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Studies of some sequences of polynomials from combinatorics and number theory

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Je dédie cette thè

À mon Père

À ma Mère

À mon frère Abdelmouneim

À ma belle famille

À ma femme

À ma petite prince

À tous me

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Introduction

The Dedekind eta function was introduced by Dedekind in 1877, it is defined by

$$\eta(t) = e^{2\pi it/24} \prod_{n=1}^{\infty}(1 - e^{2\pi int}), \qquad Im(t) > 0. \tag{0.0.1}$$

This function is not only important in Mathematics, but we can encounter it in physics and Chemistry as well. According to M. Atiyah [4], it is one of the most studied function in Mathematics. For a good illustration of its omnipresence in the whole Mathematics see [19]. The papers [59, 8, 52] are noexaustive examples of its appearance in Physics and Chemistry. Let $q = e^{2\pi it}$, where $t \in \mathbb{H}$, the upper half plane. For $z \in \mathbb{C}$, consider

$$\left(q^{-\frac{1}{24}}\eta(t)\right)^{-z} = \prod_{n=1}^{\infty}(1 - q^n)^{-z} = \sum_{n=0}^{\infty} P_n(z) q^n. \tag{0.0.2}$$

Writing the polynomials P_n as

$$P_n(z) = \frac{z}{n!}\sum_{k=0}^{n-1} N(n,k) z^k,$$

the coefficients $N(n,k)$ are positive integers. The values

$$N(n,0) = (n-1)!\sigma(n),\ N(n,n-2) = \frac{3n(n-1)}{2},\ N(n,n-1) = 1,\ n \in \mathbb{N},$$

where $\sigma(n)$ is as usual, the sum of the divisors of n, were calculated by Newman [43]. These polynomials may be also defined by

$$P_n(z) = \frac{z}{n}\sum_{k=1}^{n} \sigma(k) P_{n-k}(z). \tag{0.0.3}$$

For $n \leq 9$, these $P_n(z)$ have only real zeros. Serre [43, 54] used them to prove a density theorem for the powers of the function η of Dedekind.

Nevertheless, and due to the irregularities of the function σ, little is known about $P_n(z)$ for general n, despite the fact that for special values of z, many famous sequence are linked to the $P_n(z)$, for example $(P_n(1))_{n=1}^{\infty} = (p_n)_{n=1}^{\infty}$ is the sequence of partition numbers and $(P_{n-1}(-24))_{n=1}^{\infty} = (\tau(n))_{n=1}^{\infty}$ is the sequence of Ramanujan numbers.

The function τ is named after Ramanujan, who calculated its values for $1 \leq n \leq 30$, and conjectured its multiplicativity. Mordell [41] showed the truth of this conjecture.

Lehmer [38] showed that $\tau(n) \neq 0$ for all $n < 3316799$, then conjectured that $\tau(n) \neq 0$ for all n. Despite its simplicity, this conjecture is unsolved yet, and may be regarded as one among the oldest ones, sinces it goes back to the year (1947, and there is no significant advance till now). In the last two decades, we have seen a very active and extensive research to deal with this conjecture. There are mainly two directions in these research, the first one, is the continuation and extensions of the results of Newman and Serre (the vanishing properties of the rth power of the Dedekind eta function) see [26]. The second one, which seem to be a more general approach towards the resolution of the conjecture is due to Heim and his collaborators. This approach consists in considering polynomials generalizing the $P_n(z)$. For this type of results, see [22] and the references therein.

The subject of this thesis is in the frame of the second line of research, initiated by Heim and his collaborators. In a series of papers, Heim and his collaborators (mainly Neuhauser) developed an ingenious strategy to understand and eventually settle Lehmer conjecture. For this, let g and h be two arithmetic functions. In [25, 27], the authors defined the following sequence of polynomials:

$$P_n^{g,h}(x) = \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{n-k}^{g,h}(x), \quad P_0^{g,h}(x) = 1. \tag{0.0.4}$$

Setting $h(n) = n$ and $g(n) = \sigma(n)$ in (0.0.4) yields (0.0.3). Varying h and g gives very interesting results, and may have important repercussions on equation (0.0.3).

A very important case considered by Heim and Neuhauser [24], is when $h(n) = n$, and $g(n) = \phi_1(n) = n$ and $g(n) = \phi_2(n) = n^2$, this yields the two sequences of polynomials $(P_n^{\phi_i}(x))$, $1 \leq i \leq 2$

(for simplicity, we omitted h from the notation)

$$\begin{aligned} P_n^{\phi_i}(x) &= \frac{x}{n}\sum_{k=1}^{n}\phi_i(k)P_{n-k}^{\phi_i}(x) \\ &= \frac{x}{n!}\sum_{k=0}^{n-1}A_k^n(\phi_i)x^k,\ P_0^{\phi_i}(x)=1. \end{aligned}$$

Since for every integer $n > 1$, $n < \sigma(n) < n^2$, the two polynomials interpolate well P_n^{σ}, and thus furnishes many information about it.

They calculated the coefficients $(A_k^n(\phi_i))_{k=0}^{n-1}$ and proved their log-concavity. Based on numerical evidence, they conjectured that the polynomials $P_n^{\phi_i}(x),\ i=1,\ 2$ have only real zeros. In this thesis, we prove this conjecture and prove some other results linked to the polynomials $P_n^{\phi_i}(x),\ i=1,\ 2$. The first chapter is devoted to the background needed to the understanding of the rest of the thesis. Chapter 2 is consecrated to the proof of Heim- Neuhauser conjecture. In the last chapter, in order to continue the study of the unimodality of the coefficients of the polynomial $P_n^{\phi_2}$, we show that the sequence $(A_k^n(\phi_2))_{k=0}^{n-1}$ has a unique maximum. This is done by showing that a diophantine equation has no solutions for $n \geq 9$. The chapters 2-3 are essentially the material of the paper [62].

List of abbreviations and symbols

Symboles	Meaning	Page
$GL_n(R)$	The group of $n-$by$-n$ matrices with determinant in R^*. (The general linear group)	8
$SL_2(\mathbb{Z})$	The full congruence subgroup.	8
$\Gamma_0(N)$	Matrices in $SL_2(\mathbb{Z})$ which are upper triangular modulo N.	9
$\Gamma_1(N)$	Matrices in $\Gamma_0(N)$ which have 1's on the diagonal modulo N.	9
$\Gamma(N)$	Matrices in $SL_2(\mathbb{Z})$ which are congruent to the identity modulo N.	9
Γ	Generic congruence subgroup.	9
$\bar{\mathbb{C}}$	The Riemann sphere, $\mathbb{C} \cup \{\infty\}$.	12
$P^1_{\mathbb{C}}$	Projective $1-$space over $\mathbb{C}$.	12
H	Poincarè upper half plane.	13
$M_k(SL_2(\mathbb{Z}))$	Space of modular forms for $SL_2(\mathbb{Z})$ of weight k.	16
$S_k(SL_2(\mathbb{Z}))$	Space of cusp forms for $SL_2(\mathbb{Z})$ of weight k.	16
$M_k(\Gamma)$	Space of modular forms for Γ of weight k.	18
$S_k(\Gamma)$	Space of cusp forms for Γ of weight k.	18
$\eta(t)$	Dedekind η function.	21
Δ	Normalized cusp form of weight 12 and level 1.	23
$\tau(n)$	Ramanujan τ function.	24

Chapter 1

Short background on modular groups and modular forms

Contents

The theory of modular forms is an advanced topic and needs a strong background in algebra to understand all the fine details. It appears in many ways in number theory. They play a central role in the theory of quadratic forms. In particular, they are generating functions for the number of representations of integers by positive definite quadratic forms (for example, see [20]). They are also crucial in the recent spectacular proof of Fermat's Last Theorem see for example, [9, 12]. Koblitz's book [31] is also a very nice introduction to the theory of elliptic curves and modular forms (and includes a lot of information about the congruent number problem). Chapter 5 in Milne's book [40] contains a good, concise overview of the subject.

Modular forms are now at the center of a huge research activity. The material in this interoductory chapter is taken almost verbatim from the book of Koblitz [31].

1.1. Modular Group

The modular group is omnipresent in Mathematics. We can find it in many branches of pure mathematics **Gareth Jones.**[30]

Definition 1.1. *For any commutative ring R, the general linear group $GL_n(R)$ is the group of $n-$by$-n$ matrices with determinant in R^* (the multiplicative group of invertible elements of R).*

It is easy to see that $GL_n(R)$ is a group.

Definition 1.2. *The special linear group $SL_n(R)$ is defined to be the subgroup of $GL_n(R)$ consisting of matrices of determinant* 1.

In this section, we will be concerned with the cases $R = \mathbb{R}$, $R = \mathbb{Z}$, $R = \mathbb{Z}/N\mathbb{Z}$ (for a positive integer N), and $n = 2$.

1.1.1. Full modular group

Definition 1.3. *The full modular group $SL_2(\mathbb{Z})$ (also known simply as the modular group) to be the group of $2-$by-2 matrices with integer entries and determinant 1,*

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big/ \ a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}.$$

This group is occasionally called the unimodular group in the literature.

The full modular group is generated by the two matrices,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.1.1}$$

Theorem 1.1. *[42] The matrices S and T generate $SL_2(\mathbb{Z})$.*

1.1.2. Congruence subgroups

In this subsection, we will introduce some interesting subgroups of $SL_2(\mathbb{Z})$, those are useful for the study of modular forms.

Definition 1.4. *Let N be a positive integer. We define the following subgroups of $SL_2(\mathbb{Z})$:*

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \Big/ \; c \equiv 0 \pmod N \right\}.$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \Big/ a \equiv d \equiv 1 \pmod N \right\}.$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \Big/ b \equiv 0 \pmod N \right\}.$$

Definition 1.5. *Any subgroup of $SL_2(\mathbb{Z})$ which contains $\Gamma(N)$ for some N is called a congruence subgroup.*

Clearly, $\Gamma_0(1) = \Gamma_1(1) = \Gamma(1) = SL_2(\mathbb{Z})$. We call $\Gamma(N)$ the principal congruence subgroup of level N. If $\Gamma(N)$ is the largest principal congruence subgroup contained within a congruence subgroup N, then we say that Γ has level N. For instance, $SL_2(\mathbb{Z})$ has level 1, and $\Gamma_0(2)$ has level 2. Notice that a congruence subgroup of level N also has level M for any multiple M of N. This is because $\Gamma(M) \subseteq \Gamma(N)$[1] Alternatively, $SL_2(\mathbb{Z})$ can also be defined as $\Gamma_0(N)$.

Proposition 1.1. *Let N be a positive integer. Then*

1) The natural map $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ given by $A \mapsto A \pmod N$, is a surjective homomorphism with kernel $\Gamma(N)$.

2) The map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d \pmod N$, is a surjective homomorphism with kernel $\Gamma_1(N)$

3) The map $\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto b \pmod N$, is a surjective homomorphism with kernel $\Gamma(N)$.

Proof. For a proof of this proposition, see Murty [42, pp. 162, 167]. □

We have the following chain of inclusions, for any $N \in \mathbb{N}$:

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq SL_2(\mathbb{Z}). \tag{1.1.2}$$

We will call the congruence subgroups in this chain the standard congruence subgroups.

Proposition 1.2 ([48]). *Let $N \geq 1$ and let M be a positive divisor of N. Then*

$$\Gamma_0(N) \subseteq \Gamma_0(M), \quad \Gamma_1(N) \subseteq \Gamma_1(M) \quad \textit{and} \quad \Gamma(N) \subseteq \Gamma(M). \tag{1.1.3}$$

[1]See Murty [42, solution of exercise 2.5.2 p. 170] we have $\Gamma(N) \bigcap \Gamma(M) = \Gamma(lcm(N, M))$.

Proposition 1.3.

1) Let N be a positive integer. Then $\Gamma(N)$ is a normal subgroup of $SL_2(\mathbb{Z})$ of finite index.

2) Let p be a prime. Then $SL_2(\mathbb{Z}/p\mathbb{Z})$ is a normal subgroup of $GL_2(\mathbb{Z}/p\mathbb{Z})$ of index $p-1$.

Proof. For a proof of these results, see Murty [42, solution of exercise 2.2.2, 2.2.5 pp. 163–164]. □

Proposition 1.4. *Let p be a prime. The order of $GL_2(\mathbb{Z}/p\mathbb{Z})$ is given by $(p^2-1)(p^2-p)$.*

Proof. See Murty [42, solution of exercise 2.2.4 p. 163]. □

Corollary 1.1. *Let p be a prime. The order of $SL_2(\mathbb{Z}/p\mathbb{Z})$ is given by $p(p^2-1)$.*

Proof. Using propositions 1.4 and 1.3 . □

Proposition 1.5.

1) Let p be a prime, Let n be a positive integer. The order of $SL_2(\mathbb{Z}/p^n\mathbb{Z})$ is given by $p^{3n-2}(p^2-1)$.

2) Let N be a positive integer. The order of $SL_2(\mathbb{Z}/N\mathbb{Z})$ is given by $N^3 \prod_{p|N}(1-\frac{1}{p^2})$.

Proof. The book of Murty [42] contains the proofs of these assertions (pp. 164–165). □

Proposition 1.6. *Let N be a positive integer. Then*

1) $[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|n}(1-\frac{1}{p^2})$.

2) $[SL_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{p|n}(1-\frac{1}{p^2})$.

3) $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|n}(1+\frac{1}{p})$.

4) $[\Gamma_0(N) : \Gamma(N)] = N^2 \prod_{p|n}(1-\frac{1}{p})$.

5) $[\Gamma_0(N) : \Gamma_1(N)] = \phi(N)$, *where ϕ is the Euler totient function from number theory.*

6) $[\Gamma_1(N) : \Gamma(N)] = N$.

Proof. These results are classical. The proofs of the two first ones are to be found in the book of Murty (pp. 165–167). For 3) and 4), see the book of Kilford (pp. 23–24). The last assertions are proved in the book of Diamond & Shurman [16]. □

We will often call an unspecified congruence subgroup Γ, this follows the notation in some of the literature, but we note that in Koblitz (1993) [31, p. 99] the group $SL_2(\mathbb{Z})$ is called Γ. A natural further convention is to use γ for an element of Γ.

1.1.3. Groups acting on topological spaces

Definition 1.6. *Let G be a group and X a topological space. We say that there is a left action of G on X if for each element $g \in G$, we have a continuous map $x \mapsto gx \ \in X$, for $x \in X$ satisfying the conditions:*

a) $1x = x, \qquad \forall x \in X.$

b) $(gh)x = g(hx), \qquad \forall g, h \in G,\ x \in X.$

From these axioms, we see that $x \mapsto g^{-1}x$ is the inverse map of $x \mapsto gx$. Thus each element $g \in G$ gives rise to a topological automorphism of X.

It may be clearer to define an action as a pairing $G \times X \to X$, where we use the notation $g.x$ to denote the image of $(g,\ x)$ under this mapping. We omit the symbol "$\cdot$" from the notation to follow what is usual in the literature.

Let us the recall the basic definitions

Definition 1.7. *For each element $x \in X$, we define the stabilizer subgroup G_x as*

$$G_x = \{g \in G : \ gx = x\}. \tag{1.1.4}$$

An element $x \in X$ is called a fixed point of $g \in G$ if $gx = x$.

Proposition 1.7. *If G acts on X and $g \in G$, $x \in X$, we have*

$$G_{gx} = gG_xg^{-1}.$$

Proof. See Murty [42, solution of exexice 2.4.1 p. 168]. □

For each element $x \in X$, we define the $G-$orbit of x to be

$$Gx = \{gx/\ g \in G\}. \tag{1.1.5}$$

The set of all $G-$orbits in X will be denoted G/X. We say two elements $x, y \in X$ are $G-$equivalent if there is a $g \in G$ such that $gx = y$. In particular, $y \in Gx$. This is clearly an equivalence relation on X and we can decompose X as a union of disjoint orbits:

$$X = \bigcup Gx.$$

If there is only one orbit, we say G acts transitively on X. This is equivalent to saying that any two elements of X are G-equivalent. That is, given any two elements $x, y \in X$, there is a $g \in G$ such that $gx = y$.

Theorem 1.2. *If G acts on a set X and $x \in X$, then*

$$|Gx| = [G : G_x],$$

the index of the stabilizer G_x in G.

Proof. See Rotman [49, proof of Theorem 2.141 p. 197]. □

Proposition 1.8. *If G acts on a set X and $x \in X$, $g, h \in G$, then $gx = hx$ if and only if g and h lie in the same left coset of G_x.*

Proof. See Murty [42, solution of exercice 2.4.3.(a) p. 169]. □

Corollary 1.2. *If G and X are both finite, we deduce that*

$$|X| = \sum_i [G : G_{x_i}] = \sum_i |Gx_i|,$$

where the sum is over a complete set of representatives of the orbits of G.

Proof. See Rotman [49, p. 196], or see Murty [42, solution of exercise 2.4.3.(b) p. 169]. □

1.2. Modular forms

There are five basic operations in arithmetic: addition, subtraction, multiplication, division, and modular forms **Martin Eichler.**[11]

Let $\bar{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$ (i.e., the complex plane with a point at infinity, or equivalently the complex projective line $P^1_{\mathbb{C}}$ also known as the Riemann sphere). Given an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ and a point $z \in \mathbb{C}$, we define

$$\gamma z = \frac{az+b}{cz+d}, \qquad \gamma\infty = \lim_{x\to\infty} \gamma z = \frac{a}{c}. \tag{1.2.1}$$

(Thus, $\gamma(-\frac{d}{c}) = \infty$, and if $c = 0$, then $\gamma\infty = \infty$.) These maps $z \mapsto \gamma z$ are called fractional linear (**Möbius**) transformations of the Riemann sphere $\bar{\mathbb{C}}$. It is easy to check that (1.2.1) defines a group action on the set $\bar{\mathbb{C}}$, in other words:

$$Iz = z \qquad for\ all\ z \in \bar{\mathbb{C}}.$$

$$(\gamma_1\gamma_2)z = \gamma_1(\gamma_2 z) \qquad for\ all\ \gamma_1, \gamma_2 \in SL_2(\mathbb{R}),\ z \in \bar{\mathbb{C}}.$$

Thus, the quotient group $SL_2(\mathbb{R})/\pm I$, which is sometimes denoted $PSL_2(\mathbb{R})$ acts faithfully on $\mathbb{C}$, in other words, each element other than the identity acts nontrivially.

Let $H \subset \mathbb{C}$ denote the **upper half-plane** $H = \{z \in \mathbb{C}/\ Im(z) > 0\}$. It is important to note that any $\gamma \in PSL_2(\mathbb{R})$ preserves H, i.e., $Im(z) > 0$ implies $Im(\gamma z) > 0$. This is because

$$Im(\gamma z) = \frac{Im(z)}{|cz + d|^2}.$$

Thus, the group $SL_2(\mathbb{R})$ acts on the set H by the transformations (1.2.1) .

We shall denote $\bar{\Gamma} = SL_2(\mathbb{Z})/\pm I$, $\bar{\Gamma}$ acts faithfully on H. It is one of the basic groups arising in number theory and other branches of mathematics.

Whenever we have a subgroup G of $SL_2(\mathbb{R})$, we shall let $\bar{G}$ denote $G/\pm I$ if G contains $-I$, if $-I \notin G$, we set $\bar{G} = G$. Notice that $\bar{\Gamma}(2) = \Gamma(2)/\pm I$, whereas for $N > 2$ we have $\bar{\Gamma}(N) = \Gamma(N)$, because $-1 \not\equiv 1(mod\ N)$, and so $-I$ is not in $\Gamma(N)$.

Whenever a group acts on a set, it divides the set into equivalence classes, where two points are said to be in the same equivalence class if there is an element of the group which takes one to the other. In particular, if Γ is a subgroup of $SL_2(\mathbb{Z})$, we say that two points $z_1,\ z_2 \in H$ are $\Gamma-$**equivalent** if there exists $\gamma \in \Gamma$ such that $z_2 = \gamma z_1$.

Let F be a closed region in H (usually, F will also be simply connected).

Definition 1.8. *We say that F is a **fundamental domain** for the subgroup Γ of $SL_2(\mathbb{Z})$ if every $z \in H$ is $\Gamma-$equivalent to a point in F, but no two distinct points $z_1,\ z_2$ in the interior of F are $\Gamma-$equivalent (two boundary points are permitted to be $\Gamma-$equivalent).*

The most famous example of a fundamental domain is shown in Fig.1.1

$$F = \{z \in H/\ -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}\ and\ |z| \geq 1\}. \tag{1.2.2}$$

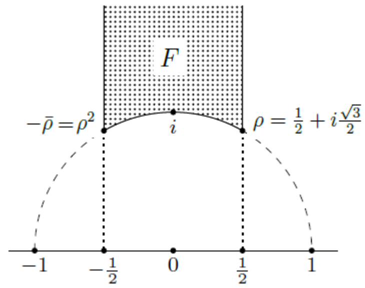

Figure 1.1: Fundamental Domain for $SL_2(Z)$.

Proposition 1.9. *The region F defined in 1.2.2 is a fundamental domain for $SL_2(\mathbb{Z})$.*

Proof. See Koblitz [31, Ch. III, Proposition 1 p. 100] □

From the proof of Proposition 1.9, we deduce two other facts.

Proposition 1.10. *Two distinct points z_1, z_2 on the boundary of F are $SL_2(\mathbb{Z})-$equivalent only if $Re(z_1) = \pm\frac{1}{2}$ and $z_2 = z_1 \pm 1$, or if z_1 is on the unit circle and $z_2 = -\frac{1}{z_1}$.*

In the next proposition, we use the notation $SL_2(\mathbb{Z})_z$. By definition 1.1.4, for each element $z \in F$

$$SL_2(\mathbb{Z})_z = \{\gamma \in SL_2(\mathbb{Z}) : \ \gamma z = z\}.$$

Proposition 1.11. *If $z \in F$, then $SL_2(\mathbb{Z})_z = \pm I$ except in the following three cases:*

(1) $\pm\{I,\ S\}$ *if* $z = i$.

(2) $\pm\{I,\ TS,\ (TS)^2 = -(TS)^{-1}\}$ *if* $z = \rho = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

(3) $\pm\{I,\ ST,\ (ST)^2 = -(ST)^{-1}\}$ *if* $z = \rho^2 = -\bar{\rho} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

Another by-product of the proof of Proposition 1.9 is the following useful fact.

Proposition 1.12. *The group $\bar{\Gamma} = SL_2(\mathbb{Z})/\pm I$ is generated by the two elements S and T (see* (1.1.1) *). In other words, any fractional linear transformation can be written as a word in S (the negative-reciprocal map) and T (translation by 1) and their inverses.*

Proof. See Koblitz [31, Ch. III, Proposition 4, pp 102]. □

Let f be a meromorphic function defined on the upper half-plane H, periodic of period w for some integer w. Hence

$$f(t) = g\big(e^{2\pi it/w}\big),$$

where $g(q)$ is a meromorphic function in the punctured unit disk $0 < |q| < 1$.

Definition 1.9 ([11]). *We will say that f is meromorphic or holomorphic at infinity if g is meromorphic or holomorphic at 0, respectively. If f is holomorphic at infinity, the value $g(0)$ is denoted $f(i\infty)$ and is called the value of f at infinity, and we say that f vanishes at infinity if $g(0) = 0$.*

Thus, if f is meromorphic at infinity, there exists some integer n_0 such that we can write $f(t) = \sum\limits_{n\geq n_0} a_n e^{2\pi int/w}$ for some coefficients a_n, the Fourier coefficients of f. The function f is holomorphic at infinity if and only if we can take $n_0 \geq 0$ (i.e., $a_n = 0$ for $n < 0$, see Koblitz [31]), and in that case $f(i\infty) = a_0$. Thus, f vanishes at infinity if and only if we can write $f(t) = \sum\limits_{n\geq 1} a_n e^{2\pi int/w}$ (i.e., $a_n = 0$ for $n \leq 0$, see Kilford [30]).

1.2.1. Modular forms for $SL_2(\mathbb{Z})$

Let $f(t)$ be a meromorphic function on the upper half-plane H, and let k be an integer. Suppose that $f(t)$ satisfies the relation

$$f(\gamma t) = (ct + d)^k f(t), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.2.3}$$

In particular, for the elements $\gamma = T$ and $\gamma = S$, (1.2.3) gives

$$f(t + 1) = f(t). \tag{1.2.4}$$

$$f(-1/t) = (-t)^k f(t). \tag{1.2.5}$$

Since all of $\bar{\Gamma} = SL_2(\mathbb{Z})/\pm I$ is generated by S and T (see proposition 1.12), this means that (1.2.4) and (1.2.5) imply (1.2.3).

Further suppose that $f(t)$ is meromorphic at infinity. Recall that this means that the Fourier series

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi int}, \tag{1.2.6}$$

has at most finitely many nonzero an with $n < 0$. Then $f(t)$ is called a **modular function** of weight k for $\Gamma = SL_2(\mathbb{Z})$.

If, in addition, $f(t)$ is actually holomorphic on H and at infinity (i.e., $a_n = 0$ for all $n < 0$), then $f(t)$ is called a **modular form** of weight k for $\Gamma = SL_2(\mathbb{Z})$ [2]. The set of such functions is denoted $M_k(\Gamma)$.

If we further have $a_0 = 0$, i.e., the modular form vanishes at infinity, then $f(t)$ is called a **cusp-form** of weight k for $\Gamma = SL_2(\mathbb{Z})$. The set of such functions is denoted $S_k(\Gamma)$ (some authors write it as $M_k^0(\Gamma)$. The use of the letter S is traditional, and comes from the German Spitzenform for **cusp-form**. Cusp-forms are sometimes also called **parabolic forms**).

We say that a modular form is **normalized** if the first non-zero coefficient of its $q-$expansion is equal to 1.

Proposition 1.13. *Let k be an even integer*

a) The only modular forms of weight 0 for $SL_2(\mathbb{Z})$ are constants, (ie., $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$.)

b) $M_k(SL_2(\mathbb{Z})) = \{0\}$ if $k < 0$ or $k = 2$.

c) $S_k(SL_2(\mathbb{Z})) = \{0\}$ if $k < 12$ or $k = 14$, $S_{12}(SL_2(\mathbb{Z})) = \Delta\mathbb{C}$. and for $k > 14$, $S_k(SL_2(\mathbb{Z})) = \Delta M_{k-12}(SL_2(\mathbb{Z}))$ (i.e., the cusp forms of weight k are obtained by multiplying modular forms of weight $k - 12$ by the discriminant $\Delta(t)$ function).

Proof. The proofs of the two first ones are to be found in the book of Murty [42, p. 189]. The last assertion is proved in the book of Koblitz [31, Ch. III, Proposition 9.d, p. 118]. □

Proposition 1.14. *Let k, k' be integers.*

a) If f, $g \in M_k(SL_2(\mathbb{Z}))$ Then, for all α, $\beta \in \mathbb{C}$, the function $\big(\alpha f + \beta g\big) \in M_k(SL_2(\mathbb{Z}))$. Therefore, $M_k(SL_2(\mathbb{Z})$ is a vector space over $\mathbb{C}$.

b) $S_k(SL_2(\mathbb{Z})$ is a $\mathbb{C}-$linear subspace of $M_k(SL_2(\mathbb{Z})$.

c) If $f \in M_k(SL_2(\mathbb{Z}))$ (Resp $f \in S_k(SL_2(\mathbb{Z})))$ and $g \in M_{k'}(SL_2(\mathbb{Z}))$(Resp $g \in S_k(SL_2(\mathbb{Z})))$, then $fg \in M_{k+k'}(SL_2(\mathbb{Z}))$(Resp $fg \in S_{k+k'}(SL_2(\mathbb{Z})))$.

Proof. See Murty [42, solution of exercice 4.1.5 p. 186]. □

[2] Some authors for instance see Murty [42] say that f has level 1

Let $\gamma \in SL_2(\mathbb{Z})$, let $f(t)$ be a function on the extended upper half-plane $H^* = H \cup \mathbb{Q} \cup \{\infty\}$ with values in $\mathbb{C} \cup \{\infty\}$, and let $k \in \mathbb{Z}$. We introduce the notation $f|[\gamma]_k$ to denote the function whose value at t is $(ct+d)^{-k} f((at+b)/(ct+d)))$. We denote the value of $f|[\gamma]_k$ at t by $f(t)|[\gamma]_k$,

$$f(t)|[\gamma]_k = (ct+d)^{-k} f(\gamma t) \quad for\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.2.7}$$

More generally, let $GL_2^+(\mathbb{R})$ denote the subgroup of $GL_2(\mathbb{R})$ consisting of matrices with positive determinant. Then we define

$$f(t)|[\gamma]_k = det^{k/2}(\gamma)(ct+d)^{-k} f(\gamma t) \quad for\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}). \tag{1.2.8}$$

For example, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a scalar matrix, we have $f|[\gamma]_k = f$ unless $a < 0$ and k is odd, in which case $f|[\gamma]_k = -f$.

Then the modularity condition (1.2.3) is equivalent to

$$f|[\gamma]_k = f, \qquad \forall \gamma \in SL_2(\mathbb{Z}).$$

Proposition 1.15. *For all* $\gamma_1,\ \gamma_2 \in GL_2^+(\mathbb{R})$, *we have*

$$f|[\gamma_1\gamma_2]_k = (f|[\gamma_1]_k)|[\gamma_2]_k. \tag{1.2.9}$$

Proof. See Murty [42, solution of exercice 4.1.3 p. 186]. □

1.2.2. Modular forms for congruence subgroups

We are now ready to define modular functions, modular forms, and cusp forms for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$.

Definition 1.10 ([31]). *Let $f(t)$ be a meromorphic function on H, and let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup of level N. Let $k \in \mathbb{Z}$. We call $f(t)$ a **modular function** of weight k for Γ if*

$$f|[\gamma]_k = f \quad for\ all\ \gamma \in \Gamma, \tag{1.2.10}$$

and if, for any $\gamma_0 \in SL_2(\mathbb{Z})$,

$$f|[\gamma_0]_k \text{ has the form } \sum a_n e^{2\pi i z/N} \text{ with } a_n = 0 \text{ for } n \ll 0. \tag{1.2.11}$$

We call such an $f(t)$ *a* ***modular form*** *of weight* k *for* Γ *if it is holomorphic on* H *and if for all* $\gamma_0 \in SL_2(\mathbb{Z})$ *we have* $a_n = 0$ *for all* $n < 0$ *in* (1.2.11). *We call a modular form a* ***cusp-form*** *if in addition* $a_n = 0$ *in* (1.2.11) *for all* $\gamma_0 \in SL_2(\mathbb{Z})$. *We now explain this further.*

The first condition (1.2.10) is the obvious analog of the first condition (1.2.3) for modular functions for $SL_2(\mathbb{Z})$. The second condition (1.2.11) is called meromorphicity at the cusps (holomorphicity if $a_n = 0$ for $n < 0$, vanishing if $a_n = 0$ for $n \leq 0$).

Let $g = f|[\gamma_0]_k$ for some fixed $\gamma_0 \in GL_2(\mathbb{Q})$. If f is invariant under Γ, i.e., if $f|[\gamma]_k = f$ for $\gamma \in \Gamma$, then it follows from 1.15 that g is invariant under the group $\gamma_0^{-1}\Gamma\gamma_0$: for all $\gamma_0^{-1}\gamma\gamma_0 \in \gamma_0^{-1}\Gamma\gamma_0$ we have $g|[\gamma_0^{-1}\gamma\gamma_0]_k = (f|[\gamma_0]_k)|[\gamma_0^{-1}\gamma\gamma_0]_k = f|[\gamma\gamma_0]_k = (f|[\gamma]_k)|[\gamma_0]_k = f|[\gamma_0]_k = g$. In particular, if $\gamma_0 \in SL_2(\mathbb{Z})$ and $\Gamma(N) \subset \Gamma$, then $\gamma_0^{-1}\Gamma\gamma_0$ also contains $\Gamma(N)$ (since $\Gamma(N)$ is normal in $SL_2(\mathbb{Z})$), and so $g = f|[\gamma_0]_k$ is invariant under $\Gamma(N)$. Because $T^N \in \Gamma(N)$, we have $g(t + N) = g(t)$, and so $g = f|[\gamma_0]_k$ has a Fourier series expansion in powers of $q_N = e^{2\pi i z/N}$. The content of condition (1.2.11) is that this expansion has only finitely many negative powers of q_N (no negative powers for holomorphicity, only positive powers for vanishing).

It may happen that $g = f|[\gamma_0]_k$ is invariant under a smaller translation T^h, where $h|N$, i.e., $g(t + h) = g(t)$. In that case the only powers of q_N that appear in the Fourier series are powers of $q^h = q_N^{N/h}$.

We let $M_k(\Gamma)$ and $S_k(\Gamma)$ denote the set of modular forms of weight k for Γ and the set of cusp-forms of weight k for Γ, respectively. As in the case $\Gamma = SL_2(\mathbb{Z})$ treated in the last subsection, it is easy to see that these are a $\mathbb{C}-$vector spaces, that $f \in M_k(\Gamma)$ and $g \in M_{k'}(\Gamma)$ implies $fg \in M_{k+k'}(\Gamma)$, and that the vector space of weight zero modular functions for Γ is a field.

It is immediate from the definition that if $\Gamma' \subset \Gamma$, then a modular function/ modular form/ cusp-form for Γ is also a modular function/ modular form/ cusp-form for Γ'. The next proposition generalizes Proposition 1.13.a) in the last section. Like Proposition 1.13.a), it is useful in proving equality of two modular forms from information about their zeros.

Chapter 2

A conjecture of Heim and Neuhauser

Contents

Introduction

Let $\eta(t)$ be the Dedekind function, and $q = e^{2\pi it}$, where $t \in H$, the upper half plane. For $z \in \mathbb{C}$, consider

$$\left(q^{-\frac{1}{24}}\eta(t)\right)^{-z} = \prod_{n=1}^{\infty}(1-q^n)^{-z} = \sum_{n=0}^{\infty} P_n(z)q^n. \tag{2.0.1}$$

Writing the polynomials P_n as

$$P_n(z) = \frac{z}{n!}\sum_{k=0}^{n-1} N(n,k)z^k, \tag{2.0.2}$$

the coefficients $N(n,k)$ are positive integers. The values

$$N(n,0) = (n-1)!\sigma(n),\ N(n,n-2) = \frac{3n(n-1)}{2},\ N(n,n-1) = 1,\ n \in \mathbb{N},$$

where $\sigma(n)$ is as usual, the sum of the divisors of n, were calculated by Newman [43] (Note that, $N(n,k)$ is to equal $A_{n-1-k}(n)$ in the paper [43] i.e., $N(n,n-1-k) = A_k(n) = \sum_{i=1}^{k} c_i(k)\binom{n}{k+i}$ where $c_{i+1}(k) = \sum_{j=1}^{k-i} j!\ \sigma(j+1)c_i(k-j)\binom{k+i}{j}$, $1 \le i \le k-1$, $c_1(k) = k!\ \sigma(k+1)$). These polynomials may be also defined by

$$P_n(z) = \frac{z}{n} \sum_{k=1}^{n} \sigma(k) P_{n-k}(z). \tag{2.0.3}$$

For $n \leq 10$, these $P_n(z)$ may be calculated recursively and have only real zeros (for $n \leq 9$), see [43, 54] . Also, they were used by Serre [54] to prove a density theorem for the powers of the function η of Dedekind. But, and because of the chaotic behaviour of the function σ, little is known about $P_n(z)$ for general n. Note that $P_n(z)$ are related to many famous sequences, for example $(P_n(1))_{n=1}^{\infty} = (p_n)_{n=1}^{\infty}$ is the sequence of partition numbers and $(P_{n-1}(-24))_{n=1}^{\infty} = (\tau(n))_{n=1}^{\infty}$ is the sequence of Ramanujan numbers.

After calculating the first values of $\tau(n)$, $1 \leq n \leq 30$, Ramanujan conjectured that τ is a multiplicative function, a fact proved later by Mordell [41], using advanced tools of modular forms. According to [60], there is no known elementary proof of this result. For the importance of the η function in general, let us cite Atiyah [4]: *The Dedekind function is one of the most famous and well-studied function in mathematics, particularly in relation to elliptic curves and modular forms.* The Dedekind eta function is surprisingly almost everywhere! a good reference showing this quite unexpected presence is [19]. For its incursions in physics, see [59, 8, 52].

Lehmer [38] showed that $\tau(n) \neq 0$ for all $n < 3316799$, and that the least integer m where $\tau(m) = 0$ (if it exists) must be a prime number. He then conjectured that $\tau(n) \neq 0$ for all n. Due to the importance of the eta function (and what is around, its powers, the coefficients of certain of these powers such that $(\tau(n))_{n\geq 1}$) the research to settle this conjecture is very active, especially in these last two decades. These studies are mainly in two directions: First, the extensions of the results of Newman and Serre (the vanishing properties of the rth power of the Dedekind eta function) see [26]. Second, a general approach towards the resolution of the conjecture, by considering sequences of polynomials generalizing the $P_n(z)$. For this type of results, see [22] and the references therein.

In a series of papers, Heim and his collaborators (mainly Neuhauser) developed an ingenious strategy to understand and eventually settle Lehmer conjecture. For this, let g and h be two arithmetic functions. In [25, 27], the authors defined the following sequence of polynomials:

$$P_n^{g,h}(x) = \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{n-k}^{g,h}(x), \quad P_0^{g,h}(x) = 1. \tag{2.0.4}$$

Setting $h(n) = n$ and $g(n) = \sigma(n)$ in (2.0.4) yields (2.0.3). Varying h and g gives very interesting results, and may have important repercussions on equation (2.0.3).

For example Heim and Neuhauser [24], with $h(n) = n$, and the two functions $g(n) = \phi_1(n) = n$ and $g(n) = \phi_2(n) = n^2$, obtained the two sequences of polynomials $(P_n^{\phi_i}(x))$, $1 \leq i \leq 2$ (for simplicity, we omitted h from the notation)

$$\begin{aligned} P_n^{\phi_i}(x) &= \frac{x}{n}\sum_{k=1}^{n}\phi_i(k)P_{n-k}^{\phi_i}(x) \\ &= \frac{x}{n!}\sum_{k=0}^{n-1}A_k^n(\phi_i)x^k,\ P_0^{\phi_i}(x) = 1. \end{aligned}$$

Since for every integer $n > 1$, $n < \sigma(n) < n^2$, the two polynomials interpolate well $P_n^{\sigma}(x)$, and thus furnishes many information about it. Also, they determined the coefficients $(A_k^n(\phi_i))_{k=0}^{n-1}$ and proved their log-concavity. Based on numerical evidence, they conjectured that the polynomials $P_n^{\phi_i}(x)$, $i = 1, 2$ have only real zeros.

The aim of this chapter is to prove this conjecture and prove some results linked to the polynomials $P_n^{\phi_i}(x)$, $i = 1, 2$. In the next section, we prove the conjecture. In the third section, we consider a polynomial related to Fibonacci polynomial and use it to prove some identities and find (new?) ones. The last section is devoted to the unimodality of the sequence considered in the third section.

2.1. Dedekind's eta-function

In many applications of elliptic modular functions to number theory the eta function plays a central role. It was introduced by Dedekind[1] in 1877 and is defined in the half-plane H by

$$\eta(t) = q^{1/24}\prod_{n\geq 1}(1-q^n), \quad q = exp(2\pi it). \tag{2.1.1}$$

If $t \in H$ then $|q| < 1$ so the infinite product $\prod\limits_{n\geq 1}(1-q^n)$ converges absolutely and is nonzero. Moreover, since the covergence is uniform on compact subset of H, $\eta(t)$ is analytic on H.

[1]Julius Wilhelm Richard Dedekind (6 October 1831 12 February 1916) was a German mathematician who made important contributions to number theory, abstract algebra (particularly ring theory), and the axiomatic foundations of arithmetic. His best known contribution is the definition of real numbers through the notion of Dedekind cut. He is also considered a pioneer in the development of modern set theory and of the philosophy of mathematics known as Logicism.

Theorem 2.1. *If $t \in H$, we have*

$$\eta\left(\frac{at+b}{ct+d}\right) = \varepsilon(ct+d)^{1/2}\eta(t). \tag{2.1.2}$$

where ε[1] *depends on* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, *and* $\varepsilon^{24} = 1$.

Proof. See [11, Theorem 5.8.1 p.191]. □

For the generator T we have

$$\eta(t+1) = e^{\pi i(t+1)/12}\prod_{n\geq 1}(1 - e^{2\pi i n(t+1)}) = e^{\pi i/12}\eta(t). \tag{2.1.3}$$

Consequently, for any integer b, we have

$$\eta(t+b) = e^{\pi i b/12}\eta(t).$$

For the other generator S we have the following theorem

Theorem 2.2. *If $t \in H$, we have*

$$\eta(-1/t) = \sqrt{-it}\;\eta(t). \tag{2.1.4}$$

The function $\sqrt{\ }$ is the branch of the square root function which has non-negative real part.

Proof. See chapter 3 of Apostol [3] gives two different proofs. □

Proposition 2.1 ([11]). *The function $\eta(t)$ has the Fourier expansion*

$$\begin{aligned}\eta(t) &= q^{1/24}\left(1 + \sum_{k\geq 1}(-1)^k\left(q^{k(3k-1)/2} + q^{k(3k+1)/2}\right)\right)\\ &= \sum_{k\in\mathbb{Z}}(-1)^k q^{(6k+1)^2/24} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)q^{n^2/24}.\end{aligned}$$

Proof. See [11, Corollary 2.1.24, p. 33]. □

[1] The explicit definition of ε, which depends on a, b, c, d, is somewhat complicated, so we do not give it here, but it is given in Theorem 5.8.1.(b) of [11, p. 191].

The last formula of this proposition shows that $\eta(t)$ is essentially a theta function of weight 1/2 with character $\left(\frac{12}{n}\right)$.

Proposition 2.2 ([11]). *The function $\eta^3(t)$ has the Fourier expansion*

$$\begin{aligned}\eta^3(t) &= q^{1/8}\sum_{k\geq 1}(-1)^k(2k+1)q^{k(k+1)/2}\\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}(-1)^k(2k+1)q^{(2k+1)^2/8} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{-4}{n}\right)nq^{n^2/8}.\end{aligned}$$

Proof. See [11, Corollary 2.1.25, p.33]. □

Once again, the last formula of this proposition shows that $\eta^3(t)$ is essentially a theta function of weight 3/2 with character $\left(\frac{-4}{n}\right)$.

The relations (2.1.2) say that $\eta(t)$ is a modular form of weight 1/2 for the modular group $SL_2((Z))$.

The eta function is closely related to the discriminant $\Delta(t)$ (See Apostol [3, Chapter 1, p. 14].), We can write the Δ function in terms of η

Theorem 2.3. *Let $t \in H$, and $q = e^{\pi it}$ as usual. Then the following equivalences hold:*

$$\Delta(t) = q\prod_{n\geq 1}(1-q^n)^{24} = \eta(t)^{24}. \tag{2.1.5}$$

Proof. See Kilford [30, p. 99]. □

Remark 2.1. *We note that in some of the literature (see Koblitz [31, p. 122], Diamond & Shurman [16, p. 75], Apostol [3, p. 47]) they used*

$$\Delta(t) = (2\pi)^{12}q\prod_{n\geq 1}(1-q^n)^{24} = (2\pi)^{12}\eta(t)^{24} \qquad t\in H,\ q=e^{2\pi it}.$$

Theorem 2.4. *If $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ then*

$$\Delta\left(\frac{at+b}{ct+d}\right) = (ct+d)^{12}\Delta(t), \qquad t\in H.$$

In particular

$$\Delta(t+1) = \Delta(t) \quad and \quad \Delta(-1/t) = t^{12}\Delta(t).$$

Proof. See Apostol [3, p. 50]. □

The discriminant function Δ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$.

2.2. The Ramanujan tau-function

In his fundamental paper of 1916, Ramanujan[2] introduced the $\tau-$function as being the coefficients in the power series expansion of the infinite product

$$\sum_{n\geq 1}\tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = \eta(t)^{24} = (2\pi)^{-12}\Delta(t). \tag{2.2.1}$$

where $q = e^{2\pi i t}$ with $t \in H$. Ramanujan observed, but did not prove, the following three properties of $\tau(n)$:

a) if $(n,m) = 1$ then $\tau(nm) = \tau(n)\tau(m)$ (meaning that $\tau(n)$ is a multiplicative function).

b) if p is prime and $\alpha \geq 0$ then $\tau(p^{\alpha+2}) = \tau(p)\tau(p^{\alpha+1}) - p^{11}\tau(p^{\alpha})$.

c) if p is prime then $|\tau(p)| \leq 2p^{11/2}$.

The first two items were proved by Mordell[3] [41] in 1917, using advanced tools of modular forms. A year after Ramanujan's paper appeared. But the third conjecture defied attempts by many eminent mathematicians until 1974, when Deligne[4] [14] resolved it as a corollary of his solution to another problem due to Weil (specifically, he deduced it by applying them to a Kuga-Sato variety).

[2]Srinivasa Ramanujan, (born December 22, 1887, Erode, Indiadied April 26, 1920, Kumbakonam), Indian mathematician he made substantial contributions to mathematical analysis, number theory, infinite series, and continued fractions, including solutions to mathematical problems then considered unsolvable.

[3]Louis Joel Mordell (28 January 1888 12 March 1972) was an American-born British mathematician, known for pioneering research in number theory. He was born in Philadelphia, United States.

[4]Pierre René, Viscount Deligne (3 October 1944) is a Belgian mathematician. He is best known for work on the Weil conjectures, leading to a complete proof in 1973. He is the winner of the 2013 Abel Prize, 2008 Wolf Prize, 1988 Crafoord Prize, and 1978 Fields Medal.

The coefficients $\tau(n)$ in the Fourier expansion of $\Delta(\tau)$ have also been extensively tabulated by Lehmer [37] and others. The first ten entries in Lehmer's table are repeated here:

$$\tau(1) = 1 \qquad \tau(6) = -6048$$
$$\tau(2) = -24 \qquad \tau(7) = -16744$$
$$\tau(3) = 252 \qquad \tau(8) = 84480$$
$$\tau(4) = -1472 \qquad \tau(9) = -113643$$
$$\tau(5) = 4830 \qquad \tau(10) = -115920.$$

Lehmer[5] [38] showed that $\tau(n) \neq 0$ for all $n < 3316799$, and that the least integer m where $\tau(m) = 0$ (if it exists) must be a prime number. He then conjectured that $\tau(n) \neq 0$ for all n. The following table 2.1 summarizes progress on finding successively larger values of N for which this condition holds for all $n < N$.

N	Reference
3316799	Lehmer 1947 [38]
214928639999	Lehmer 1949 [3, Apostol p. 22]
10^{15}	Serre 1973 [53, p. 98]
1213229187071998	Jennings 1993 [28, p. 09]
22689242781695999	Jordan & Kelly 1999 [29]
22798241520242687999	Bosman 2007 [7]
982149821766199295999	Zeng & Yin 2013 [61, p. 1470]
816212624008487344127999	Derickx, Maarten, Mark, and Zeng 2013 [15]

Table 2.1: $\tau(n) \neq 0$ for all $n < N$

[5]Derrick Henry Dick Lehmer (February 23, 1905 May 22, 1991), almost always cited as Lehmer, was an American mathematician significant to the development of computational number theory. Lehmer refined Édouard Lucas' work in the 1930s and devised the LucasLehmer test for Mersenne primes. His peripatetic career as a number theorist, with him and his wife taking numerous types of work in the United States and abroad to support themselves during the Great Depression, fortuitously brought him into the center of research into early electronic computing.

2.3. The sequences $A_k^n(\phi_1)$ and $A_k^n(\phi_2)$

Almost all of the material in this chapter (and the remaining ones) are to be found in the paper [62]. As noted in the previous section, in the case where the arithmetic functions are $\phi_1(n) = n$ and $\phi_2(n) = n^2$, with fixed $h(n) = n$ in (2.0.4), Heim and Neuhauser [24] obtained the following polynomials:

$$\widetilde{P}_n^{\phi_1}(x) = \sum_{k=0}^{n-1} A_k^n(n)x^k, \ \widetilde{P}_n^{\phi_2}(x) = \sum_{k=0}^{n-1} A_k^n(n^2)x^k.$$

In the next proposition, we give the explicit formula of $A_k^n(\phi_1)$, and $A_k^n(\phi_2)$

Proposition 2.3. *Let $0 \leq k \leq n-1$, we have*

a.) $A_k^n(\phi_1) = \frac{n!}{(k+1)!}\binom{n-1}{k}$.

b.) $A_k^n(\phi_2) = \frac{n!}{(k+1)!}\binom{n+k}{2k+1}$.

Proof [24]. Before we prove Proposition 2.3.a) we show the following useful property.

Lemma 2.1. *For $n \leq k \leq 1$, we obtain*

$$\sum_{m_1+\ldots+m_k=n} 1 = \binom{n-1}{k-1}, \tag{2.3.1}$$

where $m_i \geq 1, \ 1 \leq i \leq k$.

Proof. We prove (2.3.1) by induction. For $k = 1$ we have $\binom{n-1}{k-1} = \binom{n-1}{0} = 1$. Suppose the formula holds for k then

$$\begin{aligned}\sum_{m_1+\ldots+m_k=n} 1 &= \sum_{m_{k+1}=1}^{n+1-(k+1)} \ \sum_{m_1+\ldots+m_k=n-m_{k+1}} 1 \\ &= \sum_{m_{k+1}=1}^{n-k} \binom{n-m_{k+1}-1}{k-1} \\ &= \sum_{m_{k+1}=k}^{n-1} \binom{m_{k+1}-1}{k-1} = \binom{n-1}{k}.\end{aligned}$$

The proof (again by induction) of the identity $\sum_{m=k}^{n-1} \binom{m-1}{k-1} = \binom{n-1}{k}$. □

Now, for the proof of Proposition 2.3.a), we start with the identity [24]

$$\sum_{n=0}^{\infty} \widetilde{P}_n^{\phi_1}(x) q^n = exp\left(x \sum_{n=1}^{\infty} q^n \right). \tag{2.3.2}$$

It is useful to substitute $\left(\sum_{n=1}^{\infty} q^n\right)^k$ by the multi-index sum

$$\sum_{m_1,\ldots,m_k=1}^{\infty} q^{m_1+\ldots+m_k}$$

Comparing the coefficients of q^n in (2.3.2) leads to

$$\begin{aligned}\widetilde{P}_n(x) &= \sum_{k=1}^{n} \frac{1}{k!} \left(\sum_{m_1+\ldots+m_k=n} 1 \right) x^k \\ &= \frac{x}{n!} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!} \left(\sum_{m_1+\ldots+m_{k+1}=n} 1 \right) x^k.\end{aligned}$$

In the next step we apply Lemma 2.1 and obtain

$$\frac{x}{n!} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!} \binom{n-1}{k} x^k,$$

which yields the desired result.

Before we prove Proposition 2.3.b) we show the following useful property.

Lemma 2.2. *For $n \geq k \geq 1$ we obtain*

$$\sum_{m_1+\ldots+m_k=n} m_1 m_2 \ldots m_k = \binom{n+k-1}{2k-1}. \tag{2.3.3}$$

Proof. The proof is by induction on n and k. For $k = 1$ we have $n = \binom{n}{1} = \binom{n+k-1}{2k-1}$. Let now $N \geq K \geq 2$ be fixed. We assume that formula (2.3.3) holds for $k < K$. For $k = K$ we also assume that the formula (2.3.3) holds for $n < N$. Then we will prove formula (2.3.3) for $n = N$ and $k = K$.

We have

$$\begin{aligned}
\sum_{m_1+\ldots+m_K=N} m_1 m_2 \ldots m_K &= \sum_{m_K=1}^{N+1-K} m_K \sum_{m_1+\ldots+m_{K-1}=N-m_K} m_1 m_2 \ldots m_{K-1} \\
&= \sum_{m_K=1}^{N+1-K} \sum_{m_1+\ldots+m_{K-1}=N-m_K} m_1 m_2 \ldots m_{K-1} \\
&\quad + \sum_{m_K=1}^{N-K} m_K \sum_{m_1+\ldots+m_{K-1}=N-1-m_K} m_1 m_2 \ldots m_{K-1} \\
&= \sum_{m_K=1}^{N+1-K} \binom{N-m_K+(K-1)-1}{2(K-1)-1} + \sum_{m_1+\ldots+m_K=N-1} m_1 m_2 \ldots m_K \\
&= \sum_{m_K=2K-2}^{N+K-2} \binom{m_K-1}{2K-3} + \binom{N-1+K-1}{2K-1} \\
&= \binom{N+K-2}{2K-2} + \binom{N+K-2}{2K-1} = \binom{N+K-1}{2K-1}.
\end{aligned}$$

Note that here we again used the identity $\sum_{m=k}^{n-1} \binom{m-1}{k-1} = \binom{n-1}{k}$. □

Proof of Proposition 2.3.b). We obtain

$$\begin{aligned}
\exp\left(X \sum_{n=1}^{\infty} \frac{n^2}{n} q^n\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \left(\sum_{n=1}^{\infty} n q^n\right)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} X^k \left(\sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} m_1 \cdots m_k q^{m_1+\cdots+m_k}\right) \\
&= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k!} X^k \left(\sum_{m_1+\cdots+m_k=n} m_1 m_2 \cdots m_k\right) q^n.
\end{aligned}$$

In the next step we apply Lemma 2.2 and obtain

$$\begin{aligned}
&1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{k!} \binom{n+k-1}{2k-1} X^k q^n \\
=&1 + \sum_{n=1}^{\infty} \frac{X}{n!} \sum_{k=0}^{n-1} \frac{n!}{(k+1)!} \binom{n+k}{2k+1} X^k q^n.
\end{aligned}$$

which yields the desired result. □

Note that in [24], the polynomials $P_n^{\phi_1}(x) = \frac{x}{n!}\widetilde{P}_n^{\phi_1}(x)$ and $P_n^{\phi_2}(x) = \frac{x}{n!}\widetilde{P}_n^{\phi_2}(x)$ are given explicitly for $1 \leq n \leq 7$. In this section, we prove the conjecture stated by the authors of [24]. The proof we give is based on the following classical theorem of Laguerre.

Theorem 2.5 (Laguerre)**.** *Let $A(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial having only real zeros. Then the polynomial $\sum_{k=0}^{n} \frac{a_k}{k!} x^k$ has only real zeros .*

Proof. Before we prove Theorem 2.5 we show the following useful property.

Lemma 2.3. *Assume that the equation*

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0,$$

has only real roots. Then the polynomial

$$a_0 P(x) + a_1 P^{(1)}(x) + a_2 P^{(2)}(x) + ... + a_n P^{(n)}(x),$$

has not more non-real zeros than the polynomial $P(x)$ itself.

Proof. A proof of this result may be found in the book of Pólya and Szegö [46, Vol. II, Part V, problem 63] □

Now, Since $A(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ has only real zeros it follows that $A^r(x) = a_n + a_{n-1} x + a_{n-2} x^2 + ... + a_0 x^n$ also does and hence also 2.3

$$\begin{aligned}\frac{1}{n!}\left(a_n x^n + a_{n-1}\frac{dx^n}{dx} + a_{n-2}\frac{d^2 x^n}{dx^2} + \cdots\right) \\ = \frac{a_n}{n!}x^n + \frac{a_{n-1}}{(n-1)!}x^{n-1} + \frac{a_{n-2}}{(n-2)!}x^{n-2} + \cdots\end{aligned}$$

which yields the desired result.

For a very short and simple proof of Laguerre Theorem, see [10]. There, the proof is based essentially on two facts: if a polynomial has real zeros then so does its reciprocal polynomial. And, if the polynomial $T(x)$ has only real zeros, the polynomial $T'(x) - \alpha T(x)$ also has this property for every real number α. □

The following corollary is crucial for the proof of the main theorem, even it is a straightforward consequence from Laguerre Thorem.

Corollary 2.1. *Let* $A(x) = \sum\limits_{k=0}^{n} a_k x^k$ *be a real polynomial having only real zeros. Then for every* $l \geq 0$, *the polynomial* $\sum\limits_{k=0}^{n} \frac{a_k}{(k+l)!} x^k$ *has only real zeros .*

Proof. Consider the polynomial $B(x) = x^l A(x) = \sum\limits_{i=0}^{n+l} b_i x^i$, with $b_i = 0$ for $0 \leq i \leq l-1$. By the previous theorem, the polynomial

$$C(x) = \sum_{i=0}^{n+l} \frac{b_i}{i!} x^i = \sum_{i=0}^{l-1} \frac{b_i}{i!} x^i + \sum_{i=l}^{n+l} \frac{b_i}{i!} x^i = x^l \sum_{k=0}^{n} \frac{a_k}{(k+l)!} x^k,$$

has only real zeros. The proof is finished. □

Before starting the proof of the theorem, let us recall the definition of Jacobsthal-Fibonacci polynomials, $JF_n(x)$, (see [33, p. 489]), those are defined as follows

$$JF_n(x) = JF_{n-1}(x) + xJF_{n-2}(x), \; n \geq 2, \tag{2.3.4}$$

with the boundary conditions: $JF_0(x) = 0, \; JF_1(x) = 1$. Using the relation (2.3.4) we prove the explicit formula of $JF_n(x)$

Proposition 2.4. *For* $n \geq k \geq 0$, *we have*

$$JF_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^k = \frac{1}{\sqrt{4x+1}} \left(\left(\frac{1+\sqrt{4x+1}}{2} \right)^n - \left(\frac{1-\sqrt{4x+1}}{2} \right)^n \right). \tag{2.3.5}$$

Proof [6]. Write the relation (2.3.4) in matrix form, as follows: $\binom{JF_n(x)}{JF_{n-1}(x)} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix} \binom{JF_{n-1}(x)}{JF_{n-2}(x)}$. We deduce

$$\begin{pmatrix} JF_n(x) \\ JF_{n-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} JF_1(x) \\ JF_0(x) \end{pmatrix} = \begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvalues of the matrix $\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix}$ are

$$\lambda_1 = \frac{1+\sqrt{4x+1}}{2}, \; \lambda_2 = \frac{1-\sqrt{4x+1}}{2},$$

and two eigenvectors of A are $V_1 = \binom{\lambda_1}{1}$ and $V_2 = \binom{\lambda_2}{1}$. Now the matrix A may be written

$$\begin{pmatrix} 1 & x \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}^{-1}.$$

From this, we obtain

$$\begin{aligned}\begin{pmatrix}1 & x\\ 1 & 0\end{pmatrix}^{n-1} &= \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}\begin{pmatrix}\lambda_1^{n-1} & 0\\ 0 & \lambda_2^{n-1}\end{pmatrix}\begin{pmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{pmatrix}^{-1}\\ &= \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\lambda_1^n-\lambda_2^n & -\lambda_1^n\lambda_2+\lambda_2^n\lambda_1\\ \lambda_1^{n-1}-\lambda_2^{n-1} & -\lambda_1^{n-1}\lambda_2+\lambda_2^{n-1}\lambda_1\end{pmatrix}.\end{aligned}$$

The vector $\binom{JF_n(x)}{JF_{n-1}(x)}$ is now

$$\begin{pmatrix}JF_n(x)\\ JF_{n-1}(x)\end{pmatrix} = \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\lambda_1^n-\lambda_2^n & -\lambda_1^n\lambda_2+\lambda_2^n\lambda_1\\ \lambda_1^{n-1}-\lambda_2^{n-1} & -\lambda_1^{n-1}\lambda_2+\lambda_2^{n-1}\lambda_1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

So,

$$JF_n(x) = \frac{1}{\lambda_1-\lambda_2}\left(\lambda_1^n-\lambda_2^n\right).$$

Since $\lambda_1-\lambda_2=\sqrt{4x+1}$, we finally obtain

$$JF_n(x) = \frac{1}{\sqrt{4x+1}}\left(\left(\frac{1+\sqrt{4x+1}}{2}\right)^n - \left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right).$$

This is the desired result. □

Note that $JF_n(1)=F_n$, the Fibonacci number.

Let us recall the definition of the unsigned Lah numbers $Lh(n,k)$ (this is sequence A048854 in [[56]).

Definition 2.1. *The rising factorial* $x^{\bar{n}}$ *and the falling factorial* $x^{\underline{n}}$ *are defined by*

$$\begin{aligned}x^{\bar{n}} &= x(x+1)\dots(x+n-1)\\ x^{\underline{n}} &= x(x-1)\dots(x-n+1).\end{aligned}$$

Since, $\left(x^{\bar{k}}\right)_0^n$ and $\left(x^{\underline{k}}\right)_0^n$ are bases of the vector space of polynomials of degree $\leq n$. So, we can expand the rising factorial with respect to the falling factorial and vice versa by some connecting coefficients.

Definition 2.2. *The connecting coefficients, denoted by $Lh(n,k)$, such that*

$$x^{\bar{n}} = \sum_{k=0}^{n} Lh(n,k)\ x^{\underline{k}},$$

are called Lah numbers.

Example 2.1.

$$\begin{aligned} x^{\bar{3}} &= x(x+1)(x+2) \\ &= 6x + 6x(x-1) + x(x-1)(x-2) \\ &= 6x^{\underline{1}} + 6x^{\underline{2}} + 1x^{\underline{3}}. \end{aligned}$$

So, we have

$$Lh(3,0) = 0,\ Lh(3,1) = 6,\ Lh(3,2) = 6,\ Lh(3,3) = 1.$$

Proposition 2.5. *Let $n \geq k \geq 1$, we have*

$$Lh(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

Proof. A combinatorial proof of this result may be found in the book Mezo [39, p. 67]. □

Definition 2.3 (Combinatorial interpretation)**.** *$Lh(n,k)$ is the total number of ways one can partition an $n-$element set into k nonempty totally ordered subsets.*

Example 2.2. *The total number of ways one can partition the set $\{1,\ 2,\ 3\}$ into two nonempty totally ordered subsets is $Lh(3,2) = 6$. The partitions are*

$$1|2\ 3,\ \ 1|3\ 2,\ \ 2|1\ 3,\ \ 2|3\ 1,\ \ 3|1\ 2,\ \ 3|2\ 1.$$

Corollary 2.2. *Let $n \geq k \geq 1$, we have*

$$Lh(n+1,k) = (n+k)Lh(n,k) + Lh(n,k-1),$$

where $Lh(n,n) = 1,\ Lh(n,0) = 0,$ and $Lh(n,k) = 0$ for all $k > n$.

Proposition 2.6. *The Lah polynomials $Lh_n(x) = \sum_{k=0}^{n} Lh(n,k)x^k$, $n \geq 1$ have only real zeros, they have $n-1$ negative zeros and one zero at $x = 0$.*

Proof. For a very short and simple proof of this fact see [39, p. 114]. □

Now, we can prove our main result:

Theorem 2.6. *The polynomials $\widetilde{P}_n^{\phi_1}(x)$ and $\widetilde{P}_n^{\phi_2}(x)$ have only real zeros.*

Proof. We have

$$Lh(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

Note that

$$P_n^{\phi_1}(x) = \frac{x}{n!}\sum_{k=1}^{n} Lh(n,k)x^{k-1},$$

is nothing but Lah polynomial $Lh_n(x)$ (up to a multiplicative factor), which is known to be real rooted. As it is stated in [21], the polynomial $\widetilde{P}_n^{\phi_1}(x) = (n-1)!L_{n-1}^1(-x)$ is a generalized Laguerre polynomial (these polynomials are orthogonal), so its zeros are real, and it is irreducible over $\mathbb{Z}$. For the sake of completeness, let us prove directly that $\widetilde{P}_n^{\phi_1}(x)$ has only real zeros: the polynomial $(1+x)^{n-1}$ has only real zeros and then by Corollary 2.1, the polynomial $\widetilde{P}_n^{\phi_1}(x) = \sum_{k=0}^{n-1} \frac{n!}{(k+1)!}\binom{n-1}{k}x^k$ has the same property too ($l = 1$).

For $\widetilde{P}_n^{\phi_2}(x)$, it suffices to prove that $\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k$ has only real zeros. But this is true. In fact, the last polynomial is just the reciprocal of $JF_{2n}(x)$, call it $JF_{2n}^r(x)$:

$$\begin{aligned}
JF_{2n}^r(x) &= x^{n-1}JF_{2n}(1/x) = x^{n-1}\sum_{k=0}^{n-1}\binom{2n-k-1}{k}\frac{1}{x^k} = \sum_{k=0}^{n-1}\binom{2n-k-1}{k}x^{n-k-1} \\
&= \frac{x^{n-1}}{\sqrt{\frac{4}{x}+1}}\left(\left(\frac{1+\sqrt{\frac{4}{x}+1}}{2}\right)^{2n} - \left(\frac{1-\sqrt{\frac{4}{x}+1}}{2}\right)^{2n}\right) \\
&= \frac{(\sqrt{x})^{2n}}{x\sqrt{\frac{4}{x}+1}}\left(\left(\frac{1+\sqrt{\frac{4}{x}+1}}{2}\right)^{2n} - \left(\frac{1-\sqrt{\frac{4}{x}+1}}{2}\right)^{2n}\right) \\
&= \frac{1}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right).
\end{aligned}$$

Now, the coefficient of x^k in $JF_{2n}^r(x)$ is

$$\binom{2n-(n-1-k)-1}{n-1-k} = \binom{n+k}{n-1-k} = \frac{(n+k)!}{(n-k-1)!(2k+1)!} = \binom{n+k}{2k+1}.$$

It follows then that

$$\begin{aligned} JF_{2n}^r(x) &= \sum_{k=0}^{n-1} \binom{n+k}{2k+1} x^k \\ &= \frac{1}{\sqrt{x(x+4)}} \left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2} \right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2} \right)^{2n} \right). \end{aligned} \tag{2.3.6}$$

Finally, by Corollary 2.1 applied to the polynomial $JF_{2n}^r(x)$ with $l=1$, we deduce that $\widetilde{P}_n^{\phi_2}(x)$ has only real zeros. □

Remark 2.2. *The polynomials $\widetilde{P}_n^{\phi_2}(x)$ are irreducible, this was proved in [18].*

In [22, 26], the sequence of polynomials defined by (2.0.4) is investigated. Many interesting results are obtained, especially for particular arithmetic functions h and g. For example for $h(n)=1$ and $g(n)=n$, $Q_n^g(x)$ in [21] has only real zeros.

The polynomials satisfying a three term recursion or more (see [21]) are generally real rooted. So, we can apply Laguerre theorem to these polynomials to obtain new real rooted polynomials (the kth coefficient is divided by $k!$).

In Example 2. 7. 1 of [23], the polynomial $Q_n^g(x) = xJF_{2n}^r(x) = xU(x/2+1)$, where $U(x)$ is the Chebyshev polynomial of the second kind. In fact since the discriminant $D = D(x) = x(x+4)$, then the roots of the characteristic equation are given by

$$\lambda_{1,2} = \frac{x+2\pm\sqrt{D}}{2}.$$

It follows then that

$$Q_n^g(x) = \alpha\left(\frac{x+2+\sqrt{D}}{2}\right)^n + \beta\left(\frac{x+2-\sqrt{D}}{2}\right)^n,\ n \geq 1,$$

with $Q_0^g(x)=1, Q_1^g(x)=x, Q_2^g(x)=x(x+2)$, we deduce α and β, so

$$Q_n^g(x) = \frac{x}{\sqrt{D}}\left(\left(\frac{x+2+\sqrt{D}}{2}\right)^n - \left(\frac{x+2-\sqrt{D}}{2}\right)^n\right).$$

But,

$$\frac{x+2\pm\sqrt{D}}{2}=\left(\frac{\sqrt{x}\pm\sqrt{x+4}}{2}\right)^2.$$

So,

$$\begin{aligned}Q_n^g(x) &= \frac{x}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{x+4}}{2}\right)^{2n}-\left(\frac{\sqrt{x}-\sqrt{x+4}}{2}\right)^{2n}\right)\\ &= xJF_{2n}^r(x).\end{aligned}$$

The polynomials considered in this paper (Jacobsthal-Fibonacci, Lah) are closely linked to Laguerre and Chebyshev (orthogonal) polynomials, may be this highlights the appearance of Fibonacci and Lucas numbers in the context of modular forms. For the sake of completeness, we give without proof, the generating function of the sequence $P_n^{\phi_2}(x)$.

Theorem 2.7. *The generating function of the sequence $A_k^n(n^2)$ is given by*

$$\sum_{n\geq 0}P_n^{\phi_2}(x)z^n=\exp\left(\frac{xz}{(1-z)^2}\right).$$

Proof. We have

$$\begin{aligned}\sum_{n\geq 0}P_n^{\phi_2}(x)z^n &= 1+\sum_{n\geq 1}\frac{x}{n!}\left(\sum_{k\geq 0}\frac{n!}{(k+1)!}\binom{n+k}{2k+1}x^k\right)z^n\\ &= 1+\sum_{n\geq 1}\sum_{k\geq 0}\frac{1}{(k+1)!}\binom{n+k}{2k+1}x^{k+1}z^n\\ &= 1+\sum_{k\geq 0}\frac{1}{(k+1)!}x^{k+1}z^{-k}\sum_{n\geq 1}\binom{n+k}{2k+1}z^{n+k}\\ &= 1+\sum_{k\geq 0}\frac{1}{(k+1)!}x^{k+1}z^{-k}\frac{z^{2k+1}}{(1-z)^{2k+2}}\\ &= 1+\sum_{k\geq 0}\frac{1}{(k+1)!}\,\frac{(xz)^{k+1}}{(1-z)^{2k+2}}.\end{aligned}$$

Now write

$$\begin{aligned}\sum_{n\geq 0} P_n^{\phi_2}(x)z^n &= 1+\sum_{k\geq 0}\frac{1}{(k+1)!}\left(\frac{xz}{(1-z)^2}\right)^{k+1}\\ &= \sum_{k\geq 0}\frac{1}{k!}\left(\frac{xz}{(1-z)^2}\right)^{k}\\ &= \exp\left(\frac{xz}{(1-z)^2}\right).\end{aligned}$$

This finishes the proof. □

Remark 2.3. *Dilcher [17] deduced the formula*

$$F_{4n} = 3\sum_{k=0}^{n-1}\binom{n+k}{2k+1}5^k \tag{2.3.7}$$

using the hypergeometric functions, and said that he was unable to find it in the literature. In fact this is known and proved in Swamy [57]. Note that (2.3.7) *is easily deduced by setting* $x = 5$ *in* (2.3.6)*, and remembering that*

$$\left(\frac{\sqrt{5}\pm 3}{2}\right) = \left(\frac{1\pm\sqrt{5}}{2}\right)^2.$$

2.4. Some identities involving Fibonacci and Lucas numbers

In the paper [5], Benoumhani used the explicit formulas of the Fibonacci and Lucas polynomials to find some new identities (and rediscover some old ones) involving Fibonacci and Lucas numbers. In this section, we use the same elementary technique i.e., explicit formulas of Fibonacci and Lucas polynomials, their derivatives and antiderivatives to prove some identities, missed in [5].

Let us start with

$$JF_{2n}^r(x) = \sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^k = \frac{1}{\sqrt{x(x+4)}}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} - \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right). \tag{2.4.1}$$

Integrating $JF_{2n}^r(x)$, we obtain

$$\int_0^x JF_{2n}^r(t)dt = \sum_{k=0}^{n-1}\binom{n+k}{2k+1}\frac{x^{k+1}}{k+1} = \frac{1}{n}\left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2}\right)^{2n} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2}\right)^{2n}\right) - \frac{2}{n}. \tag{2.4.2}$$

Using this, we prove the

Proposition 2.7. *For every $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \frac{1}{2(2n+1)} \left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2} \right)^{4n+2} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2} \right)^{4n+2} \right) \tag{2.4.3}$$

Proof. We will prove the result using only (2.4.1) and (2.4.2). We have

$$\frac{\binom{2n+k+1}{2k}}{2n+k+1} = \frac{\binom{2n+k}{2k-1}}{2k}, \; k \neq 0, \tag{2.4.4}$$

and then

$$\sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \frac{1}{2n+1} + \sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1},$$

$$\sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} = \sum_{k=1}^{2n+1} \binom{2n+k}{2k-1} \frac{x^k}{2k}.$$

This last sum may be written as

$$\begin{aligned} \sum_{k=1}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} &= \sum_{k=1}^{2n+1} \binom{2n+k}{2k-1} \frac{x^k}{2k} \\ &= \frac{1}{2} \sum_{k=0}^{2n} \binom{2n+k+1}{2k+1} \frac{x^{k+1}}{k+1}. \end{aligned}$$

Finally, using (2.4.2) we obtain

$$\begin{aligned} \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{x^k}{2n+k+1} &= \frac{1}{2n+1} + \frac{1}{2} \sum_{k=0}^{2n} \binom{2n+k+1}{2k+1} \frac{x^{k+1}}{k+1} \\ &= \frac{1}{2(2n+1)} \left(\left(\frac{\sqrt{x}+\sqrt{4+x}}{2} \right)^{4n+2} + \left(\frac{\sqrt{x}-\sqrt{4+x}}{2} \right)^{4n+2} \right). \end{aligned}$$

The proof is finished. □

Before giving the next result, recall the sequence of Lucas numbers, $(L_n)_{n\geq 0}$:

$$L_0 = 2, \; L_1 = 1, \; L_n = L_{n-1} + L_{n-2}, \; n \geq 2.$$

Corollary 2.3. *For every $n \in \mathbb{N}$, we have the identities*

$$a) F_{2n+1} = \pm\frac{2(2n+1)}{\sqrt{5}} \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{(\pm\sqrt{5}-2)^k}{2n+k+1}$$

$$b) F_{2n+1} = (2n+1) \sum_{k=0}^{2n+1} (-1)^{k+1} \binom{2n+k+1}{2k} \frac{F_{3k}}{2n+k+1}$$

$$c) F_{4n+2} = 2 \sum_{k=1}^{2n+1} \frac{k}{2n+k+1} \binom{2n+k+1}{2k}$$

$$d) L_{(4n+2)l} = 2(2n+1) \sum_{k=0}^{2n+1} (-1)^{l(2n+k+1)} \binom{2n+k+1}{2k} \frac{5^k F_l^{2k}}{2n+k+1}, \ l \geq 1$$

$$e) \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{(-4)^k}{2n+k+1} = -\frac{1}{2n+1}$$

$$f) \sum_{k=0}^{2n+1} \binom{2n+k+1}{2k} \frac{(-2)^k}{2n+k+1} = 0.$$

Proof. Relation (a) is relation (7.1) in Dilcher [17]. There, is it deduced via the hypergeometric representations of the Fibonacci numbers. Now, set $x = \pm\sqrt{5} - 2$, in the last relation in the proof of Proposition 2.7, and divide both sides by $\sqrt{5}$ to get (a).

Relation (b) also appears in the paper [17], it is obtained by adding the two identities in (a). The identities (c) and (d) seem to be new. To obtain (c), differentiate (2.4.1), then set $x = 1$. For (d), remember the identity

$$5F_l^2(-1)^l + 4 = (-1)^l L_l^2,$$

then put $x = 5F_l^2(-1)^l = 5F_l^2 i^{2l}$ in (2.4.3). The remaining are obvious. □

Chapter 3

On the modes of two sequences related to $A_k^n(\phi_2)$

Contents

In this chapter, we will determine the modes of two sequences related to $A_k^n(\phi_2)$. The following facts will be needed in the sequel.

Definition 3.1. *A sequence of positive real numbers* $(a_k)_{k=0}^n$ *is said to be unimodal, if there exist integers* $k_0,\ k_1$ $(k_0 \leq k_1)$ *such that*

$$a_0 \leq a_1 \leq \ldots \leq a_{k_0} = a_{k_0+1} = \ldots = a_{k_1} \geq a_{k_1+1} \geq \ldots \geq a_n.$$

It is log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$, for $1 \leq k \leq n-1$. *The integers* $k_0,\ k_0+1,\ ...,\ k_1$ *are the modes of the sequence.*

A real sequence $(a_k)_{k=0}^n$ is said to be with no internal zeros (NIZ), if $i < j, a_i \neq 0,\ a_j \neq 0$ then $a_l \neq 0$ for every $l, i \leq l \leq j$. A (NIZ) log-concave sequence is obviously unimodal, but the converse is not true. The sequence $1,\ 1,\ 4,\ 2$ is unimodal but not log-concave. Note the importance of (NIZ): the sequence $1,\ 0,\ 0,\ 1$ is log-concave but not unimodal. A real polynomial is unimodal (log-concave, symmetric, respectively) provided that the sequence of its coefficients is unimodal (log-concave, symmetric, respectively).

If inequalities in the log-concavity definition are strict, then the sequence is called strictly log-concave (SLC for short), and in this case, it has at most two consecutive modes. The following result may be helpful in proving unimodality, this goes back to Newton:

Theorem 3.1. *[35] If the polynomial* $\sum_{k=0}^{n} a_k x^k$ *associated with the sequence* $(a_k)_{k=0}^n$ *has only real zeros then*[1]

$$a_k^2 \geq \frac{k+1}{k}\frac{n-k+1}{n-k}a_{k-1}a_{k+1}, \text{ for } 1 \leq k \leq n-1 \tag{3.0.1}$$

By the previous theorem, if the positive sequence $(a_k)_{k=0}^n$, is such that its generating polynomial has only real zeros, then it is SLC, and it has at most two peaks. A result of Darroch [13] states that if the polynomial $\sum_{k=0}^n a_k x^k$, $a_k \geq 0$, has only negative real zeros, then every mode k_0 of the sequence $(a_k)_{k=0}^n$ satisfies the inequality

$$\left\lfloor \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rceil. \tag{3.0.2}$$

3.1. The modes of the sequences $A_k^n(\phi_2)$

The explicit form of the coefficients of the polynomial $P_n^{\phi_1}(x)$ allows us to find the exact value of the modes, we can even see where a double maximum occurs. For the polynomial $P_n^{\phi_2}(x)$, the coefficients are also explicit, but it is not easy to find the exact value of the mode(s). Heim and Neuhauser [24] gave the following estimate of the mode:

$$k_0 \sim 2^{-\frac{2}{3}} n^{\frac{2}{3}}.$$

For the sake of simplicity, let $A_k^n(\phi_2) = A_{n,k}$. A double maximum occurs if and only if

$$A_{n,k} = A_{n,k+1},$$

which is equivalent to

$$n^2 = 4k^3 + 19k^2 + 28k + 13.$$

[1] Inequality (3.0.1) is also called as Newton's inequality after Isaac Newton who intended to solve the problem of counting the number of imaginary roots of the algebraic equation $a_0 x^n + a_1 x^{n-1} + ... + a_1 x + a_0$. During this attempt, he claimed (but gave no proof) that the number of imaginary roots cannot be less than the sign changes in the sequence $a_0^2, \left(\frac{a_1}{\binom{n}{1}}\right)^2 - \frac{a_2}{\binom{n}{2}}\frac{a_0}{\binom{n}{0}}, \ldots, \left(\frac{a_{n-1}}{\binom{n}{n-1}}\right)^2 - \frac{a_n}{\binom{n}{n}}\frac{a_{n-2}}{\binom{n}{n-2}}, a_n^2$. If all the solutions of the above equation are real, then all the entries in the above sequence must be non-negative. From here, (3.0.1) follows. More details can be found in [44].

According to a classical theorem of Selberg, this equation has only a finite number of solutions. Next, we prove that the sequence $A_{n,k}$ has a unique mode for $n > 8$.

Theorem 3.2. *The equation*

$$n^2 = 4k^3 + 19k^2 + 28k + 13,$$

has a unique solution in positive integers: $(n,\ k) = (8,\ 1)$*, thus, the sequence* $A_{n,k}$ *has a plateau for the single value* $n = 8$*, where* $A_{8,1} = A_{8,2}$*.*

Proof. The equation

$$n^2 = 4k^3 + 19k^2 + 28k + 13,$$

is an elliptic curve. Using Magma, we have all the solutions of this equation, namely:

$$(n,\ k) = (8,\ 1),\ (n,\ k) = (-8,\ 1),\ (n,\ k) = (0,\ -1).$$

The only positive integer where the sequence has a plateau is $n = 8$, and in this case $k_0 = 1$. We can check that $A_{8,1} = A_{8,2}$. □

In Chapter 4, we give an elementary proof of the previous theorem.

3.2. The modes of the sequences $(a_{n,k})_{k=0}^{2n+1}$

In this section, we will determine the modes of the sequence $a_{n,k}$. The fact that $a_{n,k}$ is log-concave is a consequence of Proposition 2.7. The techniques used are essentially due to Tanny and Zucker [58].

Now, let us consider the sequence $a_{n,k} = \frac{\binom{2n+k+1}{2k}}{2n+k+1}$. According to Proposition 2.7, this sequence is SLC and it has at most two consecutive modes. In fact we have the following result

Theorem 3.3. *The sequence* $(a_{n,k})_{k=0}^{2n+1}$ *is SLC and it has at most two consecutive maxima. Namely, if* k_n *is the integer where the maximum occurs, then*

$$k_n = \left\lceil \frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5} \right\rceil.$$

Proof. Proposition 2.7 shows that the generating polynomial associated with the sequence $(a_{n,k})_{k=0}^{2n+1}$

has only real zeros. It follows then that the sequence $(a_{n,k})_{k=0}^{2n+1}$ is SLC. Let k_n be the integer satisfying

$$a_{n,k_n} > a_{n,k_{n-1}}, \text{ and } a_{n,k_n} \geq a_{n,k_{n+1}}.$$

We follow the steps of Tanny and Zuker [58]. The previous inequalities may be written as

$$5k_n^2 - 4k_n - 4n^2 - 4n < 0 \tag{3.2.1}$$

$$5k_n^2 + 6k_n + 2 - (2n+1)^2 \geq 0 \tag{3.2.2}$$

Let $k_n = (2n+1)\alpha,\ 0 < \alpha < 1$, then we have

$$5(\alpha(2n+1))^2 - 4\alpha(2n+1) - 4n^2 - 4n < 0$$

and

$$5(\alpha(2n+1))^2 + 6\alpha(2n+1) + 2 - (2n+1)^2 \geq 0.$$

Put

$$P(x) = 5(2n+1)^2x^2 - 4(2n+1)x - 4n^2 - 4n,$$

and

$$Q(x) = 5(2n+1)^2x^2 + 6(2n+1)x + 2 - (2n+1)^2.$$

This yields $P(\alpha) < 0$ and $Q(\alpha) \geq 0$. So we deduce that α is between the roots of P, namely

$$\frac{2 \pm \sqrt{5(2n+1)^2 - 1}}{5(2n+1)},$$

and outside the roots of Q:

$$\frac{-3 \pm \sqrt{5(2n+1)^2 - 1}}{5(2n+1)}.$$

We deduce

$$\frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5(2n+1)} \leq \alpha < \frac{2 + \sqrt{5(2n+1)^2 - 1}}{5(2n+1)},$$

furthermore

$$\frac{-3 + \sqrt{5(2n+1)^2 - 1}}{5} \leq k_n < \frac{2 + \sqrt{5(2n+1)^2 - 1}}{5},$$

The difference between the bounds is exactly one, hence the wanted result. □

Corollary 3.1. *The integer k_n satisfies*

$$k_n = \left\lfloor \frac{\sqrt{5}(2n+1)}{5} \right\rfloor \ \text{or}\ \left\lceil \frac{\sqrt{5}(2n+1)}{5} \right\rceil,$$

Proof. Note that

$$\frac{2+\sqrt{5(2n+1)^2-1}}{5} \le \frac{2+\sqrt{5}(2n+1)}{5} = \frac{2}{5} + \frac{\sqrt{5}(2n+1)}{5},$$

and

$$\frac{-4}{5} + \frac{\sqrt{5}(2n+1)}{5} = \frac{-4+\sqrt{5}(2n+1)}{5} \le \frac{-3+\sqrt{5(2n+1)^2-1}}{5(2n+1)}.$$

So, we get

$$\frac{-4}{5} + \frac{\sqrt{5}(2n+1)}{5} \le k_n < \frac{2}{5} + \frac{\sqrt{5}(2n+1)}{5}.$$

Since $\frac{4}{5} < 1$ and $\frac{2}{5} < 1$, we get the result. □

In some cases, the sequence $a_{n,k}$ has a double maximum. For example, for $n=0$, we have

$$a_{0,k} = \frac{\binom{k+1}{2k}}{k+1},$$

and $a_{0,0} = a_{0,1} = 1$. Next, we determine all the integers where a double maximum occurs for $a_{n,k}$, $n \ge 1$. We have the following

Theorem 3.4. *The sequence $(a_{n,k})_{k=0}^{2n+1}$ has a double maximum if and only if*

$$n_j = \frac{F_{12j+9}-2}{4},$$

and in this case, the smallest mode k_j is given by

$$k_j = \frac{L_{12j+9}-6}{10}.$$

Proof. We use relation (2.4.3)

$$a_{n,k} = \frac{\binom{2n+k}{2k-1}}{2k},\ k \ge 1.$$

The equation

$$a_{n,k} = a_{n,k+1},$$

is equivalent to

$$5k^2 + 6k + 2 - (2n+1)^2 = 0.$$

The reduced discriminant of this last equation is

$$d' = 5(2n+1)^2 - 1,$$

and must be a square. So, let $y = 2n+1$, our equation becomes $x^2 = 5y^2 - 1$, or

$$x^2 - 5y^2 = -1,$$

which is the so called negative Pell equation. The solutions of this equation are given by

$$\begin{aligned} x_n &= \frac{1}{2}\left[(2+\sqrt{5})^{2n+1} + (2-\sqrt{5})^{2n+1}\right] \\ y_n &= \frac{1}{2\sqrt{5}}\left[(2+\sqrt{5})^{2n+1} - (2-\sqrt{5})^{2n+1}\right]. \end{aligned}$$

More details are to be found in [1, Sect 3.6. p. 46] . Note that

$$2 \pm \sqrt{5} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^3,$$

so we obtain

$$\begin{aligned} x_n &= \frac{1}{2} L_{6n+3} \\ y_n &= \frac{1}{2} F_{6n+3}. \end{aligned}$$

The numbers

$$k_j = \frac{-3 + \frac{1}{2} L_{6j+3}}{5},\ n_j = \frac{F_{6j+3} - 2}{4},$$

are not always integers. It is easy to see that numbers k_j and n_j are integers if and only if $j = 2l+1$.

So, the wanted solutions are given by

$$k_j = \frac{-3 + \frac{1}{2}L_{12j+9}}{5}, \ n_j = \frac{F_{12j+9} - 2}{4}.$$

In other words,

$$a_{n_j,k_j} = a_{n_j,k_j+1} \iff n_j = \frac{F_{12j+9} - 2}{4} \text{ and } k_j = \frac{-3 + \frac{1}{2}L_{12j+9}}{5}.$$

This proves the theorem. □

Corollary 3.2. *For every integer $j \geq 0$, we have*

$$\begin{aligned} \left\lfloor \frac{F_{12j+9}F_{F_{12j+9}}}{2L_{F_{12j+9}}} \right\rfloor &= \frac{L_{12j+9} - 6}{10} \\ \left\lfloor \frac{F_{4j+1}F_{F_{4j+1}-1}}{L_{F_{4j+1}}} \right\rfloor &= F_{2j}^2 \\ \left\lfloor \frac{F_{4j+3}L_{F_{4j+3}-1}}{5F_{F_{4j+3}-1}} - \frac{2}{5} \right\rfloor &= \frac{L_{4j+3} - 4}{5}. \end{aligned}$$

Proof. If the positive sequence $(a_k)_{k=0}^n$ has a generating polynomial with only real zeros then, by Darroch Theorem [13], every mode k_0 of $(a_k)_{k=0}^n$ satisfies (3.0.2). By Proposition 2.7, the generating polynomial $B_n(x)$ of the sequence

$$a_{n,k} = \frac{\binom{2n+k+1}{2k}}{2n+k+1},$$

has only real zeros. So we have

$$\left\lfloor \frac{\sum_{k=1}^{n} k a_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum_{k=1}^{n} k a_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rceil.$$

In the case where we have two consecutive modes, $k_0, \ k_0 + 1$, necessarily

$$k_0 = \left\lfloor \frac{\sum_{k=0}^{n} k a_{n,k}}{\sum_{k=0}^{n} a_{n,k}} \right\rfloor.$$

Again, by Proposition 2.7 and Theorem 3.4 ,

$$k_j = \left\lfloor \frac{\sum_{k=0}^{n_j} k a_{n_j,k}}{\sum_{k=0}^{n_j} a_{n_j,k}} \right\rfloor = \left\lfloor \frac{B'_{n_j}(1)}{B_{n_j}(1)} \right\rfloor = \left\lfloor \frac{(2n_j+1)F_{4n_j+2}}{L_{4n_j+2}} \right\rfloor.$$

Replace n_j and k_j by their values yields the wanted result. For the second relation, we use the results of the paper [6], in which the properties of the the sequence

$$c_{n,k} = \frac{n}{n-k}\binom{n-k}{k},$$

are studied. The generating polynomial of the sequence $c_{n,k}$ is given by

$$C_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k}x^k = \left(\frac{1+\sqrt{4x+1}}{2}\right)^n + \left(\frac{1-\sqrt{4x+1}}{2}\right)^n.$$

All the zeros of $C_n(x)$ are real. The integers where a double maximum occurs are $n_j = F_{4j+1}$, and in this case, the first mode is $k_j = F_{2j}^2$. Also,

$$\frac{C'_n(1)}{C_n(1)} = \frac{nF_{n-1}}{L_n}.$$

Finally

$$k_j = \left\lfloor \frac{C'_{n_j}(1)}{C_{n_j}(1)} \right\rfloor = \left\lfloor \frac{n_jF_{n_j-1}}{L_{n_j}} \right\rfloor.$$

Replacing n_j and k_j by their values yields the second relation. The last relation is obtained exactly as the previous one, using the results of [58]. □

Chapter 4

Elementary proof of a diophantine equation

As mentioned in Chapter 3, the aim of this chapter is an elementary solution of the equation in Theorem 3.2. For this, we need to solve some preliminary equations.

4.1. Binary quadratic Diophantine equation

Let the general diophantine quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{4.1.1}$$

where $a, b, c, d, e \in \mathbb{Z}$. While solving the diophantine equation (4.2.1), we encounter many quadratic diophantine equations. For this, let us recall some facts about these equations. More than two centuries ago, Lagrange [36] investigated this equation above and gave a solution for the cases when the coefficients are small integers. Lagrange's method works perfectly well, and may be this is the reason why it has not been improved so much since then.

If we put

$$D = b^2 - 4ac,\ E = bd - 2ae,\ F = d^2 - 4af$$

, Lagrange realized that (4.1.1) could be written as

$$DY^2 = (Dy + E)^2 + DF - E^2, \tag{4.1.2}$$

where $Y = 2ax + by + d$. Clearly, if we put $N = E^2 - DF = 4a(ebd + 4acf - ae^2 - fb^2 - cd^2)$, then (4.1.2) can be written as

$$X^2 - DY^2 = N, \tag{4.1.3}$$

where $X = Dy + E$.

We will examine several cases of the last equation see [2, 51].

1) If $D < 0$, then (4.1.3) can only have a finite number of solutions, and these can be determined by making use of the algorithm of Cornacchia (see, Nitaj [45]).

2) If $D > 0$ is a perfect integral square, we have the following cases:

   2.1) If $N = 0$, in this case we get an infinitude of solutions of (4.1.3), but they are very easily characterized.

   2.2) If $N \neq 0$, in this case the problem of solving (4.1.3) reduces to that of factoring N, once again the equation (4.1.3) have a finite number of solutions.

3) If $D > 0$ is not a perfect integral square, we have the following cases

   3.1) If $N = 0$, in this case the equation (4.1.3) has no solutions.

   3.2) If $N \neq 0$, in this case (4.1.3) has an infinitude of solutions, if it has at least one (The general Pell-type equation).

The case of $N \neq 0$, $D > 0$ and D not a perfect integral square. The authors of paper [51] they said "In this case (4.1.1) has an infinitude of solutions, if it has at least one." (Page 38. Line 20), It can simply be ensured that equations $32x^2 - y^2 + 7x - y = 0$, $512x^2 - y^2 + 23x - y = 0$, and $128x^2 - y^2 + 23x + 1 = 0$ do not have solutions. In this chapter, we try give Elementary proof of a diophantine equation (4.2.1).

4.2. The equation $4k^3 + 19k^2 + 28k + 13 = n^2$

The equation

$$(k+1)\left(4k^2 + 15k + 13\right) = n^2, \tag{4.2.1}$$

has a unique solution in positive integers : $(k,\ n) = (1,\ 8)$.

Proof. We have

$$\gcd\left(k+1,\ 4k^2 + 15k + 13\right) = 1 \ OR\ 2.$$

We distinguish the two cases: 1) If $\gcd(k+1,\ 4k^2+15k+13) = 1$, we obtain the systems

$$(S) : \begin{cases} k+1 = d^2, \\ 4k^2+15k+13 = r^2. \end{cases}$$

Observe that $k = d^2 - 1$, hence we have

$$\begin{aligned} 4k^2+15k+13 &= 4\left(d^2-1\right)^2 + 15\left(d^2-1\right) + 13 \\ &= 4d^4 + 7d^2 + 2. \end{aligned}$$

For all integers d, we find

$$d^2 \equiv 0 \pmod 8 \quad \text{or} \quad d^2 \equiv 1 \pmod 8 \quad \text{or} \quad d^2 \equiv 4 \pmod 8.$$

This implies

$$4d^4 + 7d^2 + 2 \equiv 2 \pmod 8 \quad \text{or} \quad 4d^4 + 7d^2 + 2 \equiv 5 \pmod 8$$
$$\text{or} \quad 4d^4 + 7d^2 + 2 \equiv 6 \pmod 8.$$

Since $4d^4 + 7d^2 + 2$ is not a perfect square, then $4k^2 + 15k + 13$ is not a perfect square. Therefore $(k+1)(4k^2+15k+13)$ is not a perfect square too. Finally, in this case the Equation (4.2.1) is not solvable in $\mathbb{Z} \times \mathbb{Z}$.

2) If $\gcd(k+1,\ 4k^2+15k+13) = 2$, then k is an odd integer and $k = 2m+1$, for a certain integer m. We get

$$(k+1)\left(4k^2+15k+13\right) = 4(m+1)\left(8m^2+23m+16\right).$$

This implies

$$\gcd\left(m+1,\ 8m^2+23m+16\right) = 1.$$

The following cases arise

2.1) The first case m an odd number.

Therefore there is an integer q such that $m = 2q-1$. So, have

$$4(m+1)\left(8m^2+23m+16\right) = 4(2q)\left(32q^2+14q+1\right).$$

Since, $(k+1)\left(4k^2+15k+13\right)$ a perfect square, is also $(2q)\left(32q^2+14q+1\right)$ a perfect square . Observe that

$$\gcd\left(2q,\ 32q^2+14q+1\right)=1,$$

hence both $\left(32q^2+14q+1\right)$ and $(2q)$ are perfect squares. Furthermore, $32q^2+14q+1$ is an odd number and $2q$ is an even one, then there are two integers a and b such that

$$(S_1):\begin{cases} 2q=(2a)^2, \\ 32q^2+14q+1=(2b+1)^2. \end{cases}$$

Now, since $q=2a^2$, hence we deduce

$$32\left(2a^2\right)^2+14\left(2a^2\right)+1=(2b+1)^2.$$

This is equivalent to

$$4\left(32a^4+7a^2\right)+1=4b^2+4b+1.$$

which is equivalent to

$$a^2\left(32a^2+7\right)=b(b+1). \tag{4.2.2}$$

The last equation is not solvable in $\mathbb{Z}^*\times\mathbb{Z}$ (except the trivial solution). Thus, the only possible solutions are for $a=0$ and $b=0,-1$, and in this case the our original equation Equation has the unique solution $(k=-1,\ n=0)$.

2.2) The second case m an even integer.

Let q be such that $m=2q$. So we find

$$4(m+1)\left(8m^2+23m+16\right)=4(2q+1)\left(32q^2+46q+16\right).$$

Since

$$(k+1)\left(4k^2+15k+13\right),$$

is a perfect square, so

$$(2q+1)\left(32q^2+46q+16\right),$$

is also a perfect square. Observe that

$$\gcd\left(2q+1,\ 32q^2+46q+16\right) = 1.$$

Hence both $(32q^2+46q+16)$ and $(2q+1)$ are perfect squares. Furthermore, $32q^2+46q+16$ is an even number and $2q+1$ is an odd one. So there are two integers $a,\ b$ such that

$$(S_2) : \begin{cases} 2q+1 = (2a+1)^2, \\ 32q^2+46q+16 = (2b)^2. \end{cases}$$

Since $q = 2a(a+1)$, hence we have

$$32(2a(a+1))^2 + 46(2a(a+1)) + 16 = (2b)^2.$$

The last equation could be written as

$$32a^4 + 64a^3 + 55a^2 + 23a + 4 = b^2.$$

Because $32a^4+64a^3+55a^2+23a+4$ is even, then there is an integer c such that, $b^2 = (2c)^2$. So we find

$$a(32a^3+64a^2+55a+23) = 4(c^2-1).$$

a) If $|a| \leq 4$. This equation has only the solution solution $a = 0,\ |c| = 1$. In this case the Equation (4.2.1) has a unique solution $(k = 1,\ |n| = 8)$.

b) If $|a| > 4$ and c even, i.e., $(gcd(c-1,\ c+1) = 1)$. So we get

$$a\left(32a^3+64a^2+55a+23\right) \equiv 0 \pmod 4.$$

We must have

$$a \equiv 0 \pmod 4 \ \text{or}\ a \equiv -1 \pmod 4.$$

b.1) If $a \equiv 0 \pmod 4$. Then, there is an integer $d \neq 0$ such that $a = 4d$. We get

$$4d\left(32(4d)^3 + 64(4d)^2 + 55(4d) + 23\right) = 4\,(c-1)\,(c+1)\,.$$

This is equivalent to

$$d\left(2048d^3+1024d^2+220d+23\right)=(c-1)(c+1).$$

Since, $(2048d^3+1024d^2+220d+23)$ and $(c-1)(c+1)$ odd number. We must have d an odd integer. Therefore there is integer $e\neq 0$ such that $d=2e-1$, we find

$$(2e-1)\left(2048(2e-1)^3+1024(2e-1)^2+220(2e-1)+23\right)=(c-1)(c+1).$$

The last equation could be written as

$$\left(16e^2-14e+3\right)\left(128\left(16e^2-14e+3\right)+23\right)=(c-1)(c+1). \tag{4.2.3}$$

This last equation is not solvable in $\mathbb{Z}\times\mathbb{Z}$.

b.2) If $a\equiv -1 \pmod 4$. There is an integer $d\neq 0$ such that $a=4d-1$. We get

$$(4d-1)\left(32(4d-1)^3+64(4d-1)^2+55(4d-1)+23\right)=4(c-1)(c+1).$$

We can write this equation as

$$\left(4d^2-d\right)\left(128\left(4d^2-d\right)+23\right)=(c-1)(c+1), \tag{4.2.4}$$

and this is not solvable in $\mathbb{Z}\times\mathbb{Z}$.

c) If $|a|>4$ and c odd, i.e., $(gcd(c-1,\ c+1)=2)$. We get

$$a\left(32a^3+64a^2+55a+23\right)\equiv 0 \pmod 8.$$

From this last equation, we must have

$$a\equiv 0 \pmod 8 \ \mathit{or}\ a\equiv -1 \pmod 8.$$

c.1) If $a\equiv 0 \pmod 8$. There is an integer $d\neq 0$ such that $a=8d-1$. We get

$$8d\left(32(8d)^3+64(8d)^2+55(8d)+23\right)=4\left(c-1\right)\left(c+1\right),$$

which is equivalent to

$$2d\left(16384d^3 + 4096d^2 + 440d + 23\right) = (c-1)(c+1).$$

Since, $(16384d^3 + 4096d^2 + 440d + 23)$ is an odd integer and $(c-1)\frac{(c+1)}{2}$ is an even one, the number d must be an even integer: $d = 2e$. So, we have

$$2(2e)\left(16384(2e)^3 + 4096(2e)^2 + 440(2e) + 23\right) = (c-1)(c+1).$$

The last equation could be written as

$$4\left(16e^2 + e\right)\left(512\left(16e^2 + e\right) + 23\right) = (c-1)(c+1).$$

Since, c is an odd number, write $c = 2b + 1$. So we get

$$\left(16e^2 + e\right)\left(512\left(16e^2 + e\right) + 23\right) = b(b+1). \tag{4.2.5}$$

The last equation is not solvable in $\mathbb{Z}^* \times \mathbb{Z}$.

c.2) If $a \equiv -1 \pmod 8$, there is an integer $d \neq 0$ such that $a = 8d - 1$. So

$$(8d-1)\left(32(8d-1)^3 + 64(8d-1)^2 + 55(8d-1) + 23\right) = 4(c-1)(c+1).$$

Rearrange and simplify to get

$$\left(8d^2 - d\right)\left(512\left(8d^2 - d\right) + 46\right) = (c-1)(c+1).$$

Since c is an odd integer, there is integer b such that $c = 2b + 1$. So we get

$$\left(8d^2 - d\right)\left(512\left(8d^2 - d\right) + 46\right) = 2b(2b+2),$$

and this is equivalent to

$$\left(8d^2 - d\right)\left(256\left(8d^2 - d\right) + 23\right) = 2b(b+1).$$

Since, $256\,(8d^2 - d) + 23$ is odd and $2b(b+1)$ is even number, $8d^2 - d$ must be even and then d is an even number. So write $d = 2e$, for an integer e. This yields

$$\left(8(2e)^2 - 2e\right)\left(256\left(8(2e)^2 - 2e\right) + 23\right) = 2b(b+1),$$

or which is the same:

$$\left(16e^2 - e\right)\left(512\left(16e^2 - e\right) + 23\right) = b(b+1). \tag{4.2.6}$$

The last equation is not solvable in $\mathbb{Z}^* \times \mathbb{Z}$. Thus, the set S of all solutions of this equation (4.2.1) in $\mathbb{Z} \times \mathbb{Z}$, is $S = \{(-1,\ 0),\ (1,\ -8),\ (1,\ 8)\}$.
The proof is finished.

□

Conclusion

The work of this thesis is essentially composed of two main parts. In the first part we proved Heim- Neuhauser conjecture, then we deduce some identities involving Fibonacci and Lucas numbers. Some of these identities seem to be new, others are easily proved using just the explicit formula of the Fibonacci polynomials.

The third chapter is devoted to a very detailed study of a unimodal sequence arising from the sequence considered in the previous chapter. The modes are determined accurately.

The last chapter supplies an elementary proof of the finiteness of the number of solutions of a diophantine equation.

The central result of the thesis is the proof of Heim -Neuhauser conjecture related to some polynomials. Basically, this study developed by Heim and Neuhauser is made to help understanding Lehmer conjecture, and eventually to settle it. So, the logical continuation of our future research will be in this direction, that is, we will consider other sequences of polynomials, by varying h and g in the relation

$$P_n^{g,h}(x) = \frac{x}{h(n)} \sum_{k=1}^{n} g(k) P_{n-k}^{g,h}(x), \quad P_0^{g,h}(x) = 1.$$

An other promising perspective is the use of the explicit formulas of Fibonacci and Lucas polynomials, to reprove old formulas for Fibonacci and Lucas numbers, or finding new ones, in the same vein as in [5].

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الملخص: قام هايم و نيوهاوزر بالتحقيق في بعض كثيرات الحدود المتعلقة بدالة ديديكيند. لقد أثبتوا التقعر اللوغاريتمي لهذه كثيرات الحدود وخمنوا أن لديهم أصفارا حقيقية فقط. في هذه الأطروحة، أثبتنا هذا التخمين، واستنتجنا بعض المتطابقات لأرقام فيبوناتشي، وحددنا المنوال لمتتاليات اخرى متعلقة بمتعدد حدود فيبوناتشي. لقد درسنا أيضا حلول بعض معادلات الديوفنتية.

الكلمات المفتاحية: معادلات ديوفنتية. التقعر اللوغاريتمي. كثير حدود. متتالية. أصفار.

Abstract: Heim and Neuhauser investigated some polynomials related to the Dedekind function. They proved the log-concavity of these polynomials and conjectured that they have only real zeros. In this thesis, we prove this conjecture, and deduce some identities for Fibonacci numbers and determine the modes of another sequence related to Fibonacci polynomials. We have also studied solution of some diophantine equations.

Keywords: Diophantine equations. Log–concave. Polynomial. Sequence. Zeros.

Résumé: Heim et Neuhauser ont étudié certains polynômes liés à la fonction Dedekind. Ils ont prouvé la log-concavité de ces polynômes et ont supposé qu'ils n'avaient que des zéros réels. Dans cette thèse, nous prouvons cette conjecture, en déduisons quelques identités pour les nombres de Fibonacci et déterminons les modes d'une autre séquence liée aux polynômes de Fibonacci. Nous avons également étudié la solution de certaines équations diophantienne.

Mots-clés: Équations diophantienne. Log–concave. Polynôme. Suite. Zéros.