apply the same subdivision which is applied in the method of trapezoids on the interval $[x_i, x_{i+1}]$, $i = 0, ..., j-2$. exept in the last subinterval, we introduce the nodes noted by $x_{j-\frac{3}{4}}$, $x_{j-\frac{1}{2}}$ and $x_{j-\frac{1}{4}}$ called intermediate points between x_{j-1} and x_j on the subinterval $[x_{j-1}, x_j]$ as follows.

$$\underset{x_{j-1}}{*} - - - \underset{x_{j-\frac{3}{4}}}{*} - - - \underset{x_{j-\frac{1}{2}}}{*} - - - \underset{x_{j-\frac{1}{4}}}{*} - - - \underset{x_j}{*}$$

The approximation of Volterra integral is given by

$$\begin{aligned}\int_a^x K(x_j, t, \varphi(t))dt = & \sum_{i=1}^{j-2} \int_{x_i}^{x_{i+1}} K(x_j, t, \varphi(t))dt + \int_{x_{j-1}}^{x_{j-\alpha}} K(x_j, t, \varphi(t))dt \\ + & \int_{x_{j-\alpha}}^{x_j} K(x_j, t, \varphi(t))dt,\end{aligned}$$

where $\alpha = \frac{1}{4}$, $\alpha = \frac{1}{2}$ or $\alpha = \frac{3}{4}$; $x \in [a, b]$.

2. Description of the method

First we apply the Newton–Kantorovich method for solving the nonlinear integral equation we get

$$(2) \qquad \begin{cases} \varphi_k(x) = \varphi_{k-1}(x) + y_{k-1}(x) \\ y_{k-1}(x) = r_{k-1}(x) + \int\limits_a^x K'_\varphi(x, t, \varphi_{k-1}(t))y_{k-1}(x)dt \\ r_{k-1}(x) = f(x) - \varphi_{k-1}(x) + \int\limits_a^x K(x, t, \varphi(t))dt. \end{cases}$$

From (2), we have

$$(3) \qquad \begin{cases} y_{k-1}(x) = f(x) - \varphi_{k-1}(x) + \int\limits_a^x K(x, t, \varphi_{k-1}(t))dt \\ + \int\limits_a^x K'_\varphi(x, t, \varphi_{k-1}(t))y_{k-1}(t)dt \ . \end{cases}$$

In this step we use the adaptive trapezoidal method, we approximate the two integrals on the right-hand side using this method we obtain

$$\begin{aligned}
\int_a^x K(x,t,\varphi_{k-1}(t))dt &= \sum_{i=0}^{j-2}\int_{x_i}^{x_{i+1}} K(x_j,t,\varphi_{k-1}(t)) + \int_{x_{j-1}}^{x_{j-\frac{1}{2}}} K(x_j,t,\varphi_{k-1}(t) \\
&\quad + \int_{x_{j-\frac{1}{2}}}^{x_j} K(x_j,t,\varphi_{k-1}(t) \\
&= \sum_{i=0}^{j-2}\frac{1}{2}(K(x_j,t_{i+1},\varphi_{k-1}(t_{i+1})) + K(x_j,t_i,\varphi_{k-1}(t_i)))h \\
&\quad + \frac{1}{2}(K(x_j,t_{j-\frac{1}{2}},\varphi_{k-1}(t_{j-\frac{1}{2}})) + K(x_j,t_{j-1},\varphi_{k-1}(t_{j-1})))\frac{h}{2} \\
&\quad + \frac{1}{2}(K(x_j,t_j,\varphi_{k-1}(t_j)) + K(x_j,t_{j-\frac{1}{2}},\varphi_{k-1}(t_{j-\frac{1}{2}})))\frac{h}{2} \\
&= \sum_{i=0}^{j-2}\frac{h}{2}K_{j,i+1,i+1} + \sum_{i=0}^{j-2}\frac{h}{2}K_{j,i,i} \\
&\quad + \frac{h}{4}(K_{j,j-1,j-1} + 2K_{j,j-\frac{1}{2},j-\frac{1}{2}} + K_{j,j,j})
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^x K_\varphi^{'}(x,t,\varphi_{k-1}(t))y_{k-1}(t)dt = \sum_{i=0}^{j-2}\int_{x_i}^{x_{i+1}} K_\varphi^{'}(x_j,t,\varphi_{k-1}(t))y_{k-1}(t)dt \\
&+ \int_{x_{j-1}}^{x_{j-\frac{1}{2}}} K_\varphi^{'}(x_j,t,\varphi_{k-1}(t)y_{k-1}(t)dt + \int_{x_{j-\frac{1}{2}}}^{x_j} K_\varphi^{'}(x_j,t,\varphi_{k-1}(t)y_{k-1}(t)dt \\
= &\sum_{i=0}^{j-2}\frac{1}{2}(K_\varphi^{'}(x_j,t_{i+1},\varphi_{k-1}(t_{i+1}))y_{k-1}(t_{i+1}) + K_\varphi^{'}(x_j,t_i,\varphi_{k-1}(t_i))y_{k-1}(t_i))h \\
&+ \frac{1}{2}(K_\varphi^{'}(x_j,t_{j-\frac{1}{2}},\varphi_{k-1}(t_{j-\frac{1}{2}}))y_{k-1}(t_{j-\frac{1}{2}}) + K_\varphi^{'}(x_j,t_{j-1},\varphi_{k-1}(t_{j-1}))y_{k-1}(t_{j-1}))\frac{h}{2} \\
&+ \frac{1}{2}(K_\varphi^{'}(x_j,t_j,\varphi_{k-1}(t_j))y_{k-1}(t_j) + K_\varphi^{'}(x_j,t_{j-\frac{1}{2}},\varphi_{k-1}(t_{j-\frac{1}{2}}))y_{k-1}(t_{j-\frac{1}{2}}))\frac{h}{2} \\
= &\sum_{i=0}^{j-2}\frac{h}{2}K_{\varphi j,i+1,i+1}^{'}y_{k-1,i+1} + \sum_{i=0}^{j-2}\frac{h}{2}K_{\varphi j,i,i}^{'}y_{k-1,i} \\
&+ \frac{h}{4}(K_{\varphi j,j-1,j-1}^{'}y_{k-1,j-1} + 2K_{\varphi j,j-\frac{1}{2},j-\frac{1}{2}}^{'}y_{k-1,j-\frac{1}{2}} + K_{\varphi j,j,j}^{'}y_{k-1,j})
\end{aligned}$$

So (3) will be

$$y_{k-1}(x) = f(x) - \varphi_{k-1}(x) + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i+1,i+1} + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i,i}$$
$$+\frac{h}{4}(K_{j,j-1,j-1} + 2K_{j,j-\frac{1}{2},j-\frac{1}{2}} + K_{j,j,j})$$
$$+\sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i+1,i+1} y_{k-1,i+1} + \sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i,i} y_{k-1,i}$$
$$+\frac{h}{4}(K'_{\varphi j,j-1,j-1} y_{k-1,j-1} + 2K'_{\varphi j,j-\frac{1}{2},j-\frac{1}{2}} y_{k-1,j-\frac{1}{2}} + K'_{\varphi j,j,j} y_{k-1,j})$$

It remains to replace $y_{k-1}(x)$ by $\varphi_k(x) - \varphi_{k-1}(x)$ and x by x_j for $j = 0, 1, 2, ..., n$, we obtain

$$\begin{aligned}
\varphi_{kj} \quad = \quad & f_j + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i+1,i+1} + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i,i} \\
& +\frac{h}{4}(K_{j,j-1,j-1} + 2K_{j,j-\frac{1}{2},j-\frac{1}{2}} + K_{j,j,j}) \\
& +\sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i+1,i+1}(\varphi_{k,i+1} - \varphi_{k-1,i+1}) + \sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i,i}(\varphi_{k,i} - \varphi_{k-1,i}) \\
& +\frac{h}{4}(K'_{\varphi j,j-1,j-1}(\varphi_{k,j-1} - \varphi_{k-1,j-1}) \\
& +2K'_{\varphi j,j-\frac{1}{2},j-\frac{1}{2}}(\varphi_{k,j-\frac{1}{2}} - \varphi_{k-1,j-\frac{1}{2}}) + K'_{\varphi j,j,j}(\varphi_{k,j} - \varphi_{k-1,j})),
\end{aligned}$$

after some simplification we conclude the last expression

$$(1 - \frac{h}{4} K'_{\varphi j,j,j})\varphi_{k,j} = f_j + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i+1,i+1} + \sum_{i=0}^{j-2} \frac{h}{2} K_{j,i,i}$$
$$+\frac{h}{4}(K_{j,j-1,j-1} + 2K_{j,j-\frac{1}{2},j-\frac{1}{2}} + K_{j,j,j})$$
$$+\sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i+1,i+1}(\varphi_{k,i+1} - \varphi_{k-1,i+1}) + \sum_{i=0}^{j-2} \frac{h}{2} K'_{\varphi j,i,i}(\varphi_{k,i} - \varphi_{k-1,i})$$
$$+\frac{h}{4}(K'_{\varphi j,j-1,j-1}(\varphi_{k,j-1} - \varphi_{k-1,j-1})$$
$$+2K'_{\varphi j,j-\frac{1}{2},j-\frac{1}{2}}(\varphi_{k,j-\frac{1}{2}} - \varphi_{k-1,j-\frac{1}{2}}) - K'_{\varphi j,j,j}\varphi_{k-1,j})),$$

so

$$\varphi_{k,j} = \frac{1}{(1-\frac{h}{4}K'_{\varphi j,j,j})}[f_j + \sum_{i=0}^{j-2}\frac{h}{2}K_{j,i+1,i+1} + \sum_{i=0}^{j-2}\frac{h}{2}K_{j,i,i}$$
$$+\frac{h}{4}(K_{j,j-1,j-1} + 2K_{j,j-\frac{1}{2},j-\frac{1}{2}} + K_{j,j,j})$$
$$+\sum_{i=0}^{j-2}\frac{h}{2}K'_{\varphi j,i+1,i+1}(\varphi_{k,i+1} - \varphi_{k-1,i+1}) + \sum_{i=0}^{j-2}\frac{h}{2}K'_{\varphi j,i,i}(\varphi_{k,i} - \varphi_{k-1,i})$$
$$+\frac{h}{4}(K'_{\varphi j,j-1,j-1}(\varphi_{k,j-1} - \varphi_{k-1,j-1})$$
$$+2K'_{\varphi j,j-\frac{1}{2},j-\frac{1}{2}}(\varphi_{k,j-\frac{1}{2}} - \varphi_{k-1,j-\frac{1}{2}}) - K'_{\varphi j,j,j}\varphi_{k-1,j}))],$$

by considering an initial solution $\varphi_1(x)$ and constructing the $\varphi(i,1)$ we can solve this equation and find an approximate solution. But before this we should determine the term $\varphi_k(x(j-1/2))$ by substituting $x(j)$ by $x(j-1/2)$ in the last expression, and we suggest to take the average of $\varphi_k(x(j-1))$ and $\varphi_k(x(j))$ in order to reduce the error.

3. Numerical Results

In this section, implementation of the methods mentioned in this paper will be done. This implementation consists of computer programmes written in MATLAB, which approximates solutions to some example nonlinear Volterra integral equations of the second kind.

Example 1. Consider the following nonlinear Volterra integral equation

$$\varphi(x) - \int_0^x \varphi^2(t)dt = \sin x + \frac{1}{4}\sin 2x - \frac{1}{2}x, \qquad 0 \le x,t \le 1,$$

such that the initial value of the approximate solution is chosen to be the function $f(x)$, where the exact solution is given by

$$\varphi(x) = \sin x.$$

Example 2. Consider the following nonlinear Volterra integral equation

$$\varphi(x) - \int_0^x te^{-x}\varphi^2(t)dt = x^2 - \frac{1}{6}x^6e^{-x}, \qquad 0 \le x,t \le 1,$$

where the exact solution is given by

$$\varphi(x) = x^2.$$

Example 3. Consider the following linear Volterra integral equation

$$\varphi(x) - \int_0^x e^{-x-t}\varphi(t)^2dt = \sqrt{x} + (x+1)e^{-2x} - e^{-x}, \qquad 0 \le x,t \le 1,$$

where the exact solution is given by

$$\varphi(x) = \sqrt{x}.$$

4. Table and Figure caption position

Table 1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error is compared with the ones treated in [7].

x	Exact φ	App $\widetilde{\varphi}$	Error	Error [7]
0	0	0	0	0
0.1	9.9833e-02	9.9832e-02	4.4657e-07	3.3327e-04
0.2	1.9867e-01	1.9866e-01	4.2686e-06	1.5351e-03
0.3	2.9552e-01	2.9550e-01	1.4978e-05	6.6506e-03
0.4	3.8942e-01	3.8938e-01	3.6372e-05	1.3901e-02
0.5	4.7943e-01	4.7935e-01	7.2687e-05	2.4222e-02
0.6	5.6464e-01	5.6451e-01	1.2879e-04	4.6982e-02
0.7	6.4422e-01	6.4400e-01	2.1039e-04	6.8954e-02
0.8	7.1736e-01	7.1703e-01	3.2430e-04	9.5916e–02
0.9	7.8333e-01	7.8284e-01	4.7870e-04	1.4380e–01
1	8.4147e-01	8.4078e-01	6.8345e-04	1.8539e–01

Table 2. We present the exact and the approximate solutions of the equation in the example 2 in some arbitrary points.

x	Exact s φ	App $\widetilde{\varphi}$	Error
0	0	0	0
0.1	1.0000e-02	1.0000e-02	7.5406e-08
0.2	4.0000e-02	4.0001e-02	1.0921e-06
0.3	9.0000e-02	9.0005e-02	5.0094e-06
0.4	1.6000e-01	1.6001e-01	1.4369e-05
0.5	2.5000e-01	2.5003e-01	3.1925e-05
0.6	3.6000e-01	3.6006e-01	6.0453e-05
0.7	4.9000e-01	4.9010e-01	1.0271e-04
0.8	6.4000e-01	6.4016e-01	1.6153e-04
0.9	8.1000e-01	8.1023e-01	2.3987e-04
1	1.0000e+00	1.0003e+00	3.4105e-04

5. Conclusion

We had present a numerical method for solving nonlinear Volterra integral equations, based on Newton-Kontorovich quadrature methods where we had choose as a quadrature method the adaptive trapezoidal method and it was compared with Simpson method which was used by [7], as we had show that the choose of the odd term $\varphi_k(x(j-1/2))$ will effect on the approximate solution.

TABLE 3. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points.

x	Exact φ	App $\widetilde{\varphi}$	Error (M2)
0	0	0	0
0.1	3.1623e-01	3.1476e-01	1.4603e-03
0.2	4.4721e-01	4.4560e-01	1.6051e-03
0.3	5.4772e-01	5.4608e-01	1.6424e-03
0.4	6.3246e-01	6.3082e-01	1.6338e-03
0.5	7.0711e-01	7.0550e-01	1.5978e-03
0.6	7.7460e-01	7.7305e-01	1.5434e-03
0.7	8.3666e-01	8.3518e-01	1.4761e-03
0.8	8.9443e-01	8.9302e-01	1.4003e-03
0.9	9.4868e-01	9.4736e-01	1.3191e-03
1	1.0000e+00	9.9876e-01	1.2352e-03

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