PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

Mohamed Boudiaf University of M'sila
Faculty of Mathematics and Informatics
Mathematics Department

Master of Mathematics

Faculty of : Mathematics and Informatics
Specialty : Mathematics
Option : Mathematical and Numerical Analysis

Theme

Modified homotopy perturbation method for solving linear Fredholm integral equations

presented by:
Zaiter Wafa

Publicly presented on: 26/06/2022.

In front of the jury :

NADIR Mostefa	Prof.,	University of M'sila	**Prsident.**
KHIRANI Amina	M.C.A,	University of M'sila	**Supervisor .**
GAGUI Bachir	M.C.A,	University of M'sila	**Examiner .**

University Year 2021/2022

Dedication

To my parents who always guides me with their endless love and encouragement...,To my faithful sister and brothers..., to everyone who has ever been through these words... .
To all of them, I dedicate this work.

Acknowledgement

First of all, all my thanks and gratitude go to Allah who enabled me to accomplish this work.

I would like to express my highly gratefulness to my supervisor Prof. **khirani Amina** who gave me so much of her time and knowledge, She overwhelmed me with her understanding, assistance and support throughout the period of accomplishing this work.

my thanks towards all people who have, in one way or another, supported during this humble journey of my research.

Also, my thanks go to my teachers and the members of the department of mathematics at mohamed bodiaf University.

Notations

- $C[a,b]$: set of all continuous real functions in the interval $a \leq x \leq b$.
- H : homotopy.
- p:homotopy parameter
- $K(x,t)$: the kernel of the integral equation.
- $H(u,p)$:Homotopy perturbation method.
- $H(u,p,m)$: Modified homotopy perturbation.
- FIE:Fredholm integral equations .

Contents

Introduction

The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations .Among these integral equations we mention Fredholm integral equations.

In recent years, the use of the Fredholm integral equations has increased in many physical applications, e.g. potential theory and Dirichlet problems, electrostatic, mathematical problems of radiative equilibrium. Some valid methods for solving Fredholm integral equations have been developed such as quadrature methods, Adomian's decomposition method and the Homotopy perturbation method,This later was proposed first by He which is, in fact, a coupling of the traditional perturbation method and homotopy in topology. in this work, we made a comparisons between the standard HPM and the modified one wich made by introducing accelerating parameters.

the thesis is organized in three chapters.

Chapter one provides the basic definitions and introductory concepts: Linear operator, compactness, fixed point principle, the concept of the homotopy method.

Chapter two discusses the concept of integral equations and their types, some theories that demonstrate the existence and uniqueness of the solution to the integral equation.

Chapter three deals with the homotopy perturbation , then the modified homotopy perturbation method, and its application to the first and second kind of fredholm integral equation

Chapter 1

Basics and Preliminaries

In this chapter, we cover some fundamental notions and theorems that we should know for use in our topic.

1.1 Some definitions and fundamental theorems

Definition 1.1.1. (Norm)
Let E be a vector space over the field $\mathbb{K}(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$. A norm on E, is a map $\| . \|: E \to \mathbb{R}$ satisfying the following conditions:

1. $\| x \| \geq 0$ and $\| x \| = 0 \Leftrightarrow x = 0$,
2. $\| \lambda x \| = | \lambda | \| x \|$, for all $x \in E$ and for all $\lambda \in \mathbb{K}$,
3. $\| x + y \| \leq \| x \| + \| y \|$, for all $x, y \in E$.

Remark 1.1.1.
The vector space E endowed with a norm, is called normed space, denoted by $(E, \| . \|)$.

Definition 1.1.2. (Suite by Cauchy)
Let E be a normed vector space. A sequence $(x_n)_{n\in\mathbb{N}}$ of E is Cauchy if and only if,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \ / \ \forall n > m \geq N \Longrightarrow \| x_n - x_m \| < \varepsilon.$$

Definition 1.1.3. (Full space)
Let E be a normed vector space. We say that E is complete if any Cauchy sequence in E is convergent in E.

Definition 1.1.4. (Banach space)
A Banach space is a complete normed space.

Definition 1.1.5. (Topological space)
A topology on a nonempty set X is a collection of subsets of X, called open sets, such that:

(1) the empty set $\varnothing$ and the set X are open,

(2) the union of an arbitrary collection of open sets is open,

(3) the intersection of a finite number of open sets is open.

A subset A of X is a closed set if and only if its complement,$A^{\complement} = X/A$, is open.
More formally, a collection τ of subsets of X is a topology on X if:

(1) $\varnothing, X \in \tau$,

(2) if $G_\alpha \in \tau$ for all $\alpha \in \mathcal{A}$, then $\cup_{\alpha\in\mathcal{A}} G_\alpha \in \tau$,

(3) if $G_i \in \tau$ for $i = 1, 2, .., n$, then $\cap_{i=1}^{n} G_i \in \tau$.

We call the pair (X, τ) a topological space, if τ is clear from the context, then we often refer to X as a topological space.

Linear operator

Let $(X, \parallel . \parallel)$ and $(Y, \parallel . \parallel)$ be two normed linear spaces over the same scalar field F

Definition 1.1.6.
A function (or mapping or transformation) $T : X \longrightarrow Y$ is called a linear operator if

(i) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$ *and*

(ii) $T(\alpha x) = \alpha T(x)$, for all scalars $\alpha \in F$ and for all $x \in X$.

Remark 1.1.2.
The property (i) and (ii) can be combined together to a single property, that is $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all scalars $\alpha, \beta \in F$ and for all $x, y \in X$.

Definition 1.1.7. (Bounded operator)
A linear operator $T : X \longrightarrow Y$ is called bounded if there is a positive constant M such that $\parallel T(x) \parallel \leq M \parallel x \parallel$, for all $x \in X$, in other words

$$\frac{\parallel T(x) \parallel}{\parallel x \parallel} \leq M$$

for all non zero member $x \in X$

Definition 1.1.8. (Completely Continuous Operator)
A linear operator A defined from a normed space E into a normed space F is called a compact linear operator or completely continuous linear operator if for every bounded subset Ω of E, the image $A(\Omega)$ is relatively compact in F. In other words, the closure $\overline{A(\Omega)}$ is compact.

Compactness in $C(K)$

The set $C(K)$ designates the space of continuous real or complex valued functions defined in a compact set $K \subset \mathbb{R}^n$, furnished with the maximum norm

$$\parallel \varphi \parallel_\infty = max_{x \in K} \mid \varphi(x) \mid .$$

Theorem 1.1.1. (Ascoli–Arzelà's)
A set $G \subset C(K)$ is relatively compact if and only if it is bounded and equicontinuous. Say, if there exists a constant M such that
$\mid \varphi(x) \mid \leq M$ for all $x \in K$ and all $\varphi \in G$.
Further, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\varphi \in G$, we have
$\mid \varphi(x) - \varphi(y) \mid < \varepsilon$ for all $x, y \in K$ with $\mid x - y \mid < \delta$.

Theorem 1.1.2.
The integral operator A defined from $C(\Omega)$ into $C(\Omega)$

$$A\varphi(x) = \int_{\Omega} k(x,y)\varphi(y)dy, \quad x,y \in \Omega$$

with continuous kernel $k(x,y)$ is a compact operator.

Proof.
Let G be a bounded set of $C(\Omega)$ then, for each $\varphi \in G$, there exists $M > 0$, such that

$$\| \varphi \| \leq M,$$

besides, for all $x \in \Omega$ and $\varphi \in G$, we get

$$\begin{aligned} | A\varphi(x) | &= | \int_{\Omega} k(x,y)\varphi(y)dy | \\ &\leq max_{x,y\in\Omega} | k(x,y) | Mmes(\Omega). \end{aligned}$$

It follows that $A(G)$ is bounded.
By assumption, the kernel $k(x,y)$ is continuous over the compact $\Omega \ \ \Omega$, thus it is uniformly continuous and therefore

$$\forall\varepsilon > 0, \exists\delta > 0, \forall x,y,z \in \Omega, | x-y | < \delta \Longrightarrow | k(x,z) - k(y,z) | < \frac{\varepsilon}{Mmes(\Omega)}.$$

Hence, for each $\varphi \in G$and $x,y \in \Omega$, with$| x-y | < \delta$

$$\begin{aligned} | A\varphi(x) - A\varphi(y) | &= | \int_{\Omega} (k(x,z) - k(y,z))\varphi(z)dz | \\ &< \frac{\varepsilon}{Mmes(\Omega)} Mmes(\Omega) = \varepsilon. \end{aligned}$$

This relation expresses that $A(G)$ is equicontinuous. Hence $A(G)$ is relatively compact, so by Arzela-Ascoli's theorem A is compact. □

1.2 Fixed point principle

Definition 1.2.1.
Let T be an operator defined in a Banach space E in itself, then for all $x \in E$, such that $x = T(x)$, is called a fixed point of the operator T.

Definition 1.2.2.
Let T be an operator of a Banach space E in itself, T is a contraction (or contracting application), if there is a constant $0 \leq k < 1$ such that, for all $x,y \in E$, we have

$$\| T(x) - T(y) \| \leq K \| x - y \| \tag{1.1}$$

Remark 1.2.1.

- If $k \geq 0$ in the relation (1.1), T is said lipshchitziènne.
- The Lipshchitziènne application is necessarily continuous.
- If $k < 1$ in the relation (1.1), T is said to be contraction.
- If $k = 1$ in the relation (1.1), T is said to be nonexpansive.

Theorem 1.2.1.
Let F be a closed subset in a Banach space and let $T : F \to F$ a contracting application, then

- The equation $Tx = x$, has only one solution.
- The unique solution x can be obtained by the limit of the sequence x_n of F defined by $x_n = Tx_{n-1}, n = 1, 2, ...$, where x_0 is an arbitrary element of F,

$$x = \lim_{n \to \infty} T^n x_0$$

Proof. see [5] □

1.3 Concept of the homotopy

In this section, we present the concept of the homotopy, a fundamental concept in topology. Basic definitions with illustrative examples are given.

A homotopy is a kind of continuous transformation from one function to another. In particular, we consider the following definition.

Definition 1.3.1.
A homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $H : X \ \ [0, 1] \to Y$ from the product of the space X with the unit interval $[0, 1]$ to Y such that, if $x \in X$ then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Let $C[a, b]$ denote a set of all continuous real functions in the interval $a \leq x \leq b$. If a continuous function $f \in C[a, b]$ can be deformed continuously into another continuous function $g \in C[a; b]$, one can construct a homotopy

$$H : f(x) \sim g(x),$$

in the way

$$H(x,p) = (1-p)f(x) + pg(x), x \in [a,b], p \in [0,1].$$

or

$$H(x,p) = f(x) + p[g(x) - f(x)], x \in [a,b], p \in [0,1].$$

For this homotopy a first-order homotopy-derivative is defined as

$$\frac{\partial H(x,p)}{\partial p} = g(x) - f(x), p \in (0,1).$$

Definition 1.3.2.
The embedding parameter $p \in [0,1]$ in a homotopy of functions or equations is called **homotopy parameter** .
To illustrate the idea, let us construct a homotopy between the identity function

$$f(x) = x$$

and the zero function

$$g(x) = 0$$

in the interval $X = [0,1]$. Since f and g are continuous in interval $[0,1]$
then $H : f(x) \sim g(x)$. Define $H : X \quad [0,1] \to Y$ by:

$$H(x,p) = (1-p)f(x) + pg(x),$$

$$= (1-p)x, \qquad \forall x \in X, \quad \forall p \in [0,1].$$

Then H is a continuous function and

$$H(x,0) = f(x) = x, \forall x \in X.$$

$$H(x,1) = g(x) = 0, \forall x \in X.$$

The first order homotopy-derivative is

$$\frac{\partial H(x,p)}{\partial p} = g(x) - f(x),$$

$$= -x, \quad p \in (0,1),$$

1.4 Perturbation Theory

Perturbation theory comprises mathematical methods which are used to find the approximate solution to a problem which cannot be a solved accurately, by starting from the accurate solution of a related problem. Perturbation theory leads to an expression for the desired solution in terms of a formal power series in small parameter (ε)-known as perturbation series that quantifies the deviation from the exactly solvable problem and further terms describe the deviation in the solution.
Consider,

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$$

Here x_0 be the known solution to the exactly solvable initial problem and $x_1, x_2, \ldots$ are the higher order terms. For small these higher order terms are successively smaller. An appro imate "perturbation solution" is obtained by truncating series usually by keeping only the first two terms.

Chapter 2

Introductory Concepts of Integral Equations

In this chapter, we give the definition of the integral equations and their classifications, some theories that prove the existence and uniqueness of solutions of integral equations .

2.1 Definition of the integral equations

Definition 2.1.1.
An integral equation is defined as an equation in which the unknown function $u(x)$ to be determined appear under the integral sign. A typical form of an integral equation in u(x) is of the form

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x, t, u(t))dt \tag{2.1}$$

where $K(x, t, u(t))$ is called the kernel of the integral equation (2.1), and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $u(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x, t, u(t))$ and the function $f(x)$ in equation (2.1) are given functions, and λ is a constant parameter.

2.2 Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation. The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations. Of course, we have to classify them as homogeneous or non-homogeneous; and also linear or nonlinear. In some practical problems, we come across singular equations also. In this text, we shall distinguish four major types – the two main classes and two related types - of integral equations. In particular, the four types are given below:

1. Volterra integral equations.

2. Fredholm integral equations.

3. Integro-differential equations.

4. Singular integral equations.

In our study, we will focus on Friedholm integral equations

Fredholm integral equations

The most standard form of linear Fredholm integral equations is given by the form

$$\phi(x)u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{2.2}$$

where the limits of integration a and b are constants and the unknown function $u(x)$ appears linearly under the integral sign. If the function $\phi(x) = 1$, then (2.2) becomes simply

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \tag{2.3}$$

and this equation is called Fredholm integral equation of second kind, whereas if $\phi(x) = 0$, then (2.2) yields

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0 \tag{2.4}$$

which is called Fredholm integral equation of the first kind.

Remark 2.2.1.
If the unknown function $u(x)$ appearing under the integral sign is given in the functional form $F(u(x))$ such as the power of $u(x)$ is no longer unity, e.g. $F(u(x)) = u^n(x)$,$n \neq 1$, or $sin(u(x))$ etc., then the Fredholm integral equations are classified as nonlinear integral equations. As for examples, the following integral equations are nonlinear integral equations:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u^2(t)dt$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)sin(u(t))dt$$

$$u(x) = f(x) + \lambda \int_a^b K(x,t)ln(u(t))dt$$

Next, if we set $f(x) = 0$, in Fredholm integral equations, then the resulting equation is called a homogeneous integral equation, otherwise it is called nonhomogeneous integral equation.

2.3 The existence and uniqueness of solutions of integral equations

2.3.1 Riesz's theory and Fredholm's alternative

In this paragraph, we designate by $A : X \to X$ the compact linear operator in a normed space in itself. We present the basic theory for an equation

$$\varphi - A\varphi = f.$$

We define the operator L, by $L = I - A$ where I designates the identity operator.

Theorem 2.3.1. (Riesz's First Theorem)
The null space of the operator$L$, i.e. the kernel of the operator L

$$N(L) = Ker(L) = \{\varphi \in X : L\varphi = 0\}$$

is a closed and finite dimensional subspace.

Proof.
deed $Indeed$, it is known that the kernel $N(L)$ of a bounded operator L is a closed subspace of X, since, for all sequence φ_n in $N(L)$ converges to φ in X, we obtain φ in $N(L)$. Really, due to the boundedness of L we have

$$L\varphi_n = 0 \Longrightarrow \lim_{n \longrightarrow \infty} L\varphi_n = 0$$

or still

$$L(\lim_{n \longrightarrow \infty} \varphi_n) = 0 \Longrightarrow L(\varphi) = 0$$

Hence, the null-space $N(L)$ is closed.
On the other hand, all functions $\varphi \in N(L)$ must satisfy the equation $L\varphi = \varphi - A\varphi = 0$, or still

$$A\varphi = \varphi.$$

Noting that, the restriction of the operator A to the subspace $N(L)$ coincides with the identity operator on $N(L)$. The operator A is compact from X to X and therefore also compact from $N(L)$ to $N(L)$ since $N(L)$ is closed. Evidently, for all bounded sequence φ_n in X in particular in $N(L)$ the sequence $A\varphi_n = \varphi_n$ contains a convergent subsequence $A\varphi_{nk} = \varphi_{n_k}$ in the closed $N(L)$. Hence, the compact operator A represents the identity operator on the subspace $N(L)$ and therefore the subspace $N(L)$ is of finite dimensional.
□

Theorem 2.3.2. (Riesz's Second Theorem)
The image of the operator L, i.e.

$$R(L) = Im(L) = \{L\varphi : \varphi \in X\}$$

is a closed linear subspace of finite co-dimension.

Proof.
The image of the operator L is a subspace. Let f be an element of $\overline{L(X)}$, then it exists a sequence (φ_n) of X such that $L\varphi_n \to f, n \to \infty$, we choose the best approximation χ_n, i.e.

$$\| \varphi_n - \chi_n \| = \inf_{\chi_n \in Im(L)} \| \varphi_n - \chi_n \|,$$

we define the sequence

$$\varphi_n^{\sim} = \varphi_n - \chi_n, n \in \mathbb{N},$$

which is bounded.
We assume that the sequence $(\varphi_n^{\sim})$ is not bounded, then we can extract a subsequence $(\varphi_{n(k)}^{\sim})$, such that $\| \varphi_{n(k)}^{\sim} \| \geq k$, for any $k \in \mathbb{N}$, now we pose

$$\psi_k = \frac{\varphi_{n(k)}^{\sim}}{\| \varphi_{n(k)}^{\sim} \|}, k \in \mathbb{N},$$

with $\| \psi_k \| = 1$ and A is compact, then there is a subsequence $\psi_{k(j)}$ such that $A\psi_{k(j)} \to \psi, j \to \infty$, on the other hand

$$L\psi_k = \frac{\| L\varphi_{n(k)}^{\sim} \|}{\| \varphi_{n(k)}^{\sim} \|} \leq \frac{\| L\varphi_{n(k)}^{\sim} \|}{k} \to 0, k \to \infty,$$

since the sequence $(L\varphi_n)$ is converged and therefore bounded. Consequently

$$L\psi_{k(j)} \to 0, j \to \infty,$$

then, we get

$$\psi_{k(j)} = L\psi_{k(j)} + A\psi_{k(j)} \to \psi, j \to \infty$$

and since L is bounded, and by the two preceding equations we conclude that $L\varphi = 0$. But like

$$\chi_{n(k)} + \| \varphi_{n(k)}^{\sim} \| \chi \in Im(L), \forall k \in \mathbb{N},$$

we find

$$\begin{aligned} \| \psi_k - \psi \| &= \frac{1}{\| \varphi_{n(k)}^{\sim} \|} \| \varphi_{n(k)} - \{\chi_{n(k)} + \| \varphi_{n(k)}^{\sim} \|\}\psi \| \\ &\geq \frac{1}{\| \varphi_{n(k)}^{\sim} \|} \inf_{\chi \in Im(L)} \| \varphi_{n(k)} - \chi_n \| \\ &= \frac{1}{\| \varphi_{n(k)}^{\sim} \|} \| \varphi_{n(k)} - \chi_{n(k)} \| \\ &= 1 \end{aligned}$$

This contradicts the fact that this $\psi_k(j) \to \psi, I \to \infty$. Therefore $(\varphi_n^\sim)$ is bounded, and since A is compact, we can extract a subsequence $(\varphi_{n(k)}^\sim)$ such that $(A\varphi_{n(k)}^\sim)$ converges for $k \to \infty$. Because $L\varphi_{n(k)}^\sim \to f, k \to \infty$, and by

$$\varphi_{n(k)}^\sim = L\varphi_{n(k)}^\sim + A\varphi_{n(k)}^\sim$$

We observe that $\varphi_{n(k)}^\sim \to \varphi \in X, k \to \infty$, but $L\varphi_{n(k)}^\sim \to L\varphi \in X, k \to \infty$. □

Theorem 2.3.3. (Riesz's Third Theorem)
There is a unique $r \in \mathbb{N}$ called the Riesz number of the operator L such that:
$0 = Ker(L^0) \subset Ker(L^1) \subset ...... \subset Ker(L^r) = Ker(L^{r+1})$
$E = Im(L^0) \supset Im(L^1) \supset .... \supset Im(L^r) = Im(L^{r+1})$.
And we have the direct sum

$$E = Ker(L^r) \oplus Im(L^r).$$

Proof. see [12] □

Theorem 2.3.4.
Of the same conditions in the preceding theorems, then

(i) $I - A$ is injective if and only if it is surjective.

(ii) If $I - A$ is injective, then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.

Proof.
It is known that, for all Riesz number $r = p = q$, The subspaces $N(L^r)$ and $R(L^r)$ are supplementary. Say

$$E = N(L^r) \oplus R(L^r).$$

- The injection of the operator L implies the one of L^r. Hence the surjection of the operator L^r which it assures us the surjection of the operator L.
- The surjection of the operator L implies the one of L^r. Hence the injection of the operator L^r which it assures us the injection of the operator L.
- The injection of the operator L or its surjection implies the bijection of this operator $L = (I - A)$. Hence the boundedness of its inverse $L^{-1} = (I - A)^{-1}$.

See [13] □

Corollary 2.3.1.

i- If the homogeneous equation

$$\varphi - A\varphi = 0 \tag{2.5}$$

admits only the trivial solution $\varphi = 0$, then for any $f \in X$ the equation

$$\varphi - A\varphi = f \tag{2.6}$$

admits a single solution $\varphi \in X$ and this solution is depends on the continuity of f.

ii- If the homogeneous equation (2.5) does not admit the trivial solution $\varphi = 0$, then it has only a finite number $m \in \mathbb{N}$ of solutions $\varphi_1, \varphi_2, ..., \varphi_n$ of X are linearly independents and the non-homogeneous equation (2.6) is unsolvable or its solution is the general form

$$\varphi = \varphi^{\sim} + \sum_{k=1}^{m} \alpha_k \varphi_k,$$

where $\alpha_1, \alpha_2, ..., \alpha_k$ are complex arbitrary numbers and $\varphi^{\sim}$a particular solution of the non-homogeneous equation.

Proof. see [12] □

2.3.2 Fixed point theory

Theorem 2.3.5.
Let be the following equation

$$\varphi(x) - \lambda \int_a^b k(x,y)\varphi(y)dy = g(x)$$

admits a unique solution $\varphi \in L^2([a,b])$, with the kernel k is continuous over the interval $[a,b]$,

$$g \in L^2([a,b]) \ \ and \ \ |\lambda|k < 1, \quad with \quad k = \sqrt{\int_a^b \int_a^b \mid k(x,y) \mid^2 dxdy}$$

Proof.
We consider the equation

$$(T\varphi)(x) = g(x) + \lambda \int_a^b k(x,y)\varphi(y)dy, \tag{2.7}$$

since $g \in L^2([a,b])$,$T\varphi \in L^2([a,b])$. If

$$\int_a^b k(x,y)\varphi(y)dy \in L^2([a,b]).$$

Using Schwartz inequality, so

$$\begin{aligned} \mid \int_a^b k(x,y)\varphi(y)dy \mid &\leq \int_a^b \mid k(x,y)\varphi(y) \mid dy \\ &\leq \left(\int_a^b \mid k(x,y) \mid^2 \varphi(y)dy\right)^{\frac{1}{2}} \left(\int_a^b \mid \varphi(y) \mid^2 dy\right)^{\frac{1}{2}} \end{aligned}$$

therfore

$$\mid \int_a^b k(x,y)\varphi(y)dy \mid^2 \leq \left(\int_a^b \mid k(x,y) \mid^2 \varphi(y)dy\right)\left(\int_a^b \mid \varphi(y) \mid^2 dy\right)$$

then

$$\begin{aligned}\int_a^b \mid \int_a^b k(x,y)\varphi(y)dy \mid^2 dx &\leq \int_a^b \left(\int_a^b \mid k(x,y) \mid^2 \varphi(y)dy\right)\left(\int_a^b \mid \varphi(y) \mid^2 dy\right) dx \\ &\leq \int_a^b \int_a^b \mid k(x,y) \mid^2 dydx \int_a^b \mid \varphi(y) \mid^2 dy\end{aligned}$$

since

$$\int_a^b \int_a^b \mid k(x,y) \mid^2 dydx < \infty \quad and \quad \int_a^b \mid \varphi(y) \mid^2 dy < \infty$$

then equation (2.7) is satisfactory and T of $L^2([a,b])$ in itself.
Note that the demonstration below is also that the operator defined by

$$(A\varphi)(x) = \int_a^b k(x,y)\varphi(y)dy$$

is bounded, so by theorem 1.2.1 the equation $T\varphi = \varphi$ admits a unique solution, for $\mid \lambda \mid k < 1$ □

We mention that, the Fredholm integral equation of the first kind is very often ill-posed and the solution $u(x)$ is very sensitive to any change in the data $f(x)$. In other words, a very small change on the data $f(x)$ can give a large change in the solution $u(x)$. For all these reasons, the Fredholm integral equations of the first kind is ill-posed that may have no solution, or if a solution exists it is not unique and may not depend continuously on the data.

Remark 2.3.1.
To find the explicit solutions of fredholm linear integral equations, many powerful methods have been used, such as

- The Adomian Decomposition Method
- The Variational Iteration Method
- The Successive Approximations Method
- The Homotopy Perturbation Method
- The Modified Homotopy Perturbation Method

In the next chapter, we will apply the standard homotopy perturbation method and the modified one

Chapter 3

Solving Fredholm integral equations using the modified Homotopy perturbation method

Various kinds of analytical methods were used to solve integral equations. In this chapter, we apply the standard homotopy perturbation method and the modified one to solve linear fredholm integral equations.

3.1 The Homotopy perturbation method

Homotopy perturbation method has been recently intensively studied by scientists and engineers and used for solving nonlinear problems. This method was proposed first by He which is, in fact, a coupling of the traditional perturbation method and homotopy in topology. HPM method yields a very rapid convergence of the solution series in most cases, usually only few iterations leading to very accurate solutions. This method has been applied to many problems , specially in integral equations .

Consider the Fredholm integral equation:

$$\gamma(x) = f(x) + \int_a^b K(x,t)\gamma(t)dt, \quad a \le x \le b, \tag{3.1}$$

let

$$L(u) = u(x) - f(x) - \int_a^b K(x,t)u(t)dt = 0, \tag{3.2}$$

with solution $u(x) = \gamma(x)$, we define the homotopy $H(u,p)$ by

$$H(u,0) = F(u), H(u,1) = L(u)$$

where $F(u)$ is a functional operator with solution, say,u_0. We may choose a convex homotopy

$$H(u,p) = (1-p)F(u) + pL(u) = 0, \tag{3.3}$$

and continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(\gamma,1)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(u) = 0$ continuously deformed to the original problem $L(u) = 0$. The parameter p can be considered as an expanding parameter . In fact HPM uses the homotopy parameter p as an expanding parameter to obtain

$$u = u_0 + pu_1 + p^2u_2 + ..., \tag{3.4}$$

when $p \longrightarrow 1$, (3.4) corresponds to (3.3) and gives an approximation to the solution of (3.2) as follows:

$$\gamma = \lim_{p \longrightarrow 1} u_0 + pu_1 + p^2u_2 + ..., \tag{3.5}$$

The series (3.5) converges in most cases, and the rate of convergence depends on $L(u)$. Taking $F(u) = u(x) - f(x)$, and substituting (3.4) in (3.3) and equating the terms with

identical power of p, we obtain
$p^0 : u_0 - f(x) = 0 \Rightarrow u_0 = f(x)$,
$p^1 : u_1 - \int_a^b K(x,t)u_0(t)dt = 0$,
$u_1 = \int_a^b K(x,t)u_0(t)dt$,
.
.
.
and in general we have
$u_0(x) = f(x)$,
$u_{n+1}(x) = \int_a^b K(x,t)u_n(t)dt, \quad n = 0, 1, ...$
which is the standard Adomian's decomposition method.
Now consider Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)\gamma(t)dt$$

and the convex homotopy:

$$H(u,p) = (1-p)u(x) + p[\int_a^b K(x,t)u(t)dt - f(x)] = 0, \tag{3.6}$$

with the starting point $H(0,0)$ and the solution $H(\gamma,1)$, by similar operations as above, we obtain

$$u_0 = 0, u_1 = f(x),$$
$$u_{n+1} = u_n - \int_a^b K(x,t)u_n(t)dt,$$

or

$$u_{n+1} = (I-K)u_n(t),$$
$$= (I-K)^n u_1(t),$$

where the operator K is defined as

$$K = \int_a^b K(x,t)(.)dt,$$

and for convergence we should have $\parallel I - K \parallel < 1$.

3.2 The modified Homotopy perturbation method

In this section we propose a scheme to accelerate the rate of convergence of HPM applied to linear Fredholm integral equations with kernels of the form $\sum_{i=1}^{N} g_i(x)h_i(t)$. We define a new convex homotopy perturbation as follows:

$$H(u,p,m) = (1-p)F(u) + pL(u) + p(1-p)[\sum_{i=1}^{N} m_i g_i(x)] = 0, \tag{3.7}$$

where $m = [m_i]$ and m_i, $i = 1, 2, ..., N$ are called the accelerating parameters, and for $m_i = 0$, $i = 1, 2, ..., N$, we define $H(u, p, 0) = H(u, p)$,
which is the standard HPM. It is easy to verify that the solutions $H(u_0, 0)$ and $H(\gamma, 1)$ also satisfy (3.7).

3.2.1 Application of H(u,p,m) to second kind FIE

We first assume that $k(x,t) = g(x)h(t)$, so for equation

$$\gamma(x) = f(x) + \int_a^b k(x,t)\gamma(t)dt, a \leq x \leq b,$$

we consider (3.7) as follows:

$$H(u,p,m) = (1-p)F(u) + pL(u) + p(1-p)[mg(x)] = 0,$$

where

$$F(u) = u(x) - f(x) \quad and \quad L(u) = u(x) - f(x) - \int_a^b g(x)h(t)u(t)dt = 0,$$

hence we can write

$$(1-p)(u-f) + p[u - f - \int_a^b g(x)h(t)u(t)dt] + mp(1-p)g(x) = 0,$$

or

$$u - f - pg(x)\int_a^b h(t)u(t)dt + mpg(x) - mp^2 g(x) = 0, \tag{3.8}$$

now by using (3.4) in (3.8), and equating the terms with identical power of p, we obtain
$p^0 : u_0 - f(x) = 0 \Rightarrow u_0 = f(x)$,
$p^1 : u_1 + mg(x) - g(x) \int_a^b h(t)u_0(t)dt = 0$,
$u_1 = (c - m)g(x)$, where $c = \int_a^b h(t)f(t)dt$,
$p^2 : u_2 - mg(x) - g(x) \int_a^b h(t)u_1(t)dt = 0$,
$u_2 = [m + (c - m)\alpha]g(x)$, where $\alpha = \int_a^b h(t)g(t)dt$,
$p^3 : u_3 = \int_a^b h(t)u_2(t)dt$
and in general
$u_{n+1} = \int_a^b h(t)u_n(t)dt, n = 2, 3, ...,$
now we find m such that $u_2 = 0$, since if $u_2 = 0$ then $u_3 = u_4 = ... = 0$, and the exact solution will be obtained as $u(x) = u_0(x)+u_1(x)$, hence for all values of x we should have

$$m + (c - m)\alpha = 0$$

or

$$m = \frac{c\alpha}{\alpha - 1} = \frac{[\int_a^b h(t)f(t)dt][\int_a^b h(t)g(t)dt]}{\int_a^b h(t)g(t)dt - 1}$$

or

$$m = \frac{\int_a^b k(t,t)dt}{\int_a^b k(t,t)dt - 1} \int_a^b h(t)f(t)dt, \tag{3.9}$$

provided that

$$\int_a^b k(t,t)dt \neq 1$$

Now consider the general case

$$k(x,t) = \sum_{i=1}^{N} g_i(x)h_i(t),$$

here we choose the convex homotopy as follows:

$$H(u,p,m) = (1 - p)F(u) + pL(u) + p(1 - p)[\sum_{i=1}^{N} m_i g_i(x)] = 0,$$

by doing similar manipulations, we obtain
$u_0 = f$,
$u_1 = -\sum_{i=1}^{N} m_i g_i(x) + \sum_{i=1}^{N} g_i(x) \int_a^b h_i(t)u_0(t)dt$,

$= \sum_{i=1}^{N}[\int_a^b h_i(t)f(t)dt - m_i]g_i(x)$

$u_2 = \sum_{i=1}^{N} m_i g_i(x) + \sum_{i=1}^{N} g_i(x) \int_a^b h_i(t)u_1(t)dt,$

$= \sum_{i=1}^{N}[m_i + \int_a^b h_i(t)u_1(t)dt]g_i(x)$

.

.

.

$u_{n+1} = \sum_{i=1}^{N} g_i(x) \int_a^b h_i(t)u_n(t)dt, \quad n = 2, 3, ...,$
we try to find the parameters m_i,$i = 1, 2, ..., N$, such that

$$u_2 = u_3 = u_4 = 0,$$

hence from u_2 we should have

$$m_i + \int_a^b h_i(t)u_1(t)dt = 0, \quad \forall x \in [a, b], \tag{3.10}$$

now by substituting u_1 in (3.10), we obtain

$$m_i + \sum_{j=1}^{N} \int_a^b h_i(t)[\int_a^b h_j(t)f(t)dt - m_j]g_j(t)dt = 0,$$

let

$$c_j = \int_a^b h_j(t)f(t)dt, b_{ij} = \int_a^b h_i(t)g_j(t)dt,$$

then

$$m_i + \sum_{j=1}^{N} b_{ij}(c_j - m_j) = 0, \quad i = 1, 2, ..., N \tag{3.11}$$

under certain condition,the values of m_i, $i = 1, 2, ..., N$, can be obtained from the system of linear equations in (3.11). Let the matrix B and the vectors m and c be defined as follows:

$$B = [bij], \quad m = [m_j], \quad c = [c_j]$$

therefore from (3.11) we can write

$$(B - I)m = Bc,$$

and if $(B-I)$ is nonsingular then

$$m=(B-I)^{-1}Bc, \tag{3.12}$$

In the case of non-degenerate kernels,by using Taylor expansion for functions of two variables, we can write $k(x,t)$ (if possible) as follows:

$$k(x,t)=\sum_{i=1}^{N} g_i(x)h_i(t),$$

and by applying $H(u,p,m)$ method we can approximate the solution of the given integral equation.

3.2.2 Application of H(u,p,m) to first kind FIE

Consider the integral equation:

$$f(x)=\int_a^b k(x,t)\gamma(t)dt, a\leq x\leq b, \tag{3.13}$$

let $k(x,t)=g(x)h(t)$, then from (3.13):we observe

$$f(x)=cg(x),$$

using (3.7) we can write

$$(1-p)F(u)+pL(u)+p(1-p)[mg(x)]=0,$$

where

$$F(u)=u, \quad L(u)=\int_a^b g(x)h(t)u(t)dt-f,$$

by similar operations we obtain

$p^0: u_0=0,$

$p^1: u_1=(c-m)g(x),$

$p^2: u_2=u_1+mg(x)-g(x)\int_a^b h(t)u_1(t)dt,$

$\quad =[c-(c-m)\alpha]g(x), \quad$ where $\quad \alpha=\int_a^b h(t)g(t)dt=\int_a^b k(t,t)dt,$

$u_3=u_2-g(x)\int_a^b h(t)u_2(t)dt,$

and in general
$u_{n+1} = u_n - g(x)\int_a^b h(t)u_n(t)dt, \quad n = 2, 3, ...,$
for $u_2 = u_3 = u_4 = ... = 0$, from u_2 we should have

$$c - (c - m)\alpha = 0, \quad \forall x \in [a, b]$$

and consequently

$$m = c - \frac{c}{\alpha}, (\alpha \neq 0). \tag{3.14}$$

Now assume $k(x,t) = \sum_{i=1}^{N} g_i(x)h_i(t)$, then

$$f(x) = \sum_{i=1}^{N} c_i g_i(x),$$

in this case , we choose the following convex homotopy

$$(1-p)u + p[\int_a^b \sum_{i=1}^{N} g_i(x)h_i(t)dt - f] + p(1-p)[\sum_{i=1}^{N} m_i g_i(x)] = 0, \tag{3.15}$$

by substituting (3.4) in (3.15) and equating the coefficients of the identical powers of p, we obtain
$u_0 = 0,$
$u_1 = \sum_{i=1}^{N}[c_i - m_i]g_i(x),$
$u_2 = \sum_{i=1}^{N}[c_i - \sum_{j=1}^{N}(c_j - m_j)b_{ij}]g_i(x), \quad where \quad b_{ij} = \int_a^b h_i(t)g_j(t)dt,$

.

.

.

$u_{n+1} = u_n - \sum_{i=1}^{N} g_i(x)\int_a^b h_i(t)u_n(t)dt, \quad n = 2, 3, ...,$

as above for $u_2 = u_3 = u_4 = ... = 0$, parameters $m_i, i = 1, 2, ..., N$, should satisfy the following linear system of equations

$$c_i - \sum_{j=1}^{N}(c_j - m_j)b_{ij} = 0, \quad i = 1, 2, ..., N,$$

or in matrix form we have

$$Bm = (B - I)c,$$

where

$$B = [bij], \quad m = [m_j], \quad c = [c_j], \quad j = 1, 2, ..., N,$$

or

$$m = B^{-1}(B - I)c, \tag{3.16}$$

provided that (B) is not singular. For non-degenerate kernels $k(x,t)$, using Taylor expansion, we can write

$$k(x,t) = \sum_{i=1}^{N} g_i(x)h_i(t), \quad f(x) = \sum_{i=1}^{N} c_i g_i(x)$$

and then by applying $H(u,p,m)$ method, we can approximate the solution of the given problem.

3.3 Numerical results

example 1 Consider the equation

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt,$$

with exact solution $u(x) = e^x$. We apply $H(u,p)$ and $H(u,p,m)$ methods to approximate the solution as follows

$\mathbf{H(u,p)}$ **method**:

We have

$$\begin{aligned} u_0(x) &= f(x), \\ &= e^x - 1 \end{aligned}$$

$u_{n+1}(x) = \int_a^b k(x,t)u_n(t)dt, \quad n = 1, 2, ...,$

so we obtain

$$\begin{aligned} u_1 &= \int_0^1 tu_0(t)dt \\ &= \int_0^1 t(e^t - 1)dt, \\ &= \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} u_2 &= \int_0^1 \frac{1}{2} t dt \\ &= \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} u_3 &= \int_0^1 \frac{1}{4} t dt \\ &= \frac{1}{8}, \end{aligned}$$

$$\begin{aligned} u_4 &= \int_0^1 \frac{1}{8} t dt \\ &= \frac{1}{16}, \\ &. \\ &. \\ &. \end{aligned}$$

Hence the solution will be as follows:

$$\begin{aligned} u(x) &= u_0 + u_1 + u_2 + ..., \\ &= e^x - 1 + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... \right) \\ &= e^x. \end{aligned}$$

$\mathbf{H(u, p, m)}$ method:

In this case we have:

$$u_0 = f(x),$$

$$u_1 = (c - m)g(x), \quad where \quad c = \int_a^b h(t)f(t)dt,$$

and

$$u(x) = u_0 + u_1$$

For this problem $g(x) = 1$ and $h(t) = t$, hence

$$\begin{aligned} u_0 &= e^x - 1, \\ u_1 &= \left[\int_0^1 t(e^t - 1)dt - m\right], \\ &= \left[\frac{1}{2} - m\right], \end{aligned}$$

and from (3.9):

$$\begin{aligned} m &= \frac{\int_0^1 tdt}{\int_0^1 tdt - 1} \int_0^1 t(e^t - 1)dt \\ &= \frac{\frac{1}{2}}{\frac{1}{2} - 1} \left(\frac{1}{2}\right) \\ &= \frac{-1}{2}, \end{aligned}$$

hence we obtain

$$\begin{aligned} u_1 &= \left[\frac{1}{2} + \frac{1}{2}\right] \\ &= 1, \end{aligned}$$

and the solution will be as follows:
$u(x) = u_0 + u_1 = e^x$.
As we observe, after two terms the exact solution is obtained.

example 2 Consider the equation

$$u(x) = arcsin(x) + \left(\frac{\pi}{2} - 1\right) x - \int_0^1 xu(t)dt,$$

with exact solution $u(x) = arcsin(x)$. We apply $H(u, p)$ and $H(u, p, m)$ methods to approximate the solution as follows

$\mathbf{H(u, p)}$ method:

We have

$$\begin{aligned} u_0(x) &= f(x), \\ &= arcsin(x) + \left(\frac{\pi}{2} - 1\right) x \end{aligned}$$

$u_{n+1}(x) = \int_a^b k(x,t)u_n(t)dt, \quad n = 1, 2, ...,$
so we obtain

$$\begin{aligned} u_1 &= -x\left[\int_0^1 \left(arcsin(t) + \left(\frac{\pi}{2} - 1\right)t\right)dt\right], \\ &= -x\left[\int_0^1 arcsin(t)dt + \int_0^1 \left(\frac{\pi}{2} - 1\right)tdt\right], \\ &= -x\left[arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right], \end{aligned}$$

$$\begin{aligned} u_2 &= x\left[\int_0^1 \left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right)tdt\right] \\ &= \frac{1}{2}x\left[arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right], \end{aligned}$$

$$\begin{aligned} u_3 &= -x\left[\int_0^1 \frac{1}{2}\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right)tdt\right] \\ &= -\frac{1}{4}x\left[arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right], \end{aligned}$$

.

.

.

Hence the solution will be as follows:

$u(x) = u_0 + u_1 + u_2 + ...,$

$= arcsin(x) + \left(\frac{\pi}{2} - 1\right)x - x\left[arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right) - \frac{1}{2}arcsin(1) + \frac{1}{2} - \frac{1}{4}\left(\frac{\pi}{2} - 1\right)\right]$
$-x\left[\frac{1}{4}arcsin(1) - \frac{1}{4} + \frac{1}{8}\left(\frac{\pi}{2} - 1\right)\right] + ...$

$= arcsin(x) - x[arcsin(1)\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ...\right) + \left(-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - ...\right) + \left(\frac{\pi}{2} - 1\right)\left(-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - ...\right)]$

$= arcsin(x) - x\left[arcsin(1)\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ...\right) + \frac{\pi}{2}\left(-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - ...\right)\right]$

$\mathbf{H(u,p,m)}$ method:

In this case we have:

$$u_0 = f(x),$$
$$u_1 = (c-m)g(x), \quad where \quad c = \int_a^b h(t)f(t)dt,$$

and

$$u(x) = u_0 + u_1$$

For this problem $g(x) = -x$ and $h(t) = 1$, hence

$$\begin{aligned} u_0 &= arcsin(x) + \left(\frac{\pi}{2} - 1\right) x, \\ u_1 &= -\left[\int_0^1 arcsin(t) + \left(\frac{\pi}{2} - 1\right) tdt - m\right] x, \\ &= -\left[\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right) - m\right] x, \end{aligned}$$

and from (3.9):

$$\begin{aligned} m &= \frac{\int_0^1(-t)dt}{\int_0^1(-t)dt - 1} \int_0^1 arcsin(t) + \left(\frac{\pi}{2} - 1\right) tdt \\ &= \frac{-\frac{1}{2}}{-\frac{1}{2} - 1}\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right) \\ &= \frac{1}{3}\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right), \end{aligned}$$

hence we obtain

$$\begin{aligned} u_1 &= -\left[\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right) - \frac{1}{3}\left(arcsin(1) - 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right)\right)\right] x \\ &= -\left[\frac{2}{3}arcsin(1) - \frac{2}{3} + \frac{2}{6}\left(\frac{\pi}{2} - 1\right)\right] x, \end{aligned}$$

and the solution will be as follows:
$u(x) = u_0 + u_1 = arcsin(x) + \left(\frac{\pi}{3} - \frac{2}{3}arcsin(1)\right) x.$
As we observe,H(u,p,m) method gives (after two terms) a result that is closer for exact solution than H(u,p) method .

example 3 Consider the equation

$$(e-1)e^{3x} = \int_0^1 e^{3x-4t} u(t)dt,$$

where the exact solution is $u(x) = e^{3x+1}$ or $u(x) = e^{5x}$. We apply $H(u,p)$ and $H(u,p,m)$ methods to approximate the solution as follows

$\mathbf{H(u,p)}$ **method**:

Notice that

$$\mid 1 - \int_0^1 k(t,t)dt \mid = e^{-1} < 1$$

so

$$\begin{aligned} u_0 &= 0, \\ u_1 &= f(x), \\ &= (e-1)e^{3x}, \end{aligned}$$

$$\begin{aligned} u_{n+1} &= u_n - \int_a^b K(x,t)u_n(t)dt, \\ u_2 &= (e-1)e^{3x} - \int_0^1 e^{3x-4t}(e-1)e^{3t}dt, \\ &= (e-1)e^{3x-1}, \end{aligned}$$

$$\begin{aligned} u_3 &= (e-1)e^{3x-1} - \int_0^1 e^{3x-4t}(e-1)e^{3t-1}dt, \\ &= (e-1)e^{3x-2}, \end{aligned}$$

$$\begin{aligned} u_4 &= (e-1)e^{3x-2} - \int_0^1 e^{3x-4t}(e-1)e^{3t-2}dt, \\ &= (e-1)e^{3x-3}, \end{aligned}$$

.

.

.

Hence the solution will be as follows

$$u(x) = u_0 + u_1 + u_2 + ...,$$

$$
\begin{aligned}
&= (e-1)e^{3x}\left(1+e^{-1}+e^{-2}+e^{-3}+...\right) \\
&= (e-1)e^{3x}\left(\frac{e}{e-1}\right) \\
&= ee^{3x} \\
&= e^{3x+1}
\end{aligned}
\tag{3.17}
$$

.

in fact we could find the solution as a limit of (3.17).
Now applying $\mathbf{H(u,p,m)}$, we can easily find the solution as following

$$g(x) = e^{3x}, h(t) = e^{-4t}, c = e-1 \ \ and \ \ f(x) = (e-1)e^{3x}$$

and from (3.14), we have

$$m = c - \frac{c}{\alpha}, \quad \alpha = \int_0^1 e^{-t}dt = (-e^{-1}+1)$$

so we obtain $m = -1$, and consequently

$$
\begin{aligned}
u_1 &= (e-1+1)e^{3x} \\
&= e^{3x+1}
\end{aligned}
$$

and the solution is

$$
\begin{aligned}
u(x) &= u_0 + u_1 \\
&= e^{3x+1}.
\end{aligned}
$$

example 4 For integral equation

$$u(x) = x^2 - x^3 + \int_0^1 (1+xt)\, u(t)dt,$$

with exact solution $u(x) = -x^3 + x^2 - \frac{1}{6}x - \frac{29}{90}$. We apply $H(u,p)$ and $H(u,p,m)$ methods to approximate the solution as follows

$\mathbf{H(u,p)}$ **method**:

We have

$$
\begin{aligned}
u_0(x) &= f(x), \\
&= x^2 - x^3
\end{aligned}
$$

$$u_{n+1}(x) = \int_a^b k(x,t)u_n(t)dt, \quad n = 0, 1, ...,$$

so we obtain

$$\begin{aligned} u_1 &= \int_0^1 (1+xt)(t^2 - t^3)dt \\ &= \frac{1}{20}x + \frac{1}{12}, \end{aligned}$$

$$\begin{aligned} u_2 &= \int_0^1 (1+xt)(\frac{1}{20}t + \frac{1}{12})dt \\ &= \frac{7}{120}x + \frac{13}{120}, \end{aligned}$$

$$\begin{aligned} u_3 &= \int_0^1 (1+xt)(\frac{7}{120}t + \frac{13}{120})dt \\ &= \frac{53}{720}x + \frac{11}{80}, \end{aligned}$$

$$\begin{aligned} u_4 &= \int_0^1 (1+xt)(\frac{53}{720}t + \frac{11}{80})dt \\ &= \frac{403}{4320}x + \frac{251}{1440}, \\ &. \\ &. \\ &. \end{aligned}$$

Hence the solution will be as follows:

$$\begin{aligned} u(x) &= u_0 + u_1 + u_2 + u_3 + ..., \\ &= x^2 - x^3 + \frac{1}{20}x + \frac{1}{12} + \frac{7}{120}x + \frac{13}{120} + \frac{53}{720}x + \frac{11}{80} + ... \end{aligned}$$

as we observe the series is diverging and no convergent solution will be obtained by this method.

$\mathbf{H(u,p,m)}$ method:

we have

$g_1(x) = x,\ h_1(t) = t,\ g_2(x) = 1,\ h_2(t) = 1$. then

$c_1 = \int_0^1 t(t^2 - t^3)dt = \frac{1}{20}$,

$c_2 = \int_0^1 (t^2 - t^3)dt = \frac{1}{12}$,

$b_{11} = \int_0^1 t^2 dt = \frac{1}{3}$,

$b_{12} = \int_0^1 t dt = \frac{1}{2}$,

$b_{21} = \int_0^1 t dt = \frac{1}{2}$,

$b_{22} = \int_0^1 1 dt = 1$,

by applying (3.12) and informations of the problem, we obtain

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{2}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{20} \\ \frac{1}{12} \end{bmatrix}$$

consequently

$$\begin{aligned} u_1 &= (c_1 - m_1)g_1(x) + (c_2 - m_2)g_2(x), \\ &= \left(\frac{1}{20} - \frac{52}{240}\right)x + \left(\frac{1}{12} - \frac{292}{720}\right)1, \\ &= -\frac{1}{6}x - \frac{29}{90} \end{aligned}$$

finally the solution is as follows:

$$u(x) = u_0 + u_1 = x^2 - x^3 - \frac{1}{6}x - \frac{29}{90}$$

example 5 Consider the equation

$$e^x + 1 = \int_0^1 (4te^x + 3)u(t)dt,$$

where the exact solution is $u(x) = \dfrac{1}{6(3-e)}e^x + \dfrac{7-3e}{6(3-e)}$ or $u(x) = x^2$. We apply $H(u,p)$ and $H(u,p,m)$ methods to approximate the solution as follows

$\mathbf{H(u,p)}$ **method**:

Notice that

$$\mid 1 - \int_0^1 k(t,t)dt \mid = 6 > 1$$

so

$$\begin{aligned} u_0 &= 0, \\ u_1 &= f(x), \\ &= e^x + 1, \end{aligned}$$

$$\begin{aligned} u_{n+1} &= u_n - \int_a^b K(x,t)u_n(t)dt, \\ u_2 &= e^x + 1 - \int_0^1 (4te^x + 3)(e^t + 1)dt, \\ &= -5e^x - 3e + 1, \end{aligned}$$

$$\begin{aligned} u_3 &= -5e^x - 3e + 1 - \int_0^1 (4te^x + 3)(-5e^t - 3e + 1)dt, \\ &= 13e^x + 6ee^x + 21e - 17, \\ &= (13 + 6e)e^x + 21e - 17, \end{aligned}$$

$$\begin{aligned} u_4 &= (13 + 6e)e^x + 21e - 17 - \int_0^1 (4te^x + 3)((13 + 6e)e^t + 21e - 17)dt, \\ &= (-5 - 60e)e^x - 18e^2 - 63e + 73, \\ &. \\ &. \\ &. \end{aligned}$$

Hence the solution will be as follows

$$u(x) = u_0 + u_1 + u_2 + ...,$$

$$= e^x + 1 - 5e^x - 3e + 1 + (13 + 6e)e^x + 21e - 17 + (-5 - 60e)e^x - 18e^2 - 63e + 73 + ... \quad (3.18)$$

$$= (4 - 54e)e^x - 18e^2 - 45e + 58 + ...$$

as we observe the series is diverging and no convergent solution will be obtained by this method.

Now applying $\mathbf{H}(\mathbf{u}, \mathbf{p}, \mathbf{m})$ we have

$g_1(x) = x,\ h_1(t) = 4t,\ g_2(x) = 1,\ h_2(t) = 3,\ c_1 = 1,\ c_2 = 1.$ then

$b_{11} = \int_0^1 4te^t dt = 4,$

$b_{12} = \int_0^1 4t dt = 2,$

$b_{21} = \int_0^1 3e^t dt = 3e - 3,$

$b_{22} = \int_0^1 3dt = 3,$

by applying (3.16) and informations of the problem, we obtain

$$B = \begin{bmatrix} 4 & 2 \\ 3e-3 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & 2 \\ 3e-3 & 3 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3e-3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \dfrac{17-6e}{18-6e} \\ \dfrac{11-3e}{18-6e} \end{bmatrix}$$

consequently

$$u_1 = (c_1 - m_1)g_1(x) + (c_2 - m_2)g_2(x),$$

$$= \left(1 - \frac{17-6e}{18-6e}\right) e^x + \left(1 - \frac{11-3e}{18-6e}\right) 1,$$

$$= \frac{1}{6(3-e)} e^x + \frac{7-3e}{6(3-e)}$$

finally the solution is as follows:

$$u(x) = u_0 + u_1 = \frac{1}{6(3-e)}e^x + \frac{7-3e}{6(3-e)}$$

example 6 For integral equation

$$u(x) = -2 - 3x + \int_0^1 (3x+t)\, u(t)dt,$$

with exact solution $u(x) = 6x$. We apply $H(u,p)$ and $H(u,p,m)$ methods to approximate the solution as follows

$\mathbf{H(u,p)}$ **method**:

We have

$$u_0(x) = f(x), \\ = -2 - 3x$$

$$u_{n+1}(x) = \int_a^b k(x,t)u_n(t)dt, \quad n = 0, 1, ...,$$

so we obtain

$$u_1 = \int_0^1 (3x+t)(-2-3t)dt \\ = -\frac{21}{2}x - 2,$$

$$u_2 = \int_0^1 (3x+t)(-\frac{21}{2}t - 2)dt \\ = -\frac{87}{4}x - \frac{9}{2},$$

$$u_3 = \int_0^1 (3x+t)(-\frac{87}{4}t - \frac{9}{2})dt \\ = -\frac{369}{8}x - \frac{19}{2},$$

$$\begin{aligned} u_4 &= \int_0^1 (3x+t)(-\frac{369}{8}t - \frac{19}{2})dt \\ &= -\frac{1563}{16}x - \frac{161}{8}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

Hence the solution will be as follows:

$$\begin{aligned} u(x) &= u_0 + u_1 + u_2 + u_3 + ..., \\ &= -2 - 3x - \frac{21}{2}x - 2 - \frac{87}{4}x - \frac{9}{2} - \frac{369}{8}x - \frac{19}{2} - \frac{1563}{16}x - \frac{161}{8} + ... \\ &= -\frac{2865}{16}x - \frac{305}{8} \end{aligned}$$

as we observe the series is diverging and no convergent solution will be obtained by this method.

H(u, p, m) method:

we have

$g_1(x) = 3x,\ h_1(t) = 1,\ g_2(x) = 1,\ h_2(t) = t.$ then

$c_1 = \int_0^1 1(-2-3t)dt = -\frac{7}{2},$

$c_2 = \int_0^1 t(-2-3t)dt = -2,$

$b_{11} = \int_0^1 3tdt = \frac{3}{2},$

$b_{12} = \int_0^1 1dt = 1,$

$b_{21} = \int_0^1 3t^2dt = 1,$

$b_{22} = \int_0^1 tdt = \frac{1}{2},$

by applying (3.12) and informations of the problem, we obtain

$$B = \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{7}{2} \\ -2 \end{bmatrix}$$

thus

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -\frac{13}{2} \\ -4 \end{bmatrix}$$

consequently

$$\begin{aligned} u_1 &= (c_1 - m_1)g_1(x) + (c_2 - m_2)g_2(x), \\ &= \left(-\frac{7}{2} + \frac{13}{2}\right) 3x + (-2 + 4)\, 1, \\ &= 9x + 2 \end{aligned}$$

finally the solution is as follows:

$$u(x) = u_0 + u_1 = 6x$$

Conclusion

In this work, we discussed the standard homotopy perturbation method and the modified one, then applied them to solve linear Fredholm integral equations of the first and second kind. We Note through the comparision between HPM and MHPM that in contrast to the standard one, H(u,p,m) method is a simple and very effective tool for calculating the exact solutions and in most cases it can give solution to some problems which could not be solved by H(u,p) method. As it is clarified by the example (four, five and six) in the numerical application part.

Bibliography

[1] S. Abbasbandy, Application of He's homotopy perturbation method to functional integral equations. Chaos, Solitons and Fractals 2007;31(5):1243–7.

[2] Z. Avazzadeh, M. Heydari, G.B. Loghmani, Numerical solution of Fredholm integral equations of the second kind by using integral mean value theorem. Yazd University, Yazd, Iran,22 November 2010.

[3] C. Chun, Integration using He's homotopy perturbation method. Chaos, Solitons and Fractals 2007;34(4):1130–4.

[4] L. Cveticanin, Homotopy perturbation method for pure nonlinear differential equation. Chaos, Solitons and Fractals 2006;30(5):1221–30.

[5] L. Debnath and P. Mikusninski, Introduction to Hilbert spaces with application, Academic press, New York, 1990.

[6] B. GAGUI, Sur les équations intégrales dans les espaces d'Orlicz. université de Msila , 03 03 2015.

[7] A.A. Golbabai *, B. Keramati, Modified homotopy perturbation method for solving Fredholm integral equations. 23 October 2006.

[8] J-H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int Nonlinear Mech 2000;35(1):37–43.

[9] J-H. He, Comparison of homotopy perturbation method and homotopy analysis method. Appl Math Comput 2004;156:527–39.

[10] J-H. He, Homotopy perturbation technique. Comput Methods Appl Mech Eng 1999;173(3–4):257–62.

[11] J-H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl Math Comput 2004;151:287–92.

[12] R. Kress, Linear Integral Equations, Springer-Verlag, New York, 1999.

[13] M. NADIR, Cours d'analyse fonctionnelle, université de Msila 2004.

[14] T. Ozis, A. Yildirim, A note on He's homotopy perturbation method for van der Pol oscillator with very strong nonlinearity.Chaos, Solitons and Fractals 2007;34(3):989–91.

[15] M. Rahman, Integral Equations and their Applications. Dalhousie University, Canada.

[16] AM. Siddiqui, R. Mahmood,QK. Ghori , Homotopy perturbation method for thin film flow of a third fluid down an inclined plane. Chaos, Solitons and Fractals 2008;35(1):140–7.

[17] S. Turq, Homotopy type methods for nonlinear differential equations.May 2020.

[18] A-M. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications.

Abstract

In this thesis, we study Fredholm's linear integral equations of the first and second kind, particularly whose the general form is (which are written in the form)

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (Second\ kind),$$

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0 \qquad (First\ kind),$$

where $k(.,.)$and $f(.)$ are known continuous functions with $k(x,t) = g(x)h(t)$.
To solve this type of equation we use the homotopy perturbation method and the modified one. and then we compare them through selected examples.

Keywords:Integral equations, Fredholm linear integral equations, Homotopy perturbation method, The modified homotopy perturbation,Fredholm integral equations of the first kind, of the second kind.

Résumé

Dans cette thèse, nous étudions les équations intégrales linéaires de Fredholm du premier et du deuxième type, en particulier dont la forme générale est (qui sont écrites dans la forme)

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (deuxime\ type),$$

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0 \qquad (Premier\ type),$$

où $k(.,.)$and $f(.)$ sont des fonctions continues connues avec $k(x,t) = g(x)h(t)$.
Pour résoudre ce type d'équation, nous utilisons la méthode de perturbation d'homotopie et la méthode modifiée. puis nous les comparons à travers des exemples sélectionnés.

Mots clés: équations intégrales, équations intégrales linéaires de Fredholm, méthode de perturbation d'homotopie, perturbation d'homotopie modifiée, équations intégrales de Fredholm du premier type, du second type.

ملخص

في هذه الأطروحة، قمنا بدراسة معادلات فريدهولم التكاملية الخطية من النوع الأول والثاني، بالأخص منها تلك التي يكون شكلها العام كالتالي:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$,(النوع الثاني)

$$f(x) + \lambda \int_a^b K(x,t)u(t)dt = 0$$,(النوع الأول)

حيث $k(.,.)$ و $f(.)$ هي دوال مستمرة معروفة مع $k(x,t) = g(x)h(t)$.
لحل هذا النوع من المعادلات استخدمنا طريقة الاضطراب المتجانس والطريقة المعدلة. ثم قمنا بتطبيق كل منهما على بعض الامثلة المختارة ومقارنة النتائج المحصل عليها من كل طريقة مع الحل الدقيق للمعادلة.

الكلمات المفتاحية: المعادلات التكاملية ، معادلات فريدهولم التكاملية الخطية ، طريقة اضطراب هوموتوبي ، اضطراب هوموتوبي المعدل ، معادلات فريدهولم التكاملية من النوع الأول ، من النوع الثاني.