



الجمهورية الجزائرية الديمقراطية الشعبية

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITE DE MOHAMED BOUDIAF-M'SILA

FACULTE DES MATHEMATIQUES ET DE L'INFORMATIQUE

DEPARTEMENT DE MATHEMATIQUES

THÈSE

Présentée pour l'obtention du diplôme de doctorat en sciences

SPECIALITE: MATHEMATIQUES

OPTION: ANALYSE FONCTIONNELLE

Par

Abdelaziz BELAADA

Intitulée

Idéaux d'opérateurs non linéaires et théorèmes de factorisation

"Ideals of non-linear operators and factorization theorems"

Soutenue publiquement le 13 /03/2018, devant le jury:

M. MOUSSAI	Prof.	Université de M'sila	Président
A. TIAIBA	M. C. A.	Université de M'sila	Rapporteur
M. HAFAYED	Prof.	Université de Biskra	Examineur
K. SAADI	M. C. A.	Université de M'sila	Examineur
F. MOKHTARI	M. C. A.	Université d'Alger 1	Examineur
B. GHERBAL	M. C. A.	Université de Biskra	Examineur

إهداء

إلى الوالدين الكريمين،

إلى زوجتي آمال،

إلى أبنائي عبد الرؤوف، ايمان ، قدس و أنس عبد الودود،

إلى كلّ أفراد العائلة،

إلى كلّ من مدّ لي يد العون من قريب أو بعيد.

Acknowledgments

First, I would like to thank my supervisor, professor **Abdelmoumen TIAIBA**, who has entrusted me by supervising me in this work. I would like to thank him for being so gentle and for the advice he gave me during the fulfillment of this thesis.

I am so grateful to professor **Khalil SAADI**, who has continuously helped me and supported me; I thank him too for his modesty and his valuable remarks.

Again, I would like to express my gratitude to professor **Madani MOUSSAI**, the president of the jury and Professors **Fares MOKHTARI**, **Mokhtar HAFAYED** and **Boulakhras GHERBAL**, who have accepted to be part of the jury.

Table of contents

Introduction		5
1 Preliminaries		6
1.1 Basic concepts		6
1.1.1 Bounded operators on Hilbert spaces $\mathcal{B}(H_1; H_2)$		6
1.1.2 The polar decomposition		7
1.2 The ideals of linear mappings		8
1.2.1 The ideal of p -summing linear operators.		10
1.2.2 The ideal of (Cohen) strongly p -summing linear operators.		11
1.2.3 The ideal of (Cohen) p -nuclear linear operators.		12
1.2.4 The ideal of p -integral linear operators.		12
1.2.5 The ideal of p -factorable operators		13
1.3 Some based notions on multilinear mappings		13
2 The ideal of Schatten class operators and linear mappings generated by \mathcal{S}_p		16
2.1 Schatten class operators $\mathcal{S}_p(H)$		16
2.1.1 Hilbert-Schmidt operators $\mathcal{S}_2(H)$		17
2.1.2 The trace class operators $\mathcal{S}_1(H)$		19
2.1.3 Factorization of the trace class operators		20
2.2 Schatten class operators $\mathcal{S}_p(H_1; H_2)$		23
2.3 Properties of the class $\mathcal{S}_2(H_1; H_2)$		25
2.4 Linear mappings generated by \mathcal{S}_p		27

2.4.1	Linear mappings of type $\mathcal{S}_p \circ \mathcal{B}$	28
2.4.2	Relation to ideals of linear mappings	31
2.4.3	Connection with linear mappings of type $\mathcal{B}(\mathcal{S}_p)$	36
2.5	Characterization of the classes of type $\mathcal{S}_2 \circ \mathcal{B}$ and $\mathcal{B}(\mathcal{S}_2)$	38
3	On the composition ideals of Schatten class type mappings	41
3.1	Ideals of multilinear mappings	41
3.1.1	Cohen strongly p -summing multilinear operators	43
3.1.2	Factorization of Hilbert-Schmidt multilinear mappings	43
3.2	Multilinear mappings generated by \mathcal{S}_p	45
3.2.1	Multilinear operators of type $\mathcal{L}(\mathcal{S}_p)$	45
3.2.2	Multilinear operators of type $\mathcal{S}_p \circ \mathcal{L}$	47
3.2.3	Connection with Cohen strongly p -summing multilinear operators . .	49
3.2.4	Connection with Hilbert-Schmidt multilinear mappings	51
3.3	Factorization of Schatten class type mappings	52
4	Composition ideals of polynomials generated by Schatten class	57
4.1	Definitions and auxiliary results	57
4.1.1	Cohen strongly p -summing m -homogeneous polynomials	61
4.1.2	Hilbert-Schmidt polynomials	62
4.2	Polynomials mappings of type $\mathcal{S}_p \circ \mathcal{P}$	63
4.2.1	Relation to Cohen strongly p -summing m -homogeneous polynomials .	65
4.2.2	Connection with Hilbert-Schmidt polynomials	65
4.2.3	On factorization of Schatten class type polynomials	66
5	Characterization of positive p-summing sublinear operators using repre-	
	sentable mappings	68
5.1	Preliminaries	68
5.2	Positive p -summing operators	71
5.2.1	Positive p -summing linear operators	71
5.2.2	Positive p -summing sublinear operators	72
5.3	Properties of the positive p -summing sublinear operators	73

5.4	Some inclusion and coincidence results	76
5.5	On the positive p -summing sublinear operators on the space $L_p(\mu)$	80
5.6	Characterization of positive p -summing sublinear operators using representable mappings	82
	Bibliography	87

Résumé: Les travaux de cette thèse s'inscrivent dans le cadre de la théorie des opérateurs non linéaires. Ce travail se divise en deux parties. Dans la première, nous étudions les idéaux multilinéaires (multi-idéaux) et idéaux polynomiales engendrés par les classes de Schatten \mathcal{S}_p , en présentant quelques théorèmes de coïncidence aux opérateurs multilinéaires Cohen fortement p -sommants et aux polynômes homogènes Cohen fortement p -sommants, puis le théorème d'inclusion aux opérateurs multilinéaires de Hilbert Schmidt et aux polynômes homogènes de Hilbert Schmidt. A la fin de cette partie, nous donnons des résultats de factorisation sémilaires à ceux donnés par Lindenstrauss-pelczynski des opérateurs linéaires de Hilbert-Schmidt en 1968 et J. Diestel, H. Jarchow et A. Tonge en 1995. En deuxième partie, l'idée a été inspirée de l'article de O. Blasco son intitulé "*Positive p -summing operators on L_p -spaces*" en 1987. Par conséquent, on a traité le concept des opérateurs sous-linéaires positivement p -sommants en généralisant certaines propriétés de ces opérateurs comme dans le cas linéaire. Nous avons donné aussi quelques résultats de caractérisation des opérateurs sous-linéaires positivement p -sommants $T : L_{p'}(\mu) \rightarrow Y$ où $\frac{1}{p} + \frac{1}{p'} = 1$, en utilisant la représentation de l'application $u \in \nabla T$ où l'espace Y possède la propriété de Radon-Nikodym et autres résultats sans cette propriété. A la fin de cette partie, nous donnons une condition nécessaire à la propriété de Radon-Nikodym.

Mots-clés: Ideaux multilinéaires engendrés par les classes de Schatten, opérateur sous-linéaire positivement p -sommant, propriété de Radon-Nikodym, théorèmes de factorisation.

Abstract: The work of this thesis is situated within the framework of the non-linear operators theory, and it consists on the development of some theorems of factorization in the multilinear and polynomial mappings. Also some characterizations of positive p -summing sublinear operators using representable mappings. Our work is divided into two main parts. In the first part, we study the composition ideals of multilinear and polynomial mappings generated by Schatten classes. We give some coincidence theorems for Cohen strongly p -summing multilinear and Cohen strongly p -summing polynomial mappings. We also present inclusion theorem for Hilbert Schmidt multilinear and Hilbert Schmidt polynomial mappings. In the end of this part, we give factorization results like that given by Lindenstrauss-Pelczński for Hilbert Schmidt linear operators. The idea of the second part was inspired by O. Blasco's article entitled "*Positive p -summing operators on L_p -spaces*".

(American Mathematical Society, 1987). Therefore, we have treated the concept of the positive p -summing sublinear operators. Let $T : X \longrightarrow Y$ be a positive p -summing sublinear operator, we shall establish analogous results of the linear case studied by O. Blasco. Firstly, if $X = C(\Omega)$, we prove some coincidence theorems and properties. Secondly, if $X = L_{p'}(\mu) \left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$, we use the representation of $u \in \nabla T$ for characterization of positive p -summing sublinear operators. And deduce that T is positive p -summing sublinear operator if, and only if, for all $u \in \nabla T$, u is positive p -summing operator. In the end of this part, we give necessary condition that Y has the Radon-Nikodym property.

Keywords : Banach lattice, Cohen strongly p -summing operators, factorization theorems, homogeneous polynomials, multilinear mappings, positive p -summing sublinear, Schatten class type mappings, the Radon-Nikodym property.

Notations

\mathbb{K}	The field of real or complex numbers.
X^*	The topological dual of X .
B_X	The closed unit ball of X .
$L(X; Y)$	The set of all linear operators.
$\mathcal{B}(X; Y)$	The set of all continuous linear operators.
$\mathcal{S}_p \circ \mathcal{B}(X; H)$	The space of all linear mappings of type $\mathcal{S}_p \circ \mathcal{B}$.
\mathcal{I}	The ideal of all linear operators.
\mathcal{M}	The ideal of all multilinear operators.
$\mathcal{S}_p(H_1; H_2)$	the p -th Schatten class ($1 \leq p < \infty$).
$\mathcal{S}_2(H_1; H_2)$	The space of all Hilbert-Schmidt linear operators.
$\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$	The space of all Hilbert-Schmidt multilinear operators
$\mathcal{P}({}^m X; Y)$	The space of all continuous m -homogeneous polynomials.
$\mathcal{D}_p^m(X_1, \dots, X_m; Y)$	The class of all Cohen strongly p -summing multilinear operators ($1 < p \leq \infty$).
$\mathcal{S}_p \circ \mathcal{P}({}^m X, H)$	The space of all polynomials of type $\mathcal{S}_p \circ \mathcal{P}$.
$\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H)$	The space of all multilinear mappings of type $\mathcal{S}_p \circ \mathcal{L}$.
$\mathcal{P}_{Coh}^p({}^m X; Y)$	The space of all Cohen strongly p -summing m -homogeneous polynomials.
u^*	The adjoint operator of u .
$\mathcal{L}(X_1, \dots, X_m; Y)$	The space of all bounded m -linear operators.
\tilde{T}	The linearization of the operator T .
\tilde{P}	The linearization of the polynomial P .
$L_p(\mu, Y)$	The space of measurable functions on Ω with $\ f\ = \left(\int_{\Omega} \ f(t)\ ^p d\mu \right)^{\frac{1}{p}}$.

i_m	The canonical multilinear $i_m : X_1 \times \dots \times X_m \rightarrow X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$.
δ_m	The canonical polynomial $\delta_m : X \rightarrow \widehat{\otimes}_{\pi,s}^m X$.
$C(\Omega)$	The set of all continuous functions on the compact set Ω .
$\mathcal{K}(H_1; H_2)$	The set of all compact linear operators.
$\mathcal{L}_f(X_1, \dots, X_m; Y)$	The space of all finite rank multilinear operators.
$\mathcal{F}(X; Y)$	The set of all finite rank linear operators.
$\mathcal{L}_{p\text{-fat}}(X_1, \dots, X_m; Y)$	The space of all p -factorable multilinear operators.
$\Pi_p(X; Y)$	The class of p -summing linear operators ($1 \leq p < \infty$).
$\Pi_p^+(X; Y)$	The class of all positive p -summing linear operators ($1 \leq p < \infty$).
$\Pi_{s-p}^+(X; Y)$	The class of all positive p -summing sublinear operators ($1 \leq p \leq \infty$).
$\mathcal{D}_p(X; Y)$	The class of all (Cohen) strongly p -summing linear operators ($1 < p \leq \infty$).
$\mathcal{SL}(X; Y)$	The set of all sublinear operators.
$\mathcal{SB}(X; Y)$	The set of all bounded sublinear operators.
$\Pi_{s-p}(X; Y)$	The class of all p -summing sublinear operators ($1 \leq p < \infty$).
$I_p(X; Y)$	The set of all p -integral linear operators ($1 \leq p < \infty$).
$\mathcal{N}_p(X; Y)$	The set of all p -nuclear linear operators ($1 \leq p < \infty$).
$C_p(X; Y)$	The set of all p -concave sublinear operators ($1 \leq p < \infty$).

List of publications

1. Belaada, A., Saadi, K., & Tiaiba, A. *On the Composition Ideals of Schatten Class Type Mappings*. Journal of Mathematics, 2016.

Introduction

The concept of an absolutely 1-summing linear operator was established first by Grothendieck [24] in the 1950's. Later on, in 1967 the German mathematician A. Pietsch in [37] generalized it for all p strictly positive. In 1968 Lindenstrauss-Pełczyński have given the characterization result of Hilbert-Schmidt linear operators through factorization by \mathcal{L}_∞ -space and by \mathcal{L}_1 -space in [28]. In 1973 J. S. Cohen [18] has introduced a characterization of the conjugates of p^* -summing linear operators. Later, this theory triggered a lot of motivations for study and development by giving characterization and factorization results concerning the ideal of p -summing linear operators (sublinear, multilinear and polynomial case) and their conjugates. In this direction, the multilinear ideals were mainly introduced in 1983 by Pietsch's paper entitled "Ideals of multilinear functions" [39]. In his work, Pietsch has proposed two methods to construct multilinear ideals from a given linear ideal, namely the composition and the factorization methods, some classes of multilinear operators can be interpreted via these methods, namely the class of p -dominated, weakly compact and compact multilinear operators. Since then, many researchers have been interested in this field, and further, the ideals of Pietsch were generalized to homogeneous polynomials.

On the other hand, the notion of positive p -summing linear operators has been introduced and studied in 1987 by O. Blasco [9, 10]. We focus our work on the paper entitled "*Positive p -summing operators on L_p -spaces.*" (American Mathematical Society, 1987). In [9] he has given the following characterization: the positive p -summing operators $u : L_{p'}(\mu) \longrightarrow Y$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ is representable by a function $f \in L_p(\mu, Y)$, $(1 \leq p \leq \infty)$ if and only if, the space Y has the Radon Nikodym property. In addition, O. Blasco has proved for all $1 \leq p \leq \infty$, the coincidence $\Pi_p^+(L_{p'}(\mu); Y) = \Pi_1^+(L_{p'}(\mu); Y)$. D. Popa in [40] and [41] has given a characterization of the positive p -summing operators on the $L_p(\mu)$ and the Köthe

spaces respectively. In 2007 D. Achour, L. Mezrag and A. Tiaiba have given generalization to sublinear operators in [4]. In 2015 Q. Bu and C. A. Labuschagne have presented in [16] a generalization of some results given in [9] on the multilinear case.

The work of this thesis is situated within the framework of the non-linear operators theory, and they consist of the development of some theorems of factorization in the multilinear and polynomial case, also some characterizations of sublinear operators using representable mappings. This work is divided into two main parts. In the first part, we study the main topic which is the composition ideals of multilinear and polynomial mappings generated by Schatten classes. In the second part, we study the positive p -summing sublinear operators and we will establish analogous results of the linear case studied by O. Blasco in [9].

The first part focuses on the development of the multilinear ideals (multi-ideals) and the polynomial ideals generated by the Schatten classes \mathcal{S}_p , these latter were introduced by R. Schatten and J. Von Neumann [1946-1948]. Note that the class \mathcal{S}_2 is that of the linear operators of Hilbert Schmidt. In non-commutative theory, the Schatten classes are considered as a fundamental example of newly introduced non commutative L_p -space for $1 \leq p < \infty$. Braunsch and Junek [14] have studied the multi-ideals generated by this linear ideal using the factorization method of Pietsch. After that, H-A Braunsch in [15] has given a generalization for Polynomials of type $\mathcal{P}(\mathcal{S}_p)$. C. A. Mendes [33] has given some factorization results of these mappings and showed that the class $\mathcal{L}(\mathcal{S}_2)$ coincides with the class of 2-dominated multilinear operators. In this part, we give an other class of m -linear operators of Schatten class type \mathcal{S}_p whose operators T can be written as $T = u \circ A$ with u belongs to \mathcal{S}_p and A is a multilinear operator (this technique of factorization is known as composition ideals, see [13]). (i.e.,

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & H \\ & \searrow A & u \uparrow \\ & & K \end{array}$$

where $u \in \mathcal{S}_p(K; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K)$ such that $T = u \circ A$). In this case, we write $T \in \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m, H)$. Also we give an other class of m -homogeneous polynomials of Schatten class type \mathcal{S}_p whose polynomials P can be written as $P = u \circ Q$ with u belongs to \mathcal{S}_p and Q is an m -homogeneous polynomial (i.e., the polynomials $P \in \mathcal{S}_p \circ \mathcal{P}(^m X; H)$, if its m -linear symmetric \widehat{P} is of type $\mathcal{S}_p \circ \mathcal{L}$). We also prove that the class $\mathcal{S}_p \circ \mathcal{L}$ is compact

and prove some of its properties. We will show that, the class $\mathcal{S}_2 \circ \mathcal{L}$ coincides with \mathcal{D}_p^m , the class of Cohen strongly p -summing multilinear operators which was introduced by Achour and Mezrag [2]. We will use p -factorable multilinear operators introduced by Martin Cerna Maguina in [31] for given factorization results like that given for Hilbert Schmidt linear operators, and C. A. Mendes for multilinear operators of the type $\mathcal{L}(\mathcal{S}_p)$. The same results are given for polynomial case, but by using \mathcal{P}_{Coh}^p the class of Cohen strongly p -summing m -homogeneous polynomials which was introduced by Achour and Saadi [5]. We shall introduce certain factorization results and the coincidence $\mathcal{S}_2 \circ \mathcal{P}({}^m X; H) = \mathcal{P}_{Coh}^p({}^m X; H)$.

In the second part. Let $T : L_{p'}(\mu) \longrightarrow Y, \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be a positive p -summing sublinear operator, we use the representation of $u \in \nabla T$ for a characterization of positive p -summing sublinear operators. And deduce that every T is positive p -summing sublinear operator if, and only if, for all $u \in \nabla T$ is positive p -summing operator. We show in Theorem 5.6.1 a characterization of positive p -summing sublinear operator $T : L_{p'}(\mu) \longrightarrow Y, \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. So, we prove that

$$\|T(f)\| \leq \int_{\Omega} |f(t)| g(t) d\mu(t) \text{ such that, } g(t) = \sup_{u \in \nabla T} g_u(t), g \geq 0 \text{ in } L_p(\mu).$$

Where $g_u(t)$ is the function of the representation of $u \in \nabla T$, where

$$\nabla T = \{\text{linear operators } u : L_{p'}(\mu) \longrightarrow Y \text{ such that } u \leq T\},$$

and we conclude that

$$\Pi_{s-p}^+(L_{p'}(\mu), Y) = \Pi_{s-1}^+(L_{p'}(\mu); Y).$$

As a main result and with the condition that Y has the Radon-Nikodym property in Theorem 5.6.2, we give other characterization using the representation of $u \in \nabla T$ and deduce that

$$\forall u \in \nabla T, u \in \Pi_p^+(L_{p'}(\mu); Y) \Leftrightarrow T \in \Pi_{s-p}^+(L_{p'}(\mu); Y).$$

Finally, we give a necessary condition that Y has the Radon-Nikodym property. as follows

$$T(f) \leq \int_{\Omega} |f(t)| g(t) d\mu(t) \text{ such that, } g(t) = \sup_{u \in \nabla T} |g_u(t)|, g \geq 0 \text{ in } L_p(\mu, Y).$$

The thesis is divided into five chapters.

Chapter 1 is devoted to the ideals of linear mappings and a recall of the basic concepts of the space $B(H_1; H_2)$ of bounded operators on Hilbert spaces with some properties of the polar decomposition, compact operators and finite rank operators. Finally in the preliminaries, we will give a general overview of the space of all (bounded) continuous m -linear operators and some useful properties. We will also give some recent results relating to this class of operators and the projective tensor product, the linearization of the operator and identification.

In Chapter 2, we start by studying the Schatten class \mathcal{S}_p . And then, the class $\mathcal{S}_p \circ \mathcal{B}(X; H)$ of linear operators u which admit a factorization $u = u_2 u_1$, where u_2 is in \mathcal{S}_p and the relation to ideals of linear mappings and some of its properties, as well as relations with different ideals of linear mapping and linear mapping of type $\mathcal{B}(\mathcal{S}_p)$. In the end of this chapter, we will give factorization results as Lindenstrauss-Pelczński in [28] and J. Diestel, H. Jarchow and A. Tonge in [19].

In Chapter 3, we study the composition ideals of multilinear mappings generated by Schatten class \mathcal{S}_p . And then, we introduce a definition of multilinear mappings ideals, we will give an overview of the Pietsch methods of construction which allow us to define a multi-ideal from a linear ideal, by concentrating on the composition method. An in-depth study of the multilinear operators of Hilbert Schmidt will be carried out by addressing the factorization theorems that concern these classes. In the linear case, a Hilbert Schmidt operator is characterized by its factorization by a \mathcal{L}_∞ -space or by \mathcal{L}_1 -space. We will show for example that this characterization is no longer verified in the multilinear case. We study the class $\mathcal{S}_p \circ \mathcal{L}$ of multilinear operators T which admit a factorization $T = u \circ A$ where u is in \mathcal{S}_p , we will show that this is a class of compact multilinear operators. In particular case, we show that the class $\mathcal{S}_2 \circ \mathcal{L}$ coincides with the class of Cohen strongly 2-summing multilinear operators. We will use p -factorable multilinear operators introduced in [31] for given factorization results like that given for Hilbert Schmidt linear operators and multilinear operators of the type $\mathcal{L}(\mathcal{S}_p)$.

We study in Chapter 4, the composition ideals of polynomials generated by Schatten class \mathcal{S}_p . First, we recall some definitions and auxiliary results concerning polynomials,

we introduce then the definition of polynomial of type $\mathcal{S}_p \circ \mathcal{P}$ and their relations to Cohen strongly p -summing m -homogeneous polynomials. At the end of this chapter, we give results on factorization of Schatten class type Polynomials $\mathcal{S}_p \circ \mathcal{P}$ analogous to the class $\mathcal{S}_p \circ \mathcal{L}$.

In the last chapter of this thesis (Chapter 5), we recall some standard notations and definitions concerning Banach lattices spaces and sublinear operators. We also cite results of some theorems that we use in this chapter, and after giving some results of coincidence theorems analogous to the work in [9], we give some properties and coincidence results for a positive p -summing sublinear operators $T : X \longrightarrow Y$ in two cases if $X = C(\Omega)$ and $X = L_{p'}(\mu)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We give the main results about the characterization of a positive p -summing sublinear operators $T : L_{p'}(\mu) \longrightarrow Y$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ by using the linear representation of Blasco without Y has the Radon-Nikodym property and condition the Radon-Nikodym property. Finally, we obtain the corollary $\Pi_{s-p}^+(L_{p'}(\mu), Y) = \Pi_{s-1}^+(L_{p'}(\mu), Y)$ and for all $u \in \nabla T$, $u \in \Pi_p^+(L_1(\mu), Y)$ if and only if, $T \in \Pi_{s-p}^+(L_1(\mu), Y)$.

Chapter 1

Preliminaries

In this chapter, we present some basic concepts of bounded operators on Hilbert spaces and also different ideals of linear mappings. Finally, we give a recall of the basic notions and the classical results concerning multilinear mappings used along this thesis.

1.1 Basic concepts

Let X, Y be Banach spaces, the space of all bounded linear operators from X into Y , which is denoted by $\mathcal{B}(X; Y)$.

We denote by B_X the closed unit ball of X , i.e., $\{x \in X : \|x\| \leq 1\}$. If $Y = \mathbb{K}$, the space $\mathcal{B}(X; \mathbb{K})$ will be called the topological dual space of X , denoted by X^* . We note that if H is a Hilbert space, then $H^* = H$.

1.1.1 Bounded operators on Hilbert spaces $\mathcal{B}(H_1; H_2)$

Consider H_1, H_2 be two Hilbert spaces. We define the space of bounded linear operators from H_1 into H_2 by

$$\mathcal{B}(H_1; H_2) = \{u : H_1 \rightarrow H_2 : u \text{ is linear and bounded}\}.$$

The space $\mathcal{B}(H_1; H_2)$ is a Banach whose norm is

$$\|u\| = \sup_{\|x\|=1} \|u(x)\|.$$

If $H_1 = H_2 = H$, we simply write $\mathcal{B}(H)$. We denote by $u^* : H_2 \rightarrow H_1$ the adjoint operator of u .

Definition 1.1.1 *Let H_1, H_2 two Hilbert spaces and $u \in \mathcal{B}(H_1; H_2)$. The operator u is called:*

- (a) *Projection if $u \circ u = u$.*
- (b) *Normal if $u^*u = uu^*$.*
- (c) *Unitary if $u^*u = Id_{H_1}$ and $uu^* = Id_{H_2}$.*
- (d) *Self-adjoint if $u = u^*$.*
- (e) *Isometric if $\|u(x)\| = \|x\|$, for every $x \in H_1$.*
- (f) *Positive (notation : $u \geq 0$) if u is self-adjoint and for every $x \in H_1 : \langle u(x), x \rangle \geq 0$.*

1.1.2 The polar decomposition

Definition 1.1.2 (1) *Let $T \in \mathcal{B}(H_1; H_2)$, the module of T is given by $|T| = \sqrt{T^*T}$.*
 (2) *A linear mapping $U \in \mathcal{B}(H_1; H_2)$ is a partial isometry if*

$$\|U(x)\| = \|x\|,$$

for all $x \in \ker(U)^\perp$.

Theorem 1.1.1 (Polar decomposition) *For all $T \in \mathcal{B}(H_1; H_2)$, there is a partial isometry $U \in \mathcal{B}(H_1; H_2)$ such that $T = U|T|$.*

Proposition 1.1.1 (a) *U is a partial isometry if and only if, U^*U and UU^* are orthogonal projections on H_1 and H_2 respectively.*

(b) *U is a partial isometry if and only if, U^* is also.*

Remark 1.1.1 *If $T = U|T|$, then $|T| = U^*T$.*

Definition 1.1.3 (Finite rank operators) *A linear operator $u \in \mathcal{B}(H_1; H_2)$ is of finite rank if it is a finite sum of operators of the form*

$$u_{h_2 \otimes \bar{h}_1} = \bar{h}_1 \otimes h_2 : x \rightarrow \langle h_1, x \rangle h_2, \quad (1.1.1)$$

where $h_1 \in H_1, h_2 \in H_2$. The class of all finite rank linear operators between Hilbert spaces is denoted by $\mathcal{F}(H_1; H_2)$.

Example 1.1.1 If $H_1 = H_2$, for every $n \in \mathbb{N}^*$, we denote by P_n the orthogonal projection on the subspace generated by $\{e_1, \dots, e_n\}$. P_n is an operator of finite rank equal to n .

Definition 1.1.4 (Compact operators) A linear operator $T \in B(H_1; H_2)$ is said to be compact if $T(B_{H_1})$ is relatively compact in H_2 . We denote the space of compact linear operators by $\mathcal{K}(H_1; H_2)$.

Proposition 1.1.2 We have the following properties.

- (1) The space $\mathcal{K}(H_1; H_2)$ is closed in $B(H_1; H_2)$.
- (2) The operator T is compact if for any bounded sequence (x_i) of H_1 , the sequence $(T(x_i))$ has a convergent sequence in H_2 .
- (3) The operator T is compact if it is the limit of a sequence of operators of finite rank.
- (4) The operator T is compact if it adheres to the space $\mathcal{F}(H_1; H_2)$
i.e., for all $\varepsilon > 0$, $\exists \mathcal{S} \in \mathcal{F}(H_1; H_2)$ such that

$$\|T - \mathcal{S}\| < \varepsilon.$$

1.2 The ideals of linear mappings

First, we present the definition $\mathcal{L}_{p,\lambda}$ -space introduced by Lindenstrauss-Pelczynski in their article "Absolutely summing operators in \mathcal{L}_p -spaces and their applications". And studied by Diestel, Jarchow and Tonge in [19].

Definition 1.2.1 Given $1 \leq p \leq \infty$ and $\lambda > 1$. The Banach space X is called to be an $\mathcal{L}_{p,\lambda}$ -space if every finite dimensional subspace $E \subset X$ is contained in a finite dimensional subspace $F \subset X$, there exists an isomorphism $u : F \longrightarrow l_p^{\dim F}$ with $\|u\| \|u^{-1}\| < \lambda$. We say that X is a \mathcal{L}_p -space, if it is $\mathcal{L}_{p,\lambda}$ -space, for some $\lambda > 1$.

Example 1.2.1 Let $(\Omega; \mu)$ a measure space, for $1 \leq p \leq \infty$, the Lebesgue spaces $L_p(\mu)$ are an \mathcal{L}_p -spaces. The space $C(\Omega)$ of continuous functions on a compact space Ω is an \mathcal{L}_∞ -space.

Proposition 1.2.1 [17] (1) If $1 < p < \infty$ and X is an \mathcal{L}_p -space, then X is isomorphic to a subspace complemented by $L_p(\mu)$.

(2) If X is an \mathcal{L}_1 -space (resp. \mathcal{L}_∞), then X^{**} is isomorphic to a subspace complemented by $L_1(\mu)$ (resp. $C(K)$).

Conversely, if X is an complemented subspace of $L_p(\mu)$ ($1 < p < \infty$), then X is an \mathcal{L}_p -space.

Proposition 1.2.2 [17] (1) If X is an $\mathcal{L}_{p,\lambda}$ -space for every $\lambda \geq 1$, then X is an $L_p(\mu)$.

(2) Any Hilbert space is an $\mathcal{L}_{2,\lambda}$ -space for all $\lambda \geq 1$.

Definition 1.2.2 [38, Page 25] (Finite rank operators) Let X, Y be Banach spaces. we called that $u \in \mathcal{B}(X; Y)$ is to have a finite rank if $u(X)$ is a finite dimensional subspace of Y . We denote by $\mathcal{F}(X; Y)$ the class of all finite rank linear operators from X into Y . An operator has rank one if and only it has the form

$$x^* \otimes y : x \longmapsto \langle x, x^* \rangle y$$

i.e., if $u \in \mathcal{F}(X; Y)$, we have

$$u = \sum_{i=1}^n x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset X^*$ and $(y_i)_{i=1}^n \subset Y$.

Definition 1.2.3 An ideal of linear mappings \mathcal{I} is a subclass of the class \mathcal{B} of all continuous linear operators between Banach spaces such that for all Banach spaces X and Y , the components $\mathcal{I}(X; Y) := \mathcal{B}(X; Y) \cap \mathcal{I}$ satisfy:

(1) $\mathcal{I}(X; Y)$ is a linear subspace of $\mathcal{B}(X; Y)$ which contains the finite rank operators.

(2) The ideal property: if $v \in \mathcal{B}(X_0; X)$, $u \in \mathcal{I}(X; Y)$ and $w \in \mathcal{B}(Y; Y_0)$, then the composition $w \circ v \circ u$ is in $\mathcal{I}(X_0; Y_0)$.

If $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \rightarrow \mathbb{R}^+$ satisfies:

(1') $(\mathcal{I}(X; Y), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space for all Banach spaces X and Y .

(2'') $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.

(3''') If $v \in \mathcal{B}(X_0; X)$, $u \in \mathcal{I}(X; Y)$ and $w \in \mathcal{B}(Y; Y_0)$, $\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|v\|_{\mathcal{I}} \|u\|$.

then $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is called a normed (Banach) operator ideal, where X_0 and Y_0 are Banach spaces.

The operator ideal \mathcal{I} is said to be closed if each $\mathcal{I}(X; Y)$ is a closed subspace of $\mathcal{B}(X; Y)$ for the sup norm.

Remark 1.2.1 [38, Theorem 1.2.2] *The finite rank linear ideals \mathcal{F} is the smallest operator ideal and \mathcal{B} the largest one.*

Let us recall the definitions of ideal of p -summing [19, page. 31], (Cohen) strongly p -summing operators, Cohen p -nuclear [18] and p -integral operators.[19, p. 95], which will be used in the sequel.

1.2.1 The ideal of p -summing linear operators.

Definition 1.2.4 *Let $1 \leq p \leq \infty$. A linear operator $u : X \rightarrow Y$ is p -summing if there exists a constant $C \geq 0$ such that, for any $x_1, \dots, x_n \in X$, we have*

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^n |\xi(x_i)|^p \right)^{\frac{1}{p}}. \quad (1.2.1)$$

The class of p -summing linear operators from X into Y , which is denoted by $\Pi_p(X; Y)$. is a Banach space for the norm $\pi_p(T)$, i.e., the smallest constant C such that the inequality (1.2.1) holds.

Remark 1.2.2 *u is p -summing if it takes weakly p -summable sequences in X to strongly p -summable sequences in Y .*

Corollary 1.2.1 [19, Page 39] *If $1 \leq p \leq q < \infty$, then $\Pi_p(X; Y) \subset \Pi_q(X; Y)$. Moreover, for $u \in \Pi_p(X; Y)$ we have $\pi_q(u) \leq \pi_p(u)$.*

Theorem 1.2.1 [19, Corollary 2.16] *Let Ω be a compact Hausdorff space. An operator $u : X \rightarrow Y$ is 2-summing if and only if there exist a regular Borel probability measure on Ω such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow i_x & & \downarrow \tilde{u} \\ \mathcal{C}(\Omega) & \xrightarrow{j_2} & L_2(\mu). \end{array}$$

Moreover, we away arrange that $\|\tilde{u}\| = \pi_2(u)$.

Theorem 1.2.2 [19, Theorem 3.1, Page 60] If X is an $\mathcal{L}_{1,\lambda}$ -space and Y is an Hilbert space, then every operator $u \in \mathcal{B}(X; Y)$ is 1-summing. i.e.,

$$\mathcal{B}(X; Y) = \Pi_1(X; Y),$$

with $\pi_1(u) \leq k_G \|u\|$, k_G is the constant of Grothendieck's inequality.

Theorem 1.2.3 [19, Theorem 3.7, Page 64] If X is an \mathcal{L}_∞ -space and Y is an \mathcal{L}_p -space with $1 \leq p \leq 2$. then every operator $u \in \mathcal{B}(X; Y)$ is 2-summing. i.e.,

$$\mathcal{B}(X; Y) = \Pi_2(X; Y),$$

with $\pi_2(u) \leq k_G \|u\|$. k_G is the constant of Grothendieck's inequality.

1.2.2 The ideal of (Cohen) strongly p -summing linear operators.

Definition 1.2.5 Let $1 \leq p \leq \infty$. A linear operator $u : X \rightarrow Y$ is (Cohen) strongly p -summing if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n \in X$, and any $y_1^*, \dots, y_n^* \in Y^*$, we have

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}. \quad (1.2.2)$$

The class of strongly p -summing linear operators from X into Y , which is denoted by $\mathcal{D}_p(X; Y)$, is a Banach space for the norm $d_p(u)$, i.e., the smallest constant C such that the inequality (1.2.2) holds.

Remark 1.2.3 u is (Cohen) strongly p -summing if its adjoint operator $u^* : Y^* \rightarrow X^*$ is p^* -summing.

Corollary 1.2.2 The following results due to Cohen, for proof (see [18]).

- (1) If $1 \leq p \leq q < \infty$, then $\mathcal{D}_q(X; Y) \subset \mathcal{D}_p(X; Y)$.
- (2) In general, we have $\mathcal{D}_{p^*}(X; Y) \neq \Pi_p(X; Y)$.
- (3) If $u : X \rightarrow Y$, $u \in \Pi_p(X; Y)$ if, and only if, $u^* \in \mathcal{D}_{p^*}(Y^*; X^*)$ and $\pi_p(u) = d_{p^*}(u^*)$.
- (4) If $u \in \mathcal{D}_p(X; Y)$ if and only if, $u^{**} \in \mathcal{D}_p(X^{**}; Y^{**})$.
- (5) If $1 \leq p < \infty$, then $\Pi_p(X; Y) \subseteq \mathcal{D}_{p^*}(X; Y)$ when X is an \mathcal{L}_{p^*} -space.

(6) If $1 \leq p < \infty$, then $\mathcal{D}_{p^*}(X; Y) \subseteq \Pi_p(X; Y)$ when Y is an \mathcal{L}_p -space.

(7) If $1 \leq p < \infty$, then $\mathcal{D}_{p^*}(X; Y) = \Pi_p(X; Y)$ when X is an \mathcal{L}_{p^*} -space and Y is an \mathcal{L}_p -space.

1.2.3 The ideal of (Cohen) p -nuclear linear operators.

Definition 1.2.6 [19, Page 64] Let $1 \leq p \leq \infty$. A linear operator $u : X \longrightarrow Y$ is (Cohen) p -nuclear, if there exists a constant $C \geq 0$ such that for all finite sets $x_1, \dots, x_n \in X$, and for $y_1^*, y_2^*, \dots, y_n^* \in Y^*$, we have

$$\left| \sum_{i=1}^n \langle T(x_i), y_i^* \rangle \right| \leq C \sup_{x^* \in B_{X^*}} \|((x_i(x^*)))\|_{l_p^n} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}. \quad (1.2.3)$$

The class of (Cohen) p -nuclear linear operators from X into Y , which is denoted by $\mathcal{N}_p(X; Y)$, is a Banach space for the norm $n_p(u)$, i.e., the smallest constant C such that the inequality (1.2.3) holds.

Remark 1.2.4 $\mathcal{N}_p(X; Y)$ is an Banach ideal in $\mathcal{B}(X; Y)$ (i.e., $\forall u \in \mathcal{N}_p(X; Y), u \in \mathcal{B}(X; Y)$ and $\|u\| \leq n_p(T)$).

Corollary 1.2.3 If $1 \leq p \leq q < \infty$, then $I_p(X; Y) \subset I_q(X; Y)$. Moreover, for $u \in \mathcal{N}_p(X; Y)$, we have $i_q(T) \leq i_p(T)$.

1.2.4 The ideal of p -integral linear operators.

Definition 1.2.7 [19, Page 95] Let $1 \leq p \leq \infty$. A linear operator $u : X \longrightarrow Y$ is p -integral if there are operators such that the following diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{K_Y} & Y^{**} & & \\ \downarrow v & & & \nearrow w & & & \\ L_\infty(\mu) & \xrightarrow{i_p} & L_p(\mu) & & & & \end{array} \quad (1.2.4)$$

We denote by $I_p(X; Y)$ the class of (Cohen) of all p -integral linear operators ($1 \leq p < \infty$).

Corollary 1.2.4 If $1 \leq p \leq q < \infty$, then $I_p(X; Y) \subset I_q(X; Y)$. Moreover, for $u \in I_p(X; Y)$ we have $i_q(T) \leq i_p(T)$.

To close this section, we introduce the definition of p -factorable operators ideals.

1.2.5 The ideal of p -factorable operators

Definition 1.2.8 [19, Page154] Let $1 \leq p \leq \infty$ and X, Y be Banach spaces.

The operator $u : X \rightarrow Y$ is said p -factorable if there exist a measure space (Ω, Σ, μ) , $v \in \mathcal{B}(L_p(\mu); Y^{**})$ and $w \in \mathcal{B}(X; L_p(\mu))$ such that

$$K_Y u : X \xrightarrow{w} L_p(\mu) \xrightarrow{v} Y^{**}.$$

In other words

$$K_Y \circ u = v \circ w,$$

where K_Y is the isometric embedding of Y into Y^{**} . We denote by $\Gamma_{p\text{-fat}}(X; Y)$ the space of all p -factorable linear operators, which is a Banach space.

Corollary 1.2.5 An operator $u : X \rightarrow Y$ belongs to $\Gamma_{2\text{-fat}}(X; Y)$ if and only if, it has a factorization

$$X \xrightarrow{w} H \xrightarrow{v} Y^{**},$$

where H is a Hilbert space.

1.3 Some based notions on multilinear mappings

In this section, we recall briefly some basic notions of the m -linear operators and their properties. Let now $m \in \mathbb{N}$ and X_1, \dots, X_m, Y be Banach spaces over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). A mapping $T : X_1 \times \dots \times X_m \rightarrow Y$ is multilinear or m -linear if it is linear for each component. It is bounded (continuous) if there exists a constant $C > 0$ such that for any $(x_1, \dots, x_m) \in X_1 \times \dots \times X_m$, we have

$$\|T(x_1, \dots, x_m)\| \leq C \|x_1\| \dots \|x_m\|. \quad (1.3.1)$$

We denote by $\mathcal{L}(X_1, \dots, X_m; Y)$ the space of all (bounded) continuous m -linear operators from $X_1 \times \dots \times X_m$ into Y . Banach whose norm is the smallest verifying constant (1.3.1).

It can be expressed

$$\|T\| = \sup_{\|x_j\| \leq 1; 1 \leq j \leq m} \|T(x_1, \dots, x_m)\|.$$

If $Y = \mathbb{K}$, we write shortly $\mathcal{L}(X_1, \dots, X_m; \mathbb{k}) = \mathcal{L}(X_1, \dots, X_m)$.

The projective tensor product

Let X_1, \dots, X_m be Banach spaces. We denote $X_1 \otimes \dots \otimes X_m$ the algebraic tensor product of X_1, \dots, X_m . We define the projective norm by

$$\|v\|_\pi = \inf \left\{ \sum_{i=1}^n \prod_{j=1}^m \|x_i^j\| \right\}, \quad (1.3.2)$$

where the infimum relates to all possible representations of v of the form

$$v = \sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m.$$

As usual, $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$ stands for the (complete) projective tensor product of the Banach spaces X_1, \dots, X_m . If $X_1 = \dots = X_m = X$, we simply write $\widehat{\otimes}_\pi^m X$.

The linearization of an operator

If $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, we denote by \widetilde{T} the linearization of T , which is the linear map $\widetilde{T} : X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \rightarrow Y$ given by

$$\widetilde{T} \left(\sum_{i=1}^n x_i^1 \otimes \dots \otimes x_i^m \right) = \sum_{i=1}^n T(x_i^1, \dots, x_i^m),$$

for all $x_i^j \in X_j$ ($n \in \mathbb{N}, 1 \leq i \leq n, 1 \leq j \leq m$). This linear operator is well defined because it does not depend on a chosen representation (see [42]). We also have

$$T \text{ is bounded} \Leftrightarrow \widetilde{T} \text{ is bounded.}$$

Furthermore, $\|\widetilde{T}\| = \|T\|$.

Identification

Given the following mapping

$$\begin{aligned} \Psi & : \mathcal{L}(X_1, \dots, X_m; Y) \rightarrow \mathcal{B}(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m; Y) \\ T & \quad \rightarrow \quad \Psi(T) = \widetilde{T}. \end{aligned}$$

It is easy to see that this mapping is surjective isometric. So we have the following isometric identification

$$\mathcal{L}(X_1, \dots, X_m; Y) = \mathcal{B}(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m; Y).$$

Particular case. The dual of $X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m$ is identified with the space of bounded multilinear forms

$$(X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m)^* = \mathcal{L}(X_1, \dots, X_m). \quad (1.3.3)$$

For Banach spaces X_1, \dots, X_m, Y , we have the isometric identification

$$\mathcal{L}(X_1, \dots, X_m; Y^*) = (X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \widehat{\otimes}_\pi Y^*). \quad (1.3.4)$$

Adjoint operator. To each multilinear operator $T : X_1 \times \dots \times X_m \rightarrow Y$, we associate the following adjoint operator

$$T^* : Y^* \rightarrow \mathcal{L}(X_1, \dots, X_m).$$

Which is defined by $y^* \rightarrow T^*(y^*) : X_1 \times \dots \times X_m \rightarrow \mathbb{K}$, where

$$T^*(y^*)(x^1, \dots, x^m) = y^*(T(x^1, \dots, x^m)).$$

The canonical multilinear mapping

Consider the canonical multilinear mapping

$$\begin{aligned} i_m : X_1 \times \dots \times X_m &\rightarrow X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \\ (x^1, \dots, x^m) &\rightarrow x^1 \otimes \dots \otimes x^m. \end{aligned}$$

We have the next diagram which is commute

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & Y \\ & \searrow i_m & \widetilde{T} \uparrow \\ & & X_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi X_m \end{array}$$

In the other words

$$T = \widetilde{T} \circ i_m. \quad (1.3.5)$$

Chapter 2

The ideal of Schatten class operators and linear mappings generated by \mathcal{S}_p

In this chapter, we study the composition ideals of linear mappings generated by Schatten class. Firstly, we study the Schatten class \mathcal{S}_p and some properties. And then present a relation to ideals of linear mappings (p -summing, (Cohen) strongly p -summing, Cohen p -nuclear and p -integral operators), especially coincidence theorem for Cohen strongly 2-summing linear operators and connection with linear mappings of type $\mathcal{B}(\mathcal{S}_p)$. Finally, we give some factorization results like that given by Lindenstrauss-Pełczyński and J. Diestel, H. Jarchow and A. Tonge, for Hilbert Schmidt linear operators.

2.1 Schatten class operators $\mathcal{S}_p(H)$

In this section, we study the Schatten class \mathcal{S}_p . (\mathcal{S}_p considered as a fundamental example of newly introduced non commutative L_p -space for $1 \leq p < \infty$). Throughout this section, H is a separable Hilbert space. It is important to begin the definitions of Schatten class by \mathcal{S}_2 , it serves us to demonstrate several results, in particular those concerning the trace class operators \mathcal{S}_1 .

2.1.1 Hilbert-Schmidt operators $\mathcal{S}_2(H)$

The Schatten class $\mathcal{S}_2(H)$ is no other than Hilbert-Schmidt operators. Indeed, we define $\mathcal{S}_2(H)$ as the class of all operators T of $B(H)$ who verify

$$\sum_{i \in I} \langle |T|^2 e_i, e_i \rangle < \infty. \quad (2.1.1)$$

That means

$$\begin{aligned} \sum_{i \in I} \langle |T|^2 e_i, e_i \rangle &= \sum_{i \in I} \langle T^* T e_i, e_i \rangle = \sum_{i \in I} \langle T e_i, T e_i \rangle \\ &= \sum_{i \in I} \|T(e_i)\|^2 < \infty. \end{aligned}$$

This class is equipped with the following Hilbert-Schmidt norm

$$\|T\|_{\mathcal{S}_2} = \left(\sum_{i \in I} \|T(e_i)\|^2 \right)^{\frac{1}{2}}.$$

It is quickly to verify that this norm does not depend on the choice of the orthonormal basis of H .

First, we show that T and T^* have the same norm of Hilbert Schmidt. Indeed

$$\begin{aligned} \|T\|_{\mathcal{S}_2}^2 &= \sum_{i \in I} \|T(e_i)\|^2 = \sum_{i \in I} \sum_{i \in I} |\langle e_i, T(e_i) \rangle|^2 \\ &= \sum_{i \in I} \sum_{i \in I} |\langle T^*(e_i), e_i \rangle|^2 \\ &= \sum_{i \in I} \|T^*(e_i)\|^2 = \|T^*\|_{\mathcal{S}_2}^2. \end{aligned}$$

Now given $(e_i)_{i \in I}, (f_j)_{j \in J}$ two orthonormal basis of H , we have

$$\begin{aligned} \|T\|_{\mathcal{S}_2}^2 &= \|T^*\|_{\mathcal{S}_2}^2 = \sum_{i \in I} \|T^*(e_i)\|^2 = \sum_{i \in I} \sum_{j \in J} |\langle f_j, T^*(e_i) \rangle|^2 \\ &= \sum_{j \in J} \sum_{i \in I} |\langle T(f_j), e_i \rangle|^2 \\ &= \sum_{j \in J} \|T(f_j)\|^2. \end{aligned}$$

Remark 2.1.1 (1) $\mathcal{S}_2(H)$ is a Hilbert space, its inner product is given by

$$\forall T_1, T_2 \in \mathcal{S}_2(H) : \langle T_1, T_2 \rangle_{\mathcal{S}_2} = \sum_{i \in I} \langle T_1 e_i, T_2 e_i \rangle.$$

(2) Let $T \in \mathcal{S}_2(H)$. If x in H and its norm is 1. We can find an orthonormal basis of H which contains x . It follows that

$$\|T(x)\| \leq \|T\|_{\mathcal{S}_2},$$

therefore

$$\|T\| \leq \|T\|_{\mathcal{S}_2}.$$

Corollary 2.1.1 *It is concluded from above, then*

(a) $T \in \mathcal{S}_2(H) \Leftrightarrow T^* \in \mathcal{S}_2(H)$.

(b) For all $T \in \mathcal{S}_2(H)$, we have $\|T\|_{\mathcal{S}_2}^2 = \text{Tr}(T^*T) = \text{Tr}(TT^*)$.

(c) For all $T_1, T_2 \in \mathcal{S}_2(H)$, we have $\text{Tr}(T_1T_2) = \text{Tr}(T_2T_1)$.

(d) (Ideal property) Let $T \in \mathcal{S}_2(H)$ and $u, v \in B(H)$, we have $uTv \in \mathcal{S}_2(H)$.

In addition

$$\|uTv\|_{\mathcal{S}_2} \leq \|u\| \|T\|_{\mathcal{S}_2} \|v\|.$$

Proof. (a) Immediate, since $\|T\|_{\mathcal{S}_2}^2 = \|T^*\|_{\mathcal{S}_2}^2$.

(b) Let $T \in \mathcal{S}_2(H)$, we have

$$\begin{aligned} \text{Tr}(T^*T) &= \sum_{i \in I} \langle T^*T e_i, e_i \rangle \\ &= \sum_{i \in I} \langle T e_i, T e_i \rangle = \sum_{i \in I} \|T(e_i)\|^2 = \sum_{i \in I} \|T^*(e_i)\|^2 \\ &= \sum_{i \in I} \langle T^* e_i, T^* e_i \rangle = \sum_{i \in I} \langle TT^* e_i, e_i \rangle = \text{Tr}(TT^*). \end{aligned}$$

(c) We use (b) and the polarization identity.

(d) Let $T \in \mathcal{S}_2(H)$ and $u \in B(H)$, then

$$\begin{aligned} \|uT\|_{\mathcal{S}_2}^2 &= \sum_{i \in I} \|uT(e_i)\|^2 \\ &\leq \sum_{i \in I} \|u\|^2 \|T(e_i)\|^2 = \|u\|^2 \sum_{i \in I} \|T(e_i)\|^2 = \|u\|^2 \|T\|_{\mathcal{S}_2}^2, \end{aligned}$$

therefore, $uT \in \mathcal{S}_2(H)$ and $\|uT\|_{\mathcal{S}_2} \leq \|u\| \|T\|_{\mathcal{S}_2}$.

Now, let $v \in B(H)$, by (a) the adjoint operator $T^*u^* \in \mathcal{S}_2(H)$. Then, $v^*T^*u^* \in \mathcal{S}_2(H)$, by a second application of (a) this indicates that $uTv \in \mathcal{S}_2(H)$. ■

Proposition 2.1.1 *The operators of Hilbert-Schmidt are compact, i.e., $\mathcal{S}_2(H) \subset \mathcal{K}(H)$.*

Proof. Let $(e_i)_{i \in I}$ be an orthonormal basis of H and $T \in \mathcal{S}_2(H)$. For all n in \mathbb{N}^* , we denote by P_n the orthogonal projection on the subspace generated by $\{e_1, \dots, e_n\}$. The operator $T_n = TP_n$ is of finite rank equal to n and we have

$$T_n(e_i) = \begin{cases} T(e_i) & \text{if } i \leq n \\ 0 & \text{if } i \geq n \end{cases}$$

So

$$(T - T_n)(e_i) = \begin{cases} 0 & \text{if } i \leq n \\ T(e_i) & \text{if } i \geq n + 1 \end{cases}$$

Since $T - T_n \in \mathcal{S}_2(H)$, it follows that

$$\|T - T_n\|^2 \leq \|T - T_n\|_{\mathcal{S}_2}^2 = \sum_{i=n+1}^{\infty} \|(T - T_n)(e_i)\|^2 = \sum_{i=n+1}^{\infty} \|T(e_i)\|^2.$$

The second member is the remainder of a convergent series, so it tends to zero when n tends to ∞ . Therefore the operator T is the limit of a sequence of the finite rank operators. ■

2.1.2 The trace class operators $\mathcal{S}_1(H)$

The class $\mathcal{S}_1(H)$ which we will define plays an important role in the Schatten class. Its dual is the entire space $B(H)$. We will also see that its elements are characterized by the decomposition into two elements of $\mathcal{S}_2(H)$. We start by this follow definition.

Definition 2.1.1 *We denote by $\mathcal{S}_1(H)$ the space of all the operators $T \in \mathcal{B}(H)$ with $Tr(|T|) < \infty$ (where $|T| = (T^*T)^{\frac{1}{2}}$), i.e.,*

$$\mathcal{S}_1(H) = \left\{ T \in B(H) : \sum_{i \in I} \langle |T| e_i, e_i \rangle < \infty \right\}. \quad (2.1.2)$$

The space $\mathcal{S}_1(H)$ of the following norm

$$\|T\|_{\mathcal{S}_1} = Tr(|T|).$$

Which makes it a Banach space.

Proposition 2.1.2 *Let $T \in \mathcal{S}_1(H)$, by the polar decomposition of T , $T = U|T|$. Then, $|T|^{\frac{1}{2}}$ and $U|T|^{\frac{1}{2}}$ are in $\mathcal{S}_2(H)$.*

Proof. We put $x = |T|^{\frac{1}{2}}$ and $y = U|T|^{\frac{1}{2}}$, then

$$\begin{aligned} \|x\|_{\mathcal{S}_2} &= \sum_{i \in I} \langle |T|^{\frac{1}{2}} e_i, |T|^{\frac{1}{2}} e_i \rangle \\ &= \sum_{i \in I} \langle |T| e_i, e_i \rangle = \|T\|_{\mathcal{S}_1} < \infty. \end{aligned}$$

For y , we know that U^*U is a projection on H with an image $\{|T|(h) : h \in H\}$, then

$$U^*U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} \|y\|_{\mathcal{S}_2} &= \sum_{i \in I} \langle U|T|^{\frac{1}{2}} e_i, U|T|^{\frac{1}{2}} e_i \rangle \\ &= \sum_{i \in I} \langle |u|^{\frac{1}{2}} U^*U|T|^{\frac{1}{2}} e_i, e_i \rangle \\ &= \sum_{i \in I} \langle |T| e_i, e_i \rangle = \|T\|_{\mathcal{S}_1} < \infty. \quad \blacksquare \end{aligned}$$

2.1.3 Factorization of the trace class operators

In this paragraph, we will study some relations between the two classes \mathcal{S}_1 and \mathcal{S}_2 . It will be seen that each element of \mathcal{S}_1 is decomposed into two elements of \mathcal{S}_2 . This permits us to give another version of the norm of \mathcal{S}_1 .

Proposition 2.1.3 *Let $T \in \mathcal{B}(H)$. Then, $T \in \mathcal{S}_1(H)$ if and only if, there are two elements $u, v \in \mathcal{S}_2(H)$ such that*

$$T = uv.$$

Proof. Let $u, v \in \mathcal{S}_2(H)$. Then, it is shown that $uv \in \mathcal{S}_1(H)$. Indeed

$$\begin{aligned} \text{Tr}(|uv|) &= \sum_{i \in I} \langle |uv| e_i, e_i \rangle \leq \sum_{i \in I} \|uv(e_i)\| \\ &\leq \|v\|_{\mathcal{S}_2} \|u\|_{\mathcal{S}_2} < \infty. \end{aligned}$$

The reciprocal is immediate by Proposition 2.1.2. \blacksquare

Corollary 2.1.2 (1) For any $T \in B(H)$. Then, $Tr(|T|) \leq 1 \Leftrightarrow T$ is written as follows $T = uv$ with $u, v \in B_{\mathcal{S}_2(H)}$.

(2) If T is positive then, $T \in \mathcal{S}_1(H) \Leftrightarrow T^{\frac{1}{2}} \in \mathcal{S}_2(H)$.

Proof. According to the previous Proposition

$$\|u\|_{\mathcal{S}_2} = \|v\|_{\mathcal{S}_2} = Tr(|T|) \leq 1. \quad \blacksquare$$

Proposition 2.1.4 For any $T \in \mathcal{S}_1(H)$, we have

$$\|T\|_{\mathcal{S}_1} = \min \{ \|u\|_{\mathcal{S}_2} \|v\|_{\mathcal{S}_2} \},$$

where the minimum deals with all the decompositions of T , where $T = uv$ with $u, v \in \mathcal{S}_2(H)$.

Proof. By Proposition 2.1.3, for any $T = v_1v_2$, we have

$$\|T\|_{\mathcal{S}_1} = \|uv\|_{\mathcal{S}_1} \leq \|u\|_{\mathcal{S}_2} \|v\|_{\mathcal{S}_2},$$

therefore

$$\|T\|_{\mathcal{S}_1} \leq \min \{ \|u\|_{\mathcal{S}_2} \|v\|_{\mathcal{S}_2} \}.$$

We obtain immediately the reciprocal implication by taking the polar decomposition of T .

■

Proposition 2.1.5 (1) For any $u \in \mathcal{B}(H)$ and $T \in \mathcal{S}_1(H)$, we have

$$Tr(uT) = Tr(Tu).$$

(2) (Ideal Property) Let $T \in \mathcal{S}_1(H)$, then

$$uTv \in \mathcal{S}_1(H),$$

for all $u, v \in B(H)$.

Proof. (1) By Proposition 2.1.3, for any $T = xy$, we obtain

$$\begin{aligned} Tr(uT) &= Tr(u(xy)) = Tr((ux)y) \\ &= Tr(y(ux)) = Tr((yu)x) \\ &= Tr(x(yu)) = Tr(Tu). \end{aligned}$$

(2) Let $T \in \mathcal{S}_1(H)$ and $u, v \in B(H)$

$$\begin{aligned} \text{Tr}(|uTv|) &= \sum_{i \in I} \langle |uTv| e_i, e_i \rangle \\ &\leq \|u\| \|v\| \sum_{i \in I} \langle |T| e_i, e_i \rangle \\ &\leq \|u\| \|v\| \text{Tr}(|T|) < \infty. \quad \blacksquare \end{aligned}$$

Generalization of the trace on $\mathcal{S}_1(H)$

We can generalize the definition of the trace on the entire space $\mathcal{S}_1(H)$ by the following way.

Definition 2.1.2 (Trace on $\mathcal{S}_1(H)$) Let H be a Hilbert space and $(e_i)_{i \in I}$ an orthonormal basis of H . Let $T \in \mathcal{S}_1(H)$, we define the trace of T by

$$\text{Tr}(T) = \sum_{i \in I} \langle T e_i, e_i \rangle.$$

This is a linear form on $\mathcal{S}_1(H)$. Indeed, for any $T_1, T_2 \in \mathcal{S}_1(H)$ and $\alpha_1, \alpha_2 \in \mathbb{K}$, we have

$$\begin{aligned} \text{Tr}(\alpha_1 T_1 + \alpha_2 T_2) &= \sum_{i \in I} \langle (\alpha_1 T_1 + \alpha_2 T_2) e_i, e_i \rangle \\ &= \sum_{i \in I} \langle \alpha_1 T_1 e_i + \alpha_2 T_2 e_i, e_i \rangle \\ &= \alpha_1 \sum_{i \in I} \langle T_1 e_i, e_i \rangle + \alpha_2 \sum_{i \in I} \langle T_2 e_i, e_i \rangle. \end{aligned}$$

Proposition 2.1.6 For any $T \in \mathcal{S}_1(H)$, we have

$$|\text{Tr}(T)| \leq \text{Tr}(|T|).$$

Proof. By Proposition 2.1.2, if $T = U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}} = xy$, we have

$$\begin{aligned} |\text{Tr}(T)| &= \left| \sum_{i \in I} \langle (xy) e_i, e_i \rangle \right| \\ &= \left| \sum_{i \in I} \langle y e_i, x^* e_i \rangle \right| = |\langle y, x^* \rangle_{\mathcal{S}_2}| \leq \|y\|_{\mathcal{S}_2} \|x\|_{\mathcal{S}_2}. \end{aligned}$$

By taking the infimum on all T decompositions of two Hilbert-Schmidt elements, we find

$$|\text{Tr}(T)| \leq \text{Tr}(|T|). \quad \blacksquare$$

Remark 2.1.2 The application trace on $\mathcal{S}_1(H)$ is well defined, because $\text{Tr}(T) = \langle y, x^* \rangle_{\mathcal{S}_2}$ is a inner product on $\mathcal{S}_2(H)$, that is independent of the choice of the orthonormal basis of H .

Corollary 2.1.3 The space $\mathcal{S}_1(H)$ contains all the finite rank operators, i.e., $\mathcal{F}(H) \subset \mathcal{S}_1(H)$.

2.2 Schatten class operators $\mathcal{S}_p(H_1; H_2)$

In this section, we will study the Schatten class operators $\mathcal{S}_p(H_1; H_2)$.

Definition 2.2.1 Let H_1, H_2 be Hilbert spaces and $1 \leq p < \infty$.

We denote by $\mathcal{S}_p(H_1; H_2)$ the p -th Schatten class of all compact operators

$$T : H_1 \rightarrow H_2,$$

such that $\text{Tr}_{H_1}(|T|^p) < \infty$, equipped with the norm

$$\|T\|_{\mathcal{S}_p} = (\text{Tr}_{H_1}(|T|^p))^{\frac{1}{p}}.$$

$$\text{i.e., } \mathcal{S}_p(H_1; H_2) = \{T \in B(H_1; H_2) : |T|^p \in \mathcal{S}_1(H_1)\}.$$

Which makes it a Banach space.

Remark 2.2.1 In the case $0 < p < 1$, the space $\mathcal{S}_p(H_1; H_2)$ is quasinormed (p -normed).

Remark 2.2.2 For any $1 \leq p < \infty$, we have $\mathcal{S}_p(H_1; H_2) \subset \mathcal{K}(H_1; H_2)$.

Indeed, the projections P_n on H_1 are taken as in example 1.1.1. We pose $T_n = T \circ P_n$ (Finite rank operators), and it is shown that $T_n \rightarrow T$.

Proposition 2.2.1 [19, Theorem 4.6] An operator $u : H_1 \rightarrow H_2$ is compact if, and only if, There is $(\lambda_n(T))_{n \in \mathbb{N}}$ scalars sequences (tends to zero) such that

$$T = \sum_{n \in \mathbb{N}} \lambda_n(T) \overline{h_n} \otimes k_n, \tag{2.2.1}$$

where $(h_n)_{n \in \mathbb{N}}$ is an orthonormal in H_1 and $(k_n)_{n \in \mathbb{N}}$ is an orthonormal in H_2 .

Proposition 2.2.2 *Let $1 \leq p < \infty$. The operator $T \in \mathcal{S}_p(H_1; H_2)$ if and only if, the sequence $(\lambda_n(T))_{n \in \mathbb{N}} \in \ell_p$. Furthermore, we have*

$$\|T\|_{\mathcal{S}_p} = \left(\sum_{n \in \mathbb{N}} |\lambda_n|^p \right)^{\frac{1}{p}}. \quad (2.2.2)$$

Proof. Let $1 \leq p < \infty$. First, it should be noted that if $|T| = \sum_{n \in \mathbb{N}} \lambda_n (|T|) \bar{h}_n \otimes h_n$ then,

$$|T|^p = \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) \bar{h}_n \otimes h_n.$$

We have

$$\begin{aligned} \text{Tr}(|T|^p) &= \sum_{i \in I_1} \langle |T|^p e_i, e_i \rangle = \sum_{i \in I_1} \left\langle \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) \langle h_n, e_i \rangle h_n, e_i \right\rangle \\ &= \sum_{i \in I_1} \left\langle \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) h_n^i h_n, e_i \right\rangle = \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) \sum_{i \in I_1} h_n^i \langle h_n, e_i \rangle \\ &= \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) \sum_{i \in I_1} (h_n^i)^2 = \sum_{n \in \mathbb{N}} \lambda_n^p (|T|) \|h_n\|^2 = \sum_{n \in \mathbb{N}} \lambda_n^p (|T|). \end{aligned}$$

What it shows the equivalence. ■

Proposition 2.2.3 *Let $1 \leq p \leq q < \infty$. Then*

$$\mathcal{S}_p(H_1; H_2) \subseteq \mathcal{S}_q(H_1; H_2). \quad (2.2.3)$$

Proof. Let $T \in \mathcal{S}_p(H_1; H_2)$, we have

$$\begin{aligned} \left(\sum_{n \in \mathbb{N}} |\lambda_n|^q \right)^{\frac{1}{q}} &\leq \left(\sum_{n \in \mathbb{N}} |\lambda_n|^p \right)^{\frac{1}{p}} \\ &= \|T\|_{\mathcal{S}_p} < \infty. \end{aligned}$$

Then, T is in $\mathcal{S}_q(H_1; H_2)$. ■

Duality for Schatten classes

Theorem 2.2.1 [19, Theorem 6.3] *Suppose that $1 < p, q_1, q_2 < \infty$ are such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$.*

Let $T \in \mathcal{B}(H_1; H_2)$. The following properties are equivalent.

- (1) *The operator T belongs to $\mathcal{S}_p(H_1; H_2)$.*
- (2) *There exist a Hilbert space H and operators $u \in \mathcal{S}_{q_1}(H_1; H)$ and $v \in \mathcal{S}_{q_2}(H; H_2)$ such that $T = uv$.*

Theorem 2.2.2 [19, Theorem 6.4] *Let H_1, H_2 be Hilbert spaces.*

i) If $1 < p < \infty$, then an isometric isomorphism

$$\Phi : \mathcal{S}_{p^*}(H_2; H_1) \rightarrow [\mathcal{S}_p(H_1; H_2)]^*$$

is obtained by setting

$$\forall v \in \mathcal{S}_{p^*}(H_2; H_1), \forall u \in \mathcal{S}_p(H_1; H_2) \quad \Phi(v)(u) = \text{Tr}_{H_2}(vu).$$

ii) In the same way, $\mathcal{S}_1(H_1; H_2)$ is isometrically isomorphic to $[\mathcal{K}(H_1; H_2)]^$ and $\mathcal{B}(H_2; H_1)$ is isometrically isomorphic to $[\mathcal{B}(H_2; H_1)]^*$.*

2.3 Properties of the class $\mathcal{S}_2(H_1; H_2)$

This section is devoted to representing the Schatten classes \mathcal{S}_2 by a tensor writing. First, we define the tensor product of Hilbert.

Hilbert's tensor product. Let H_1, H_2 be Hilbert spaces. The algebraic tensor product $H_1 \otimes H_2$ of the inner product defined by

$$\forall h = (h_1 \otimes h_2), k = (k_1 \otimes k_2) \in H_1 \otimes H_2 : \langle h, k \rangle = \langle h_1, k_1 \rangle \langle h_2, k_2 \rangle.$$

We denote by $\|\cdot\|_2$ the corresponding norm and $H_1 \widehat{\otimes}_2 H_2$ the completed Hilbert space. Let $(e_{k_j})_{k_j \in I_j}$ an orthonormal basis of H_j ($1 \leq j \leq 2$). It can be seen without difficulty that the system

$$\{e_{k_1} \otimes e_{k_2}\}_{\substack{k_j \in I_j \\ 1 \leq j \leq 2}},$$

forms an orthonormal basis of $H_1 \widehat{\otimes}_2 H_2$.

We know the class $\mathcal{S}_2(H_1; H_2)$ is that Hilbert-Schmidt linear operators. thus let's start with the following result on factorization of $\mathcal{S}_2(H_1; H_2)$.

Theorem 2.3.1 (Lindenstrauss-Pelczński [28]) *Let H_1, H_2 be two Hilbert spaces and $u \in \mathcal{B}(H_1; H_2)$. The following properties are equivalent.*

- (a) *The operator u belongs to $\mathcal{S}_2(H_1; H_2)$.*
- (b) *u factors through an \mathcal{L}_∞ -space.*
- (c) *u factors through an \mathcal{L}_1 -space.*

A more general result due to Diestel, Jarchow and Tonge, on the following form.

Theorem 2.3.2 [19, Theorem 19.2] *Let H_1, H_2 be two Hilbert spaces and $u \in \mathcal{B}(H_1; H_2)$.*

The following properties are equivalent:

- (1) *The operator u belongs to $\mathcal{S}_2(H_1; H_2)$.*
- (2) *For any Banach space G , there is $v_1 \in \mathcal{B}(H_1; G)$ and $v_2 \in \mathcal{B}(G; H_2)$ such that $u = v_2 v_1$.*

Theorem 2.3.3 [19, Theorem 5.30] *Let H_1 and H_2 be Hilbert spaces.*

- (a) *If $1 < p < \infty$, then $\mathcal{I}_p(H_1; H_2) = \mathcal{N}_p(H_1; H_2) = \mathcal{S}_2(H_1; H_2)$ isomorphically and isometrically if $p = 2$.*
- (b) *$\mathcal{I}_1(H_1; H_2) = \mathcal{N}_1(H_1; H_2) = \mathcal{S}_1(H_1; H_2)$ isometrically.*

Pełczyński in [36] has proved that $\mathcal{S}_2(H_1; H_2) = \Pi_p(H_1; H_2)$. After that J. S. Cohen in [18, Theorem 4.1.1] has given us the following result.

Theorem 2.3.4 *Let H_1, H_2 be two Hilbert spaces and $u \in \mathcal{B}(H_1; H_2)$. For $1 \leq p < \infty$ and $1 < p^* \leq \infty$, we have*

$$\mathcal{S}_2(H_1; H_2) = \Pi_p(H_1; H_2) = \mathcal{D}_{p^*}(H_1; H_2). \quad (2.3.1)$$

We show that the dual of the Hilbert tensor product Hilbert coincides with the class \mathcal{S}_2 .

Theorem 2.3.5 *Let H_1, H_2 be Hilbert spaces. Then, $H_1 \widehat{\otimes}_2 H_2$ isometrically identifies with $\mathcal{S}_2(H_1; H_2)$.*

Proof. Let $u \in H_1 \widehat{\otimes}_2 H_2$ such that $u = \sum_{i=1}^n h_i^1 \otimes h_i^2$. The following application is defined

$$\begin{aligned} \Phi & : H_1 \widehat{\otimes}_2 H_2 \rightarrow \mathcal{S}_2(H_1; H_2) \\ u & \mapsto \Phi(u) \end{aligned}$$

where $\Phi(u)$ is defined by

$$\begin{aligned} \Phi(u) & : H_1 \rightarrow H_2 \\ h & \mapsto \Phi(u)(h) = \sum_{i=1}^n \langle h_i^2, h \rangle h_i^1 \end{aligned}$$

First, we verify that $\Phi(u) \in \mathcal{S}_2(H_1; H_2)$. Indeed

$$\begin{aligned} \|\Phi(u)\|_{\mathcal{S}_2}^2 &= \sum_{i \in I} \|\Phi(u)(e_i)\|^2 = \sum_{i \in I} \langle \Phi(u)(e_i), \Phi(u)(e_i) \rangle \\ &= \sum_{j_2=1}^n \sum_{j_1=1}^n \sum_{i \in I} \langle h_{j_1}^2, e_i \rangle \langle h_{j_2}^2, e_i \rangle \langle h_{j_1}^1, h_{j_2}^1 \rangle \\ &= \sum_{j_2=1}^n \sum_{j_1=1}^n \langle h_{j_1}^2, h_{j_2}^2 \rangle \langle h_{j_1}^1, h_{j_2}^1 \rangle = \|u\|_{\mathcal{S}_2}. \end{aligned}$$

We also conclude that Φ is isometric. It remains to see that Φ is surjective.

Let $T \in \mathcal{S}_2(H_1; H_2)$, we put

$$u_T = \sum_{i \in I} T(e_i) \otimes \bar{e}_i.$$

It is verified that $u_T \in H_1 \widehat{\otimes}_2 H_2$ and $\Phi(u_T) = T$.

Indeed

$$\begin{aligned} \|u_T\|_2^2 &= \left\langle \sum_{i \in I} T(e_i) \otimes \bar{e}_i, \sum_{i \in I} T(e_i) \otimes \bar{e}_i \right\rangle \\ &= \sum_{i_1 \in I} \sum_{i_2 \in I} \langle T(e_{i_1}) \otimes \bar{e}_{i_1}, T(e_{i_2}) \otimes \bar{e}_{i_2} \rangle \\ &= \sum_{i_1 \in I} \sum_{i_2 \in I} \langle T(e_{i_1}), T(e_{i_2}) \rangle \langle \bar{e}_{i_1}, \bar{e}_{i_2} \rangle \\ &= \sum_{i \in I} \|T(e_i)\|^2 = \|T\|_{\mathcal{S}_2} < \infty. \end{aligned}$$

On other hand

$$\begin{aligned} \Phi(u_T)(h) &= \sum_{i \in I} \langle e_i, h \rangle T(e_i) \\ &= T\left(\sum_{i \in I} h_i e_i\right) = T(h), \end{aligned}$$

this ends the proof. ■

2.4 Linear mappings generated by \mathcal{S}_p

At the beginning of this paragraph, we present the class multilinear mappings of type $\mathcal{L}(\mathcal{S}_p)$ for $m = 1$ (i.e., linear mappings of type $B(\mathcal{S}_p)$). The definition was introduced by Braunsch and Junek in [14].

Definition 2.4.1 Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space.

The linear operator $u : H \rightarrow X$ is said to be of type $\mathcal{B}(\mathcal{S}_p)$, in symbols $u \in \mathcal{B}(\mathcal{S}_p)(H; X)$, if it factors through a Hilbert space K

$$\begin{array}{ccc} H & \xrightarrow{u} & X \\ u_1 \searrow & & \nearrow u_2 \\ & K & \end{array} \quad (2.4.1)$$

where $u_1 \in \mathcal{S}_p(K; H)$. The space $\mathcal{B}(\mathcal{S}_p)(H; X)$ is a normed space with the following norm

$$\|u\|_{\mathcal{B}(\mathcal{S}_p)} = \inf \|u_2\| \|u_1\|_{\mathcal{S}_p},$$

where the infimum is taken over all possible factorizations of the form (2.4.1).

Proposition 2.4.1 Let $1 \leq p \leq q \leq \infty$. We have for a Banach space X and a Hilbert space H .

$$\mathcal{B}(\mathcal{S}_p)(H; X) \subset \mathcal{B}(\mathcal{S}_q)(H; X). \quad (2.4.2)$$

2.4.1 Linear mappings of type $\mathcal{S}_p \circ \mathcal{B}$

Definition 2.4.2 Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space.

The linear operator $u : X \rightarrow H$ is said to be of type $\mathcal{S}_p \circ \mathcal{B}$, in symbols $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, if it factors through a Hilbert space K

$$\begin{array}{ccc} X & \xrightarrow{u} & H \\ u_1 \searrow & & \nearrow u_2 \\ & K & \end{array} \quad (2.4.3)$$

where $u_2 \in \mathcal{S}_p(K; H)$. The space $\mathcal{S}_p \circ \mathcal{B}(X; H)$ is a Banach space with the following norm

$$\|u\|_{\mathcal{S}_p \circ \mathcal{B}} = \inf \|u_2\|_{\mathcal{S}_p} \|u_1\|,$$

where the infimum is taken over all possible factorizations of the form (2.4.3). We have

$$\|u\| \leq \|u\|_{\mathcal{S}_p \circ \mathcal{B}}.$$

Remark 2.4.1 If $X = G$ is a Hilbert space, for $1 \leq p < \infty$. Because $\mathcal{S}_p(K; H)$ is a Banach ideal, we have

$$\mathcal{S}_p \circ \mathcal{B}(G; H) = \mathcal{S}_p(G; H). \quad (2.4.4)$$

Indeed, if $u \in \mathcal{S}_p \circ \mathcal{B}(G; H)$ then

$$u = u_2 u_1 : G \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. By the ideal property of the Schatten class \mathcal{S}_p . We obtain

$$u = u_2 \circ u_1 \in \mathcal{S}_p(G; H) = \mathcal{S}_p \circ \mathcal{B}(G; H).$$

Remark 2.4.2 (Same argument) If $X = G$ is a Hilbert space, for $1 \leq p < \infty$. Because $\mathcal{S}_p(K; H)$ is a Banach ideal, we have

$$\mathcal{B}(\mathcal{S}_p)(G; X) = \mathcal{S}_p(G; X).$$

Proposition 2.4.2 Let X a Banach space and H a Hilbert space. Let $1 \leq p < \infty$. Every operator in the space $\mathcal{S}_p \circ \mathcal{B}(X; H)$ is compact.

Proof. Let $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. Then by (2.4.3), we have

$$u = u_2 u_1,$$

with $u_2 \in \mathcal{S}_p(K; H)$. Since u_2 is compact, there exists a sequence of finite rank operators $(u_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \|u_2 - u_n\| = 0.$$

Consider $(u_n u_1)_{n \in \mathbb{N}}$ the sequence of linear operators of finite rank, we have

$$\begin{aligned} \|u - u_n u_1\| &= \|u_2 u_1 - u_n u_1\| \\ &= \|(u_2 - u_n) u_1\| \\ &\leq \|u_2 - u_n\| \|u_1\| \xrightarrow{n} 0. \quad \blacksquare \end{aligned}$$

Proposition 2.4.3 *Let $1 \leq p \leq q \leq \infty$. We have for a Banach space X and a Hilbert space H .*

$$\mathcal{S}_p \circ \mathcal{B}(X; H) \subset \mathcal{S}_q \circ \mathcal{B}(X; H). \quad (2.4.5)$$

Proof. Let $1 \leq p \leq q \leq \infty$. Suppose that $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, by (2.4.3) then $u = u_2 u_1$ when $u_2 \in \mathcal{S}_p(K; H)$. According to (2.2.3), we have $u_2 \in \mathcal{S}_q(K; H)$, then $u_2 u_1 \in \mathcal{S}_q \circ \mathcal{B}(X; H)$. Consequently $u \in \mathcal{S}_q \circ \mathcal{B}(X; H)$. ■

Ideal property of $\mathcal{S}_p \circ \mathcal{B}$

Proposition 2.4.4 *Let H, G be Hilbert spaces and X, Y be Banach spaces. Let $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. v be a operator in $\mathcal{B}(Y; X)$ and w be a operator in $\mathcal{B}(H; G)$. Then, the operator wuv belongs to $\mathcal{S}_p \circ \mathcal{B}(Y; G)$. In addition*

$$\|wuv\|_{\mathcal{S}_p \circ \mathcal{B}} \leq \|w\| \|u\|_{\mathcal{S}_p \circ \mathcal{B}} \|v\|.$$

Proof. Let $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, by (2.4.3) then

$$u = u_2 u_1 : X \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. Suppose that $v \in \mathcal{B}(Y; X)$ and $w \in \mathcal{B}(H; G)$. Then

$$\begin{array}{ccccc} Y & \xrightarrow{v} & X & \xrightarrow{u} & H \\ & & u_1 \downarrow & \nearrow u_2 & w \downarrow \\ & & K & \xrightarrow{w_1} & G \end{array}$$

We have

$$\begin{aligned} wuv &= (wu_2)(u_1v) \\ &= w_1 w_2, \end{aligned}$$

with $w_1 \in \mathcal{S}_p(K; G)$ and $w_2 \in \mathcal{B}(Y; K)$. Therefore $wuv \in \mathcal{S}_p \circ \mathcal{B}(Y; G)$.

On other hand, we also have

$$\|wuv\|_{\mathcal{S}_p \circ \mathcal{B}} \leq \|w\| \|u\|_{\mathcal{S}_p \circ \mathcal{B}} \|v\|. \quad \blacksquare$$

And we deduce the following result.

Corollary 2.4.1 *If X_0 is a subspace of X and $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. Then, the restriction mapping $u|_{X_0} : X_0 \xrightarrow{u} H$ is also of type $\mathcal{S}_p \circ \mathcal{B}$*

$$\text{i.e., } u|_{X_0} \in \mathcal{S}_p \circ \mathcal{B}(X_0; H),$$

and

$$\|u|_{X_0}\|_{\mathcal{S}_p \circ \mathcal{B}} \leq \|u\|_{\mathcal{S}_p \circ \mathcal{B}}.$$

Injectivity of $\mathcal{S}_p \circ \mathcal{B}$

Proposition 2.4.5 *If H_0 is a subspace of H and $i : H_0 \longrightarrow H$ to be the inclusion mapping. Then, The next properties are equivalent.*

- (a) *The operator u belongs to $\mathcal{S}_p \circ \mathcal{B}(X; H)$.*
- (b) *The operator iu belongs to $\mathcal{S}_p \circ \mathcal{B}(X; H_0)$.*

In this case

$$\|iu\|_{\mathcal{S}_p \circ \mathcal{B}} = \|u\|_{\mathcal{S}_p \circ \mathcal{B}}.$$

Proof. Immediately by ideal property of linear mappings of type $\mathcal{S}_p \circ \mathcal{B}$. ■

2.4.2 Relation to ideals of linear mappings

We have some inclusion and coincidence situations with old classes.

Proposition 2.4.6 *Let $1 \leq q \leq 2$, we have*

- (1) $\mathcal{S}_q \circ \mathcal{B}(X; H) \subseteq \mathcal{D}_p(X; H)$, for all p .
- (2) $\mathcal{S}_q \circ \mathcal{B}(X; H) \subseteq \Pi_p(X; H)$, for all p .
- (3) $\mathcal{S}_q \circ \mathcal{B}(X; H) \subseteq I_p(X; H)$, for all $p > 1$.
- (4) $\mathcal{S}_q \circ \mathcal{B}(X; H) \subseteq \mathcal{N}_p(X; H)$, for all $p > 1$.

Proof. (1) Let $1 \leq q \leq 2$ and $u \in \mathcal{S}_q \circ \mathcal{B}(X; H)$, by Proposition 2.4.3 then $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$. There is a Hilbert space K such that $u = u_2 \circ u_1$ with $u_2 \in \mathcal{S}_2(K; H)$. By Theorem 2.3.4 we have $u_2 \in \mathcal{D}_p(K; H)$, for all p . We conclude the proof by the ideal property of (Cohen) strongly p -summing operators.

(2) Same argument with (1).

(3) Let $1 \leq q \leq 2$ and $u \in \mathcal{S}_q \circ \mathcal{B}(X; H)$, by Proposition 2.4.3 then $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$. There is a Hilbert space K such that $u = u_2 \circ u_1$ with $u_2 \in \mathcal{S}_2(K; H)$. By Theorem 2.3.3 this implies that $u_2 \in I_p(K; H)$, for all $p > 1$. We conclude the proof by the ideal property of p -integral operators.

(4) Same argument with (3).

Proposition 2.4.7 *Let X a Banach space and H a Hilbert space, we have*

(a) $\mathcal{S}_1 \circ \mathcal{B}(X; H) \subseteq I(X; H)$.

(b) $\mathcal{S}_1 \circ \mathcal{B}(X; H) \subseteq \mathcal{N}(X; H)$.

Proof. (a) Let $u \in \mathcal{S}_1 \circ \mathcal{B}(X; H)$. So, There is a Hilbert space K such that $u = u_2 \circ u_1$ with $u_2 \in \mathcal{S}_1(K; H)$. Then from Theorem 2.3.3 (b), we have $u_2 \in I(K; H)$, by the ideal property of p -integrale, we obtain $u_2 \circ u_1 \in I(X; H)$, therefore $u \in I(X; H)$.

(b) Same argument with (a). ■

Remark 2.4.3 *If X is a Banach space isomorphic to a Hilbert space, it is not difficult to show that the coincidence in the situations (1) in proposition 2.4.6 (respectively (a), (b) in proposition 2.4.7)*

Proposition 2.4.8 *Let X be a Banach space and H be a Hilbert space. Then*

If $1 < p < \infty$. The following statements are equivalent:

(a) *The operator u belongs to $\mathcal{S}_p \circ \mathcal{B}(X; H)$.*

(b) *The operator u^{**} belongs to $\mathcal{S}_p \circ \mathcal{B}(X^{**}; H)$.*

In this case

$$\|u^{**}\|_{\mathcal{S}_p \circ \mathcal{B}} = \|u\|_{\mathcal{S}_p \circ \mathcal{B}}.$$

Proof. (a) \Rightarrow (b) : Let $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, then

$$u = u_2 u_1 : X \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{S}_p(K; H)$. The adjoint of u is given by

$$u^* = u_1^* u_2^* : H \xrightarrow{u_2^*} K \xrightarrow{u_1^*} X^*.$$

The second adjoint of u is given by

$$u^{**} = u_2^{**} u_1^{**} : X^{**} \xrightarrow{u_1^{**}} K \xrightarrow{u_2^{**}} H,$$

then $u_2^{**} \in \mathcal{S}_p(K; H)$ and consequently $u^{**} \in \mathcal{S}_p \circ \mathcal{B}(X^{**}; H)$.

(b) \Rightarrow (a) : By the next factorization

$$\begin{array}{ccc} X & \xrightarrow{u} & H \\ k_X \downarrow & & k_H \downarrow \\ X^{**} & \xrightarrow{u^{**}} & H^{**} \end{array}$$

$$\text{i.e., } u = k_H^{-1} u^{**} k_X,$$

we can show that $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. On the other hand, we have

$$\begin{aligned} \|u^{**}\|_{\mathcal{S}_p \circ \mathcal{B}} &= \|u_2^{**} u_1^{**}\|_{\mathcal{S}_p \circ \mathcal{B}} \\ &\leq \|u_2^{**}\|_{\mathcal{S}_p} \|u_1^{**}\| \\ &\leq \|u_2^{**}\|_{\mathcal{S}_p} \|u_1\|, \end{aligned}$$

so

$$\begin{aligned} \|u\|_{\mathcal{S}_p \circ \mathcal{B}} &= \|k_H^{-1} u^{**} k_X\|_{\mathcal{S}_p \circ \mathcal{B}} \\ &\leq \|k_H^{-1}\| \|u^{**}\|_{\mathcal{S}_p \circ \mathcal{B}} \|k_X\| \\ &\leq \|u^{**}\|_{\mathcal{S}_p \circ \mathcal{B}}. \end{aligned}$$

Consequently

$$\|u^{**}\|_{\mathcal{S}_p \circ \mathcal{B}} = \|u\|_{\mathcal{S}_p \circ \mathcal{B}}. \quad \blacksquare$$

From the above result and the ideal property, we deduce the following consequence.

Corollary 2.4.2 *If $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, if and only if, $k_H u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. Moreover,*

$$\|k_H u\|_{\mathcal{S}_p \circ \mathcal{B}} = \|u\|_{\mathcal{S}_p \circ \mathcal{B}}.$$

Remark 2.4.4 For $1 < p < \infty$, the adjoint of $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, he can not be of type $\mathcal{S}_p \circ \mathcal{B}$.

Lemma 2.4.1 Let X, Y be Banach spaces such that Y is reflexive. Let $u : X \rightarrow Y$ be a (Cohen) strongly 2-summing linear operator. Then, u factors through a Hilbert space, i.e., \exists a Hilbert space H and two linear operators v_1, v_2 such that

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ v_1 \searrow & & \nearrow v_2 \\ & H & \end{array} \quad (2.4.6)$$

In other words

$$u = v_2 \circ v_1,$$

with $v_2 \in \mathcal{D}_2(H; Y)$.

Proof. Let $u \in \mathcal{D}_2(X; Y)$. By Corollary 1.2.2,(3), its adjoint $u^* : Y^* \rightarrow X^*$ is 2-summing. Then, u^* factors through a Hilbert space H (see the Pietsch Factorization Theorem) Theorem 1.2.1, i.e.,

$$u^* = v_2 v_1 : Y^* \xrightarrow{v_1} H \xrightarrow{v_2} X^*,$$

where v_1 is 2-summing (i.e., its adjoint v_1^* is strongly 2-summing). The second adjoint of u is given by

$$u^{**} = v_1^* v_2^* : X^{**} \xrightarrow{v_2^*} H^* \xrightarrow{v_1^*} Y^{**}.$$

We use the elementary identity $u^{**} k_X = k_Y u$ which comes to the next diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ k_X \downarrow & & k_Y \downarrow \\ X^{**} & \xrightarrow{u^{**}} & Y^{**} \end{array}$$

with k_Y is bijective (Y is reflexive). Consequently, we obtain

$$u = (k_Y^{-1} v_1^*) (v_2^* k_X). \quad \blacksquare$$

There is an interesting relationship between (Cohen) strongly p -summing linear operators and linear operators of type $\mathcal{S}_2 \circ \mathcal{B}$.

Theorem 2.4.1 *Let X be a Banach space and H be a Hilbert space. For all $2 \leq p \leq \infty$, we have*

$$\mathcal{S}_2 \circ \mathcal{B}(X; H) = \mathcal{D}_p(X; H). \quad (2.4.7)$$

Proof. Let $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$. By the factorization (2.4.3), $u = u_2 \circ u_1$ with $u_2 \in \mathcal{S}_2(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. By (2.3.1) and the ideal property, we have

$$u \in \mathcal{D}_p(X; H).$$

Reciprocally, let $2 \leq p \leq \infty$ and $u \in \mathcal{D}_p(X; H)$. By Corollary 1.2.2 (1), u is in $\mathcal{D}_2(X; H)$. According to Lemma 2.4.1, u factors through a Hilbert space

$$u = u_2 \circ u_1 : X \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{D}_2(K; H) (= \mathcal{S}_2(K; H))$. Therefore, $u = u_2 u_1 \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$. ■

Duality for $\mathcal{S}_p \circ \mathcal{B}(X; H)$

Proposition 2.4.9 *Let $1 \leq p, r, q < \infty$ and X a Banach space and H a Hilbert space. $T \in \mathcal{B}(X; H)$. The following properties are equivalent.*

- (1) *The operator T belongs to $\mathcal{S}_r \circ \mathcal{B}(X; H)$.*
- (2) *The operator T is factored $T = uv$ with $u \in \mathcal{S}_q \circ \mathcal{B}(X; K_1)$ and $v \in \mathcal{S}_p(K_1; H)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

Proof. (1) \Rightarrow (2) : Let $T \in \mathcal{S}_r \circ \mathcal{B}(X; H)$. Then by (2.4.3), we have

$$T = u_2 u_1,$$

with $u_2 \in \mathcal{S}_r(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. By Theorem 2.2.1, the operator $u_2 = v_2 v_1$ where $v_2 \in \mathcal{S}_p(K_1; H)$ and $v_1 \in \mathcal{S}_q(K; K_1)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

$$\text{i.e., } u_2 = v_2 v_1 : K \xrightarrow{v_1} K_1 \xrightarrow{v_2} H.$$

Then, $T = v_2 v_1 u_1$, by the ideal property of the Schatten class \mathcal{S}_q , we have

$$v_1 u_1 = w \in \mathcal{S}_q \circ \mathcal{B}(X; K_1), \text{ i.e., } w = v_2 u_1 : X \xrightarrow{u_1} K \xrightarrow{v_2} K_1$$

with $v_1 \in \mathcal{S}_q(K; K_1)$ and $u_1 \in \mathcal{B}(X; K)$. So

$$T = v_2 w, \text{ where } v_2 \in \mathcal{S}_p(K_1; H) \text{ and } w \in \mathcal{S}_q \circ \mathcal{B}(X; K_1).$$

(2) \Rightarrow (1) : Suppose that $T = uv$, where $u \in \mathcal{S}_q \circ \mathcal{B}(X; K_1)$ and $v \in \mathcal{S}_p(K_1; H)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Then $u = u_2 u_1$, with $u_2 \in \mathcal{S}_q(K; H)$ and $u_1 \in \mathcal{B}(X; K)$, by Theorem 2.2.1 therefore

$$u_2 v = w \in \mathcal{S}_r(K; H).$$

Then

$$w u_1 \in \mathcal{S}_r(K; H). \quad \blacksquare$$

2.4.3 Connection with linear mappings of type $\mathcal{B}(\mathcal{S}_p)$

Proposition 2.4.10 *Let X be a Banach space, H be a Hilbert space and $1 < p < \infty$, then the following statement are equivalent:*

- (1) *The operator u belongs to $\mathcal{S}_p \circ \mathcal{B}(X; H)$.*
- (2) *The operator u^* belongs to $\mathcal{B}(\mathcal{S}_p)(H; X^*)$.*

Proof.(1) \Rightarrow (2) : Let $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$, then

$$u = u_2 u_1 : X \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $u_1 \in \mathcal{B}(X; K)$, the adjoint of u is given by

$$u^* = u_1^* u_2^* : H \xrightarrow{u_2^*} K \xrightarrow{u_1^*} X^*,$$

with $u_2^* \in \mathcal{S}_p(K; H)$ and $u_1^* \in \mathcal{B}(K; X^*)$. Consequently, we obtain $u^* \in \mathcal{B}(\mathcal{S}_p)(H; X^*)$.

(2) \Rightarrow (1) : Let $u^* \in \mathcal{B}(\mathcal{S}_p)(H; X^*)$, the adjoint of u^* is given by

$$u^{**} = u_2^{**} u_1^{**} : X^{**} \xrightarrow{u_1^{**}} K \xrightarrow{u_2^{**}} H,$$

so $u_2^{**} \in \mathcal{S}_p(K; H)$ and consequently $u^{**} \in \mathcal{S}_p \circ \mathcal{B}(X^{**}; H)$, by Proposition 2.4.8, then $u \in \mathcal{S}_p \circ \mathcal{B}(X; H)$. \blacksquare

Proposition 2.4.11 *Let $1 \leq p < \infty$ and X a Banach space and H a Hilbert space. Suppose*

$\frac{1}{p} + \frac{1}{p^} = 1$, we have*

1) If $T_1 \in \mathcal{S}_p \circ \mathcal{B}(X; H)$ and $T_2 \in \mathcal{B}(\mathcal{S}_{p^})(H; X)$, then $T_1 T_2 \in \mathcal{S}_1(H; H)$.*

2) If $T_1 \in \mathcal{S}_p \circ \mathcal{B}(X; H)$ and $T_2 \in \mathcal{B}(\mathcal{S}_p)(H; X)$, then $T_1 T_2 \in \mathcal{S}_p(H; H)$.

Proof. 1) Let $T_1 \in \mathcal{S}_p \circ \mathcal{B}(X; H)$ and $T_2 \in \mathcal{B}(\mathcal{S}_{p^*})(H; X)$. Then by (2.4.3) and (2.4.1), we have

$$T_1 T_2 = u_2 u_1 v_2 v_1$$

$$\text{i.e., } T_1 T_2 : H \xrightarrow{v_1} K \xrightarrow{v_2} X \xrightarrow{u_1} K \xrightarrow{u_2} H,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $v_1 \in \mathcal{S}_{p^*}(H; K_1)$. Firstly $u_1 v_2 = w \in \mathcal{B}(K_1; K)$. By the ideal property of the Schatten class \mathcal{S}_p , then $u_2 w \in \mathcal{S}_p(K_1; H)$. By Theorem 2.2.1, the operator

$$u_2 w v_1 \in \mathcal{S}_1(H; H).$$

Consequently

$$T_1 T_2 \in \mathcal{S}_1(H; H).$$

2) Same argument with 1). ■

Since $\mathcal{S}_p(H_1; H_2)$ is a Banach ideal and by Theorem 2.2.2, we have the following results

Corollary 2.4.3 *Let $1 < p < \infty$ and H_1, H_2 be Hilbert spaces. We have the following isometric identification*

$$[\mathcal{S}_p \circ \mathcal{B}(H_1; H_2)]^* = \mathcal{S}_{p^*} \circ \mathcal{B}(H_2; H_1), \quad (2.4.8)$$

where duality is defined by

$$\forall u \in \mathcal{S}_{p^*} \circ \mathcal{B}(H_2; H_1), \forall v \in \mathcal{S}_p \circ \mathcal{B}(H_1; H_2) : \langle u, v \rangle = \text{Tr}_{H_2}(vu).$$

Corollary 2.4.4 *Let H_1, H_2 be Hilbert spaces. Then,*

$H_1 \widehat{\otimes}_2 H_2$ isometrically identifies with $\mathcal{S}_2 \circ \mathcal{B}(H_1; H_2)$.

$H_1 \widehat{\otimes}_\pi H_2$ isometrically identifies with $\mathcal{S}_1 \circ \mathcal{B}(H_1; H_2)$.

2.5 Characterization of the classes of type $\mathcal{S}_2 \circ \mathcal{B}$ and $\mathcal{B}(\mathcal{S}_2)$

Theorem 2.5.1 *Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space. The following properties are equivalent.*

(1) *The operator u belongs to $\mathcal{S}_2 \circ \mathcal{B}(X; H)$.*

(2) *There exist a linear operator $v_1 \in \mathcal{B}(\mathcal{L}_1; H)$ and $w \in \Gamma_{2-fat}(X; \mathcal{L}_1)$ such that*

$$u = v_1 \circ w.$$

Proof. (1) \Rightarrow (2) : Let $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$, Then by (2.4.3), we have

$$u = u_2 u_1,$$

with $u_2 \in \mathcal{S}_2(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. By Theorem 2.3.1, the operator u_2 factors through an \mathcal{L}_1 -space, i.e., $u_2 = v_1 \circ v_2$ with $v_2 \in \mathcal{B}(K; \mathcal{L}_1)$ and $v_1 \in \mathcal{B}(\mathcal{L}_1; H)$, thus

$$\begin{array}{ccc} X & \xrightarrow{u} & H \\ u_1 \downarrow & & \uparrow v_1 \\ K & \xrightarrow{v_2} & \mathcal{L}_1 \end{array}$$

Then

$$\begin{aligned} u &= v_1 \circ v_2 \circ u_1 \\ &= v_1 \circ w, \end{aligned}$$

where $w = v_2 \circ u_1 \in \Gamma_{2-fat}(X; \mathcal{L}_1)$.

(2) \Rightarrow (1) : Suppose that $u = v_1 \circ w$, where $w \in \Gamma_{2-fat}(X; \mathcal{L}_1)$ and $v_1 \in \mathcal{B}(\mathcal{L}_1; H)$. One can write w as

$$w = v_2 \circ u_1 : X \xrightarrow{u_1} K_1 \xrightarrow{v_2} \mathcal{L}_1.$$

By Theorem 1.2.2, the operator v_1 is 2-summing. So it factors through a Hilbert space, i.e., $v_1 = t_1 \circ t_2 : \mathcal{L}_1 \xrightarrow{t_2} K_2 \xrightarrow{t_1} H$ with $t_2 \in \Pi_2(\mathcal{L}_1; K_2)$. Therefore,

$$u = v_1 \circ w = t_1 \circ t_2 \circ v_2 \circ u_1,$$

with $t_1 \circ t_2 \circ v_2 \in \mathcal{S}_2(K_1; H)$ and $u_1 \in \mathcal{B}(X; K_1)$

$$u = t_1 \circ t_2 \circ v_2 \circ u_1 : X \xrightarrow{u_1} K_1 \xrightarrow{v_2} \mathcal{L}_1 \xrightarrow{t_2} K_2 \xrightarrow{t_1} H.$$

Then $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$, this ends the proof. ■

Theorem 2.5.2 *Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space. The following properties are equivalent.*

(1) *The operator u belongs to $\mathcal{S}_2 \circ \mathcal{B}(X; H)$.*

(2) *For every Banach space Z , there exist $v_1 \in \mathcal{B}(Z; H)$ and $w \in \Gamma_{2-fat}(X; Z)$ such that*

$$u = v_1 \circ w.$$

Proof. (1) \Rightarrow (2) : Let $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$, then by (2.4.3), we have

$$u = u_2 u_1,$$

with $u_2 \in \mathcal{S}_2(K; H)$ and $u_1 \in \mathcal{B}(X; K)$. By Theorem 2.3.2, the operator u_2 factors through a Banach space Z , i.e., $u_2 = v_1 \circ v_2$ with $v_2 \in \mathcal{B}(K; Z)$ and $v_1 \in \mathcal{B}(Z; H)$, thus

$$\begin{array}{ccc} X & \xrightarrow{u} & H \\ u_1 \downarrow & & \uparrow v_1 \\ K & \xrightarrow{v_2} & Z \end{array}$$

then

$$\begin{aligned} u &= v_1 \circ v_2 \circ u_1 \\ &= v_1 \circ w, \end{aligned}$$

where $w = v_2 \circ u_1 \in \Gamma_{2-fat}(X; Z)$.

(2) \Rightarrow (1) : Suppose that $u = v_1 \circ w$, where $w \in \Gamma_{2-fat}(X; Z)$ and $v_1 \in \mathcal{B}(Z; H)$. One can write w as

$$\begin{aligned} w &= v_2 \circ u_1 : X \xrightarrow{u_1} K_1 \xrightarrow{v_2} Z. \\ u &= v_1 \circ w = v_1 \circ v_2 \circ u_1, \end{aligned}$$

by Theorem 2.3.2, the operator $s = (v_1 \circ v_2)$ belongs to $\mathcal{S}_2(K_1; H)$. So $s \circ u_1$ belongs to $\mathcal{S}_2 \circ \mathcal{B}(X; H)$, with $u_1 \in \mathcal{B}(X; K)$ and $s \in \mathcal{S}_2(K_1; H)$. Therefore $u \in \mathcal{S}_2 \circ \mathcal{B}(X; H)$, this ends the proof. ■

The proof of the following theorems is done in the same way as the theorems previous.

Theorem 2.5.3 *Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space. The following properties are equivalent.*

- (1) *The operator u belongs to $\mathcal{S}_2 \circ \mathcal{B}(X; H)$.*
- (2) *There exist a linear operator $v_1 \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $w \in \Gamma_{2-fat}(X; \mathcal{L}_\infty)$ such that $u = v_1 \circ w$.*

Theorem 2.5.4 *Let $1 \leq p < \infty$, X a Banach space and H a Hilbert space. The following properties are equivalent.*

- (a) *The operator u belongs to $\mathcal{B}(\mathcal{S}_2)(H; X)$.*
- (b) *There exist a linear operator $v_1 \in \mathcal{B}(H; \mathcal{L}_\infty)$ and $w \in \Gamma_{2-fat}(\mathcal{L}_\infty; X)$ such that $u = w \circ v_1$.*
- (c) *There exist a linear operator $v_1 \in \mathcal{B}(H; \mathcal{L}_1)$ and $w \in \Gamma_{2-fat}(\mathcal{L}_1; X)$ such that $u = w \circ v_1$.*
- (d) *For every Banach space Z , there exist $v_1 \in \mathcal{B}(H; Z)$ and $w \in \Gamma_{2-fat}(Z; X)$ such that $u = w \circ v_1$.*

Corollary 2.5.1 *As characterization result that give by Theorem 2.3.1, we can say that linear mappings of type $\mathcal{S}_2 \circ \mathcal{B}$ and $\mathcal{B}(\mathcal{S}_2)$ are characterized by their factorizations by \mathcal{L}_1 -space and by \mathcal{L}_∞ -space.*

Chapter 3

On the composition ideals of Schatten class type mappings

The content of this chapter is based on a paper published in Journal of Mathematics [6]. In this chapter, we study the composition ideals of multilinear mappings generated by Schatten class. Firstly, we present a factorization theorem for Hilbert-Schmidt multilinear operators by \mathcal{L}_1 -space and by \mathcal{L}_∞ -space. Secondly, we introduce the definition of multilinear mappings of type $\mathcal{S}_p \circ \mathcal{L}$ and we give some coincidence theorems for Cohen strongly 2-summing multilinear operators. Finally, we present factorization results like that given by Lindenstrauss-Pelczyński for Hilbert Schmidt linear operators.

3.1 Ideals of multilinear mappings

Definition 3.1.1 (*The m -linear mappings of finite type*) A multilinear mapping $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is of finite type if it is a finite sum of operators of the form

$$T_{y \otimes_{j=1}^m x_j^*} = x_1^* \otimes \dots \otimes x_m^* \otimes y : (x^1, \dots, x^m) \rightarrow x_1^*(x^1) \dots x_m^*(x^m) y,$$

where $x_j^* \in X_j^*$ ($1 \leq j \leq m$) and $y \in Y$. We denote by $\mathcal{L}_f(X_1, \dots, X_m; Y)$ the space of all finite type multilinear operators.

Definition 3.1.2 An ideal of multilinear mappings (or multi-ideal) \mathcal{M} is a subclass of the class for all continuous multilinear mappings such that for all $m \in \mathbb{N}$ and Banach spaces X_1, \dots, X_m and Y , the component $\mathcal{M}(X_1, \dots, X_m; Y) = \mathcal{L}(X_1, \dots, X_m; Y) \cap \mathcal{M}$ satisfy:

(1) $\mathcal{M}(X_1, \dots, X_m; Y)$ is a linear subspace of $\mathcal{L}(X_1, \dots, X_m; Y)$ which contains the m -linear mappings of finite type.

(2) The ideal property: If $T \in \mathcal{M}(X_1, \dots, X_m; Y)$, $u_j \in \mathcal{B}(E_j; X_j)$ and $v \in \mathcal{B}(Y; F)$, then $v \circ T \circ (u_1, \dots, u_m)$ is in $\mathcal{M}(E_1, \dots, E_m; F)$.

If $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbb{R}^+$ satisfies

(1') $(\mathcal{M}(X_1, \dots, X_m; Y), \|\cdot\|_{\mathcal{M}})$ is a normed (Banach) for all Banach spaces X_1, \dots, X_m and Y and for all $m \in \mathbb{N}^*$.

(2') $\|A^n : \mathbb{K}^n \rightarrow \mathbb{K}; A^n(x_1, \dots, x_m) = x_1 \dots x_m\|_{\mathcal{M}} = 1$.

(3') If $T \in \mathcal{M}(X_1, \dots, X_m; Y)$, $u_j \in \mathcal{B}(E_j; X_j)$, for $j = 1, \dots, m$ and $v \in \mathcal{B}(Y; F)$, then

$$\|v \circ T \circ (u_1, \dots, u_m)\|_{\mathcal{M}} \leq \|v\| \|T\|_{\mathcal{M}} \|u_1\| \dots \|u_m\|,$$

then $(\mathcal{M}; \|\cdot\|_{\mathcal{M}})$ is called a normed (Banach) multi-ideal.

Pietsch in [39], proposed methods for construct multi-ideals from a linear ideal. The composition method and the factorization method. But we will focus our study on the first method only, because we need it later.

Definition 3.1.3 (The composition method) Let \mathcal{I} be an operator ideal, a multilinear operator $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ is said to be of type $\mathcal{I} \circ \mathcal{L}$, in symbols $T \in \mathcal{I} \circ \mathcal{L}(X_1, \dots, X_m; Y)$, if there exist a Banach space G , a linear mapping $u \in \mathcal{I}(G; Y)$ and a multilinear operator $A \in \mathcal{L}(X_1, \dots, X_m; G)$ such that the following diagram commutes

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & Y \\ & \searrow A & u \uparrow \\ & & G \end{array}$$

In other words $T = u \circ A$. If \mathcal{I} is a normed operator ideal and $T \in \mathcal{I} \circ \mathcal{L}(X_1, \dots, X_m; Y)$, we define

$$\|T\|_{\mathcal{I} \circ \mathcal{L}} = \inf \|u\|_{\mathcal{I}} \|A\|,$$

where the infimum is taken over all possible factorizations $T = u \circ A$ with $u \in \mathcal{I}$.

3.1.1 Cohen strongly p -summing multilinear operators

In this passage, we present the definition of Cohen strongly p -summing multilinear operators and result. Which we will need in the sequel.

Definition 3.1.4 [2, Definition 2.1] *Let $1 \leq p \leq \infty$. An m -linear operator $T : X_1 \times \dots \times X_m \longrightarrow Y$ is Cohen strongly p -summing if its linearization \widehat{T} is (Cohen) p -summing linear operator. The class of Cohen strongly m -linear operators from $X_1 \times \dots \times X_m$ into Y is denoted by $\mathcal{D}_p^m(X_1, \dots, X_m; Y)$, which is a Banach space.*

Proposition 3.1.1 [2, Corollary 2.5] *If $1 \leq p \leq q$. Then*

$$\mathcal{D}_q^m(X_1, \dots, X_m; Y) \subset \mathcal{D}_p^m(X_1, \dots, X_m; Y),$$

for all Banach spaces X_1, \dots, X_m, Y .

3.1.2 Factorization of Hilbert-Schmidt multilinear mappings

In this paragraph, We will prove by anti-examples that this characterization is not verified in the multilinear, opposite linear case.

The definition of the Hilbert-Schmidt multilinear mappings was introduced by Dwyer [23]. And studied by several authors including Pietsch in [39]. These mappings where also studied by Matos in [30] for the vector -valued case.

Definition 3.1.5 *Let H_1, \dots, H_m, H be Hilbert spaces. A mapping $T \in \mathcal{L}(H_1, \dots, H_m; H)$ is said to be Hilbert -Schmidt if there is an orthonormal basis $(e_{i_j})_{i_j \in I_j}$ for H_j , for each $j = 1, \dots, m$, such that*

$$\|T\|_{\mathcal{HS}} = \sum_{i_1 \in I_1, \dots, i_m \in I_m} \|T(e_{i_1}, \dots, e_{i_m})\|^2 < \infty.$$

We denote by $\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$ the space of all Hilbert-Schmidt multilinear mappings.

It is easy to show that it is a Hilbert space under the norm $\|\cdot\|_{\mathcal{HS}}$ defined by the inner product

$$\langle T_1, T_2 \rangle = \sum_{i_1 \in I_1, \dots, i_m \in I_m} \left\langle T_1(e_{i_1}, \dots, e_{i_m}), \overline{T_2(e_{i_1}, \dots, e_{i_m})} \right\rangle.$$

Definition 3.1.6 [30, Definition 5.8] It is considered on $H_1 \otimes \dots \otimes H_m$ in the inner product

$$\langle h, k \rangle = \prod_{j=1}^m \langle h_j, k_j \rangle,$$

where $h = (h_1 \otimes \dots \otimes h_m), k = (k_1 \otimes \dots \otimes k_m) \in H_1 \otimes \dots \otimes H_m$. The space $H_1 \otimes \dots \otimes H_m$ with this inner product is denoted by $H_1 \otimes_2 \dots \otimes_2 H_m$ and its completion by $H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m$. The corresponding norm is denoted by $\|\cdot\|_2$.

If e_k is an orthonormal basis for $H_k, k = 1, \dots, m$. Then

$$\{e_{k_1} \otimes \dots \otimes e_{k_m}\}_{\substack{k_j \in I_j \\ 1 \leq j \leq m}}$$

is an orthonormal basis for $H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m$.

Proposition 3.1.2 [30, Propostion 5.10] Let H_1, \dots, H_m, H be Hilbert spaces. The following properties are equivalent.

- (1) The operator T belongs to $\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$.
- (2) The operator \widetilde{T}_2 belongs to $S_2(H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m; H)$, where \widetilde{T}_2 denotes the extension of \widetilde{T} on the space $H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m$.

In this case $\|T\|_{\mathcal{HS}} = \|\widetilde{T}_2\|_{S_2}$.

Theorem 3.1.1 [32, Theorem 2.10] Let $T \in \mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$ and H_1, \dots, H_m, H be Hilbert spaces and then for all G Banach space, we have $T = u \circ A$ where $u \in \mathcal{B}(G; H)$ and $A \in \mathcal{L}(H_1, \dots, H_m; G)$.

Remark 3.1.1 In [32, Example 2.12] C. A. Mendes has given an example such that, the converse is not true in general.

By using the composition method, we give an other similar result on factorization of Hilbert-Schmidt multilinear mappings.

Theorem 3.1.2 Let H_1, \dots, H_m, H be Hilbert spaces. If $T \in \mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$, then T factors through an \mathcal{L}_∞ -space and \mathcal{L}_1 -space in the following ways:

- (1) T is represented by $T = u \circ A$ where $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $A \in \mathcal{L}(H_1, \dots, H_m; \mathcal{L}_\infty)$.
- (2) T is represented by $T = u \circ A$ where $u \in \mathcal{B}(\mathcal{L}_1; H)$ and $A \in \mathcal{L}(H_1, \dots, H_m; \mathcal{L}_1)$.

Proof. (1) Let $T \in \mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$. By Proposition 3.1.2, \tilde{T}_2 is in $\mathcal{S}_2(H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m; H)$. The Theorem 2.3.2 assure the existence of two operators $v \in \mathcal{B}(H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m; \mathcal{L}_\infty)$ and $u \in \mathcal{B}(\mathcal{L}_\infty, H)$ such that

$$\tilde{T}_2 = uv,$$

then, $T = u \circ A$ with

$$A = v \circ i_m,$$

with i_m is defined by

$$\begin{aligned} i_m : H_1 \times \dots \times H_m &\rightarrow H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m \\ (h_1, \dots, h_m) &\mapsto h_1 \otimes \dots \otimes h_m. \end{aligned}$$

Same method then the second. ■

Remark 3.1.2 *The reciprocal of the previous theorem is not true in general. Indeed, consider the following example.*

Let $A : H \times H \times H \rightarrow \mathcal{L}_\infty$, the operator define by

$$A(x, y, z) = \langle x, y \rangle z.$$

Let $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ (i.e., $u \in \Pi_2(\mathcal{L}_\infty; H)$) by Theorem 1.2.3. Then, the operator T is represented by $T = u \circ A$ where $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $A \in L(H, H, H; \mathcal{L}_\infty)$.

Now, let us check that T is not Hilbert-Schmidt. Indeed

$$\begin{aligned} \sum_{k_1 \in I, k_2 \in I, k_3 \in I} \|T(e_{k_1}, e_{k_2}, e_{k_3})\|^2 &= \sum_{k_1 \in I, k_2 \in I, k_3 \in I} \|\langle e_{k_1}, e_{k_2} \rangle u(e_{k_3})\|^2 \\ &= \sum_{k_1 \in I, k_2 \in I, k_3 \in I} |\langle e_{k_1}, e_{k_2} \rangle|^2 \|u(e_{k_3})\|^2 \\ &= \sum_{k_3 \in I} \|u(e_{k_3})\|^2 \sum_{k_2 \in I} \|e_{k_2}\|^4 = +\infty. \end{aligned}$$

3.2 Multilinear mappings generated by \mathcal{S}_p

3.2.1 Multilinear operators of type $\mathcal{L}(\mathcal{S}_p)$

In this paragraph, we present the multilinear operators of type $\mathcal{L}(\mathcal{S}_p)$ with some properties. The definition was introduced by Braunsch and Junek in [14].

Definition 3.2.1 Let H_1, \dots, H_m be Hilbert spaces and F a Banach space. A multilinear operator $T : H_1 \times \dots \times H_m \rightarrow F$ is said to be of type $\mathcal{L}(\mathcal{S}_p)$, in symbols $T \in \mathcal{L}(\mathcal{S}_p)(H_1, \dots, H_m; F)$ if, for each $i = 1, \dots, m$, there exist a Hilbert space K_i , a linear operator $u_i \in \mathcal{S}_p(K_i; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K)$ such that

$$T = \mathcal{A} \circ (u_1, \dots, u_m).$$

A norm for that space is

$$\|T\|_{\mathcal{L}(\mathcal{S}_p)} = \inf_{T=\mathcal{A} \circ (u_1, \dots, u_m)} \|\mathcal{A}\| \prod_{i=1}^m \|u_i\|_{\mathcal{S}_p}.$$

Proposition 3.2.1 If $1 \leq p \leq q < \infty$, we have

$$\mathcal{L}(\mathcal{S}_p)(H_1, \dots, H_m; H) \subset \mathcal{L}(\mathcal{S}_q)(H_1, \dots, H_m; H).$$

Corollary 3.2.1 Let H_1, \dots, H_m, H be Hilbert spaces. We have

$$\mathcal{L}(\mathcal{S}_2)(H_1, \dots, H_m; H) \subset \mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H).$$

This inclusion is strict in general.

Proposition 3.2.2 Let H_1, \dots, H_m be Hilbert spaces and F a Banach space. For all $1 \leq p \leq 2$, we have

$$\mathcal{L}(\mathcal{S}_2)(H_1, \dots, H_m; F) = \mathcal{L}_d^p(H_1, \dots, H_m; F).$$

Finally this paragraph, we present factorization of Schatten class type mappings $\mathcal{L}(\mathcal{S}_2)$ introduced by C. A. Mendes in [33].

Theorem 3.2.1 Let H_1, \dots, H_m be Hilbert spaces and F a Banach space. The following properties are equivalent:

(a) $T \in \mathcal{L}(\mathcal{S}_2)(H_1, \dots, H_m; F)$.

(b) For each $j = 1, \dots, m$, there exist Y_j is an \mathcal{L}_1 -space, an linear operator $u_j \in \mathcal{B}(H_j; Y_j)$ and $\mathcal{R} \in \mathcal{L}_d^2(Y_1, \dots, Y_m; F)$ such that

$$T = \mathcal{R} \circ (u_1, \dots, u_m).$$

(b) For each $j = 1, \dots, m$, there exist Y_j is an \mathcal{L}_∞ -space, an linear operator $u_j \in \mathcal{B}(H_j; Y_j)$ and $\mathcal{R} \in \mathcal{L}_d^2(X_1, \dots, X_m; F)$ such that

$$T = \mathcal{R} \circ (u_1, \dots, u_m).$$

Where $\mathcal{L}_d^2(X_1, \dots, X_m; F)$ the class of all 2-dominated multilinear mappings.

3.2.2 Multilinear operators of type $\mathcal{S}_p \circ \mathcal{L}$

We introduce a similar definition to the category of multilinear mappings. This procedure is a particular case of the technique called composition ideals to generated multilinear ideals from a given linear ideal.

Definition 3.2.2 *Let X_1, \dots, X_m be Banach spaces and H a Hilbert space. A multilinear operator $T : X_1 \times \dots \times X_m \rightarrow H$ is said to be of type $\mathcal{S}_p \circ \mathcal{L}$, in symbols $T \in \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m, H)$; if there exist a Hilbert space K , a linear operator $u \in \mathcal{S}_p(K; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K)$ such that the following diagram commutes*

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & H \\ & \searrow A & u \uparrow \\ & & K \end{array}$$

In other words

$$T = u \circ A. \tag{3.2.1}$$

The space $\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H)$ is a Banach space with the following norm

$$\|T\|_{\mathcal{S}_p \circ \mathcal{L}} = \inf \|u\|_{\mathcal{S}_p} \|A\|,$$

where the infimum is taken over all possible factorizations of the form (3.2.1). We have

$$\|T\| \leq \|T\|_{\mathcal{S}_p \circ \mathcal{L}}.$$

Remark 3.2.1 *If $X_i = H_i, i = 1, \dots, m$. H_i are be Hilbert spaces, for $1 \leq p < \infty$, in general case the spaces $\mathcal{S}_p \circ \mathcal{L}(H_1, \dots, H_m; H)$ and $\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$ are not coincident. Contrary to the linear case.*

The representation of compact multilinear operators by the technique of composition ideals (see [13]) is the key to the following result.

Proposition 3.2.3 *Let X_1, \dots, X_m be Banach spaces and H a Hilbert space. Let $1 \leq p < \infty$. Every operator in the space $\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H)$ is compact.*

Proof. Let $T \in \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m, H)$. Then by (3.2.1), we have

$$T = u \circ A,$$

with $u \in \mathcal{S}_p(K; H)$. Since u is compact, there exists a sequence of finite rank operators $(u_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \|u - u_n\| = 0.$$

Consider $(u_n A)_{n \in \mathbb{N}}$ the sequence of multilinear operators of finite rank, we have

$$\begin{aligned} \|T - u_n A\| &= \|u A - u_n A\| \\ &= \|(u - u_n) A\| \leq \|u - u_n\| \|A\| \xrightarrow{n} 0. \quad \blacksquare \end{aligned}$$

As in the linear case, we can prove the following.

Proposition 3.2.4 *Let X_1, \dots, X_m be Banach space and H a Hilbert space.*

If $1 \leq p \leq q < \infty$, we have

$$\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H) \subseteq \mathcal{S}_q \circ \mathcal{L}(X_1, \dots, X_m; H). \quad (3.2.2)$$

Proof. Let $T \in \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m, H)$. Then by (3.2.1) $T = u \circ A$, with $u \in \mathcal{S}_p(K; H)$.

By (2.4.5), we have $u \in \mathcal{S}_q(K; H)$, consequently $T \in \mathcal{S}_q \circ \mathcal{L}(X_1, \dots, X_m, H)$. \blacksquare

Our theorem below deals with the relation between a multilinear operator of type $\mathcal{S}_p \circ \mathcal{L}$ and its linearization.

Theorem 3.2.2 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space.*

Let $T \in \mathcal{L}(X_1, \dots, X_m; H)$. The next properties are equivalent:

- (1) *The multilinear operator T belongs to $\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H)$.*
- (2) *The linearization $\tilde{T} \in \mathcal{S}_p \circ \mathcal{B}(X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m; H)$.*

Proof. First, we suppose that T is of type $\mathcal{S}_p \circ \mathcal{L}$. Then, by the factorization (3.2.1) we have $T = u \circ A$ with $u \in \mathcal{S}_p(K; H)$. So, by using (1.3.5) we obtain $\tilde{T} = u \circ \tilde{A}$, where \tilde{A} is the linearization of A , then

$$\tilde{T} \in \mathcal{S}_p \circ \mathcal{B}(X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m; H).$$

Now, we suppose that (2) is true. We can write

$$T = \tilde{T} \circ i_m = u_2 \circ u_1 \circ i_m,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $u_1 \circ i_m \in \mathcal{L}(X_1, \dots, X_m; K)$. Therefore $T \in \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H)$. ■

Comparing the ideal multilinear operators of type $\mathcal{S}_p \circ \mathcal{L}$ with the ideals of Cohen strongly p -summing and Hilbert-Schmidt multilinear mappings.

3.2.3 Connection with Cohen strongly p -summing multilinear operators

The following result due to L. Mezrag and K. Saadi [34, Corollary 4.2], which is considered the key of the proof of our main results in this section.

Lemma 3.2.1 *Let X_1, \dots, X_m, Y be Banach spaces and $T \in \mathcal{L}(X_1, \dots, X_m; Y)$. Then, the following assertions are equivalent:*

- (1) *The operator $T \in \mathcal{D}_p^m(X_1, \dots, X_m; Y)$.*
- (2) *There exist a Banach space Z , a linear operator $u \in \mathcal{D}_p(Z; Y)$ and $A \in \mathcal{L}(X_1, \dots, X_m; Z)$ such that*

$$T = u \circ A.$$

As in the linear case (2.4.7), we can establish the relation between Cohen strongly p -summing multilinear operators and multilinear operators of type $\mathcal{S}_2 \circ \mathcal{L}$.

Theorem 3.2.3 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space. For all $2 \leq p \leq \infty$, we have*

$$\mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H) = \mathcal{D}_p^m(X_1, \dots, X_m; H). \quad (3.2.3)$$

Proof. Let $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$. By the factorization (3.2.1), we have

$$T = u \circ A,$$

with $u \in \mathcal{S}_2(K; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K)$. Because $\mathcal{S}_2(K; H) = \mathcal{D}_p(K; H)$ from (2.3.1). Then $u \in \mathcal{D}_p(K; H)$. So, by Lemma 3.2.1, we have

$$u \circ A \in \mathcal{D}_p^m(X_1, \dots, X_m; H).$$

Reciprocally, let $2 \leq p \leq \infty$ and $T \in \mathcal{D}_p^m(X_1, \dots, X_m; H)$. By Proposition 3.1.1, T is in $\mathcal{D}_2^m(X_1, \dots, X_m; H)$ and by Lemma 3.2.1, we have $T = u \circ A$, where $u \in \mathcal{D}_2(Z; H)$ and Z is a Banach space. According to the Lemma 2.4.1, u factors through a Hilbert space,

$$i.e., u = u_2 \circ u_1,$$

with $u_2 \in \mathcal{D}_2(K; H)$ ($= \mathcal{S}_2(K; H)$). Therefore,

$$T = u_2 \circ u_1 \circ A \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H). \quad \blacksquare$$

The desired result follows.

Theorem 3.2.4 *Let H_1, \dots, H_m, H be Hilbert spaces. Then,*

(1) *If $2 < p < \infty$, we have*

$$\mathcal{D}_2^m(X_1, \dots, X_m; H) \subset \mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H).$$

(2) *If $1 \leq p < 2$, we have*

$$\mathcal{S}_p \circ \mathcal{L}(X_1, \dots, X_m; H) \subset \mathcal{D}_2^m(X_1, \dots, X_m; H).$$

Proof. (1) Let $2 < p < \infty$ and $T \in \mathcal{D}_2^m(X_1, \dots, X_m; H)$. By (3.2.3), T belongs to $\mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$ and the result follows by (3.2.2).

(2) Let $1 \leq p < 2$ and T be a multilinear operator of type $\mathcal{S}_p \circ \mathcal{L}$. Then, $T = u \circ A$, with $u \in \mathcal{S}_p(K; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K)$. It follows by (2.2.3) that $u \in \mathcal{S}_2(K; H)$, from (2.3.1), we have $u \in \mathcal{D}_2(K; H)$ Thus Lemma 3.2.1 completes the proof of Theorem. \blacksquare

we use the Lemma 3.2.1 and Theorem 3.2.3, we present the following result.

Proposition 3.2.5 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space. For all $1 \leq p \leq \infty$, we have*

- (a) *H is finite dimensional.*
- (b) *The identity $id_H \in \mathcal{S}_2 \circ \mathcal{B}(H; H)$.*
- (c) *For all Banach spaces, we have*

$$\mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H) = \mathcal{L}(X_1, \dots, X_m; H).$$

Proof. (a) \Rightarrow (b) : Obviously

(b) \Rightarrow (c) : By Theorem 2.3.3, $id_H \in \mathcal{D}_p(H; H)$. From Corollary 1.2.2 (3), we have $id_H \in \Pi_{p^*}(H; H)$.

(c) \Rightarrow (d) : It is easy by using the Lemma 3.2.1. ■

3.2.4 Connection with Hilbert-Schmidt multilinear mappings

Theorem 3.2.5 *Let H_1, \dots, H_m, H be Hilbert spaces. We have*

$$\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H) \subset \mathcal{S}_2 \circ \mathcal{L}(H_1, \dots, H_m; H). \quad (3.2.4)$$

This inclusion is strict in general.

Proof. We have the following diagram

$$\begin{array}{ccc} H_1 \times \dots \times H_m & \xrightarrow{T} & H \\ i_m \downarrow & & \tilde{T}_2 \uparrow \\ H_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi H_m & \xrightarrow{j_m} & H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m \end{array}$$

where j_m is the natural inclusion and \tilde{T}_2 is the extension of \tilde{T} to $H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m$. We have

$$T = \tilde{T}_2 j_m \circ i_m.$$

By Proposition 3.1.2, \tilde{T}_2 is Hilbert Schmidt

$$(i.e., \tilde{T}_2 \in \mathcal{S}_2(H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m; H)),$$

and $j_m \circ i_m \in \mathcal{L}(H_1, \dots, H_m; H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m)$. Therefore, T is in $\mathcal{S}_2 \circ \mathcal{L}(H_1, \dots, H_m; H)$. ■

Remark 3.2.2 *Now, to show that the inclusion (3.2.4) is strict.*

Let's take the following example. Let $A : H \times H \times H \rightarrow H$ the operator define by

$$A(x, y, z) = \langle x, y \rangle z,$$

let $u \in \mathcal{S}_2(H; H)$ (i.e., u is Hilbert-Schmidt). Then

$$\begin{array}{ccc} H \times H \times H & \xrightarrow{A} & H \\ & \searrow u & \uparrow \\ & & H \end{array}$$

(i.e., the operator $T = u \circ A$ is of the type $\mathcal{S}_2 \circ \mathcal{L}$).

Now, let us check that T is not Hilbert-Schmidt. Indeed

$$\begin{aligned}
 \sum_{k_1 \in I, k_2 \in I, k_3 \in I} \|T(e_{k_1}, e_{k_2}, e_{k_3})\|^2 &= \sum_{k_1 \in I, k_2 \in I, k_3 \in I} \|\langle e_{k_1}, e_{k_2} \rangle u(e_{k_3})\|^2 \\
 &= \sum_{k_1 \in I, k_2 \in I, k_3 \in I} |\langle e_{k_1}, e_{k_2} \rangle|^2 \|u(e_{k_3})\|^2 \\
 &= \sum_{k_2 \in I} \sum_{k_3 \in I} \|u(e_{k_3})\|^2 \|e_{k_2}\|^4 \\
 &= \|u\|_{\mathcal{S}_2}^2 \sum_{k_2 \in I} \|e_{k_2}\|^4 = +\infty.
 \end{aligned}$$

3.3 Factorization of Schatten class type mappings

In the linear case, it is well known that Hilbert Schmidt operators factor through an \mathcal{L}_1 -space or an \mathcal{L}_∞ -space (i.e., Theorem 2.3.1) and also through infinite dimensional Banach spaces in Theorem 2.3.2. The converse is also true in both cases. For the multilinear and polynomial cases, every Hilbert Schmidt multilinear or polynomial mappings factor through any Banach spaces, but the converse is not true (see Theorem 3.1.1 and Remark 3.1.1). In this section, we consider the particular class $\mathcal{S}_2 \circ \mathcal{L}$ for which it is possible to obtain an extension similar to the linear case cited above.

First, we recall the definition of p -factorable multilinear operators introduced by Martin Cerna Maguina in [31] as a generalization of the one given by Diestel, Jarchow and Tonge in the linear case [19].

Definition 3.3.1 *Let $1 \leq p \leq \infty$ and X_1, \dots, X_m, Y be Banach spaces.*

*The operator $T : X_1 \times \dots \times X_m \rightarrow Y$ is said p -factorable if there exist a measure space (Ω, Σ, μ) , $u \in \mathcal{B}(L_p(\mu); Y^{**})$ and $B \in \mathcal{L}(X_1, \dots, X_m; L_p(\mu))$ such that the following diagram commutes*

$$\begin{array}{ccccc}
 X_1 \times \dots \times X_m & \xrightarrow{T} & Y & \xrightarrow{K_Y} & Y^{**} \\
 & & & & \\
 & & B \searrow & & \nearrow u \\
 & & & L_p(\mu) &
 \end{array} \tag{3.3.1}$$

In other words

$$K_Y \circ T = u \circ B,$$

where K_Y is the isometric embedding of Y into Y^{**} .

We denote by $\mathcal{L}_{p\text{-fat}}(X_1, \dots, X_m; Y)$ the space of all p -factorable multilinear operators, which is a Banach space.

Theorem 3.3.1 [31, Proposition 2.3] *Let X_1, \dots, X_m, Y be Banach spaces. Then, the following assertions are equivalent.*

- (1) *The operator T belongs to $\mathcal{L}_{2\text{-fat}}(X_1, \dots, X_m; Y)$.*
- (2) *The operator T factors through a Hilbert space, i.e., there exist a Hilbert space H , $u \in \mathcal{B}(H; Y)$ and $B \in \mathcal{L}(X_1, \dots, X_m; H)$ such that*

$$T = u \circ B.$$

Proposition 3.3.1 *Let H_1, \dots, H_m, H be Hilbert spaces, we have*

$$\mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H) \subset \mathcal{L}_{2\text{-fat}}(H_1, \dots, H_m; H).$$

Proof. Let $T \in \mathcal{L}_{\mathcal{HS}}(H_1, \dots, H_m; H)$. By the Proposition 3.1.2, $\tilde{T}_2 \in \mathcal{S}_2(H_1 \widehat{\otimes}_2 \dots \widehat{\otimes}_2 H_m; H)$. thus, it is factored by a Hilbert space. Therefore, $T \in \mathcal{L}_{2\text{-fat}}(H_1, \dots, H_m; H)$. ■

Example 3.3.1 *We have*

$$\mathcal{L}(\mathcal{L}_1, \dots, \mathcal{L}_1; H) = \mathcal{L}_{2\text{-fat}}(\mathcal{L}_1, \dots, \mathcal{L}_1; H),$$

where H is Hilbert space. Indeed, we have the following diagram

$$\begin{array}{ccc} \mathcal{L}_1 \times \dots \times \mathcal{L}_1 & \xrightarrow{T} & H \\ i_m \downarrow & & \tilde{T}_2 \nearrow \\ \mathcal{L}_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathcal{L}_1 & & \end{array}$$

let $T \in \mathcal{L}(\mathcal{L}_1, \dots, \mathcal{L}_1; H)$, then

$$T = \tilde{T}_2 \circ i_m,$$

with $\tilde{T}_2 \in \mathcal{B}(\mathcal{L}_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathcal{L}_1; H)$. Since $\mathcal{L}_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathcal{L}_1$ is \mathcal{L}_1 -space, then by Grothendieck's Theorem 1.2.2, \tilde{T}_2 is 2-summing. Hence, it is factored by a Hilbert space K ,

$$\text{i.e., } \tilde{T}_2 = uv,$$

with $v \in \mathcal{B}(\mathcal{L}_1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \mathcal{L}_1; K)$ and $u \in \mathcal{B}(K; H)$. Therefore

$$T = u(v \circ i_m).$$

We now present a factorization results for mappings of Schatten class type $\mathcal{S}_2 \circ \mathcal{L}$.

Theorem 3.3.2 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space. Then, the next properties are equivalent:*

- (1) *The operator $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$.*
- (2) *There exist a linear operator $u \in \mathcal{B}(\mathcal{L}_1; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_1)$ such that*

$$T = u \circ B.$$

Proof. (1) \Rightarrow (2) : Let $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$, then $T = u \circ A$ with $u \in \mathcal{S}_2(K; H)$. By Theorem 2.3.1, the operator u factors through an \mathcal{L}_1 -space, i.e., $u = v_1 \circ v_2$ with $v_2 \in \mathcal{B}(K; \mathcal{L}_1)$ and $v_1 \in \mathcal{B}(\mathcal{L}_1; H)$, thus

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & H \\ A \downarrow & & \uparrow v_1 \\ K & \xrightarrow{v_2} & \mathcal{L}_1 \end{array}$$

that is

$$T = v_1 \circ B,$$

where $B = v_2 \circ A \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_1)$. Then $T = u \circ B$.

(2) \Rightarrow (1) : Let $T = u \circ B$ with $u \in \mathcal{B}(\mathcal{L}_1; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_1)$. One can write B as

$$B = v \circ A : X_1 \times \dots \times X_m \xrightarrow{A} K_1 \xrightarrow{v} \mathcal{L}_1.$$

By Theorem 1.2.2, the operator u is 2-summing. So it factors through a Hilbert space, i.e., $u = s_1 \circ s_2 : \mathcal{L}_1 \xrightarrow{s_2} K_2 \xrightarrow{s_1} H$ with $s_2 \in \Pi_2(\mathcal{L}_1; K_2)$. Therefore,

$$T = u \circ B = s_1 \circ s_2 \circ v \circ A,$$

with $s_1 \circ s_2 \circ v \in \mathcal{S}_2(K_1; H)$, this ends the proof. \blacksquare

The demonstration of the following theorem is done in the same way as the previous one.

Theorem 3.3.3 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space. Then, the next properties are equivalent:*

- (a) The operator $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$.
 (b) There exist a linear operator $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_\infty)$ such that

$$T = u \circ B.$$

Proof. (a) \Rightarrow (b) : Let $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$, then $T = u \circ A$ with $u \in \mathcal{S}_2(K; H)$. The operator u factors through an \mathcal{L}_∞ -space, i.e., $u = v_1 \circ v_2$ with $v_2 \in \mathcal{B}(K; \mathcal{L}_\infty)$ and $v_1 \in \mathcal{B}(\mathcal{L}_\infty; H)$, thus

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & H \\ A \downarrow & & \uparrow v_1 \\ K & \xrightarrow{v_2} & \mathcal{L}_1 \end{array}$$

i.e., $T = v_1 \circ B$ where $B = v_2 \circ A \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_\infty)$.

(b) \Rightarrow (a) : Let $T = u \circ B$ with $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; \mathcal{L}_\infty)$. One can write B as

$$B = v \circ A : X_1 \times \dots \times X_m \xrightarrow{A} K_1 \xrightarrow{v} \mathcal{L}_\infty.$$

By Theorem 1.2.3, the operator u is 2-summing. So it factors through a Hilbert space,

$$i.e., u = s_1 \circ s_2 : \mathcal{L}_\infty \xrightarrow{s_2} K_2 \xrightarrow{s_1} H,$$

with $s_2 \in \Pi_2(\mathcal{L}_\infty; K_2)$. Therefore

$$\begin{aligned} T &= u \circ B \\ &= s_1 \circ s_2 \circ v \circ A, \end{aligned}$$

with $s_1 \circ s_2 \circ v \in \mathcal{S}_2(K_1; H)$, this ends the proof. ■

Now, we give a multilinear version of the Diestel-Jarchow-Tonge result.

Theorem 3.3.4 *Let X_1, \dots, X_m be Banach spaces and H be a Hilbert space. The following properties are equivalent.*

- (1) The operator $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$.
 (2) For every Banach space Z , there exist $u \in \mathcal{B}(Z; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Z)$ such that

$$T = u \circ B.$$

Proof. (1) \Rightarrow (2) : Let $T \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H)$.

Then $T = u \circ A$ with $u \in \mathcal{S}_2(K; H)$. The operator u factors through an \mathcal{L}_1 -space, i.e., $u = v_1 \circ v_2$ with $v_2 \in \mathcal{B}(K; Z)$ and $v_1 \in \mathcal{B}(Z; H)$, thus

$$T : X_1 \times \dots \times X_m \xrightarrow{A} K \xrightarrow{v_2} Z \xrightarrow{v_1} H,$$

i.e., $T = v_1 \circ B$ where $B = v_2 \circ A \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Z)$.

(2) \Rightarrow (1) : Let $T = u \circ B$ with $u \in \mathcal{B}(Z; H)$ and $B \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Z)$. One can write B as

$$B = v \circ A : X_1 \times \dots \times X_m \xrightarrow{A} K_1 \xrightarrow{v} Z.$$

Then

$$T = u \circ v \circ A : X_1 \times \dots \times X_m \xrightarrow{A} K_1 \xrightarrow{v} Z \xrightarrow{u} H.$$

By Theorem 2.3.2, the operator $w = u \circ v \in \mathcal{S}_2(K_1; H)$ and $A \in \mathcal{L}(X_1, \dots, X_m; K_1)$.

Therefore

$$T = w \circ A \in \mathcal{S}_2 \circ \mathcal{L}(X_1, \dots, X_m; H),$$

this ends the proof. ■

Corollary 3.3.1 *As characterization result that give by Theorem 2.3.1, we can say that multilinear mappings of type $\mathcal{S}_2 \circ \mathcal{L}$ is characterized by their factorizations by \mathcal{L}_1 -space and by \mathcal{L}_∞ -space.*

Chapter 4

Composition ideals of polynomials generated by Schatten class

The content of this chapter is based on a paper published in Journal of Mathematics [6]. The purpose of this chapter is to study the composition ideals of polynomial mappings generated by Schatten class. We give some coincidence theorems for Cohen strongly 2-summing of polynomial mappings and factorization results like that given by Lindenstrauss-Pełczyński for Hilbert Schmidt linear operators.

4.1 Definitions and auxiliary results

At the beginning of this section, we start by the following definition of multilinear symmetric.

Definition 4.1.1 *Given Banach spaces X, Y . Let $T \in \mathcal{L}({}^m X; Y)$; T is symmetric if it is invariant for any permutation of its components, i.e.,*

$$T \circ \sigma(x_1, \dots, x_m) := T(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = T(x_1, \dots, x_m),$$

for every permutation σ of the set $\{1, \dots, m\}$.

We denote by $\mathcal{L}_S({}^m X; Y)$ the space of all symmetric continuous m -linear operators from $\underbrace{X \times \dots \times X}_m$ into Y .

For any $T \in \mathcal{L}({}^m X; Y)$ we will denote by T_S its associated symmetric multilinear operator, i.e., that is $T_S \in \mathcal{L}_S({}^m X; Y)$ defined by

$$T_S = \frac{1}{m!} \sum_{\sigma} T \circ \sigma.$$

We note that, for every $x \in X$, we have

$$T_S(x, \dots, x) = T(x, \dots, x).$$

Definition 4.1.2 *A mapping $P : X \rightarrow Y$ is an m -homogeneous polynomial if there exists a unique symmetric m -linear operator $\widehat{P} : X \times \dots \times X \rightarrow Y$ such that*

$$P(x) = \widehat{P}\left(x, \binom{m}{\dots}, x\right) \text{ for every } x \in X.$$

The polynomial P is bounded on the unit ball of X if, and only if, \widehat{P} is bounded on the unit ball of $X \times \dots \times X$.

We denote by $\mathcal{P}({}^m X; Y)$, the Banach space of all continuous m -homogeneous polynomials from X into Y endowed with the norm

$$\begin{aligned} \|P\| &= \sup \{ \|P(x)\| : \|x\| \leq 1 \} \\ &= \inf \{ C : \|P(x)\| \leq C \|x\|^m, x \in X \}. \end{aligned}$$

For the general theory of polynomial on Banach spaces, we refer to [21] and [35].

If $Y = \mathbb{K}$, we simply write $\mathcal{P}({}^m X)$.

Proposition 4.1.1 *(Polarization formula) We have for all $\widehat{P} \in \mathcal{L}_S({}^m X; Y)$ [35, Theorem 1.10].*

$$\widehat{P}(x^1, \dots, x^m) = \frac{1}{m!2^m} \sum_{\substack{\epsilon_i = \pm 1 \\ 1 \leq i \leq m}} \epsilon_1 \dots \epsilon_m P\left(\sum_{j=1}^m \epsilon_j x^j\right),$$

where P is the polynomial associated with \widehat{P} . Furthermore, P is bounded on the unit ball of X if and only if \widehat{P} is bounded. The two norms verify the following inequality [35, Theorem 2.2].

$$\|P\| \leq \left\| \widehat{P} \right\| \leq \frac{m^m}{m!} \|P\|.$$

By $\widehat{\otimes}_s^m X := X \widehat{\otimes}_s^{(m)} \widehat{\otimes}_s X$, we denote the m fold symmetric tensor product of X . That is the set of all elements $u \in \widehat{\otimes}_s X$ of the norm

$$u = \sum_{i=1}^n x_i \otimes^{(m)} \otimes x_i, \quad (n \in \mathbb{N}^*, x_i \in X, 1 \leq i \leq n).$$

By $\widehat{\otimes}_{\pi,s}^m X$, we denote the closure of $\widehat{\otimes}_s^m X$ in $\widehat{\otimes}_{\pi,s}^m X$. For symmetric tensor products, we refer to [25].

If $P \in \mathcal{P}(^m X; Y)$, we define its linearization $\widetilde{P} : \widehat{\otimes}_{\pi,s}^m X \rightarrow Y$ by

$$\widetilde{P}\left(\sum_{i=1}^n x_i \otimes^{(m)} \otimes x_i\right) = \sum_{i=1}^n P(x_i),$$

where $(x_i)_{1 \leq i \leq n} \in X$. Consider the canonical polynomial

$$\begin{aligned} \delta_m : X &\rightarrow \widehat{\otimes}_{\pi,s}^m X \\ x &\rightarrow x \otimes^{(m)} \otimes x. \end{aligned}$$

We have the next diagram which is commute

$$\begin{array}{ccc} X & \xrightarrow{P} & Y \\ & \searrow^{\delta_m} & \uparrow \widetilde{P} \\ & & \widehat{\otimes}_{\pi,s}^m X \end{array}$$

In other words

$$P = \widetilde{P} \circ \delta_m. \quad (4.1.1)$$

Proposition 4.1.2 [25] 1) The correspondence $P \leftrightarrow \widetilde{P}$ establish an isometric isomorphism between $\mathcal{P}(^m X; Y)$ and $\mathcal{B}(\widehat{\otimes}_{\pi,s}^m X; Y)$.

2) The correspondence $P \leftrightarrow \widehat{P}$ establish an isometric isomorphism between $\mathcal{P}(^m X; Y)$ and $\mathcal{L}_S(^m X; Y)$.

Definition 4.1.3 [13] (Polynomial ideal) An ideal of homogeneous polynomials \mathcal{Q} is a subclass of the class for all continuous homogeneous polynomials between Banach such that for all $m \in \mathbb{N}$ and Banach spaces X and Y , the component $\mathcal{Q}(^m X; Y) = \mathcal{P}(^m X; Y) \cap \mathcal{Q}$ satisfy:

(1) $\mathcal{Q}(^m X; Y)$ is a linear subspace of $\mathcal{P}(^m X; Y)$ which contains the m -homogeneous polynomials of finite type.

(2) *The ideal property: If $u \in \mathcal{B}(E; X)$, $P \in \mathcal{Q}(^m X; Y)$, and $v \in \mathcal{B}(Y; F)$, then the composition $v \circ P \circ u$ is in $\mathcal{Q}(^m E; F)$.*

If $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathbb{R}^+$ satisfies:

(1') *$(\mathcal{Q}(^m X; Y), \|\cdot\|_{\mathcal{Q}})$ is a normed (Banach) for all X, Y and $m \in \mathbb{N}$.*

(2') *$\|P^m : \mathbb{K}^m \rightarrow \mathbb{K}; P^m(x) = x^m\|_{\mathcal{Q}} = 1$, for all $m \in \mathbb{N}^*$.*

(3') *If $u \in \mathcal{B}(E; X)$, $P \in \mathcal{Q}(^m X; Y)$ and $v \in \mathcal{B}(Y; F)$, then*

$$\|v \circ P \circ u\|_{\mathcal{Q}} \leq \|v\| \|P\|_{\mathcal{Q}} \|u\|^m.$$

Then $(\mathcal{Q}; \|\cdot\|_{\mathcal{Q}})$ is called a normed (Banach) polynomial ideal.

Polynomials of type $\mathcal{P}(\mathcal{S}_p)$

In this paragraph, we present the Polynomials of type $\mathcal{P}(\mathcal{S}_p)$ with some properties. The definition was introduced by H-A Braunss in [15].

Definition 4.1.4 *Let X be a Banach space, H be a Hilbert space and $1 \leq p \leq \infty$. A polynomial mapping $P \in \mathcal{P}(^m H; X)$ is said to be of type $\mathcal{P}(\mathcal{S}_p)$, in symbols $P \in \mathcal{P}(\mathcal{S}_p)(^m H; X)$, if there exist a Hilbert space K , a linear operator $u \in \mathcal{S}_p(H; K)$ and $Q \in \mathcal{P}(^m K; X)$ such that*

$$P = Q \circ u. \tag{4.1.2}$$

The space $\mathcal{P}(\mathcal{S}_p)(^m H; X)$ is a space with the following norm

$$\|P\|_{\mathcal{P}(\mathcal{S}_p)} = \inf_{P=Q \circ u} \|Q\| \|u\|_{\mathcal{S}_p}^m.$$

The following results due to C. A. Mendes, for proof (see [33]).

Corollary 4.1.1 *Let H, G be Hilbert spaces, we have*

$$\mathcal{P}(\mathcal{S}_2)(^m H; G) \subset \mathcal{P}_{\mathcal{HS}}(^m H; G).$$

Proposition 4.1.3 *Let X be a Banach space, H be a Hilbert space. For all $2 \leq p \leq \infty$, we have*

$$\mathcal{P}(\mathcal{S}_2)(^m H; F) = \mathcal{P}_d^p(^m H; F).$$

Theorem 4.1.1 [33, Theorem 2.10] *Let F be a Banach space, H be a Hilbert space and $1 \leq p \leq \infty$. The following properties are equivalent.*

(a) *The operator $P \in \mathcal{P}(\mathcal{S}_2)({}^m H, F)$.*

(b) *There exist Y is an \mathcal{L}_1 -space and $R \in \mathcal{B}(H, Y)$ and $Q \in \mathcal{P}_2^p({}^m Y, \mathcal{F})$ such that*

$$P = R \circ Q.$$

(c) *There exist X is an \mathcal{L}_∞ -space and $S \in \mathcal{B}(X, H)$ and $Q \in \mathcal{P}_2^p({}^m X, \mathcal{F})$ such that*

$$P = Q \circ S.$$

Where $\mathcal{P}_2^p({}^m X, \mathcal{F})$ the class of all 2-dominated m -homogeneous polynomials.

4.1.1 Cohen strongly p -summing m -homogeneous polynomials

We present the definition of Cohen strongly p -summing m -homogeneous polynomials was introduced by Achour and Saadi [5].

Definition 4.1.5 *Let $1 \leq p \leq \infty$. An m -homogeneous polynomial $P : X \rightarrow Y$ is Cohen strongly p -summing if there exists a constant $C > 0$ such that, for any $x_1, \dots, x_n \in X$ and any $y_1^*, \dots, y_n^* \in Y^*$, we have*

$$\sum_{i=1}^n |\langle P(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^n \|x_i\|^{mp} \right)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{p^*}^m. \quad (4.1.3)$$

The class of Cohen strongly p -summing m -homogeneous polynomial from X into Y , which is denoted by $\mathcal{P}_{Coh}^p({}^m X; Y)$, is a Banach space for the norm $d_p^m(P)$, i.e., the smallest constant C such that the inequality (4.1.3) holds.

Theorem 4.1.2 [5, Theorem 2.4] *Let $1 \leq p \leq \infty$. An m -homogeneous polynomial $P : X \rightarrow Y$ is Cohen strongly p -summing if and only if, its associated symmetric m -linear operator \widehat{P} is Cohen strongly p -summing multilinear operator.*

Corollary 4.1.2 [5, Corollary 2.4] *Let $1 \leq p \leq q < \infty$.*

If $P \in \mathcal{P}_{Coh}^q({}^m X; Y)$, then $P \in \mathcal{P}_{Coh}^p({}^m X; Y)$ and $d_p(P) \leq d_q(P)$.

Proposition 4.1.4 *Let X_1, \dots, X_m, Y be Banach spaces. We have (a) \Rightarrow (b), where*

(a) $T \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Y)$.

(b) $T_S \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Y)$.

Proof. Let $T \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Y)$, By Theorem 3.3.1, then *there exist a Hilbert space H , $u \in \mathcal{B}(H; Y)$ and $B \in \mathcal{L}(X_1, \dots, X_m; H)$ such that $T = u \circ B$.*

We have

$$\begin{aligned} T_S &= \frac{1}{m!} \sum_{\sigma} T \circ \sigma \\ &= \frac{1}{m!} \sum_{\sigma} u \circ B \circ \sigma \\ &= u \circ \left(\frac{1}{m!} \sum_{\sigma} B \circ \sigma \right) \\ &= u \circ B_S. \end{aligned}$$

Consequently $T_S \in \mathcal{L}_{2-fat}(X_1, \dots, X_m; Y)$.

4.1.2 Hilbert-Schmidt polynomials

Definition 4.1.6 [32, Definition 2.2] *Let H, K be two Hilbert spaces. An m -homogeneous polynomial $P : H \rightarrow G$ is a Hilbert-Schmidt Polynomial if its m -linear symmetric \widehat{P} belongs to $\mathcal{L}_{\mathcal{HS}}({}^m H; G)$. The space of such polynomials is indicated by $\mathcal{P}_{\mathcal{HS}}({}^m H; K)$ and a norm is defined by*

$$\|P\|_{\mathcal{HS}} = \|\widehat{P}\|_{\mathcal{HS}}.$$

We give the factorization result of Hilbert-Schmidt polynomials $\mathcal{P}_{\mathcal{HS}}({}^m H; K)$.

Theorem 4.1.3 *Let H, G be two Hilbert spaces. If $P \in \mathcal{P}_{\mathcal{HS}}({}^m H; G)$, then P factors through an \mathcal{L}_{∞} -space and \mathcal{L}_1 -space in the following ways:*

(i) P is represented by

$$P = u \circ Q,$$

where $u \in \mathcal{B}(\mathcal{L}_{\infty}; H)$ and $Q \in \mathcal{P}({}^m H; \mathcal{L}_{\infty})$ with $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_{\infty})$.

(ii) P is represented by

$$P = u \circ Q,$$

where $u \in \mathcal{B}(\mathcal{L}_1; H)$ and $Q \in \mathcal{P}({}^m H; \mathcal{L}_1)$ with $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_1)$.

Proof. Let $P \in \mathcal{P}_{\mathcal{HS}}({}^m H; G)$ then its m -linear symmetric \widehat{P} belongs to $\mathcal{L}_{\mathcal{HS}}({}^m H; G)$. By Theorem 3.1.2, then \widehat{P} factors through an \mathcal{L}_∞ -space and \mathcal{L}_1 -space in the following ways:

(1) \widehat{P} is represented by $\widehat{P} = u \circ \widehat{Q}$, where $u \in \mathcal{B}(\mathcal{L}_\infty; G)$ and $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_\infty)$.

(2) \widehat{P} is represented by $\widehat{P} = u \circ \widehat{Q}$, where $u \in \mathcal{B}(\mathcal{L}_1; G)$ and $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_1)$.

From (1), we have $\widehat{P} = u \circ A$, so $P = u \circ Q$ where $u \in \mathcal{B}(\mathcal{L}_\infty; G)$ and $Q \in \mathcal{P}({}^m H; \mathcal{L}_\infty)$ with $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_\infty)$.

From (2), we have $\widehat{P} = u \circ A$, so $P = u \circ Q$ where $u \in \mathcal{B}(\mathcal{L}_1; G)$ and $Q \in \mathcal{P}({}^m H; \mathcal{L}_1)$ with $\widehat{Q} \in \mathcal{L}({}^m H; \mathcal{L}_1)$. This concludes the proof of the theorem. ■

Remark 4.1.1 *The reciprocal of the previous theorem is false in general.*

4.2 Polynomials mappings of type $\mathcal{S}_p \circ \mathcal{P}$

We introduce now the definition of polynomial mapping of type $\mathcal{S}_p \circ \mathcal{P}$.

Definition 4.2.1 *Let X be a Banach space, H be a Hilbert space and $1 \leq p \leq \infty$. A polynomial mapping $P \in \mathcal{P}({}^m X; H)$ is said to be of type $\mathcal{S}_p \circ \mathcal{P}$, in symbols $P \in \mathcal{S}_p \circ \mathcal{P}({}^m X; H)$, ; if there exist a Hilbert space K , a linear operator $u \in \mathcal{S}_p(K; H)$ and $Q \in \mathcal{P}({}^m X; K)$ such that*

$$P = u \circ Q. \quad (4.2.1)$$

The space $\mathcal{S}_p \circ \mathcal{P}({}^m X, H)$ is a Banach space with the following norm

$$\|P\|_{\mathcal{S}_p \circ \mathcal{P}} = \inf \|u\|_{\mathcal{S}_p} \|Q\|,$$

where the infimum is taken over all possible factorizations of the form (4.2.1). We have

$$\|P\| \leq \|P\|_{\mathcal{S}_p \circ \mathcal{P}}.$$

Proposition 4.2.1 *Let $P \in \mathcal{P}({}^m X; H)$ and \widehat{P} its associated m -linear symmetric operator.*

The next properties are equivalent:

(a) *The polynomial P belongs to $\mathcal{S}_p \circ \mathcal{P}({}^m X; H)$.*

(b) *The multilinear operator \widehat{P} belongs to $\mathcal{S}_p \circ \mathcal{L}({}^m X; H)$.*

Proof. (a) \Rightarrow (b) : If $P \in \mathcal{S}_p \circ \mathcal{P}(^m X; H)$, then $P = u \circ Q$ with $u \in \mathcal{S}_p(K; H)$. And $Q \in \mathcal{P}(^m X; K)$ its m -linear symmetric \widehat{Q} belongs to $\mathcal{L}(^m X; K)$. Then

$$\widehat{P} = u \circ \widehat{Q}.$$

Hence $\widehat{P} \in \mathcal{S}_p \circ \mathcal{L}(^m X; H)$.

(b) \Rightarrow (a) : Let $\widehat{P} \in \mathcal{S}_p \circ \mathcal{L}(^m X; H)$, by (3.2.1) then $\widehat{P} = u \circ \widehat{Q}$ with $u \in \mathcal{S}_p(K; H)$ and $\widehat{Q} \in \mathcal{L}(^m X; K)$.

So $Q \in \mathcal{P}(^m X; K)$, by (4.2.1) then $u \circ Q \in \mathcal{S}_p \circ \mathcal{P}(^m X; H)$.

Thus $P = u \circ Q \in \mathcal{S}_p \circ \mathcal{P}(^m X; H)$. ■

From Definitions 3.2.2 and 4.1.2, we obtain the definition as the following.

Definition 4.2.2 A polynomial mapping $P \in \mathcal{P}(^m X; H)$ is said to be of type $\mathcal{S}_p \circ \mathcal{P}$, if its m -linear symmetric \widehat{P} is of type $\mathcal{S}_p \circ \mathcal{L}$.

In addition,

$$\|P\|_{\mathcal{S}_p \circ \mathcal{P}} = \|\widehat{P}\|_{\mathcal{S}_p \circ \mathcal{L}}.$$

Proposition 4.2.2 Let X be a Banach space, H be a Hilbert space and $1 \leq p \leq \infty$. Suppose that $P \in \mathcal{P}(^m X; H)$. The next properties are equivalent.

(1) The polynomial P belongs to $\mathcal{S}_p \circ \mathcal{P}(^m X; H)$.

(2) The linearization $\widetilde{P} \in \mathcal{S}_p \circ \mathcal{B}(\widehat{\otimes}_{\pi, s}^m X; H)$. (where \widetilde{P} the linearization of P).

Proof. First, we suppose that P is of type $\mathcal{S}_p \circ \mathcal{P}$. Then, by the Proposition (4.2.1), then \widehat{P} is of type $\mathcal{S}_p \circ \mathcal{L}$. From Theorem 3.2.2, we have

$$\widetilde{P}_L \in \mathcal{S}_p \circ \mathcal{B}(\widehat{\otimes}_{\pi}^m X; H),$$

where \widetilde{P}_L the linearization of \widehat{P} , by Corollary 2.4.1. Thus $\widetilde{P} \in \mathcal{S}_p \circ \mathcal{B}(\widehat{\otimes}_{\pi, s}^m X; H)$.

Now, we suppose that the second assertion is true. By (4.1.1), we can write

$$P = \widetilde{P} \circ \delta_m.$$

According to $\widetilde{P} \in \mathcal{S}_p \circ \mathcal{B}(\widehat{\otimes}_{\pi, s}^m X; H)$. Then

$$P = u_2 \circ u_1 \circ \delta_m,$$

with $u_2 \in \mathcal{S}_p(K; H)$ and $u_1 \in \mathcal{B}(\widehat{\otimes}_{\pi, s}^m X; K)$, but $u_1 \circ \delta_m \in \mathcal{P}(^m X; K)$. Therefore P belongs to $\mathcal{S}_p \circ \mathcal{P}(^m X; H)$. ■

4.2.1 Relation to Cohen strongly p -summing m -homogeneous polynomials

Theorem 4.2.1 *Let X be a Banach space and H be a Hilbert space. If $2 < p < \infty$, we have*

$$\mathcal{S}_2 \circ \mathcal{P}({}^m X; H) = \mathcal{P}_{Coh}^p({}^m X; H).$$

Proof. Firstly, $\mathcal{P}_{Coh}^p({}^m X; H) \subset \mathcal{S}_2 \circ \mathcal{P}({}^m X; H)$, if $P \in \mathcal{P}_{Coh}^p({}^m X; H)$, by Theorem 3.2.3, then its m -linear symmetric \widehat{P} belongs to $D_p^m({}^m X; Y)$. By Theorem 3.2.3 $\widehat{P} \in \mathcal{S}_2 \circ L({}^m X; H)$, and thus $P \in \mathcal{S}_2 \circ \mathcal{P}({}^m X; H)$.

Secondly, $\mathcal{S}_2 \circ \mathcal{P}({}^m X; H) \subset \mathcal{P}_{Coh}^2({}^m X; H)$, let $P \in \mathcal{S}_2 \circ \mathcal{P}({}^m X; H)$, then \widehat{P} is of type $\mathcal{S}_2 \circ L({}^m X; H)$. By Theorem 3.2.3, So that \widehat{P} belongs to $D_2^m({}^m X; Y)$, therefore $P \in \mathcal{P}_{Coh}^2({}^m X; H)$. ■

Theorem 4.2.2 *Let X be a Banach space and H be a Hilbert space. Then*

(1) *If $2 < p < \infty$, we have*

$$\mathcal{P}_{Coh}^2({}^m X; H) \subseteq \mathcal{S}_p \circ \mathcal{P}({}^m X; H).$$

(2) *If $1 \leq p < 2$, we have*

$$\mathcal{S}_p \circ \mathcal{P}({}^m X; H) \subseteq \mathcal{P}_{Coh}^2({}^m X; H).$$

Proof.(1) If $P \in \mathcal{P}_{Coh}^2({}^m X; H)$ then its m -linear symmetric \widehat{P} belongs to $\mathcal{D}_2^m({}^m X; Y)$. By Theorem 3.2.4, $\widehat{P} \in \mathcal{S}_p \circ \mathcal{L}({}^m X; H)$, and thus $P \in \mathcal{S}_p \circ \mathcal{P}({}^m X; H)$.

(2) Let $P \in \mathcal{S}_p \circ \mathcal{P}({}^m X; H)$, then \widehat{P} is of type $\mathcal{S}_p \circ \mathcal{L}$. By Theorem 3.2.4, $\widehat{P} \in \mathcal{D}_2^m({}^m X; Y)$ and thus $P \in \mathcal{P}_{Coh}^2({}^m X; H)$. ■

4.2.2 Connection with Hilbert-Schmidt polynomials

Theorem 4.2.3 *Let H_1, \dots, H_m, H be Hilbert spaces. We have*

$$\mathcal{P}_{\mathcal{HS}}({}^m H; K) \subset \mathcal{S}_2 \circ \mathcal{P}({}^m H; K). \quad (4.2.2)$$

This inclusion is strict in general.

Proof. Let $P \in \mathcal{P}_{\mathcal{HS}}({}^m H; K)$, then its m -linear symmetric \widehat{P} belongs to $\mathcal{L}_{\mathcal{HS}}({}^m H; K)$. By Theorem 3.2.5, $\widehat{P} \in \mathcal{S}_2 \circ \mathcal{L}({}^m H; K)$, and thus $P \in \mathcal{S}_2 \circ \mathcal{P}({}^m H; K)$. ■

4.2.3 On factorization of Schatten class type polynomials

In this section, we give some factorization results of Schatten class type polynomials, which is consequence of the last theorems in chapter 3 (multilinear case).

Theorem 4.2.4 *Let X be a Banach space and H be a Hilbert space. The following properties are equivalent:*

- (a) *The polynomial $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$.*
- (b) *There exist a linear operator $u \in \mathcal{B}(\mathcal{L}_\infty; H)$ and $Q \in \mathcal{P}(^m X; \mathcal{L}_\infty)$ such that*

$$P = u \circ Q \text{ with } \widehat{Q} \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_\infty).$$

Proof. (a) \Rightarrow (b) : If $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$ then $\widehat{P} \in \mathcal{S}_2 \circ \mathcal{L}(^m X; H)$.

So, by Theorem 3.3.3, \widehat{P} factors through an \mathcal{L}_∞ -space, i.e.,

$$\widehat{P} = u \circ B, \text{ with } u \in \mathcal{B}(\mathcal{L}_\infty; H) \text{ and } B \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_\infty).$$

Now, it is not difficult to show that

$$\widehat{P} = u \circ B_S \text{ and } B_S \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_\infty).$$

Then $P = u \circ Q$ where $\widehat{Q} = B_S$.

(b) \Rightarrow (a) : suppose that (b) is true.

We have $\widehat{P} = u \circ \widehat{Q}$, by Theorem 3.3.3 \widehat{P} belongs to $\mathcal{S}_2 \circ \mathcal{L}(^m X; H)$

$$\text{i.e., } P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H). \quad \blacksquare$$

Theorem 4.2.5 *Let X be a Banach space and H be a Hilbert space. The following properties are equivalent:*

- (1) *The polynomial $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$.*
- (2) *There exist a linear operator $u \in \mathcal{B}(\mathcal{L}_1; H)$ and $Q \in \mathcal{P}(^m X; \mathcal{L}_1)$ such that*

$$P = u \circ Q \text{ with } \widehat{Q} \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_1).$$

Proof. (1) \Rightarrow (2) : If $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$ then $\widehat{P} \in \mathcal{S}_2 \circ \mathcal{L}(^m X; H)$.

So, by Theorem 3.3.2, \widehat{P} factors through an \mathcal{L}_∞ -space, i.e.,

$$\widehat{P} = u \circ B, \text{ with } u \in \mathcal{B}(\mathcal{L}_\infty; H) \text{ and } B \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_\infty).$$

Now, it is not difficult to show that

$$\widehat{P} = u \circ B_S \text{ and } B_S \in \mathcal{L}_{2\text{-fat}}(^m X; \mathcal{L}_\infty).$$

Then $P = u \circ Q$ where $\widehat{Q} = B_S$.

Reciprocally, suppose that (2) is true.

We have $\widehat{P} = u \circ \widehat{Q}$, by Theorem 3.3.2, \widehat{P} belongs to $\mathcal{S}_2 \circ \mathcal{L}(^m X; H)$, i.e., $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$. ■

To close this chapter, we give a polynomial version of the Diestel-Jarchow-Tonge result.

Theorem 4.2.6 *Let X be a Banach space and H be a Hilbert space. The following properties are equivalent:*

(i) *The polynomial $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$.*

(ii) *For every Banach space Z , there exist $u \in \mathcal{B}(Z; H)$ and $Q \in \mathcal{P}(^m X; Z)$ such that*

$$P = u \circ Q \text{ with } \widehat{Q} \in \mathcal{L}_{2\text{-fat}}(^m X; Z).$$

Proof. (i) \Rightarrow (ii) : If $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$ then $\widehat{P} \in \mathcal{S}_2 \circ \mathcal{L}(^m X; H)$.

So, by Theorem 3.3.4, then for every Banach space Z , there exist $u \in \mathcal{B}(Z; H)$ and $B \in \mathcal{L}_{2\text{-fat}}(^m X; Z)$ such that $\widehat{P} = u \circ B$.

Now, it is not difficult to show that

$$\widehat{P} = u \circ B_S \text{ and } B_S \in \mathcal{L}_{2\text{-fat}}(^m X; Z).$$

Then $P = u \circ Q$ where $\widehat{Q} = B_S$.

(ii) \Rightarrow (i) : suppose that (ii) is true.

We have $\widehat{P} = u \circ \widehat{Q}$, by Theorem 3.3.4 \widehat{P} belongs to $\mathcal{S}_2 \circ \mathcal{L}(^m X; H)$, i.e., $P \in \mathcal{S}_2 \circ \mathcal{P}(^m X; H)$. ■

Corollary 4.2.1 *As characterization result that give by Theorem 2.3.1, we can say that polynomials of type $\mathcal{S}_2 \circ \mathcal{P}$ and $\mathcal{P}(\mathcal{S}_2)$ are characterized by their factorizations by \mathcal{L}_1 -space and by \mathcal{L}_∞ -space.*

Chapter 5

Characterization of positive p -summing sublinear operators using representable mappings

In this chapter, we study the class of positive p -summing sublinear operators. Thus we shall establish analogous results of the linear case studied by Blasco in [9]. Let $T : X \rightarrow Y$ be a positive p -summing sublinear operator. Firstly, in the case $X = C(\Omega)$, we prove some coincidence theorems and properties, and as a second result if $X = L_{p'}(\mu)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we use the representation of $u \in \nabla T$ for some characterizations of positive p -summing sublinear operators, which will be proven below at Theorem 5.6.1 and Theorem 5.6.2 in this chapter. We deduce that every T is positive p -summing sublinear operator if and only if, for all $u \in \nabla T$, u is positive p -summing operator. We also conclude that $\Pi_{s-p}^+(L_{p'}(\mu), Y) = \Pi_{s-1}^+(L_{p'}(\mu), Y)$. In the end of this work, we give necessary condition that Y has the Radon-Nikodym property.

5.1 Preliminaries

We begin by recalling briefly the abstract definition of Banach lattices. Let X be a Banach space. A real Banach lattice (resp, a real complete Banach lattice) X is equipped with a

lattice (resp. a complete lattice) and for all x, y in X , then

$$\begin{cases} (i) & \| |x| \| = \|x\|. \\ (ii) & |x| \leq |y| \implies \|x\| \leq \|y\|. \end{cases}$$

Where $|x| = \sup \{x, -x\}$, $|x| = x^+ + x^-$ (with $x^+ = \sup \{x, 0\}$ and $x^- = \sup \{0, -x\}$).

The spaces L_p ($1 \leq p \leq \infty$) are complete Banach lattices. The $C(K)$ is a Banach lattice.

We denote by $X^+ = \{x \in X : x \geq 0\}$. An element x of X is positive if $x \in X^+$. The dual X^* of a Banach lattice X is a complete Banach lattice with the natural order,

$$x_1^* \leq x_2^* \iff \langle x_1^*, x \rangle \leq \langle x_2^*, x \rangle, \quad \forall x \in X^+,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket.

Let X be a Banach lattice, n in \mathbb{N} and $1 \leq p \leq \infty$. We denote by $X(l_p^n)$, ($1 \leq p < \infty$), (resp. $l_p^\omega(X)$) the space of all sequences $x = (x_1, \dots, x_n)$ in X such that

$$\|x\|_{X(l_p^n)} = \left\| \left(\sum_1^n |x_i|^p \right)^{\frac{1}{p}} \right\| \text{ if } 1 \leq p < \infty.$$

$$\text{(resp. } \|x\|_{X(l_\infty^n)} = \left\| \sup_{1 \leq i \leq n} |x_i| \right\| \text{ if } p = \infty).$$

The space $X(l_p^n)$ is a Banach lattice equipped with the natural order

$$x \leq y \iff x_i \leq y_i, \quad \forall 1 \leq i \leq n.$$

Let X be a Banach space and $1 \leq p \leq \infty$. We denote by $l_p(X)$ (resp. $l_p^n(X)$) the space of all sequences (x_i) in X equipped with the norm

$$\|(x_i)\|_{l_p(X)} = \left(\sum_1^\infty \|x_i\|^p \right)^{\frac{1}{p}} < \infty.$$

$$\text{(resp. } \|(x_n)\|_{l_p^n(X)} = \left(\sum_1^n \|x_i\|^p \right)^{\frac{1}{p}},$$

and by $l_p^\omega(X)$ (resp. $l_p^\omega(X)$) the space of all sequences (x_i) in X equipped with the norm

$$\|(x_n)\|_{l_p^\omega(X)} = \sup_{\|\xi\|_{X^*}} \left(\sum_1^\infty |\langle x_i, \xi \rangle|^p \right)^{\frac{1}{p}}.$$

$$\text{(resp. } \|(x_n)\|_{l_p^\omega(X)} = \sup_{\|\xi\|_{X^*}} \left(\sum_1^n |\langle x_i, \xi \rangle|^p \right)^{\frac{1}{p}}).$$

Definition 5.1.1 A mapping T from a Banach space X into a Banach lattice Y , is said to be sublinear if for all $x, y \in X$ and $\lambda \in \mathbb{R}_+$, we have

- (i) $T(\lambda x) = \lambda T(x)$ (i.e., positively homogeneous).
- (ii) $T(x + y) \leq T(x) + T(y)$ (i.e., subadditive).

Note that the sum of two sublinear operators is a sublinear operator, and the multiplication by a positive number is also sublinear operator. Let us denote by

$$\mathcal{SL}(X; Y) = \{\text{sublinear operators } T : X \longrightarrow Y\},$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \Leftrightarrow T_1(x) \leq T_2(x), \forall x \in X.$$

Let $T \in \mathcal{SL}(X, Y)$, we say T is:

- (a) Symmetrical if for all x in X , $T(x) = T(-x)$.
- (b) Positive if for all x in X^+ , $T(x) \geq 0$.
- (c) Increasing if for all x, y in X (where X is a lattice),

$$x \leq y \implies T(x) \leq T(y).$$

The operator T is continuous (bounded). If and only if, there is a positive constant C such that for all x in X , $\|T(x)\| \leq C \|x\|$.

In this case, we put $\|T\| = \{\sup \|T(x)\| : \|x\|_{B_X} = 1\}$.

We denote by

$$\mathcal{SB}(X; Y) = \{\text{bounded sublinear operators } T : X \longrightarrow Y\},$$

and

$$\nabla T = \{u \in \mathcal{B}(X; Y) : u \leq T \text{ (i.e., } \forall x \in X, u(x) \leq T(x))\}.$$

The set ∇T is not empty if Y is a complete Banach lattice.

Remark 5.1.1 Let $X, Y,$ and Z be Banach spaces such that Y, Z are Banach lattices.

- (a) Consider T in $\mathcal{SB}(X; Y)$ and u in $\mathcal{B}(Y; Z)$ (assume that u is positive, i.e., $u(y) \geq 0$ for every $y \in Y$). Then, $u \circ T$ in $\mathcal{SB}(X; Z)$.
- (b) Consider u in $\mathcal{B}(X; Y)$ and T in $\mathcal{SB}(Y; Z)$. Then, $T \circ u$ in $\mathcal{SB}(X; Z)$.
- (c) Consider T in $\mathcal{SB}(X; Y)$ and S in $\mathcal{SB}(Y; Z)$ (increasing). Then, $S \circ T$ in $\mathcal{SB}(X; Z)$.

Let us recall the definitions of concept of p -summing and p -concave sublinear operators [1].

Definition 5.1.2 Let X, Y be Banach lattices and $1 \leq p \leq \infty$. A sublinear operator $T : X \longrightarrow Y$ is called p -concave if there is a constant C such that; for all n in \mathbb{N} . The operators

$$\begin{aligned} T_n : X(l_p^n) &\longrightarrow l_p^n(Y) \\ (x_1, \dots, x_n) &\longmapsto (T(x_1), \dots, T(x_n)) \end{aligned}$$

are uniformly bounded by C . The smallest constant C for which this holds is denoted by $C_p(T)$. We denote by $C_p(X, Y)$ the set of all p -concave sublinear operators from X into Y .

Definition 5.1.3 Let X be a Banach space and Y be a Banach lattice, A sublinear operator $T : X \longrightarrow Y$ is p -summing for $1 \leq p \leq \infty$ if there is a constant $C > 0$ such that for any $x_1, \dots, x_n \in X$, we have

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}. \quad (5.1.1)$$

We denote by $\Pi_{s-p}(X, Y)$ the class of all p -summing sublinear operators from X into Y and $\pi_{s-p}(T) = \inf\{C \text{ verifying the inequality (5.1.1)}\}$.

5.2 Positive p -summing operators

5.2.1 Positive p -summing linear operators

In this paragraph we present the definition of positive p -summing linear operators. And some properties that we need later. For more information, see [9].

Definition 5.2.1 Let X be a Banach lattice and Y be a Banach space . An operator $u : X \longrightarrow Y$ is positive p -summing for $1 \leq p \leq \infty$, if there exists a constant $C > 0$ such that for every x_1, \dots, x_n positive elements in X , we have

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}. \quad (5.2.1)$$

We denote by $\Pi_p^+(X; Y)$ the class of all positive p -summing linear operators from X into Y and $\pi_p^+(T) = \inf\{C \text{ verifying the inequality (5.2.1)}\}$.

Remark 5.2.1 [9] Let X be a Banach space. If we use the duality $(l_p)^* = l_{p'}$, we have

$$\sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i, \xi \rangle|^p \right)^{\frac{1}{p}} = \sup_{\alpha \in U_{l_{p'}}^+} \left\| \sum_{i=1}^n \alpha_i |x_i| \right\|_X. \quad (5.2.2)$$

With $U_{l_{p'}}^+ = \left\{ \alpha = (\alpha_i)_{i=1}^n, \sum_{i=1}^n \alpha_i^{p'} \leq 1, \alpha_i \geq 0 \right\}$.

Proposition 5.2.1 Let X_1, X_2 be Banach spaces and for all $1 \leq p \leq \infty$.

If $X_1 \subseteq X_2$, $\overline{X_1} = X_2$, then $\Pi_p^+(X_2; Y) \subseteq \Pi_p^+(X_1; Y)$.

Proposition 5.2.2 Let Y be a Banach space, we have

- (1) If $1 \leq p \leq \infty$, then $\Pi_p^+(L_1(\mu), Y) = \mathcal{B}(L_1(\mu), Y)$.
- (2) If $1 \leq p \leq q \leq \infty$, then $\Pi_p^+(X; Y) \subset \Pi_q^+(X; Y)$.

Theorem 5.2.1 For all $1 \leq p \leq \infty$, we have

$$\Pi_p^+(L_{p'}(\mu); Y) = \Pi_1^+(L_{p'}(\mu); Y).$$

Theorem 5.2.2 [9, Theorem 2] Let $1 < p \leq \infty$. The following assertions are equivalent:

- (a) Let Y has the Radon-Nikodym property.
- (b) $\Pi_p^+(L_{p'}(\mu), Y) = L_p(\mu, Y)$.

5.2.2 Positive p -summing sublinear operators

Now, we give the definition of positive p -summing sublinear operators and some results, for more information, see [4].

Definition 5.2.2 Let X, Y be Banach lattices. A sublinear operator $T : X \rightarrow Y$ is positive p -summing for $1 \leq p \leq \infty$ if there is a constant $C > 0$ such that, $\forall n \in \mathbb{N}^*, \forall \{x_1, \dots, x_n\} \subset X^+$. We have

$$\left(\| (T(x_i)) \|_{l_p^n(Y)} \right) \leq C \left\| (x_i) \right\|_{l_p^n(X)}. \quad (5.2.3)$$

We denote by $\Pi_{s-p}^+(X, Y)$ the class all positive p -summing sublinear operators and

$$\pi_{s-p}^+(T) = \inf \{ C, \text{ verifying the inequality (5.2.3)} \}.$$

Remark 5.2.2 (a) All sublinear p -summing operator is positive p -summing sublinear operator. Furthermore, we have

$$\pi_{s-p}^+(T) \leq \pi_{s-p}(T).$$

(b) From duality (5.2.2), we obtain the definition equivalent to (5.2.3) as follows

$$\left(\sum_1^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left\| \sum_{i=1}^n \alpha_i x_i \right\|_X, \sum_{i=1}^n \alpha_i^{p'} \leq 1, \alpha_i \geq 0 \right\}. \quad (5.2.4)$$

Theorem 5.2.3 (Pietsch's domination theorem) Let X, Y be Banach lattices. A sublinear operator $T : X \rightarrow Y$ is positive p -summing ($1 \leq p < \infty$) if and only if there exists a positive constant C and a Borel probability λ on $(B_{X^*}^+, \sigma(X^*, X))$ ($B_{X^*}^+ = B_{X^*} \cap X_+^*$) such that

$$\|T(x)\| \leq C \left(\int_{B_{X^*}^+} |\langle x, \xi \rangle|^p d\lambda(\xi) \right)^{\frac{1}{p}} \text{ for every } x \in X. \quad (5.2.5)$$

Moreover, in this case

$$\pi_{s-p}^+(T) = \inf \{ C, \text{ verifying the inequality (5.2.5)} \}.$$

Proposition 5.2.3 Let X, Y be Banach lattices and $T \in \mathcal{SL}(X, Y)$.

(a) If $1 \leq p \leq q \leq \infty$, then

$$\Pi_{s-p}^+(X, Y) \subset \Pi_{s-q}^+(X, Y) \text{ and } \pi_{s-q}^+(T) \leq \pi_{s-p}^+(T). \quad (5.2.6)$$

(b) If $T \in \Pi_{s-p}^+(X, Y)$ (where Y is a complete lattice), then

$$\forall u \in \nabla T, \quad u \in \Pi_p^+(X, Y) \text{ and we have } \pi_p^+(u) \leq 2\pi_{s-p}^+(T). \quad (5.2.7)$$

5.3 Properties of the positive p -summing sublinear operators

The Remark 5.1.1, yields the following proposition.

Proposition 5.3.1 (Ideal property) Let E be a Banach spaces, let $F, X,$ and Y be Banach lattices. Let $T \in \Pi_{s-p}^+(X; Y)$. v be a positive operator in $\mathcal{B}(E; X)$ and w be a positive operator in $\mathcal{B}(Y; F)$. Then wTv is positive p -summing sublinear operator. In addition

$$\pi_{s-p}^+(wTv) \leq \|w\| \pi_{s-p}^+(T) \|v\|.$$

And we deduce the following results.

Corollary 5.3.1 *If X_0 is a sublattice of X and $T \in \Pi_{s-p}^+(X, Y)$. Then*

$$T/X_0 \in \Pi_{s-p}^+(X_0; Y),$$

and

$$\pi_{s-p}^+(T/X_0) \leq \pi_{s-p}^+(T).$$

Corollary 5.3.2 *(Injectivity) If Y_0 is a sublattice of Y and $i : Y_0 \longrightarrow Y$. The canonical injection. Then, The next properties are equivalent:*

(a) *The operator T belongs to $\Pi_{s-p}^+(X; Y_0)$.*

(b) *The operator iT belongs to $\Pi_{s-p}^+(X; Y)$.*

In this case

$$\pi_{s-p}^+(T) = \pi_{s-p}^+(iT).$$

Lemma 5.3.1 *[3, Proposition 2.1] Let T be a sublinear operator between a Banach space X and a Banach lattice Y .*

(1) *For all x in X , we put*

$$\varphi(x) = \sup \{T(x), T(-x)\}.$$

Then, φ is symmetrical sublinear operator and we have

$$\|T\| \leq \|\varphi\| \quad \text{and} \quad \|\varphi(x)\| \leq \sup \{\|T(x)\|, \|T(-x)\|\}. \quad (5.3.1)$$

(2) *For every $(\alpha_i)_{i=1}^n \subset \mathbb{R}$, we have*

$$T\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n |\alpha_i| \varphi(x_i).$$

In addition

$$\left\| T\left(\sum_{i=1}^n \alpha_i x_i\right) \right\| \leq \sum_{i=1}^n |\alpha_i| \|\varphi(x)\|.$$

Proposition 5.3.2 *The sublinear operator $T : X \longrightarrow Y$ is p -summing for all $1 \leq p \leq \infty$ if and only if, the sublinear operator φ is p -summing. In addition*

$$\pi_{s-p}(\varphi) \leq \sqrt[p]{2} \pi_{s-p}(T).$$

Proof. By (5.3.1), we find that if φ is p -summing, then T is p -summing

$$\begin{aligned}
 \sum_1^n \|\varphi(x_i)\|^p &\leq \sup \left\{ \sum_1^n \|T(x_i)\|^p, \sum_1^n \|T(-x_i)\|^p \right\} \\
 &\leq \sum_1^n \|T(x_i)\|^p + \sum_1^n \|T(-x_i)\|^p \\
 &\leq (\pi_{s-p}(T))^p \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i, \xi \rangle|^p \right) + (\pi_p(T))^p \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle -x_i, \xi \rangle|^p \right) \\
 &\leq 2 (\pi_{s-p}(T))^p \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i, \xi \rangle|^p \right).
 \end{aligned}$$

Then

$$\left(\sum_1^n \|\varphi(x_i)\|^p \right)^{\frac{1}{p}} \leq \sqrt[p]{2} \pi_{s-p}(T) \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i, \xi \rangle|^p \right)^{\frac{1}{p}}.$$

Hence φ is p -summing, in addition

$$\pi_{s-p}(\varphi) \leq \sqrt[p]{2} \pi_{s-p}(T).$$

This end the proof. ■

Proposition 5.3.3 *The sublinear operator $T : X \rightarrow Y$ is positive p -summing for all $1 \leq p \leq \infty$ if and only if, the sublinear operator φ is positive p -summing. In addition*

$$\pi_{s-p}^+(\varphi) \leq 2\pi_{s-p}^+(T).$$

Proof. by (5.3.1), we find that if φ is positive p -summing, then T is positive p -summing.

Reciprocally, since T is positive p -summing, therefore by (5.2.5). Then

$$\forall x \in X, \|T(x)\| \leq \pi_{s-p}^+(T) \left(\int_{B_{X^*}^+} |\langle |x|, \xi \rangle|^p d\lambda(\xi) \right)^{\frac{1}{p}}.$$

From (5.3.1), for all x in X , we have

$$\begin{aligned}
 \|\varphi(x)\| &\leq \|T(x)\| + \|T(-x)\| \\
 &\leq \pi_{s-p}^+(T) \left(\int_{B_{X^*}^+} |\langle |x|, \xi \rangle|^p d\lambda(\xi) \right)^{\frac{1}{p}} + \pi_{s-p}^+(T) \left(\int_{B_{X^*}^+} |\langle |-x|, \xi \rangle|^p d\lambda(\xi) \right)^{\frac{1}{p}} \\
 &\leq 2\pi_{s-p}^+(T) \left(\int_{B_{X^*}^+} |\langle |x|, \xi \rangle|^p d\lambda(\xi) \right)^{\frac{1}{p}}.
 \end{aligned}$$

Hence φ is positive p -summing and

$$\pi_{s-p}^+(\varphi) \leq 2\pi_{s-p}^+(T).$$

This end the proof. ■

5.4 Some inclusion and coincidence results

Proposition 5.4.1 *Let X_1, X_2, Y be Banach lattices and for all $1 \leq p \leq \infty$.*

If $X_1 \subseteq X_2$, $\overline{X_1} = X_2$, then

$$\Pi_{s-p}^+(X_2; Y) \subseteq \Pi_{s-p}^+(X_1; Y).$$

Proof. We suppose that $X_1 \subseteq X_2$, then $X_2^* \subseteq X_1^*$ and $\|\xi\|_{X_1^*} \leq \|\xi\|_{X_2^*}$.

Let $T \in \Pi_{s-p}^+(X_2, Y)$ and n in \mathbb{N} , since $\forall \{x_1, \dots, x_n\} \subset X_1^+ \subset X_2^+$. We have

$$\begin{aligned} \left(\sum_1^n \|T(x_i)\|_Y^p \right)^{\frac{1}{p}} &\leq C \sup_{\|\xi\|_{X_2^*} \leq 1} \left(\sum_1^n |\langle x_i, \xi \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq C \sup_{\|\xi\|_{X_1^*} \leq 1} \left(\sum_1^n |\langle x_i, \xi \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then $T \in \Pi_{s-p}^+(X_1; Y)$. ■

The following theorem clarifies the relation between $\Pi_{s-p}^+(X; Y)$ and $C_p(X; Y)$.

Theorem 5.4.1 *Let X, Y be Banach lattices and $1 \leq p \leq \infty$, we have*

$$\Pi_{s-p}^+(X; Y) \subseteq C_p(X; Y).$$

Proof. Let T in $\Pi_{s-p}^+(X; Y)$ and $x_1, \dots, x_n \subset X$. For all $1 \leq p < \infty$, by (5.3.1), we have

$$\begin{aligned} \left(\sum_1^n \|T(x_i)\|_Y^p \right)^{\frac{1}{p}} &\leq \left(\sum_1^n \|\varphi(x_i)\|_Y^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_1^n \|\varphi(x_i^+ + (-x_i^-))\|_Y^p \right)^{\frac{1}{p}} \\ &\leq \left(\sum_1^n \|\varphi(x_i^+)\|_Y^p \right)^{\frac{1}{p}} + \left(\sum_1^n \|\varphi(-x_i^-)\|_Y^p \right)^{\frac{1}{p}}, \end{aligned}$$

by φ symmetrical, then

$$\left(\sum_1^n \|T(x_i)\|_Y^p \right)^{\frac{1}{p}} \leq \left(\sum_1^n \|\varphi(x_i^+)\|_Y^p \right)^{\frac{1}{p}} + \left(\sum_1^n \|\varphi(x_i^-)\|_Y^p \right)^{\frac{1}{p}}.$$

By Proposition 5.3.3, then

$$\begin{aligned} \left(\sum_1^n \|T(x_i)\|_Y^p \right)^{\frac{1}{p}} &\leq \pi_{s-p}^+(\varphi) \left(\sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i^+, \xi \rangle|^p \right)^{\frac{1}{p}} + \sup_{\xi \in B_{X^*}} \left(\sum_1^n |\langle x_i^-, \xi \rangle|^p \right)^{\frac{1}{p}} \right) \\ &\leq \pi_{s-p}^+(\varphi) \left(\sup_{\alpha \in U_{l^{p'}}^+} \left\| \sum_{i=1}^n \alpha_i x_i^+ \right\|_X + \sup_{\alpha \in U_{l^{p'}}^+} \left\| \sum_{i=1}^n \alpha_i x_i^- \right\|_X \right) \text{ by (5.2.2)} \\ &\leq 2\pi_{s-p}^+(\varphi) \left(\sup_{\alpha \in U_{l^{p'}}^+} \left\| \sum_{i=1}^n \alpha_i |x_i| \right\|_X \right). \end{aligned}$$

Since $\sum_{i=1}^n \alpha_i^{p'} \leq 1$ and from Hölder's inequality, we obtains that

$$\sum_{i=1}^n \alpha_i |x_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, \text{ for all } \alpha \in U_{l^{p'}}^+.$$

This implies

$$\begin{aligned} \|T(x_i)\|_{l_p^p(Y)} &\leq 2\pi_{s-p}^+(\varphi) \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\|_X \\ &\leq 4\pi_{s-p}^+(T) \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\|_X. \end{aligned}$$

Then T is p -concave. This concludes the proof of the theorem. \blacksquare

Example 5.4.1 *The identity*

$$|Id_{l_2}| : l_2 \longrightarrow l_2,$$

is 2-concave sublinear operator but is not positive 2-summing. Indeed, because for all (e_n) orthonormal basis of l_2 . We have

$$\begin{aligned} \left(\sum_1^n \|T(e_i)\|^2 \right)^{\frac{1}{2}} &= 2^{\frac{1}{2}} \quad \text{and} \quad \sup_{\|\xi\|_2 \leq 1} \left(\sum_1^n |\langle e_i, \xi \rangle|^2 \right)^{\frac{1}{2}} = \sup_{\|\xi\|_2} \|\xi\|_2 \\ &\leq \sup_{\|\xi\|_2} \|\xi\|_2 \\ &\leq 1. \end{aligned}$$

If $X = C(\Omega)$, the following theorem gives the coincidence between $\Pi_{s-p}(C(\Omega); Y)$, $\Pi_{s-p}^+(C(\Omega); Y)$ and $C_p(C(\Omega); Y)$.

Theorem 5.4.2 *Let Y be a Banach lattice and $1 \leq p \leq \infty$, we have*

$$\Pi_{s-p}(C(\Omega); Y) = \Pi_{s-p}^+(C(\Omega); Y) = C_p(C(\Omega); Y).$$

Proof. (i) $\Pi_{s-p}^+(C(\Omega); Y) \subset \Pi_{s-p}(C(\Omega); Y)$.

First, by [19, page 41], we can find a convenient form $\|(f_i)_{i=1}^n\|_p^{weak}$ for when $\{f_1, \dots, f_n\}$ in $C(\Omega)$. To each $\omega \in \Omega$ there corresponds a point mass' $\delta_\omega \in C(\Omega)^*$ given by $\langle \delta_\omega, f \rangle = f(\omega)$.

The point masses evidently a norming subset of $C(\Omega)^*$. So

$$\begin{aligned} \sup_{\xi \in C(\Omega)^*} \left(\sum_{i=1}^n |\langle f_i, \xi \rangle|^p \right)^{\frac{1}{p}} &= \sup_{\omega \in \Omega} \left(\sum_{i=1}^n |\langle \delta_\omega, f_i \rangle|^p \right)^{\frac{1}{p}} \\ &= \sup_{\omega \in \Omega} \left(\sum_{i=1}^n |f_i(\omega)|^p \right)^{\frac{1}{p}} \\ &= \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{\frac{1}{p}} \right\|_{\infty}. \end{aligned}$$

Let $T \in \Pi_{s-p}^+(C(\Omega); Y)$. From (5.3.1), we have

$$\begin{aligned} \left(\sum_1^n \|T(f_i)\|^p \right)^{1/p} &\leq \left(\sum_1^n \|\varphi(f_i)\|^p \right)^{1/p} \\ &\leq \left(\sum_1^n \|\varphi(f_i^+ - f_i^-)\|^p \right)^{1/p}, \text{ because } f = f_i^+ - f_i^- \\ &\leq \left(\sum_1^n \|\varphi(f_i^+)\|^p \right)^{1/p} + \left(\sum_1^n \|\varphi(-f_i^-)\|^p \right)^{1/p}, \end{aligned}$$

by φ is symmetrical, then

$$\left(\sum_1^n \|T(f_i)\|^p \right)^{1/p} \leq \left(\sum_1^n \|\varphi(f_i^+)\|^p \right)^{1/p} + \left(\sum_1^n \|\varphi(f_i^-)\|^p \right)^{1/p}.$$

By Proposition 5.3.3, we obtain that

$$\begin{aligned}
 \left(\sum_1^n \|T(f_i)\|^p \right)^{1/p} &\leq \pi_{s-p}^+(\varphi) \sup_{\xi \in C(\Omega)^*} \left(\sum_{i=1}^n |\langle f_i^+, \xi \rangle|^p \right)^{\frac{1}{p}} + \pi_{s-p}^+(\varphi) \sup_{\xi \in C(\Omega)^*} \left(\sum_{i=1}^n |\langle f_i^-, \xi \rangle|^p \right)^{\frac{1}{p}} \\
 &\leq \pi_{s-p}^+(\varphi) \left\| \left(\sum_{i=1}^n |f_i^+|^p \right)^{\frac{1}{p}} \right\|_{\infty} + \pi_{s-p}^+(\varphi) \left\| \left(\sum_{i=1}^n |f_i^-|^p \right)^{\frac{1}{p}} \right\|_{\infty} \\
 &\leq 2\pi_{s-p}^+(\varphi) \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{\frac{1}{p}} \right\|_{\infty} \text{ by } |f_i^+| \leq |f_i| \text{ and } |f_i^-| \leq |f_i|.
 \end{aligned}$$

So, we have

$$\begin{aligned}
 \left(\sum_1^n \|T(f_i)\|^p \right)^{1/p} &\leq 2\pi_{s-p}^+(\varphi) \sup_{\xi \in C(\Omega)^*} \left(\sum_{i=1}^n |\langle f_i, \xi \rangle|^p \right)^{\frac{1}{p}} \\
 &\leq 4\pi_{s-p}^+(T) \sup_{\xi \in C(\Omega)^*} \left(\sum_{i=1}^n |\langle f_i, \xi \rangle|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Consequently, T is p -summing and $\pi_{s-p}(T) \leq 4\pi_{s-p}^+(T)$.

(ii) $C_p(C(\Omega), Y) \subset \Pi_{s-p}^+(C(\Omega), Y)$.

Let $T \in C_p(C(\Omega), Y)$ and $\{f_1, \dots, f_n\}$ in $(C(\Omega))^+$, we have

$$\begin{aligned}
 \left(\sum_1^n \|T(f_i)\|^p \right)^{1/p} &\leq c_p \left\| \sum_1^n (|f_i|^p)^{1/p} \right\|_{C(\Omega)} \\
 &\leq c_p \sup_{\omega \in \Omega} \left(\sum_{i=1}^n |f_i(\omega)|^p \right)^{\frac{1}{p}} \\
 &\leq c_p \sup_{\xi \in B_{C(\Omega)^*}} \left(\sum_{i=1}^n |\langle f_i, \xi \rangle|^p \right)^{\frac{1}{p}}.
 \end{aligned}$$

Then T is positive p -summing and $\pi_{s-p}^+(T) \leq c_p$. Consequently

$$C_p(C(\Omega); Y) \subset \Pi_{s-p}^+(C(\Omega); Y).$$

From (i) and (ii) we obtain that

$$\Pi_{s-p}(C(\Omega); Y) = \Pi_{s-p}^+(C(\Omega); Y) = C_p(C(\Omega); Y). \quad \blacksquare$$

5.5 On the positive p -summing sublinear operators on the space $L_p(\mu)$

At the beginning of this section, we start by the following lemma.

Lemma 5.5.1 [1] *Let X be a Banach space and let Y be a complete Banach lattice. Let $T \in \mathcal{SB}(X; Y)$, then*

- (a) $\forall x \in X, \|T(x)\| \leq \sup_{u \in \nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\|.$
- (b) $\|T\| \leq \sup_{u \in \nabla T} \|u\| \leq 2\|T\|.$
- (c) *For all x in X , there is $u_x \in \nabla T$ such that*

$$T(x) = u_x(x) \text{ (i.e., } T(x) = \sup_{u \in \nabla T} u(x)\text{).}$$

Proposition 5.5.1 *Let Y be a Banach lattice and for all $1 \leq p \leq \infty$, we have*

$$\Pi_{s-p}^+(L_1(\mu); Y) = \mathcal{SB}(L_1(\mu); Y).$$

Proof. It suffices to show that $\Pi_{s-1}^+(L_1(\mu); Y) = \mathcal{SB}(L_1(\mu); Y).$

Let $T \in \mathcal{SB}(L_1(\mu), Y)$, for all $\{f_1, \dots, f_n\}$ in $(L_1(\mu))^+$ by Lemma 5.5.1 (c), we have

$$\begin{aligned} \left(\sum_{i=1}^n \|T(f_i)\| \right) &= \left(\sum_{i=1}^n \|u_{f_i}(f_i)\| \right) \\ &\leq \left(\sup_{u \in \nabla T} \|u\| \right) \sum_{i=1}^n \|f_i\|_{L_1} \\ &\leq 2\|T\| \left\| \sum_{i=1}^n f_i \right\|_{L_1} \\ &\leq 2\|T\| \sup_{\rho \in L^\infty(\mu)} \left(\sum_{i=1}^n |\langle \rho, f_i \rangle| \right), \text{ where } \|\rho\|_\infty \leq 1. \end{aligned}$$

Then T is positive 1-summing and $\pi_{1-p}^+(T) \leq 2\|T\|.$ Consequently

$$\Pi_{s-p}^+(L_1(\mu); Y) = \mathcal{SB}(L_1(\mu); Y). \quad \blacksquare$$

According to Proposition 5.2.2 and Proposition 5.5.1, we have the following result.

Corollary 5.5.1 *Let Y be a complete Banach lattice and for all $1 \leq p \leq \infty$. The following properties are equivalent:*

- (1) *The operator $T \in \Pi_{s-p}^+(L_1(\mu); Y)$.*
- (2) *For all $u \in \nabla T$, $u \in \Pi_p^+(L_1(\mu); Y)$.*

Lemma 5.5.2 *Let p be a real number in $]0, +\infty[$ and be a measure space (Ω, μ) , let I be a set of indices.*

If $\{g_i\}_{i \in I}$ is a filtering family and norm bounded in $L_p(\mu)$. Then $\{g_i\}_{i \in I}$ has a superior in $L_p(\mu)$.

Proof. It is Known that the $(L_p(\mu))^+$ is Banach complete lattice and that the family $\{g_i\}_{i \in I}$ is norm bounded, and it has a superior.

Remark 5.5.1 *Let Y be a complete Banach lattice.*

We suppose that $T \in \Pi_{s-p}^+(L_1(\mu), Y)$. By (5.2.7), we have $\forall u \in \nabla T$, $u \in \Pi_p^+(L_{p'}(\mu); Y)$ and $\pi_p^+(u) \leq 2\pi_{s-p}^+(T)$. According to [9, statement (8) in proof Theorem1]. There exists a function $g_u \geq 0$ in $L_p(\mu)$ such that

$$\|u(\psi)\| \leq \int_{\Omega} |\psi(t)| g_u(t) d\mu \text{ where } \psi \in L_{p'}(\mu).$$

Furthermore, $\|g_u\|_p = \pi_p^+(u)$, hence

$$\|g_u\|_p \leq 2\pi_{s-p}^+(T).$$

Then $\{g_u\}_{u \in \nabla T}$ is norm bounded. Therefore by the Lemma 5.5.2, we obtain an upper bound $\left(\text{i.e., } \sup_{u \in \nabla T} g_u(t) = g(t), g \in L_p(\mu) \right)$.

Proposition 5.5.2 *Let Y be a complete Banach lattice, $T \in \mathcal{SL}(L_{p'}(\mu); Y)$ and for all $1 \leq p \leq \infty$. We suppose that $\forall u \in \nabla T$, $u \in \Pi_p^+(L_{p'}(\mu); Y)$, then, there exists a function $g_u \in L_p(\mu)$ such that*

$$u(f) = \int_{\Omega} f(t) g_u(t) d\mu \text{ for every } f \in L_{p'}(\mu). \quad (5.5.1)$$

Furthermore, $\{g_u\}_{u \in \nabla T}$ is norm bounded in $L_1(\mu)$ (i.e., $\|g_u\| \leq 2\|T\|$).

Proof. Let $u \in \nabla T, u \in \Pi_p^+(L_{p'}(\mu); Y)$.

According to [9], there exists a function $g_u \in L_p(\mu)$ such that

$$u(\psi) = \int_{\Omega} f(t) g_u(t) d\mu \text{ fore very } f \in L_{p'}(\mu).$$

By Lemma 5.5.1 (c), we obtain that

$$\forall u \in \nabla T, u(f) \leq T(f),$$

hence

$$\|u(f)\| \leq \|T(f)\| + \|T(-f)\| \leq 2\|T\| \|f\|, \quad (5.5.2)$$

and

$$\|u(f)\|_Y = \left\| \int_{\Omega} f(t) g_u(t) d\mu \right\|_Y.$$

With the choice that $\psi(t) = 1_{L_{p'}(\mu)}$.

Holds true. Then

$$\|u(1)\| = \|\|g_u\|_{L_1}\|_Y = \|g_u\|_{L_1},$$

by (5.5.2), we have $\|g_u\|_{L_1} \leq 2\|T\|$.

Therefore $\{g_u\}_{u \in \nabla T}$ is norm bounded in $L_1(\mu)$.

By Lemma 5.5.2., we obtain an upper bound $\left(i.e., \sup_{u \in \nabla T} g_u(t) = g(t) \in L_1(\mu)\right)$. ■

5.6 Characterization of positive p -summing sublinear operators using representable mappings

In this section, we denote by $L_p(\mu, Y)$ the space of measurable functions on Ω with $\|f\| = \left(\int_{\Omega} \|f(t)\|^p d\mu\right)^{\frac{1}{p}}$ and Y is a complete Banach lattice .We give some characterizations of positive p -summing sublinear operators using the representation of $u \in \nabla T$ and $u \in \Pi_p^+(L_{p'}(\mu); Y)$. We introduce now the theorem of characterization without Y has the Radon-Nikodym property.

Theorem 5.6.1 *Let $1 < p \leq \infty$. The following properties are equivalent:*

(a) The operator T belongs to $\Pi_{s-p}^+(L_{p'}(\mu); Y)$.

(b) There exists a function $g \geq 0$ in $L_p(\mu)$ such that

$$\|T(f)\| \leq \int_{\Omega} |f(t)| g(t) d\mu \text{ for all } f \in L_{p'}(\mu). \quad (5.6.1)$$

(c) The operator T belongs to $\Pi_{s-1}^+(L_{p'}(\mu), Y)$.

Proof. (a) \Rightarrow (b) : Let $T \in \Pi_{s-p}^+(L_{p'}(\mu), Y)$ by (5.2.7), we have

$$\forall u \in \nabla T, u \in \Pi_p^+(L_{p'}(\mu), Y),$$

by Remark 5.5.1, we find

$$\|u(f)\| \leq \int_{\Omega} |f(t)| g_u(t) d\mu \text{ for all } f \in L_{p'}(\mu) \text{ and } g_u \geq 0 \text{ in } L_p(\mu) ..$$

By using Lemma 5.5.1 (a), we have

$$\|T(f)\| \leq \sup_{u \in \nabla T} \|u(f)\|,$$

hence

$$\begin{aligned} \|T(f)\| &\leq \sup_{u \in \nabla T} \int_{\Omega} |f(t)| g_u(t) d\mu \\ &\leq \int_{\Omega} |f(t)| \left(\sup_{u \in \nabla T} g_u(t) \right) d\mu, \text{ by Lemma 5.5.2.} \end{aligned}$$

Consequently, there exists a function $g \geq 0$ in $L_p(\mu)$ $\left(g(t) = \sup_{u \in \nabla T} g_u(t) \right)$, such that

$$\|T(f)\| \leq \int_{\Omega} |f(t)| g(t) d\mu, \text{ with } f \in L_{p'}(\mu) \text{ and } g \in L_p(\mu).$$

(b) \Rightarrow (c) : By (5.6.1), let $f_1, \dots, f_n \geq 0$ in $L_{p'}(\mu)$, we have

$$\begin{aligned} \sum_{i=1}^n \|T(f_i)\| &\leq \sum_{i=1}^n \int_{\Omega} f_i(t) g(t) d\mu \\ &\leq \int_{\Omega} g(t) \left(\sum_{i=1}^n f_i(t) \right) d\mu \\ &\leq \|g\|_{L_p(\mu)} \left\| \sum_{i=1}^n f_i(t) \right\|_{L_{p'}(\mu)} \quad (\text{by Hölder's inequality}). \end{aligned}$$

By (5.2.4), this implies that $T \in \Pi_{s-1}^+(L_{p'}(\mu), Y)$.

(c) \Rightarrow (a) : By (5.2.6). ■

From the above result, we deduce the following corollary.

Corollary 5.6.1 *Let $1 \leq p \leq \infty$. We have*

$$\Pi_{s-p}^+(L_{p'}(\mu); Y) = \Pi_{s-1}^+(L_{p'}(\mu); Y). \quad (5.6.2)$$

Proof. By (5.2.6), we have

$$\Pi_{s-1}^+(L_{p'}(\mu); Y) \subset \Pi_{s-p}^+(L_{p'}(\mu); Y).$$

From Theorem 5.6.1, we obtain

$$\Pi_{s-p}^+(L_{p'}(\mu); Y) \subset \Pi_{s-1}^+(L_{p'}(\mu); Y). \quad \blacksquare$$

The aim of the following theorem is to give other characterizations of positive p -summing sublinear operators using the Radon-Nikodym property.

Theorem 5.6.2 *Let $1 < p \leq \infty$. Suppose Y has the Radon-Nikodym property. The following statements are equivalent:*

- (1) *The operator T belongs to $\Pi_{s-p}^+(L_{p'}(\mu); Y)$.*
- (2) *For all $u \in \nabla T$, u is representable by a function g_u in $L_p(\mu, Y)$ such that*

$$u(f) = \int_{\Omega} f(t) g_u(t) d\mu \quad \text{for every simple function } f \in L_{p'}(\mu).$$

Proof. (1) \Rightarrow (2) : Let $T \in \Pi_{s-p}^+(L_{p'}(\mu); Y)$, by (5.2.7) then

$$\forall u \in \nabla T, \quad u \in \Pi_p^+(L_{p'}(\mu); Y).$$

So by Theorem 5.2.2, we have $\Pi_p^+(L_{p'}(\mu); Y) = L_p(\mu, Y)$.

Therefore u is representable by a function g_u in $L_p(\mu, Y)$. As the follows

$$u(f) = \int_{\Omega} f(t) g_u(t) d\mu, \quad \text{for every simple function } f \in L_{p'}(\mu). \quad (5.6.3)$$

(2) \Rightarrow (1) : On other hand, by (5.6.3), we have

$$\begin{aligned} \|u(f)\| &= \left\| \int_{\Omega} f(t) g_u(t) d\mu \right\| \\ &\leq \int_{\Omega} |f(t)| \|g_u\|_{L_p(\mu, Y)} d\mu \\ &\leq \|g_u\|_{L_p(\mu, Y)} \int_{\Omega} |f(t)| d\mu. \end{aligned}$$

Hence

$$\sup_{u \in \nabla T} \|u(f)\| \leq \|g_u\|_{L_p(\mu, Y)} \int_{\Omega} |f(t)| d\mu.$$

By using Lemma 5.5.1 (c), we obtain

$$\|T(f)\| \leq \|g_u\|_{L_p(\mu, Y)} \int_{\Omega} |f(t)| d\mu.$$

Therefore, this implies for all $f_1, \dots, f_n \geq 0$ in $L_{p'}(\mu)$.

$$\begin{aligned} \sum_{i=1}^n \|T(f_i)\| &\leq \sum_{i=1}^n \|g_u\|_{L_p(\mu, Y)} \int_{\Omega} f_i(t) d\mu \\ &\leq \|g_u\|_{L_p(\mu, Y)} \int_{\Omega} \left(\sum_{i=1}^n f_i(t) \right) d\mu \\ &\leq \|g_u\|_{L_p(\mu, Y)} \left\| \sum_{i=1}^n f_i(t) \right\|_{L_{p'}(\mu)}. \end{aligned}$$

Consequently $T \in \Pi_{s-1}^+(L_{p'}(\mu), Y)$. According to (5.2.6), we have $T \in \Pi_{s-p}^+(L_{p'}(\mu), Y)$. \blacksquare

From the above result, we conclude the following corollary.

Corollary 5.6.2 *Let $1 < p \leq \infty$. Suppose Y has the Radon-Nikodym property. The following statements are equivalent:*

- (1) *The operator $T \in \Pi_{s-p}^+(L_{p'}(\mu); Y)$.*
- (2) *For all $u \in \nabla T$, $u \in \Pi_p^+(L_{p'}(\mu); Y)$.*

Proof. (1) \Rightarrow (2) : By the Inequality (5.2.7).

(2) \Rightarrow (1) : Let $u \in \nabla T$, $u \in \Pi_p^+(L_{p'}(\mu); Y)$.

Then by Theorem 5.2.2, we have $\Pi_p^+(L_{p'}(\mu); Y) = L_p(\mu, Y)$. Hence

$$u(f) = \int_{\Omega} f(t) g_u(t) d\mu.$$

By Theorem 5.6.2, we obtain $T \in \Pi_{s-p}^+(L_{p'}(\mu); Y)$. ■

We close this section by showing the following result.

Proposition 5.6.1 *Let $1 < p \leq \infty$. We have (a) \Rightarrow (b), where*

(a) *The space Y has the Radon-Nikodym property.*

(b) *There exists a positive function g in $L_p(\mu, Y)$ such that*

$$T(f) \leq \int_{\Omega} |f(t)| g(t) d\mu \text{ for all } f \in L_{p'}(\mu).$$

Proof. Suppose Y has the Radon-Nikodym property. By Theorem 5.2.2, we have $\Pi_p^+(L_{p'}(\mu); Y) = L_p(\mu, Y)$. Therefore u is representable by a function in $L_p(\mu, Y)$. i.e

$$u(f) = \int_{\Omega} f(t) g_u(t) d\mu.$$

We have $u(f) \leq |u(f)|$, then

$$u(f) \leq \int_{\Omega} |f(t)| |g_u(t)| d\mu.$$

By Lemma 5.5.1 (c), we obtain that

$$T(f) = \sup_{u \in \nabla T} u(f) \leq \sup_{u \in \nabla T} |u(f)|.$$

Hence

$$\begin{aligned} T(f) &\leq \sup_{u \in \nabla T} \int_{\Omega} |f(t)| |g_u(t)| d\mu \\ T(f) &\leq \int_{\Omega} |f(t)| \sup_{u \in \nabla T} |g_u(t)| d\mu, \\ T(f) &\leq \int_{\Omega} |f(t)| g(t) d\mu, \text{ with } g(t) = \sup_{u \in \nabla T} |g_u(t)| \\ T(f) &\leq \int_{\Omega} |f(t)| g(t) d\mu, \text{ with } f \in L_{p'}(\mu) \text{ and } g \geq 0 \text{ in } L_p(\mu, Y). \end{aligned}$$

This ends the proof. ■

Remark 5.6.1 *In both Theorem 5.6.1 and Theorem 5.6.2, we could not give a representable, but only a characterization. In other words, it is not possible at least to make a representation of positive p -summing sublinear operator. For if written in the same way in linear case*

$$T(f) = \int_{\Omega} f(t) K(t) d\mu.$$

This means that T is linear.

Bibliography

- [1] D. ACHOUR AND L. MEZRAG, *Little Grothendieck's theorem for sublinear operators*. J. Math. Anal. Appl. **296** (2004), 541-552.
- [2] D. ACHOUR AND L. MEZRAG, *On the Cohen strongly p -summing multilinear operators*, J. Math. Anal. Appl. **327** (2007), no. 1, 550-563.
- [3] D. ACHOUR AND L. MEZRAG, *Sur les opérateurs sous linéaires p -lattices sommants*, Analele Universitatii Oradea Fasc. Matematica, XIV(2007),237-250.
- [4] D. ACHOUR, L. MEZRAG AND A. TIAIBA, *On the strongly p -summing sublinear operators*, Taiwanese Journal of Mathematics, 11(4)(2007), 959-973.
- [5] D. ACHOUR AND K. SAADI, *A polynomial characterization of Hilbert spaces*, Collect. Math. **61** (2010), no. 3, 291-301.
- [6] A. BELAADA, K. SAADI, K., & A. TIAIBA, *On the Composition Ideals of Schatten Class Type Mappings*. Journal of Mathematics, 2016.
- [7] O. BLASCO, *A class of operators from a Banach lattice into a Banach space*. Collect. Math. **37** (1986), 13-22.
- [8] O. BLASCO, *Boudary values of vector valued harmonic functions considerd as operators*, *studia Math.* **46**(1987), 19-33.
- [9] O. BLASCO, *Positive p -summing operators on L_p -spaces*. Proc. Amer. Math. Soc. **100**, 2 (1987), 275-280.

-
- [10] O. BLASCO, *Positive p -summing operators, vector measures and tensor products*. Proc. Edinburgh. Math. Soc. **31** (1988), 179-184.
- [11] G. BOTELHO. *Ideals of polynomials generated by weakly compact operators*. Note di Matematica 25.1 (2006): 69-102.
- [12] G. BOTELHO, D AND D. PELLEGRINO, *Two new properties of ideals of polynomials and applications*. Indagationes Mathematicae 16.2 (2005): 157-169.
- [13] G. BOTELHO, D. PELLEGRINO AND P. RUEDA, *On composition ideals of multilinear mappings and homogeneous polynomials*, Publ. RIMS, Kyoto Univ. **43** (2007), no. 4, 1139–1155.
- [14] H. A. BRAUNSS AND H. JUNEK, *On types of polynomials and holomorphic functions on Banach spaces*, Note di Math. **10** (1990), no. 1 , 47-58.
- [15] H. A. BRAUNSS, *On holomorphic mappings of Schatten class type*. Archiv der Mathematik 59.5 (1992): 450-456.
- [16] Q. BU & LABUSCHAGNE, C. C. (2015). *Positive Multiple Summing and Concave Multilinear Operators on Banach Lattices*. Mediterranean Journal of Mathematics, 12(1),77-87.
- [17] JEAN BOURGAIN, *New Classes of \mathcal{L}_p -spaces*, Springer-Verlag Berlin Heidelberg New York (1981).
- [18] J. S. COHEN, *Absolutely p -summing, p -nuclear operators and their conjugates*, Math. Ann. **201** (1973), 177-200.
- [19] J. DIESTEL, H. JARCHOW AND A. TONGE, *Absolutely summing operators*. Cambridge University Press, 1995.
- [20] J. DIESTEL & J. J. UHL, (1977). *Vector measures* (No. 15). American Mathematical Soc.
- [21] S. DINEEN, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, London, 1999.

-
- [22] A. DEFANT, K. FLORET, *Tensor Norms and Operator Ideals*. North-Holland (1993).
- [23] T. A. W. DWYER III, *Partial differential equations in Fischer-Fock spaces for the Hilbert-Schmidt holomorphy type*. Bull. Amer. Math. Soc. **77**, (1971), 725–730.
- [24] A. GROTHENDIECK, *Résumé de la théorie métrique des produits tensoriels topologiques*. Bol. Soc. Mat. Sao Paulo **8**, (1956), 1–79.
- [25] K. FLORET, "Natural norms on symmetric tensor products of normed spaces." Note Mat 17 (1997): 153-188.
- [26] K. FLORET, *Minimal ideals of n -homogeneous polynomials on Banach spaces*, Results Math. **39** (2001), 201-217.
- [27] K. FLORET, *On ideals of n -homogeneous polynomials on Banach spaces*. Topological algebras with applications to differential geometry and mathematical physics (Athens, 1999), 2002, p. 19-38.
- [28] J. LINDENSTRAUSS AND A. PELCZYNSKI, *Absolutely summing operators in \mathcal{L}_p -spaces and their applications*, Studia Math. **29** (1968), no. 3, 275-325.
- [29] J. LINDENSTRAUSS AND L. TZAFRIRI, *Classical Banach Spaces I. Sequence Spaces*. Berlin Heidelberg New York: 1977.
- [30] M. C. MATOS, *Fully absolutely summing and Hilbert-Schmidt multilinear mappings*. Collect. Math. **54**, 111–136 (2003).
- [31] B. MARTIN C. MAGUINA, *Operadores Multilineares p -fatoráveis*, Thesis of doctorate, Brazil, 2005.
- [32] C. A. MENDES, *On factorization of Hilbert-Schmidt mappings*, J. Math. Anal. Appl. **304** (2005), no. 1, 137-146.
- [33] C. A. MENDES. *On factorization of Schatten class type mappings*, Math. Nachr. **279** (2006), no. 9–10, 988-995.

- [34] L. MEZRAG AND K. SAADI, *Inclusion and coincidence properties for Cohen strongly summing multilinear operators*, Collect. Math. **64** (2013), no. 3, 395-408.
- [35] J. MUJICA, *Complex Analysis in Banach Spaces, Holomorphic Functions and Domains of holomorphy in Finite and Infinite Dimensions*, North-Holland Mathematics Studies 120, Amsterdam, 1986.
- [36] A. PELCZYNSKI, *A characterization of Hubert-Schmidt operators*. Studia Math. 28, 355-360 (1967).
- [37] A. PIETSCH, *Absolut p -summierende in Abbildungen in normierten Räumen*. Studia Math. **28** (1967), 333-353.
- [38] A. PIETSCH, *Operator Ideals*, Deutsch. Verlag Wiss., Berlin, 1978." (1980).
- [39] A. PIETSCH, *Ideals of multilinear functionals* (designs of a theory), Proceedings of the Second International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics, 185-199. Leipzig. Teubner-Texte, 1983.
- [40] D. POPA, *On the positive absolutely summing operators on the space $L_p(\mu, X)$* . Collectanea Mathematica, 1997, vol. 48, no 3, p. 315-319.
- [41] D. POPA, *Positive absolutely summing operators on the Köthe spaces*. Positivity, 2001, vol. 5, no 4, p. 383-386.
- [42] R. A. RYAN, *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. (2002).
- [43] K. ZHU, *Operator Theory in Function Spaces*. Monographs and Textbooks in Pure and Applied Mathematics, 139. (1990).

ملخص بالعربية :

يندرج عمل هذه الأطروحة ضمن إطار نظرية المؤثرات غير الخطية وينقسم إلى جزئين أساسيين. الجزء الأول خصصناه لإنشاء مثاليات خطية ، مثاليات متعددة الخطية و مثاليات لمؤثرات كثيرات الحدود انطلاقا من أصناف **Schatten**، و ذلك باستعمال طريقة التركيب المقدمة من طرف Pietsch سنة 1983. ثم برهنا بعض نظريات المطابقة والاحتواء مع فضاءات المؤثرات لـ **Cohen** قوة p -جمعية و فضاءات المؤثرات لـ **Hilbert-Schmidt** في الحالات الخطية، المتعددة الخطية و كثيرات الحدود. وفي ختام الجزء الأول اثبتنا تمييزا خاصا لهذه المثاليات، بحيث قمنا بإعطاء نتائج التفكيك المشابهة لتلك التي قدمها **Lindenstrauss-pelczynski** للمؤثرات الخطية لـ **Hilbert-Schmidt** سنة 1968. أما الجزء الثاني، ففكرته مستمدة من نتائج **Blasco** المقدمة في مقاله المعنون بـ " المؤثرات الموجبة p -جمعية على فضاءات L_p " سنة 1987. وعليه قمنا بدراسة مفهوم المؤثرات تحت الخطية الموجبة p -جمعية ثم قدمنا تعميما لبعض نتائج **Blasco** و أعطينا بعض نتائج

التمييز للمؤثرات تحت الخطية الموجبة p -جمعية $T: L_{p'} \rightarrow Y$ مع $\frac{1}{p} + \frac{1}{p'} = 1$.

وذلك باستعمال تمثيل التوابع $u \in \nabla T$ في حالة Y يملك خاصية **Radon-Nikodym** ونتائج أخرى من دون هذه الخاصية.

الكلمات المفتاحية :

فضاءات **Hilbert** ، أصناف **Schatten** ، المؤثرات المتعددة الخطية لـ **Hilbert-Schmidt** ، المثاليات المتعددة الخطية المولدة بأصناف **Schatten**. المؤثرات تحت الخطية الموجبة p -جمعية، خاصية **Radon-Nikodym**.