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الإهداء

أهدي هذا العمل إلى :

أمي و أبي حفظهما الله و أطال في عمريهما

إخوتي و أخواتي

كل أصدقائي و أحبائي

كل من علمني حرفا

كل أعضاء مخبر التحليل الدالي و هندسة الفضاءات بكلية الرياضات و الإعلام الآلي جامعة المسيلة

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Contents

List of publications

1. A. Maamra, L. Mezrag and A. Tallab, *Lipschitz p-lattice summing operators*, Advances in Operator Theory. **6** (2021), 67.

2. D. Achour, A. Maamra, L. Mezrag, A. Tallab, *The Lipschitz $\sigma(p)$-nuclear operators and its dual*, Preparation.

Introduction

Farmer and Johnson (2009) [22] have introduced the notion of *Lipschitz p-summing* and *Lipschitz p-integral* maps between pointed metric spaces and Banach spaces. They proved that it is really a good generalization of the concept of linear p-summing operators [22, Theorem 2] and proved a nonlinear version of the domination /factorization theorem and also a non-linear variant of Grotendieck's theorem which is: if T is a Lipschitz operator from a metric tree X into a Hilbert space H; then $\pi_1^L(T) \leq K_G \text{Lip}(T)$, where K_G is the Grothendieck's constant. This class of Lipschitz operators marked the beginning of the theory of nonlinear summability. Motivated by the importance of this theory, several authors, have developed and studied many concepts relating to summability. Chen and Zheng introduced in [19] *(strongly) Lipschitz p-integral* and *p-nuclear operators*. In [15] Chávez-Domínguez introduced the notion of *Lipschitz* (r, p, q)*-summing operators* and *Lipschitz* (q, p)*-mixing* in [16]. Some papers in this domain are due to Yahi, Achour and Rueda [46] and Saadi [40]. Independently, they introduced and studied the class of *Lipschitz strongly p-summing operators*. Also, Achour et al. introduced (p, σ)*-absolutely Lipschitz operators* with an appropriate factorization in [8]. They characterized also those Lipschitz operators whose Lipschitz conjugates are absolutely p-summing. Other Lipschitz versions of different types of summability of linear operators were investigated also by many authors such as [18], [4], [41], [1] and [32].

The aim of this thesis is to generalize and to extend some other classes of linear operators to nonlinear operators (Lipschitz). Among these classes, we studied the class of p-lattice summing operators and the class of $\sigma(p)$-nuclear operators in the Lipschitz cases. These are the main axes of this thesis.

In [47], Yannovskii introduced and studied the notion of 1-lattice summing linear operators. Nielson and Szulga in [37] and [44], extended this notion to p ($1 \leq p \leq \infty$). It was extended later in [31], to the theory of operator spaces (or the non-commutative case) and to sublinear operators in [6]. In the present thesis, we generalize this class of linear operators to Lipschitz operators. Also, we examine some connections with other classes of operators cited previously. This is in a first part.

In a second part, we define and study a new concept of nuclear operators in the category of Lipschitz operators, which we call *Lipschitz $\sigma(p)$-nuclear operators* ($1 \leq p < \infty$). This concept was introduced and developed by Botelho and Mujica [11] in the linear and multilinear cases. Before this, Pietsch was introduced this notion (for $p = 1$) in his book [38]. We give some characterizations of this concept in term of factorization and we prove that this space of operators is a Banach Lipschitz ideal operators. We conclude by giving its dual.

This thesis is organized as follows.

In chapter 1, we recall some basic notions and terminologies which related to Banach space theory and we introduce the main definitions and properties of the Lipschitz operators theory that we will use later. Also, we give some basic classes of Lipschitz operators.

In chapter 2, we study the notion of Lipschitz p-lattice summing operators between a pointed metric space X and a Banach lattice E, which extends the class of p-lattice summing operators. This generalization is a natural nonlinear analogous of the class of p-lattice summing linear operators. We give some properties concerning this notion and we prove certain characterizations of this type of operators. Finally, we try to give some properties in connection with the category of Lipschitz p-summing operators.

In chapter 3, we introduce a new category of Lipschitz operators which called *Lipschitz σ-nuclear operators*. At the beginning we give a reminder about the classes of σ-nuclear operators. Then, we extend the last concept to Lipschitz case and we characterize this type of operators by introducing an equivalent definition and factorization theorem. Furthermore, we prove that the definition of Lipschitz σ-nuclear operators is a good generalization and this space of operators is Banach Lipschitz operator ideals. Finally, we give another class of Lipschitz operators which called *Lipschitz σ-integral operators* as generalization of σ-integral operators and its factorization theorem.

In the last chapter, motivated by our definition of Lipschitz σ-nuclear operators, we define and study the Lipschitz $\sigma(p)$-nuclear operators for $1 \leq p < \infty$. Firstly, we recall some basic definitions concerning "$\sigma(p)$-nuclear operators". We give some characterizations of this concept in term of factorization, where we presented two factorization theorems and related properties. Among them : we prove that this space of operators is a Banach Lipschitz operator ideals. We concluded this chapter by determining the duality of this space.

Chapter 1

Preliminaries

This chapter deals with the Banach spaces and Lipschitz operators. First, we recall some basic notions and terminologies which are related to Banach space theory, next, we introduce the main definitions and properties of the Lipschitz operator theory that we will use later. Finally, we give some basic classes of Lipschitz operators.

1.1 Banach space theory

1.1.1 Notations

Unless otherwise stated E, F, G and H will denote real Banach spaces. As customary, B_E will denote the closed unit ball of E and $\mathcal{L}(E, F)$ is the linear space of bounded linear maps from E to F. For $u \in \mathcal{L}(E, F)$, we have

$$\|u\| := \sup_{x \in B_E} \|u(x)\|.$$

If $F = \mathbb{R}$, denote $\mathcal{L}(E, \mathbb{R}) = E^*$ it is called the topological linear dual of E and its norm is given by

$$\|x^*\| := \sup_{x \in B_E} \|\langle x^*, x\rangle\|.$$

A linear operator $u : E \to F$ is an *isomorphism* if it is a continuous bijection whose inverse u^{-1} is also continuous. In such case the spaces E and F are said to be *isomorphic*. Such a mapping u is an *isometric isomorphism* when $\|u(x)\|_F = \|x\|_E$ for all $x \in E$.

Given $u \in \mathcal{L}(E,F)$, we say that *adjoint* of u the operator $u^* \in \mathcal{L}(F^*, E^*)$ defined by

$$u^*(y^*)(x) = y^*(u(x)),$$

for every $y^* \in F^*$ and $x \in E$. We have

$$\|u^*\| = \|u\|.$$

Let $1 \leq p < \infty$. We denote by $\ell_p(E)$ the set of all p-summable sequences in E with the natural norm

$$\|(a_i)_{i=1}^{\infty}\|_p = \left(\sum_{i=1}^{\infty} \|a_i\|^p\right)^{\frac{1}{p}},$$

for a sequence of vectors $(a_i)_i \subset E$.

Also, we denote by $\ell_p^{\omega}(E)$ the set of all weakly p-summable sequences in E with the norm

$$\|(a_i)_{i=1}^{\infty}\|_p^{\omega} = \omega_p((a_i)_{i=1}^{\infty}) = \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} |a^*(a_i)|^p\right)^{\frac{1}{p}},$$

for a sequence of vectors $(a_i)_i \subset E$.

A sequence $(a_i)_{i=1}^{\infty} \in \ell_p^{\omega}(E)$ is *unconditionally p-summable* in E if

$$\lim_{m\to\infty} \sup_{a^* \in B_{E^*}} \left(\sum_{i=m}^{\infty} |a^*(a_i)|^p\right)^{\frac{1}{p}} = 0.$$

We denote the set of all unconditionally p-summable sequences on E by $\ell_p^u(E)$.

Recall that, from [20], a sequence $(a_i)_{i=1}^{\infty}$ in E is called *unconditionally summable* if for each bijective $\eta : \mathbb{N} \to \mathbb{N}$ the series $\sum a_{\eta(i)}$ converges. This is easily seen to be equivalent to the convergence of the net of finite sums $\sum\limits_{i\in I} a_i$.

A. Defant and K. Floret in [20, Proposition 8.3] proved that $(a_i)_{i=1}^{\infty}$ is unconditionally 1-summable if and only if $(a_i)_{i=1}^{\infty}$ is unconditionally summable.

If $p = \infty$, we are restricted to the case of bounded sequences and in $\ell_\infty(E)$ we use the sup norm, i.e.,

$$\|(a_i)_{i=1}^{\infty}\|_\infty := \sup_{1\leq i\leq\infty} \|a_i\|.$$

Then the spaces $\ell_\infty(E)$ and $\ell_\infty^\omega(E)$ coincide and

$$\|(a_i)_{i=1}^\infty\|_\infty = \omega_\infty((a_i)_{i=1}^\infty)$$

for all $(a_i)_{i=1}^\infty \in \ell_\infty(E)$.

A sequence (e_i) with $e_i \neq 0$ is called a hyperorthogonal basis of the Banach space F if its linear span is dense and if $|\gamma_i| \leq |\theta_i|$ implies $\left\|\sum_{i=1}^n \gamma_i e_i\right\| \leq \left\|\sum_{i=1}^n \theta_i e_i\right\|$ for $n \geq 1$.

1.1.2 Basic classes of linear operators

Recall that, from [38], a linear operator $u \in \mathcal{L}(E, F)$ is said to have finite rank if $u(E)$ is a finite dimensional subspace of F. The class of all finite rank linear operators between Banach spaces is denoted by $\mathcal{L}_f(E, F)$. An operator has rank one if and only if has the form $x^* \otimes y : x \mapsto \langle x^*, x\rangle y$. Then if $u \in \mathcal{L}_f(E, F)$ we have

$$u = \sum_{i=1}^n x_i^* \otimes y_i,$$

where $(x_i^*)_{i=1}^n \subset E^*$ and $(y_i)_{i=1}^n \subset F$.

An operator $u \in \mathcal{L}(E, F)$ is said to be *approximable* if is a limit of a sequence of finite rank operators from E to F in the norm of $\mathcal{L}$. We denote by $\overline{\mathcal{L}_f(E, F)}$ the class of all approximable operators between E and F.

We say that an operator $u \in \mathcal{L}(E, F)$ is compact (weakly compact) operator, if the set $u(B_E)$ is relatively compact (weakly compact) subset of F and denote by $\mathcal{K}(E, F)$ and $\mathcal{W}(E, F)$ the classes of all compact and weakly compact operators from E to F, respectively.

Now, we recall Banach spaces, which have the approximation property.
Let $1 \leq \gamma < \infty$. We say that E has the *γ-bounded approximation property* if, for every compact subset $K \subset E$ and every $\varepsilon > 0$, there is a finite rank operator $u \in \mathcal{L}_f(E, E)$ such that

$$\|x - u(x)\| \leq \varepsilon,$$

for all $x \in K$ and $\|u\| \leq \gamma$.

A Banach space is said to have the *bounded approximation property* if it has *γ-bounded approximation property* for some γ.

A Banach space is said to have the *metric approximation property* if it has 1-*bounded approximation property*.

Definition 1.1.1 (Operator ideals). *An operator ideal $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its component*

$$\mathcal{I}(E,F) := \mathcal{L}(E,F) \cap \mathcal{I}$$

satisfy

(i) $\mathcal{I}(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$ which contains the mappings of the form $x^* \otimes y$ where $x^* \in E^*$ and $y \in F$.

(ii) The ideal property: if $v \in \mathcal{L}(H,E), u \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,G)$, then the composition $w \circ u \circ v \in \mathcal{I}(H,G)$.

An operator ideal $\mathcal{I}$ is a normed (Banach) operator ideal if there is an application $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ which satisfies

(i') $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) spaces for all Banach spaces E and F.

(ii') $\|id_{\mathbb{R}}\|_{\mathcal{I}} = 1$.

(iii') If $v \in \mathcal{L}(H,E), u \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,G)$, then

$$\|w \circ u \circ v\|_{\mathcal{I}} \leq \|w\| \|u\|_{\mathcal{I}} \|v\|.$$

The ideal $\mathcal{L}_f$ of finite rank linear operators is the smallest operator ideal and $\mathcal{L}$ is the largest one (see [38]).
The operator ideal $\mathcal{I}$ is said to be closed if each $\mathcal{I}(E,F)$ is closed for the sup norm.

Proposition 1.1.2 ([38]). *The classes $\mathcal{K}$ and $\mathcal{W}$ are closed Banach operator ideals, where the ideal norm is the operator norm.*

We now recall some definitions of summability in the linear case.

Let $1 \leq p < \infty$. We say that an operator $u \in \mathcal{L}(E,F)$ is p-summing if there is a constant $C \geq 0$ such that for finite sequence $(a_i)_{1 \leq i \leq n} \subset E$ we have

$$\left(\sum_{i=1}^{n} \|u(a_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{n} |a^*(a_i)|^p \right)^{\frac{1}{p}}. \quad (1.1)$$

The space of all p-summing operators from E into F is denoted by $\Pi_p(E,F)$, which is a Banach space under the norm

$$\pi_p(u) := \inf\{C \ : \ C \ \text{satisfies (1.1)}\}.$$

Recall that from [35], an operator $u \in \mathcal{L}(E,F)$ is absolutely $\tau(p)$-summing if there exists a positive constant C such that, for all $n \in \mathbb{N}$, $(a_i)_{1\le i\le n} \subset E$ and $(b_i^*)_{1\le i\le n} \subset F^*$, we have

$$\left(\sum_{i=1}^{n} |\langle u(a_i), b_i^*\rangle|^p\right)^{\frac{1}{p}} \le C \sup_{\substack{a^*\in B_{E^*} \\ b\in B_F}} \left(\sum_{i=1}^{n} |\langle a^*, a_i\rangle \langle b, b_i^*\rangle|^p\right)^{\frac{1}{p}}. \tag{1.2}$$

We denote by $\Pi_{\tau(p)}(E,F)$ the vector space of all $\tau(p)$-summing operators u from E into F, which is a Banach space if we consider the norm $\pi_{\tau(p)}(u)$, the infimum of all C verifying Inequality (1.2).

By [11], an operator $u \in \mathcal{L}(E,F^*)$ is said absolutely quasi-$\tau(p)$-summing if there exists a positive constant C such that, for all $n \in \mathbb{N}$, $(a_i)_{1\le i\le n} \subset E$ and $(b_i)_{1\le i\le n} \subset F$, we have

$$\left(\sum_{i=1}^{n} |\langle u(a_i), b_i\rangle|^p\right)^{\frac{1}{p}} \le C \sup_{\substack{a^*\in B_{E^*} \\ b^*\in B_{F^*}}} \left(\sum_{i=1}^{n} |\langle a^*, a_i\rangle \langle b^*, b_i\rangle|^p\right)^{\frac{1}{p}}. \tag{1.3}$$

We denote by $\Pi_{q\tau(p)}(E,F^*)$ the vector space of all quasi-$\tau(p)$-summing operators u from E into F^*, which is a Banach space if we consider the norm $\pi_{q\tau(p)}(u)$, the infimum of all C verifying Inequality (1.3).

It is straightforward that $\Pi_{\tau(p)}(E,F^*) \subseteq \Pi_{q\tau(p)}(E,F^*)$ with $\pi_{q\tau(p)}(T) \le \pi_{\tau(p)}(T)$ for every F and $\Pi_{\tau(p)}(E,F^*) = \Pi_{q\tau(p)}(E,F^*)$ isometrically for F reflexive, see [35].

1.1.3 Banach lattice spaces

We recall now some standard notations and some properties concerning Banach lattices.

Definition 1.1.3. *A partially ordered Banach space E over the reals is called a Banach lattice provided*

(i) $x \le y$ *implies* $x+z \le y+z$, *for every* $x,y,z \in E$.

(ii) $\alpha x \ge 0$, *for every* $x \ge 0$ *in* E *and every non negative real* α.

(iii) For all $x,y \in E$ *there are a least upper bound* $x \vee y := \sup\{x,y\}$ *and a greatest lower bound* $x \wedge y := -\sup\{-x,-y\}$.

(iv) $\|x\| \le \|y\|$ *whenever* $|x| \le |y|$, *where the absolute value* $|x|$ *of* $x \in E$ *is defined by* $|x| := \sup\{x,-x\}$.

Definition 1.1.4. *Let E, F be two Banach lattices. A linear operator $u : E \longrightarrow F$ is called positive ($u \geq 0$) if*

$$|u(a)| \leq u(|a|) \text{ for all } a \in E.$$

We denote by $\mathcal{L}_+(E, F)$ the set of all bounded positive operators.

Let n be a natural number. For a Banach lattice E and $1 \leq p \leq \infty$, we let $E\left(l_p^n\right)$ the space of sequences $a = (a_1, ..., a_n) \in E^n$ for which

$$\|a\|_{E\left(l_p^n\right)} = \left\| \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \right\|, \quad \text{if } 1 \leq p \leq \infty$$

and

$$\|a\|_{E\left(l_\infty^n\right)} = \left\| \max_{1 \leq i \leq n} |a_i| \right\|, \quad \text{if } p = \infty.$$

The space $E\left(l_p^n\right)$ equipped with the natural order

$$a \leq b \Longleftrightarrow a_i \leq b_i, \quad \text{for all } 1 \leq i \leq n$$

is a Banach lattice.

Let E be a Banach lattice then E^* is a complete Banach lattice w.r.t the ordering $\leq$ defined by

$$a^* \geq 0 \Longleftrightarrow \langle a^*, a \rangle \geq 0 \text{ for all } a \in E_+. \tag{1.4}$$

1.2 The space of Lipschitz operators

Unless otherwise stated X, Y, Z will always denote pointed metric spaces, i.e., with a distinguished element at all time denoted by 0.

The space $\text{Lip}_0(X, E)$ is the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm $\text{Lip}(.)$ given by

$$\text{Lip}(T) = \sup_{x \neq y} \left\{ \frac{\|T(x) - T(y)\|}{d(x, y)} \right\}.$$

We denote the Lipschitz dual of X by $X^\# = \text{Lip}_0(X) = \text{Lip}_0(X, \mathbb{R})$. The closed unit ball $B_{X^\#}$ of $X^\#$ is a compact Hausdorff space for the topology of pointwise convergence on X.

For a sequence $(\lambda_i)_i$ of real numbers and $(x_i)_i$, $(y_i)_i$ of points in a metric space X, we denote their "*weak Lipschitz p-norm*" by

$$\omega_p^{\mathrm{L}}\left((\lambda_i), (x_i), (y_i)\right) = \sup_{f \in B_{X^\#}} \left\|(\lambda_i \left(f(x_i) - f(y_i)\right))_i\right\|_p .$$

1.2.1 Arens Eells space

A molecule on X is a real valued function m on X with finite support and satisfying

$$\sum_{x \in \mathrm{supp}(m)} m(x) = 0.$$

Denote by $\mathcal{M}(X)$ the real linear space of molecules on X. The condition $\sum_{i=1}^{n} m(x_i) = 0$ ensures that m can be represented by

$$m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i},$$

where $m_{x_i y_i} = \mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}$ ($\mathbf{1}_{\{x\}}$ is the indicator function of the subset $\{x\}$ of X). Put

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{i=1}^{n} |\alpha_i| \, d(x_i, y_i)\right\},$$

where the infimum is taken over all representations of $m = \sum_{i=1}^{n} \alpha_i \left(\mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}\right)$. It follows that $\|.\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}(X)$.

Denote by Æ(X) the completion of the normed space $(\mathcal{M}(X), \|\cdot\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells in 1956 [9]. Originally, the basic idea goes back to Kantorovich [26]. The terminology Arens-Eells space Æ(X) is due to Weaver [45]. The application $i_X : X \longrightarrow$ Æ(X) defined by

$$i_X(x) = m_{x0} = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{0\}} \tag{1.5}$$

is an isometric embedding of X into Æ(X).

1.2.2 Properties

For more details on the basic theory of the spaces of Lipschitz operators and their preduals, one can consult for example the papers of Godefroy and Kalton [25] or Weaver [45].

The following theorem due to [45, Theorem 2.2] is known as the linearization of Lipschitz operators.

Theorem 1.2.1. *Let $T \in \text{Lip}_0(X,E)$. Then there is a unique bounded linear operator $T_{\text{L}} \in \mathcal{L}(\text{Æ}(X), E)$ such that $T = T_{\text{L}} \circ i_X$ and $\|T_{\text{L}}\| = \text{Lip}(T)$ ($i_X : X \longrightarrow \text{Æ}(X)$ is defined as in (1.5)).*

The linear operator T_{L} is called the linearization of T. We know from the above that every molecule m is uniquely expressible in the form

$$m = \sum_{i=1}^{n} \alpha_i m_{x_i 0}$$

where the points x_i are all distinct and none equals to 0. Then, T_{L} is defined by

$$T_{\text{L}}(m) = \sum_{i=1}^{n} \alpha_i T(x_i). \tag{1.6}$$

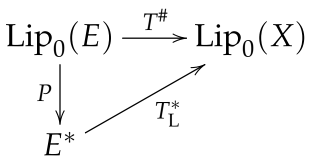
The dual E^* of E is naturally isometric to a subspace of $\text{Lip}_0(E)$. By [10, p. 37], there is a norm one projection $P : \text{Lip}_0(E) \longrightarrow E^*$. Sawashima in [42] defined the Lipschitz adjoint (or dual) $T^\# : \text{Lip}_0(E) \longrightarrow \text{Lip}_0(X)$ of a Lipschitz map $T \in \text{Lip}_0(X,E)$ by the formula

$$T^\#(f) = f \circ T, \qquad f \in \text{Lip}_0(E).$$

He showed that $T^\#$ is a continuous linear operator and

$$\|T^\#\| = \text{Lip}(T) = \|T^\# |_{E^*}\|.$$

We notice that $T^t = T^\# |_{E^*}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by Theorem 1.2.1, i.e., $T^\# |_{E^*} \cong T_{\text{L}}^*$.



1.3 Some class of Lipschitz operators

Definition 1.3.1 (Composition ideals). *Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in \mathrm{Lip}_0(X, E)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ \mathrm{Lip}_0$, denoted $T \in \mathcal{I} \circ \mathrm{Lip}_0(X, E)$, if there are a Banach space F, a Lipschitz operator $S \in \mathrm{Lip}_0(X, F)$ and an operator $u \in \mathcal{I}(F, E)$ such that $T = u \circ S$. If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal we write $\|T\|_{\mathcal{I}\circ\mathrm{Lip}_0} = \inf \|u\|_{\mathcal{I}}\mathrm{Lip}(S)$, where the infimum is taken over all u, S as above.*

The following proposition and corollary are mentioned in [7].

Proposition 1.3.2. *Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in \mathrm{Lip}_0(X, E)$:*

1. *$T \in \mathcal{I} \circ \mathrm{Lip}_0(X, E)$.*

2. *$T_L \in \mathcal{I}(Æ(X), E)$.*

If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I}\circ\mathrm{Lip}_0} = \|T_L\|_{\mathcal{I}}.$$

Corollary 1.3.3. *If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ \mathrm{Lip}_0$ is a (respectively normed, closed, Banach) Lipschitz operator ideal.*

1.3.1 Lipschitz p-summing operators

Inspired by the useful concept of absolutely summing operators, Farmer and Johnson introduced in [22] the following definition: a Lipschitz map $T : X \to E$ is called Lipschitz p-summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $n \in \mathbb{N}$ and all $x_1, ..., x_n, y_1, ..., y_n \in X$ and all non negative reals $\lambda_1, ..., \lambda_n$ we have

$$\left\| \left(\lambda_i^{\frac{1}{p}} \left(T\left(x_i\right) - T\left(y_i\right) \right) \right) \right\|_p \leq C \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right)_i, (x_i)_i, (y_i)_i \right). \tag{1.7}$$

The infimum of such constants verifying (1.7) is denoted by $\pi_p^L(T)$. This is a true generalization of the concept of linear p-summing operators, since it is shown in [22, Theorem 2] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. In the sequel, it will be useful to note that the above definition is the same if we restrict to $\lambda_i = 1$ (see [22] for an implicit proof).

We shall give another equivalent definition in terms of Lipschitz tensor product. This notion of tensor product between metric and Banach spaces; which is a nonlinear generalization of linear tensor product, was introduced by Cabrera-Padilla et al. in [13]. The Lipschitz tensor product $X \boxtimes E$ is defined as the vector subspace of $\mathrm{Lip}_0(X, E^*)'$ spanned by the set

$$\left\{\delta_{(x,y)} \boxtimes a : (x,y) \in X^2,\ a \in E\right\}$$

where $\left(\delta_{(x,y)} \boxtimes a\right)(g) = \langle g(x) - g(y), a\rangle$, for any $g \in \mathrm{Lip}_0(X, E^*)$ (whither $\delta_{(x,y)}(g) = g(x) - g(y)$). The Lipschitz injective norm on $X \boxtimes E$ is defined for $u = \sum_{i=1}^{n} \delta_{(x_i,y_i)} \boxtimes a_i$ by

$$\epsilon(u) = \sup\left\{\left|\sum_{i=1}^{n} (f(x_i) - f(y_i)) \langle \xi, a_i\rangle\right| : f \in B_{X^\#}, \xi \in E^*\right\}.$$

Remark 1.3.4. *A Lipschitz map $T : X \to E$ is Lipschitz p-summing ($1 \leq p < \infty$) if there exists a positive constant C such that for all $n \in \mathbb{N}$ and for all $x_1, ..., x_n,\ y_1, ..., y_n \in X$, we have for all linear operator $v : \ell_{p^*}^n \longrightarrow Æ(X)$, $v(e_i) = \mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}$ (namely $v = \sum_{i=1}^{n} \left(\mathbf{1}_{\{x_i\}} - \mathbf{1}_{\{y_i\}}\right) \boxtimes a_i \in X \boxtimes \ell_p^n$)*

$$\left(\sum_{i=1}^{n} \|T_{\mathrm{L}}(v(e_i))\|^p\right)^{\frac{1}{p}} \leq C \|v\|$$

where e_i denotes the unit vector basis of $\ell_{p^*}^n$ and p^* denotes the exponent conjugate to p (i.e., the one that satisfies $1/p + 1/p^* = 1$).

We have

$$\begin{aligned}
\epsilon(v) &= \sup\left\{\left|\sum_{i=1}^{n} (f(x_i) - f(y_i)) \langle \xi, a_i\rangle\right| : f \in B_{X^\#},\ \xi \in \ell_{p^*}^n\right\} \\
&= \sup\left\{\left|\sum_{i=1}^{n} \langle \xi, (f(x_i) - f(y_i))\, a_i\rangle\right| : f \in B_{X^\#},\ \xi \in \ell_{p^*}^n\right\} \\
&= \sup\left\{\left|\left\langle \xi, \sum_{i=1}^{n} (f(x_i) - f(y_i))\, a_i\right\rangle\right| : f \in B_{X^\#},\ \xi \in \ell_{p^*}^n\right\} \\
&= \sup\left\{\left(\sum_{i=1}^{n} |(f(x_i) - f(y_i))|^p\right)^{\frac{1}{p}} : f \in B_{X^\#}.\right\}
\end{aligned}$$

$$\|v\| = \epsilon(v) = \sup_{\|f\|_{X^\#}=1} \left(\sum_{i=1}^{n} |f(x_i) - f(y_i)|^p\right)^{\frac{1}{p}}. \tag{1.8}$$

Remark 1.3.4 is equivalent to

$$\left(\sum_{i=1}^{n}\|T(x_i)-T(y_i)\|^p\right)^{\frac{1}{p}}\leq C\|w\|$$

where $w : X^{\#} \longrightarrow \ell_p^n$ be the linear operator defined by

$$w(f)=(f(x_i)-f(y_i))_{1\leq i\leq n}.$$

1.3.2 Lipschitz compact and weakly compact operators

We recall now the following, as announced in [23]. The Lipschitz image of a mapping $T : X \to E$ is the set

$$\left\{\frac{T(x)-T(y)}{d(x,y)} : x,y \in X,\ x \neq y\right\}.$$

An immediate consequence, $T : X \to E$ is a Lipschitz mapping if its Lipschitz image is a bounded subset of E.

Definition 1.3.5 ([23]). *Let $T \in \mathrm{Lip}_0(X,E)$. We say that T is Lipschitz compact (Lipschitz weakly compact) if its Lipschitz image is relatively compact (resp, relatively weakly compact) in E.*

We denote by $\mathrm{Lip}_{0\mathcal{K}}(X,E)$ and $\mathrm{Lip}_{0\mathcal{W}}(X,E)$ the sets of Lipschitz compact operators and Lipschitz weakly compact operators from X to E, respectively.

Remark 1.3.6.

1. *Note that $\mathrm{Lip}_{0\mathcal{K}}(X,E)$ and $\mathrm{Lip}_{0\mathcal{W}}(X,E)$ are linear subspaces of $\mathrm{Lip}_0(X,E)$.*

2. *If X is a Banach space and $T \in \mathcal{L}(X,E)$ is a compact operator (resp. weakly compact operator), then T is a Lipschitz compact operator (resp. Lipschitz weakly compact operator) since the Lipschitz image of T is justly $T(S_X)$.*

The following result is proved in [23, Propositions 2.1 and 2.2].

Proposition 1.3.7. *Let $T \in \mathrm{Lip}_0(X,E)$ and T_L its linearization (as in Theorem 1.2.1), then T is Lipschitz compact (resp. Lipschitz weakly compact) if and only if T_L is compact (resp. weakly compact).*

1.3.3 Lipschitz finite-rank and approximable operators

Let us recall that if M is a set and E is a linear space, then a map $f : M \to E$ is said to have finite dimensional rank if the linear hull of its image is a finite dimensional subspace of E in whose case the rank of f, denoted by $rank(f)$, is defined as the dimension of $lin(f(M))$, where $lin(f(M))$ is a linear hull of $f(M)$.

The following definition was introduced by Vargas et al. in [23].

Definition 1.3.8. *Let X be a pointed metric space and E a Banach space. A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ has Lipschitz finite dimensional rank if the linear hull of its Lipschitz image is a finite dimensional subspace of E. In that case we define the Lipschitz rank $Lrank(T)$ of T to be the dimension of this subspace.*

Proposition 1.3.9. *Consider $T \in \mathrm{Lip}_0(X, E)$. The following are equivalent :*

(i) T has finite dimensional Lipschitz rank.

(ii) T has finite dimensional rank.

(i) T_L has finite rank.

In that case, $lin(T(X)) = T_L(Æ(X))$ and $Lrank(T) = rank(T) = rank(T_L)$.

We denote by $\mathrm{Lip}_{0\mathcal{F}}(X, E)$ the set of all Lipschitz finite-rank operators from X to E. Note that $\mathrm{Lip}_{0\mathcal{F}}(X, E)$ is a linear subspace of $\mathrm{Lip}_0(X, E)$.

Definition 1.3.10. *A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ is said to be approximable if it is the limit of a sequence of Lipschitz finite rank operators from X to E in the Lipschitz norm.*

We denote by $\overline{\mathrm{Lip}_{0\mathcal{F}}(X, E)}$ the class of all approximable Lipschitz operators $T \in \mathrm{Lip}_0(X, E)$ which is a Lipschitz ideal (see [7]).

Note that, if X is Banach space and T is linear operator. T is in $\overline{\mathrm{Lip}_{0\mathcal{F}}(X, E)}$ if and only if T is in $\overline{\mathcal{L}_f(X, E)}$ (where $\overline{\mathcal{L}_f(X, E)}$ the space of all approximable operators from X to E).

The following result is due to [7, Corollary 2.4].

Proposition 1.3.11. *$T \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X, E)}$ if and only if $T_L \in \overline{\mathcal{L}_f(Æ(X), E)}$.*

Chapter 2

Lipschitz p-lattice summing operators

In this chapter, we study the notion of Lipschitz p-lattice summing operators between a pointed metric space X and a Banach lattice E, which extends the class of p-lattice summing operators. This generalization is a natural nonlinear analogous of the class of p-lattice summing linear operators. We give some properties concerning this notion and we prove certain characterizations of this type of operators. Finally, we try to give some properties in connection with the category of Lipschitz p-summing operators.

2.1 Some results of p-lattice summing operators

2.1.1 Definitions

The following definition was introduced and studied in linear case by Yanovskii in [47] for $p = 1$ and generalized by Nielsen and Szulga in [37] and [44] for $p > 1$.

Definition 2.1.1. *A linear operator T from a Banach space X to a Banach lattice E is called p-lattice summing (for $1 \leq p \leq \infty$) if there is a positive constant C so that for all finite sets $\{x_1, x_2, ..., x_n\} \subset X$ we have*

$$\left\| \left(\sum_{i=1}^{n} |T(x_i)|^p \right)^{\frac{1}{p}} \right\| \leq C \sup_{x^* \in \mathcal{B}_{X^*}} \left(\sum_{i=1}^{n} |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}$$

if p is finite and

$$\left\| \sup_{1 \leq i \leq n} |T(x_i)| \right\| \leq C \sup_{1 \leq i \leq n} \|x_i\|$$

if p is infinite.

If T is p-lattice summing we denote by $\lambda_p(T)$ the smallest constant C, which is a norm on the space $\Lambda_p(X,E)$ of all p-lattice summing operators from X to E turning it into a Banach space.

Nielsen and Szulga in [37] also give us the following definition.

Definition 2.1.2. *Consider X is a Banach space and E is a Banach lattice then a bounded linear operator $T : X \longrightarrow E$ is called order bounded if there exists an element $a \in E_+$ such that*

$$|T(x)| \leq a\|x\|, \quad \text{for all } x \in X. \tag{2.1}$$

In this case, we define the order bounded norm $\|T\|_m$ by

$$\|T\|_m = \inf\{\|a\| : a \text{ satisfies } (2.1)\}$$

which is by [36] a complete norm on $\mathfrak{B}(X,E)$, the space of all order bounded operators from X into E. By Schaefer [43], the m-tensor product $X\widehat{\otimes}_m E$ is defined by the closure of $X \otimes E$ in $\mathfrak{B}(X^*,E)$.

2.1.2 Properties

The proof of the following proposition is immediate so we omit it.

Proposition 2.1.3. *$T \in \Lambda_\infty(X,E)$ if and only if T is order bounded considered as a map into E^{**}, and in that case $\lambda_\infty(T) = \|T\|_m$.*

The following results are due [37].

Theorem 2.1.4. *If $1 \leq p \leq \infty$, then T in $\Lambda_p(X,E)$ if and only if TS in $\Lambda_\infty(\ell_{p^*},E)$ for all S in $\mathcal{L}(\ell_{p^*},X)$ and in that case*

$$\lambda_p(T) = \sup\{\|TS\|_m, \|S\| \leq 1\}.$$

Theorem 2.1.5. *Let $p = 1,2$, then T in $\Lambda_p(X,E)$ if and only if for every measure μ and every positive operator $S \in \mathcal{L}_+(E, L_1(\mu))$ ST is p-summing. Further*

$$\lambda_p(T) = \sup\{\pi_p(ST), \|S\| \leq 1\}.$$

Remark 2.1.6 ([37]). *Theorem 2.1.5 is not true for $1 < p < 2$.*

Theorem 2.1.7 ([37]). *(i) If T in $\Lambda_\infty(X,E)$ then T in $\Lambda_p(X,E)$ for all $1 \le p < \infty$, and we have*

$$\lambda_p(T) \le C\lambda_\infty(T), \qquad 0 < C \le 1.$$

(ii) If T in $\Lambda_p(X,E)$ then T in $\Lambda_q(X,E)$ for all $1 \le p \le q \le 2$, and we have

$$\lambda_q(T) \le C\lambda_p(T), \qquad 0 < C \le 1.$$

(iii) If T in $\Lambda_p(X,E)$ then T in $\Lambda_2(X,E)$ for all $1 \le p \le \infty$, and we have

$$\lambda_2(T) \le C\lambda_p(T), \qquad 0 < C \le 1.$$

(iv) If T in $\Pi_p(X,E)$ then T in $\Lambda_2(X,E)$ for all $1 \le p < \infty$, and we have

$$\lambda_2(T) \le C\pi_p(T), \qquad 0 < C \le 1.$$

2.2 Lipschitz p-lattice summing operators

In this section we extend the notion of p-lattice summing operators and some results to the theory of Lipschitz operators.

The following definition is due to [17], we can see also [2] for Lipschitz p-convex operators.

Definition 2.2.1. *Let X be a arbitrary pointed metric space and let E be a Banach lattice. Consider $1 \le p \le \infty$.*

(i) A Lipschitz map $T : X \longrightarrow E$ is called Lipschitz p-convex if there exists a positive constant C such that for every $n \in \mathbb{N}$, for all $x_1, ..., x_n;\ y_1, ..., y_n \in X$ and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left\| \left(\lambda_i^{\frac{1}{p}} \left(T\left(x_i\right) - T\left(y_i\right) \right) \right) \right\|_{E\left(\ell_p^n\right)} \le C \left(\sum_{i=1}^{n} \lambda_i d\left(x_i, y_i\right)^p \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and

$$\left\| \sup_{1\leq i\leq n} \lambda_i \left| T(x_i) - T(y_i) \right| \right\| \leq C \sup_{1\leq i\leq n} \lambda_i d(x_i, y_i)$$

for $p = \infty$.

(ii) A Lipschitz map $T : E \longrightarrow X$ *is called Lipschitz* p*-concave if there exists a positive constant* C *such that for every* $n \in \mathbb{N}$, *for all* $a_1, ..., a_n; b_1, ..., b_n \in E$ *and all non negative reals* $\lambda_1, ..., \lambda_n$, *we have*

$$\left(\sum_{i=1}^{n} \lambda_i d\left(T(a_i), T(b_i)\right)^p \right)^{\frac{1}{p}} \leq C \left\| \left(\lambda_i^{\frac{1}{p}} (a_i - b_i) \right) \right\|_{E\left(\ell_p^n\right)}$$

for $1 \leq p < \infty$ *and*

$$\sup_{1\leq i\leq n} \lambda_i d\left(T(a_i), T(b_i)\right) \leq C \left\| \sup_{1\leq i\leq n} \lambda_i \left| a_i - b_i \right| \right\|$$

for $p = \infty$.

The smallest such constant C is called the Lipschitz p-convexity constant of T (resp. Lipschitz p-concavity constant of T) and is denoted by $C_p^{\mathrm{L}}(T)$ (resp. $M_p^{\mathrm{L}}(T)$). We shall also denote by $C_p^{\mathrm{L}}(X, E)$ and $M_p^{\mathrm{L}}(E, X)$ the sets of Lipschitz p-convex and Lipschitz p-concave operators respectively.

Note that this is a generalization of the linear case: a linear map $T : X \to E$ from a Banach space X to a Banach lattice E is p-convex if and only if, it is Lipschitz p-convex with the same constant. The same for Lipschitz p-concave. Because the unit ball $B_{X^\#}$ is not involved.

We know also in [17, Theorem 3.3] that a Lipschitz map $T : X \to E$ is Lipschitz p-convex if and only if, T_L :Æ$(X) \to E$ is p-convex. Moreover, in this case the p-convexity constants are the same.

If X is a Banach space, then $C_p(X) = C_p(\mathrm{Id}_X)$ and $M_p(X) = M_p(\mathrm{Id}_X)$.

Definition 2.2.2. *An operator* $T : X \longrightarrow E$ *is Lipschitz* p*-lattice (or order) summing if, there is a positive constant* C *such that for every* n *in* $\mathbb{N}$, *for all* $x_1, ..., x_n;\ y_1, ..., y_n$ *in* X *and all non negative reals* $\lambda_1, ..., \lambda_n$, *we have*

$$\left\| \left(\lambda_i^{\frac{1}{p}} \left(T(x_i) - T(y_i)\right) \right)_i \right\|_{E\left(\ell_p^n\right)} \leq C \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right)_i, (x_i)_i, (y_i)_i \right) \tag{2.2}$$

if p is finite and

$$\left\| \sup_{1\leq i\leq n} \lambda_i \left| T(x_i) - T(y_i) \right| \right\| \leq C \sup_{1\leq i\leq n} \lambda_i d(x_i, y_i) \tag{2.3}$$

if p is infinite.

We denote by $\Lambda_p^L(X, E)$ the vector space of all Lipschitz p-lattice summing operators T from X into E, which is a Banach space if we consider the norm $\lambda_p^{\mathrm{L}}(T)$, the infimum of all C verifying (2.2) and (2.3).

Remark 2.2.3. *If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then T is Lipschitz p-convex and $C_p^{\mathrm{L}}(T) \leq \lambda_p^{\mathrm{L}}(T)$ for p finite and $C_\infty^{\mathrm{L}}(T) = \lambda_\infty^{\mathrm{L}}(T)$.*

Remark 2.2.4. *If X is a Banach space and T is a bounded linear operator from X into E, then if T is p-lattice summing then T is Lipschitz p-lattice summing and $\lambda_p^{\mathrm{L}}(T) \leq \lambda_p(T)$ for all $1 \leq p < \infty$. If p is infinite, we have $\lambda_\infty^{\mathrm{L}}(T) = \lambda_\infty(T) = C_\infty(T)$.*
We do not know if the converse is true, because the unit ball $B_{X^\#}$ is involved and we do not have a factorization theorem.

Proposition 2.2.5. *Consider T in $\mathrm{Lip}_0(X, E)$, R in $\mathrm{Lip}_0(Z, X)$ and φ in $\mathcal{L}(E, F)_+$. If T is Lipschitz p-lattice summing, then $\varphi \circ T \circ R$ is Lipschitz p-lattice summing ($1 \leq p \leq \infty$) and*

$$\lambda_p^{\mathrm{L}}(\varphi \circ T \circ R) \leq \|\varphi\| \, \lambda_p^{\mathrm{L}}(T) \mathrm{Lip}(R).$$

Proof. Let $n \in \mathbb{N}$, $(z_i)_{1\leq i\leq n}$; $\left(z_i'\right)_{1\leq i\leq n} \subset Z$ and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$. Thus

$$\begin{aligned}
& \left\| \left(\lambda_i^{\frac{1}{p}} \varphi \circ T \circ R(z_i) - \varphi \circ T \circ R(z_i') \right) \right\|_{F(\ell_p^n)} \\
= \; & \left\| \left(\lambda_i^{\frac{1}{p}} \varphi \left(T \circ R(z_i) - T \circ R(z_i') \right) \right) \right\|_{F(\ell_p^n)} \\
\text{(by [29, p. 55])} \quad \leq \; & \|\varphi\| \left\| \left(\lambda_i^{\frac{1}{p}} \left(T \circ R(z_i) - T \circ R(z_i') \right) \right) \right\|_{E(\ell_p^n)} \\
\leq \; & \|\varphi\| \, \lambda_p^{\mathrm{L}}(T) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left| \left(f(R(z_i)) - f(R(z_i')) \right) \right|^p \right)^{\frac{1}{p}}. \\
\leq \; & \|\varphi\| \, \lambda_p^{\mathrm{L}}(T) \mathrm{Lip}(R) \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n \lambda_i \left| \frac{f \circ R(z_i)}{\mathrm{Lip}(R)} - \frac{f \circ R(z_i')}{\mathrm{Lip}(R)} \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\frac{f\circ R}{\mathrm{Lip}\,(R)} \in B_{Z^\#}$ we obtain

$$\begin{aligned} &\left\| \left(\sum_{i=1}^{n} \lambda_i \left| \varphi\circ T\circ R\,(z_i) - \varphi\circ T\circ R\left(z_i^{'}\right)\right|^p \right)^{\frac{1}{p}} \right\|_F \\ \leq\ & \|\varphi\|\, \lambda_p^L(T)\mathrm{Lip}\,(R) \sup_{\|g\|_{Z^\#}\leq 1} \left(\sum_{i=1}^{n} \lambda_i \left| g\,(z_i) - g\left(z_i^{'}\right)\right|^p \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that $\varphi\circ T\circ R$ is Lipschitz p-lattice summing with

$$\lambda_p^{\mathrm{L}}(\varphi\circ T\circ R) \leq \|\varphi\|\, \lambda_p^{\mathrm{L}}(T)\mathrm{Lip}\,(R)$$

and this ends the proof. □

Remark 2.2.6. *Clearly the notion of Lipschitz p-lattice summing operators is not an ideal in Pietsch's sense but it is an ideal on left.*

By Proposition 2.2.5, we have the following proposition. For the converse, we can see [40, Remark 3.3].

Proposition 2.2.7. *Consider $1 \leq p < \infty$. Let $T : X \longrightarrow E$ be a Lipschitz map and T_{L} its linearization. If T_{L} is p-lattice summing, then T is Lipschitz p-lattice summing. The converse is false.*

Proof. Assume that T_{L} is p-lattice summing, then by Remark 2.2.4 it is Lipschitz p-lattice summing from $Æ(X)$ to E. We know that $T = T_{\mathrm{L}} \circ i_X$ where $i_X \in \mathrm{Lip}_0(X, Æ(X))$, so T is Lipschitz p-lattice summing. □

Proposition 2.2.8. *Let T be in $Lip_0(X,E)$, then T is in $\Lambda_\infty^{\mathrm{L}}(X,E)$ if, and only if, T_{L} is in $\Lambda_\infty(Æ\,(X)\,,E)$ and $\lambda_\infty\,(T_{\mathrm{L}}) = \lambda_\infty^{\mathrm{L}}\,(T)$.*

Proof. Suppose that T_{L} is in $\Lambda_\infty(Æ(X)\,,E)$. Then, for every $n \in \mathbb{N}$, for all $x_1, ..., x_n, y_1, ..., y_n \in X$ and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$
\begin{aligned}
\left\|\sup_{1\le i\le n}\lambda_i\left|T(x_i)-T(y_i)\right|\right\| &= \left\|\sup_{1\le i\le n}\lambda_i\left|T_{\mathrm{L}}\left(m_{x_i0}\right)-T_{\mathrm{L}}\left(m_{y_i0}\right)\right|\right\| \\
&= \left\|\sup_{1\le i\le n}\lambda_i\left|T_{\mathrm{L}}\left(m_{x_iy_i}\right)\right|\right\| \\
&= \left\|\sup_{1\le i\le n}\left|T_{\mathrm{L}}\left(\lambda_i m_{x_iy_i}\right)\right|\right\| \\
&\le \lambda_\infty\left(T_{\mathrm{L}}\right)\sup_{1\le i\le n}\lambda_i d\left(x_i,y_i\right).
\end{aligned}
$$

This gives that T is in $\Lambda_\infty^{\mathrm{L}}(X,E)$ and $\lambda_\infty^{\mathrm{L}}(T)\le\lambda_\infty(T_{\mathrm{L}})$.

For the converse, suppose that T is in $\Lambda_\infty^{\mathrm{L}}(X,E)$. This yields by Remark 2.2.3 that, T is in $C_\infty^{\mathrm{L}}(X,E)$ and $C_\infty^{\mathrm{L}}(T)=\lambda_\infty^{\mathrm{L}}(T)$. By [17, Theorem 3.3], T_{L} is in $C_\infty(\text{Æ}(X),E)$ and hence is in $\Lambda_\infty(\text{Æ}(X),E)$. □

Proposition 2.2.9. *Let T be in $Lip_0(X,E)$. If $\left(T^t\right)^*\in\Lambda_p(\left(X^\#\right)^*,E^{**})$ then, $T\in\Lambda_p^{\mathrm{L}}(X,E)$.*

Proof. Suppose that $\left(T^t\right)^*$ is in $\Lambda_p(\left(X^\#\right)^*,E^{**})$. Then , $(T_{\mathrm{L}})^{**}$ is in $\Lambda_p(\left(X^\#\right)^*,E^{**})$ and by [39] the operator T_{L} is in $\Lambda_p(\text{Æ}(X),E)$. So we conclude by Proposition 2.2.7, that $T\in\Lambda_p^{\mathrm{L}}(X,E)$. □

Remark 2.2.10. *By Proposition 2.2.8, the converse is true for $p=\infty$. But it is not valid for $1\le p<\infty$ by Proposition 2.2.7.*

Now, we give our main result.

Theorem 2.2.11. *Let $T:X\longrightarrow E$ be a Lipschitz operator such that $T(0)=0$ and $1\le p<\infty$. Then, we have the following properties for a positive constant C.*

(i) If the operator $T\in\Lambda_p^{\mathrm{L}}(X,E)$ with $\lambda_p^{\mathrm{L}}(T)\le C$, $u\in\mathcal{L}(\ell_{p^*}^n,\text{Æ}(X))$, then for all $n\in\mathbb{N}$ and $(x_i)_{1\le i\le n},(y_i)_{1\le i\le n}\subset X$ the operator $T_{\mathrm{L}}u$ is in $\Lambda_\infty(\ell_{p^*}^n,E)$ and

$$\lambda_\infty(T_{\mathrm{L}}u)\le C\|u\|,$$

where $u(e_i)=m_{x_iy_i}$, ($(e_i)_{1\le i\le n}$ denote the unit vector basis of $\ell_{p^*}^n$).

(ii) If $\lambda_\infty(T_{\mathrm{L}}u)\le C\|u\|$ for all $n\in\mathbb{N}$ and all $u\in\mathcal{L}(\ell_{p^*}^n,\text{Æ}(X))$ then, the operator T is in $\Lambda_p^{\mathrm{L}}(X,E)$ and $\lambda_p^{\mathrm{L}}(T)\le C\|u\|$.

Proof.

(i) Suppose that T is in $\Lambda_p^{\mathrm{L}}(X,E)$. Let n be in $\mathbb{N}$ and $(x_i)_{1\leq i\leq n}, (y_i)_{1\leq i\leq n} \subset X$. Consider $u(e_i) = m_{x_i y_i}$, where $(e_i)_{1\leq i\leq n}$ denote the unit vector basis of $\ell_{p^*}^n$. By (1.8), we have

$$\|u\| = \sup_{\|f\|_{X^\#}=1} \left(\sum_{i=1}^n |f(x_i) - f(y_i)|^p \right)^{\frac{1}{p}}$$

and for $\alpha \in \ell_{p^*}^n$

$$\begin{aligned}
|T_{\mathrm{L}} u(\alpha)| &= \left| T_{\mathrm{L}} u \left(\sum_{i=1}^n \alpha_i e_i \right) \right| \\
&= \left| \sum_{i=1}^n \alpha_i T_{\mathrm{L}} \left(m_{x_i 0} - m_{y_i 0} \right) \right| \\
&= \left| \sum_{i=1}^n \alpha_i \left(T(x_i) - T(y_i) \right) \right| \\
([29, \text{Part II, page } 42]) &\leq \left(\sum_{i=1}^n |T(x_i) - T(y_i)|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |\alpha_i|^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq \left(\sum_{i=1}^n |T(x_i) - T(y_i)|^p \right)^{\frac{1}{p}} \|\alpha\|_{\ell_{p^*}^n}.
\end{aligned}$$

By [37, Proposition 1.2 and Theorem 1.3], we have

$$\begin{aligned}
\lambda_\infty(T_{\mathrm{L}} u) &\leq \left\| \left(\sum_{i=1}^n |T(x_i) - T(y_i)|^p \right)^{\frac{1}{p}} \right\| \\
&\leq \lambda_p^{\mathrm{L}}(T) \|u\| \\
&\leq C \|u\|.
\end{aligned}$$

This implies that $T_{\mathrm{L}} u$ is in $\Lambda_\infty(\ell_{p^*}^n, E)$ and $\lambda_\infty(T_{\mathrm{L}} u) \leq C\|u\|$.

(ii) Suppose that $\lambda_\infty(T_{\mathrm{L}} u) \leq C\|u\|$ for all $n \in \mathbb{N}$ and all $u \in \mathcal{L}(\ell_{p^*}^n, \text{Æ}(X))$. This implies by [37, Theorem 1.3] that $\lambda_p(T_{\mathrm{L}}) \leq C\|u\|$ and by Proposition 2.2.5, we have $\lambda_p^{\mathrm{L}}(T) \leq C\|u\|$.

□

Theorem 2.2.12. *Let X be a pointed metric space and E be a Banach lattice. Consider T in $\mathrm{Lip}_0(X, E)$. If the operator T is in $\Lambda^{\mathrm{L}}_{\infty}(X, E)$, then T is in $\Lambda^{\mathrm{L}}_{p}(X, E)$ for $1 \leq p \leq \infty$.*

Proof. Suppose that T is in $\Lambda^{L}_{\infty}(X, E)$. This implies by Proposition 2.2.8 that T_{L} is in $\Lambda_{\infty}(\text{Æ}(X), E)$. We observe by [37, Theorem 1.5] that $\Lambda_{\infty}(\text{Æ}(X), E)$ is a subspace of $\Lambda_{p}(\text{Æ}(X), E)$. Therefore $T_{\mathrm{L}} \in \Lambda_{p}(\text{Æ}(X), E)$ and consequently by Proposition 2.2.7, we conclude that $T \in \Lambda^{\mathrm{L}}_{p}(X, E)$. □

2.2.1 Lipschitz factorable p-lattice summing operators

We give now the following definition to have an equivalent relation between the operator T and its linearization T_{L}. In a way, it is the reciprocal of Proposition 2.2.7.

Definition 2.2.13. *Let $1 \leq p < \infty$ and $T : X \longrightarrow E$ is Lipschitz map from pointed metric space X into Banach space E. We say that T is Lipschitz factorable p-lattice summing if, there exists a positive constant C such that for all $n, m \in \mathbb{N}$, $(x_{ij})_{i,j}; (y_{ij})_{i,j} \in X$ and all reals $(\lambda_{ij})_{i,j}$, $(1 \leq i \leq n, 1 \leq j \leq m)$, we have*

$$\left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} \left(T\left(x_{ij}\right) - T\left(y_{ij}\right) \right) \right|^{p} \right)^{\frac{1}{p}} \right\| \leq C \sup_{f \in B_{X^{\#}}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} \left(f\left(x_{ij}\right) - f\left(y_{ij}\right) \right) \right|^{p} \right)^{\frac{1}{p}}. \quad (2.4)$$

The class of all Lipschitz factorable p-lattice summing operators is denoted by $\Lambda^{\mathrm{L},f}_{p}(X, E)$. In this case, we define $\lambda^{\mathrm{L},f}_{p}(T)$ as the infimum of all constants C verifying (2.4).

Remark 2.2.14. *Every Lipschitz factorable p-lattice summing operator is Lipschitz p-lattice summing operator, i.e., $\Lambda^{\mathrm{L},f}_{p}(X, E) \subset \Lambda^{\mathrm{L}}_{p}(X, E)$, and $\lambda^{\mathrm{L}}_{p}(T) \leq \lambda^{\mathrm{L},f}_{p}(T)$ (we take $m = 1$).*

Theorem 2.2.15. *Let X be a pointed metric space and E be a Banach lattice. Consider T in $\mathrm{Lip}_0(X, E)$. Then T is Lipschitz factorable p-lattice summing if, and only if, T_{L} is p-lattice summing, and we have*

$$\lambda^{\mathrm{L},f}_{p}(T) = \lambda_{p}(T_{\mathrm{L}}).$$

Proof. Let n, m be in $\mathbb{N}$, for $(x_{ij})_{i,j}; (y_{ij})_{i,j}$ in X and $(\lambda_{ij})_{i,j}$ in $\mathbb{R}$, $(1 \leq i \leq n, 1 \leq j \leq m)$, we have

$$\begin{aligned}
\left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij}(T(x_{ij}) - T(y_{ij})) \right|^{p} \right)^{\frac{1}{p}} \right\| &= \left\| \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} T_{\mathrm{L}}(m_{x_{ij}y_{ij}}) \right|^{p} \right)^{\frac{1}{p}} \right\| \\
&= \left\| \left(\sum_{i=1}^{n} T_{\mathrm{L}} \left(\sum_{j=1}^{m} \lambda_{ij} m_{x_{ij}y_{ij}} \right)^{p} \right)^{\frac{1}{p}} \right\| \\
&= \left\| \left(\sum_{i=1}^{n} |T_{\mathrm{L}}(m_i)|^{p} \right)^{\frac{1}{p}} \right\| \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in B_{Æ(X)^*}} \left(\sum_{i=1}^{n} |\xi(m_i)|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in B_{Æ(X)^*}} \left(\sum_{i=1}^{n} \left| \xi \left(\sum_{j=1}^{m} \lambda_{ij} m_{x_{ij}y_{ij}} \right) \right|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{\xi \in B_{Æ(X)^*}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij} \xi(m_{x_{ij}y_{ij}}) \right|^{p} \right)^{\frac{1}{p}} \\
&\leq \lambda_p(T_{\mathrm{L}}) \sup_{f \in B_{X^{\#}}} \left(\sum_{i=1}^{n} \left| \sum_{j=1}^{m} \lambda_{ij}(f(x_{ij}) - f(y_{ij})) \right|^{p} \right)^{\frac{1}{p}}.
\end{aligned}$$

This gives that T is Lipschitz factorable p-lattice summing and $\lambda_p^{\mathrm{L},f}(T) \leq \lambda_p(T_{\mathrm{L}})$. Following the inverse schema, we will have the reciprocal. □

We give here a characterization of Lipschitz factorable p-lattice summing operators.

Proposition 2.2.16. *Let $1 \leq p \leq \infty$. Then $T \in \Lambda_p^{\mathrm{L},f}(X, E)$ if and only if, $T_{\mathrm{L}}u \in \Lambda_\infty(\ell_{p^*}^n, E)$ for all $n \in \mathbb{N}$ and $u \in \mathcal{L}(\ell_{p^*}^n, Æ(X))$. In this case we have*

$$\lambda_p^{\mathrm{L},f}(T) = \sup \left\{ \lambda_\infty(T_{\mathrm{L}}u) : n \in \mathbb{N} \text{ and } u \in \mathcal{L}(\ell_{p^*}^n, Æ(X)) \right\}.$$

Proof. Consider $T \in \Lambda_p^{\mathrm{L},f}(X, E)$. This means that $T_{\mathrm{L}} \in \Lambda_p(Æ(X), E)$. As T_{L} is linear operator and according Nielsen-Szulga theorem [37, Theorem 1.3], we have $T_{\mathrm{L}}u \in \Lambda_\infty(\ell_{p^*}^n, E)$ for all $u \in \mathcal{L}(\ell_{p^*}^n, Æ(X))$ and

$$\lambda_p^{L,f}(T) = \lambda_p(T_{\mathrm{L}}) = \sup \left\{ \lambda_\infty(T_{\mathrm{L}}u) : n \in \mathbb{N} \text{ and } u \in \mathcal{L}(\ell_{p^*}^n, Æ(X)) \right\}.$$

We conclude by Theorem 2.2.15. □

2.2.2 Lipschitz p-regular operators

Let now $T : E \longrightarrow F$ be a Lipschitz operators between Banach lattices E, F. We say that T is *Lipschitz p-regular*, $(1 \leq p \leq \infty)$ and we write $T \in \rho_p^{\mathrm{L}}(E, F)$, if there is a positive constant C such that for every n in $\mathbb{N}$, for all $x_1, ..., x_n; y_1, ..., y_n$ in X and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\left\| \left(\lambda_i^{\frac{1}{p}} \left(T(x_i) - T(y_i) \right) \right) \right\|_{F\left(\ell_p^n\right)} \leq C \left\| \left(\lambda_i^{\frac{1}{p}} (x_i - y_i) \right) \right\|_{E\left(\ell_p^n\right)}$$

for p finite and

$$\left\| \sup_{1 \leq i \leq n} \lambda_i \left| T(x_i) - T(y_i) \right| \right\| \leq C \left\| \sup_{1 \leq i \leq n} \lambda_i |x_i - y_i| \right\|$$

for p is infinite.

The best possible constant will be denoted by $\rho_p^{\mathrm{L}}(T)$.

If T is linear then the notions of regular and Lipschitz regular coincide with $\rho_p(T) = \rho_p^L(T)$. It was proved by Krivine in [28] (see also [29]) that every bounded linear operator is 2-regular and every positive linear operator is p-regular for, $1 \leq p \leq \infty$. In general, the Lipschitz operators are not 2-regular, we can see [5].

Remark 2.2.17. *Let $T : X \longrightarrow E$ be a Lipschitz p-lattice summing operator. If $R : E \longrightarrow F$ is p-regular, then $R \circ T$ is Lipschitz p-lattice summing and*

$$\lambda_p^L(R \circ T) \leq \rho_p^{\mathrm{L}}(R)\, \lambda_p^{\mathrm{L}}(T).$$

Proposition 2.2.18. *Let E, F, E_1, F_1 be Banach lattices and $1 \leq p \leq \infty$. Consider a Lipschitz p-regular operator $T : E \longrightarrow F$, a positive linear bounded operator $u : F \longrightarrow F_1$ and a Lipschitz operator $R : E_1 \longrightarrow E$. Then, $u \circ T \circ R$ is Lipschitz p-regular and*

$$\rho_p^{\mathrm{L}}(u \circ T \circ R) \leq \|u\|\, \rho_p^{\mathrm{L}}(T) \mathrm{Lip}(R).$$

Proof. Since u is positive, we have

$$\begin{aligned}
\left(\sum_{i=1}^{n}\lambda_i\left|(uTR(a_i)-uTR(b_i))\right|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{n}\left|u\left(\lambda_i^{\frac{1}{p}}(TR(a_i)-TR(b_i))\right)\right|^p\right)^{\frac{1}{p}}\\
&= \sup_{(\alpha_i)\in\mathcal{B}_{\ell_{p^*}^n}} u\left(\sum_{i=1}^{n}\lambda_i^{\frac{1}{p}}(TR(a_i)-TR(b_i))\,\alpha_i\right)\\
&\leq u\left(\sup_{(\alpha_i)\in B_{\ell_{p^*}^n}}\sum_{i=1}^{n}\lambda_i^{\frac{1}{p}}(TR(a_i)-TR(b_i))\,\alpha_i\right)\\
&\leq u\left(\left(\sum_{i=1}^{n}\lambda_i\left|TR(a_i)-TR(b_i)\right|^p\right)^{\frac{1}{p}}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
\left\|\left(\lambda_i^{\frac{1}{p}}uTR(a_i)-uTR(b_i)\right)\right\|_{F_1\left(\ell_p^n\right)} &= \left\|u\left(\lambda_i^{\frac{1}{p}}(TR(a_i)-TR(b_i))\right)\right\|_{F_1\left(\ell_p^n\right)}\\
\text{(by [29, p. 55])}\quad &\leq \|u\|\left\|\left(\lambda_i^{\frac{1}{p}}(TR(a_i)-TR(b_i))\right)\right\|_{F\left(\ell_p^n\right)}\\
&\leq \|u\|\,\rho_p^{\mathrm{L}}(T)\left\|\left(\lambda_i^{\frac{1}{p}}(R(a_i)-R(b_i))\right)\right\|_{E\left(\ell_p^n\right)}\\
&\leq \|u\|\,\rho_p^{\mathrm{L}}(T)\mathrm{Lip}(R)\left\|\left(\lambda_i^{\frac{1}{p}}(a_i-b_i)\right)\right\|_{E_1\left(\ell_p^n\right)}
\end{aligned}$$

and thus the announced result is obtained. □

Remark 2.2.19. *Let E, F, F_1 be Banach lattices and $1 \leq p \leq \infty$.*

1. *Consider a Lipschitz p-regular operator $T : E \longrightarrow F$ and a Lipschitz p-concave operator $R : F \longrightarrow F_1$. Then, $R \circ T$ is Lipschitz p-concave and*

$$M_{\mathrm{Lip}}^{p}(R\circ T) \leq M_{\mathrm{Lip}}^{p}(R)\,\rho_p^{\mathrm{L}}(T).$$

Indeed,

$$\begin{aligned}\left\|\left(\lambda_i^{\frac{1}{p}}\left(R\circ T\left(a_i\right)-R\circ T\left(b_i\right)\right)\right)\right\|_{\ell_p^n(F_1)} &\leq M_{\mathrm{Lip}}^p\left(R\right)\left\|\left(\lambda_i^{\frac{1}{p}}\left(T\left(a_i\right)-T\left(b_i\right)\right)\right)\right\|_{F\left(\ell_p^n\right)}\\ &\leq M_{\mathrm{Lip}}^p\left(R\right)\rho_p^{\mathrm{L}}(T)\left\|\left(\lambda_i^{\frac{1}{p}}\left(a_i-b_i\right)\right)\right\|_{E\left(\ell_p^n\right)}.\end{aligned}$$

2. *Consider a Lipschitz p-regular operator $T : E \longrightarrow F$ and a Lipschitz p-convex operator $S : E_1 \longrightarrow E$. Then, $T\circ S$ is Lipschitz p-convex and*

$$C_{\mathrm{Lip}}^p(T\circ S)\leq C_{\mathrm{Lip}}^p\left(S\right)\rho_p^{\mathrm{L}}(T).$$

3. *If $p = 2$ and R is a bounded linear operator, we have*

$$\rho_2^{\mathrm{L}}(T\circ R)\leq K_G\rho_2^{\mathrm{L}}(T)\left\|R\right\|;$$

because every bounded linear operator R is 2-regular and $\rho_2(R)\leq K_G\left\|R\right\|$ (where K_G is the universal Grothendieck constant).

2.2.3 Lipschitz order bounded operators

Now, we define the Lipschitz order bounded operator.

Definition 2.2.20. *An operator T in $\mathrm{Lip}_0\left(X,E\right)$ is called Lipschitz order bounded if there exists an element a in E_+ such that*

$$\left|T\left(x\right)-T\left(y\right)\right|\leq ad\left(x,y\right),\quad \textit{for all } x,y\in X. \tag{2.5}$$

The space of such operators is denoted by $\mathfrak{Lip}_0\left(X,E\right)$. We let

$$\mathfrak{Lip}\left(T\right)=\inf\left\{\left\|a\right\| : a \text{ satisfies } (2.5)\right\}$$

or

$$\mathfrak{Lip}\left(T\right)=\left\|\sup_{x\neq y}\frac{\left|T\left(x\right)-T\left(y\right)\right|}{d\left(x,y\right)}\right\| \tag{2.6}$$

which exists by (1.4); because T is order bounded as a map into E^{**} (also we can take E a complete Banach lattice). We have also

$$\text{Lip}(T) \leq \mathfrak{Lip}(T). \tag{2.7}$$

Indeed, we have for all $x, y \in X$

$$\frac{|T(x) - T(y)|}{d(x,y)} \leq \sup_{x \neq y} \frac{|T(x) - T(y)|}{d(x,y)}$$

and consequently

$$\frac{\|T(x) - T(y)\|}{d(x,y)} \leq \left\| \sup_{x \neq y} \frac{|T(x) - T(y)|}{d(x,y)} \right\|$$

for all $x, y \in X$. By taking the sup, we obtain by (2.6) that $\text{Lip}(T) \leq \mathfrak{Lip}(T)$. It is clear that $\mathfrak{Lip}_0(X, E)$ is a vector space and that $\mathfrak{Lip}(.)$ is a norm on $\mathfrak{Lip}_0(X, E)$. Let us prove that the space $\mathfrak{Lip}_0(X, E)$ equipped with the norm $\mathfrak{Lip}(.)$ is a Banach space. By (2.7) and [21, Lemma 3.100], it is enough to prove that if $(T_n)_{n \geq 1} \subset \mathfrak{Lip}_0(X, E)$ such that $\sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) < +\infty$, then the operator $T = \sum_{i=1}^{n} T_n$ is in $\mathfrak{Lip}_0(X, E)$ and $\mathfrak{Lip}(T) \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n)$. In this case

$$\mathfrak{Lip}(T) = \inf \left\{ \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) \right\}$$

where the infimum is taken over all series in $\mathfrak{Lip}_0(X, E)$ summing up to T. Consider $\epsilon > 0$ and choose a sequence $(a_n) \subset E_+$ such that

$$\|a_n\| \leq \mathfrak{Lip}(T_n) + \frac{\epsilon}{2^n}, \quad \text{for } n \geq 1.$$

Then $\sum_{n=1}^{+\infty} \|a_n\| \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon$ and hence from [21, Lemma 3.100] again , we can find $a \in E_+$, such that

$$\|a\| \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon.$$

We have $\mathfrak{Lip}(T) \leq \sum_{n=1}^{+\infty} \mathfrak{Lip}(T_n) + \epsilon$. As ϵ is arbitrary, this proves the statement.

The following theorem is among our main results.

Theorem 2.2.21. *Let X be a pointed metric space and E be a Banach lattice. Consider T in $\text{Lip}_0(X, E)$. Then T is Lipschitz order bounded if, and only if, T_{L} is order bounded and $\|T_{\text{L}}\|_m = \mathfrak{Lip}(T)$.*

Proof. Suppose that T_{L} is order bounded. Then there is a in E such that

$$|T_{\mathrm{L}}(m)| \leq a \|m\| \text{ for all } m \in \mathcal{M}(X).$$

This implies for all $x, y \in X$

$$|T_{\mathrm{L}}(m_{xy})| \leq a \|m_{xy}\|$$

and hence

$$|T(x) - T(y)| \leq ad(x, y)$$

by [45, Lemma 3.5] and (1.6).

For the converse, suppose that T is Lipschitz order bounded. So there is a in E such that for all x, y in X, we have $|T(x) - T(y)| \leq ad(x, y)$. Let m be in $\mathcal{M}(X)$. For $\epsilon > 0$, consider a representation of $m = \sum_{i=1}^{n} \alpha_i m_{x_i y_i}$ such that $\sum_{i=1}^{n} |\alpha_i| d(x_i, y_i) < \|m\| + \epsilon$. We have

$$\begin{aligned} |T_{\mathrm{L}}(m)| &= \left| T_{\mathrm{L}}\left(\sum_{i=1}^{n} \alpha_i m_{x_i y_i} \right) \right| \\ &= \left| \sum_{i=1}^{n} \alpha_i (T(x_i) - T(y_i)) \right| \\ &\leq a \sum_{i=1}^{n} |\alpha_i| d(x_i, y_i) \\ &\leq a(\|m\| + \epsilon) \end{aligned}$$

and this for all $\epsilon > 0$. Thus $|T_{\mathrm{L}}(m)| \leq a \|m\|$ for all m in $\mathcal{M}(X)$ and by continuity $|T_{\mathrm{L}}(m)| \leq a \|m\|$ for all m in Æ(X). □

The following corollary is immediate.

Corollary 2.2.22. *Consider T in $\mathrm{Lip}_0(X, E)$. Then T is Lipschitz order bounded if, and only if, $(T^t)^*$ is order bounded and $\left\|(T^t)^*\right\|_m = \mathfrak{Lip}(T)$.*

Remark 2.2.23. *If $w : E \longrightarrow F$ is a positive linear operator, then $w \circ T$ is order bounded. Indeed, consider x, y in X and a in E_+ such that $|T(x) - T(y)| \leq ad(x, y)$. We have*

$$\begin{aligned} |w \circ T(x) - w \circ T(y)| &= |w(T(x) - T(y))| \\ &\leq w(|T(x) - T(y)|) \\ &\leq w(a) d(x, y). \end{aligned}$$

As w is positive, then $w(a) \in F_+$ and this implies that $w \circ T$ is order bounded.

We regard now the norm $\mathfrak{Lip}$ as Lipschitz tensor product.

Definition 2.2.24. *Let X be a pointed metric space and E be a Banach lattice. The space $X^\# \circledast E$ is defined as the linear subspace of $(X \boxtimes E^*)'$ spanned by the set*

$$\left\{ f \boxtimes a : f \in X^\#, a \in E \right\}.$$

This space is called the associated Lipschitz tensor product of X, E^*. For this concept, we can also consult [24].

The Lipschitz operators of finite rank from X into E are majorizing; that is $X^\# \circledast E \subset \mathfrak{Lip}_0(X, E)$. Indeed, consider $T = \sum_{i=1}^{n} f_i \boxtimes a_i$ in $X^\# \circledast E$. We have

$$\begin{aligned} |T(x) - T(y)| &= \left| \sum_{i=1}^{n} (f_i(x) - f_i(y)) a_i \right| \\ &\leq \sum_{i=1}^{n} |f_i(x) - f_i(y)| |a_i| \\ &\leq \left(\sum_{i=1}^{n} \mathrm{Lip}(f_i) |a_i| \right) d(x, y). \end{aligned}$$

This implies that $X^\# \circledast E$ is in $\mathfrak{Lip}_0(X, E)$ and suggests us the following definition.

Definition 2.2.25. *The $\mathfrak{Lip}$-tensor product $X^\# \widehat{\circledast}_{\mathfrak{Lip}} E$ is the closure of $X^\# \circledast E$ in $\mathfrak{Lip}_0(X, E)$.*

The following proposition generalizes, Proposition 1.2 of [37].

Proposition 2.2.26. *Consider T in $Lip_0(X, E)$. The operator T is in $\Lambda^{\mathrm{L}}_\infty(X, E)$ if, and only if, T is in $\mathfrak{Lip}_0(X, E)$. In this case, $\lambda^{\mathrm{L}}_\infty(T) = \mathfrak{Lip}(T) = \lambda_\infty(T_{\mathrm{L}}) = \|T_{\mathrm{L}}\|_m$.*

Proof. Suppose that T is in $\mathfrak{Lip}_0(X, E)$. For $\epsilon > 0$, there is a in E_+ such that for all x, y in X, we have

$$|T(x) - T(y)| \leq ad(x, y)$$

with $\mathfrak{Lip}(T) \leq \|e\| \leq (1 + \epsilon) \mathfrak{Lip}(T)$. This implies that, for every n in $\mathbb{N}$, all $x_1, ..., x_n;\ y_1, ..., y_n$ in X and all non negative reals $\lambda_1, ..., \lambda_n$, we have

$$\sup_{1 \leq i \leq n} \lambda_i |T(x_i) - T(y_i)| \leq a \sup_{1 \leq i \leq n} \lambda_i d(x_i, y_i)$$

and hence

$$\begin{aligned}\left\|\sup_{1\leq i\leq n}\lambda_i\left|T\left(x_i\right)-T\left(y_i\right)\right|\right\| &\leq \|a\|\sup_{1\leq i\leq n}\lambda_i d\left(x_i,y_i\right)\\ &\leq (1+\epsilon)\,\mathfrak{Lip}\left(T\right)\sup_{1\leq i\leq n}\lambda_i d\left(x_i,y_i\right).\end{aligned}$$

As ϵ is arbitrary, then $\lambda^{\mathrm{L}}_{\infty}(T)\leq\mathfrak{Lip}\left(T\right)$.

Consider now T in $\Lambda^{\mathrm{L}}_{\infty}(X,E)$. By Proposition 2.2.8, this gives that T_{L} *is* in $\Lambda_{\infty}(\text{Æ}(X),E)$ and consequently for all $x_1,...,x_n;\ y_1,...,y_n$ in X, we have

$$\left\|\sup_{1\leq i\leq n}\left|T_{\mathrm{L}}\left(m_{x_iy_i}\right)\right|\right\|\leq\lambda_{\infty}(T)\sup_{1\leq i\leq n}\left\|m_{x_iy_i}\right\|.$$

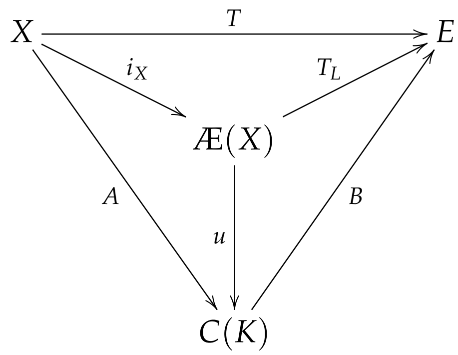
By [43, Proposition 3.4], there exists a in E_+, $\|a\|\leq\lambda_{\infty}(T)$ such that for each x,y in X, one has $\left|T_{\mathrm{L}}\left(\frac{m_{xy}}{\|m_{xy}\|}\right)\right|\leq a$. We conclude that for all $x,y\in X$, $|T\left(x\right)-T\left(y\right)|\leq ad\left(x,y\right)$ and thus by (2.5), T is Lipschitz order bounded and $\mathfrak{Lip}\left(T\right)\leq\lambda^{\mathrm{L}}_{\infty}(T)$. □

Corollary 2.2.27. *The $\mathfrak{Lip}$-tensor product $X^{\#}\widehat{\circledast}_{\mathfrak{Lip}}E$ is a subspace of $\Lambda^{\mathrm{L}}_{p}(X,E)$ for all $1\leq p\leq\infty$.*

Proof. We have by definition $X^{\#}\widehat{\circledast}_{\mathfrak{Lip}}E\subset\mathfrak{Lip}_0(X,E)$ and by Proposition 2.2.26 $\mathfrak{Lip}_0(X,E)=\Lambda^{\mathrm{L}}_{\infty}(X,E)$ with $\lambda^{\mathrm{L}}_{\infty}(T)=\mathfrak{Lip}\left(T\right)$. If T is in $\Lambda^{\mathrm{L}}_{\infty}(X,E)$ then T_{L} is in $\Lambda_{\infty}(\text{Æ}\left(X\right),E)$ which is included in $\Lambda_p(\text{Æ}\left(X\right),E)$ by [37, Theorem 1.5(i)] and consequently $T\in\Lambda^{\mathrm{L}}_{p}(X,E)$. □

Lemma 2.2.28. *Let X be a pointed metric space and E be a Banach lattice. Consider T in $\mathrm{Lip}_0(X,E)$. Then, T is Lipschitz order bounded operator if, and only if, there are a compact Haussdorff space K and operators $A\in\mathrm{Lip}_0(X,C(K))$ and $B\in\mathcal{L}_+(C(K),E)$ so that $\mathrm{Lip}(A)\leq 1$, $\|B\|_m=\mathfrak{Lip}(T)$ and $T=B\circ A$.*

Proof. Let $T\in\mathfrak{Lip}_0(X,E)$. This means by Theorem 2.2.21 that T_L is order bounded operator. Then, by a classical result we can find a compact Hausdorff space K and operators $u\in\mathcal{L}(\text{Æ}\left(X\right),C(K))$ and $B\in\mathcal{L}_+(C(K),E)$ so that $\|u\|\leq 1$, $\|B\|_m=\|T_{\mathrm{L}}\|_m$, $T_L=B\circ u$ and the following diagram is commutative



Note that $A = u \circ i_X \in \text{Lip}_0(X, C(K))$ and $\text{Lip}(A) \leq 1$. Indeed,

$$\text{Lip}(A) = \text{Lip}(u \circ i_X) \leq \|u\| \leq 1.$$

Conversely, suppose that $A \in \text{Lip}_0(X, C(K))$ and $B \in \mathcal{L}_+(C(K), E)$ such that $T = B \circ A$. Let $x, y \in X$ and $a \in E_+$. We have

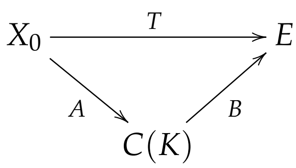
$$\begin{aligned} |T(x) - T(y)| &= |B \circ A(x) - B \circ A(y)| \\ &= |B(A(x) - A(y))| \\ &\leq a \|A(x) - A(y)\| \\ &\leq a\text{Lip}(A) d(x, y). \end{aligned}$$

This implies that $T \in \mathfrak{Lip}_0(X, E)$. ☐

We give the following extension theorem.

Theorem 2.2.29. *Let X_0 be a subspace of a pointed metric space X which contained the distinguished element and E be a Banach lattice space. Then, each operator T in $\mathfrak{Lip}_0(X_0, E)$ admits a Lipschitz order bounded extension $\widetilde{T} : X \longrightarrow E$, with $\mathfrak{Lip}(\widetilde{T}) = \mathfrak{Lip}(T)$.*

Proof. Consider T in $\mathfrak{Lip}_0(X_0, E)$. By Lemma 2.2.28, T has the following factorization



where $A \in \text{Lip}_0(X_0, C(K))$ and $B \in \mathcal{L}_+(C(K), E)$ such that $\text{Lip}(A) \leq 1$, $\|B\|_m = \mathfrak{Lip}(T)$. By the non linear Hahn-Banach theorem [10, Proposition 1.2], A admits an extension $\widetilde{A} \in$

$\mathrm{Lip}_0(X, C(K))$ with $\mathrm{Lip}(\widetilde{A}) = \mathrm{Lip}(A)$. Hence, T admits an extension $\widetilde{T}$ such that $\widetilde{T} = B \circ \widetilde{A}$. We deduce from Lemma 2.2.28, that $\widetilde{T}$ is Lipschitz order bounded operator with

$$\mathfrak{Lip}(\widetilde{T}) \leq \|B\|_m \mathrm{Lip}(\widetilde{A}) = \mathfrak{Lip}(T)\mathrm{Lip}(A) \leq \mathfrak{Lip}(T).$$

For the reverse inequality, note that $T = \widetilde{T} \circ i$ ($i : X_0 \longrightarrow X$ is the natural isometric embedding) and hence *

$$\mathfrak{Lip}(T) = \mathfrak{Lip}(\widetilde{T} \circ i) \leq \mathfrak{Lip}(\widetilde{T}).$$

□

2.3 Relationships between Lipschitz p-lattice summing and Lipschitz p-summing operators

In this section, we try to make the connection between Lipschitz p-lattice summing and Lipschitz p-summing operators.

Theorem 2.3.1. *Consider $1 \leq p < \infty$. Let $T : X \longrightarrow E$ be a Lipschitz map. If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then for all Lipschitz p-concave operator $S : E \longrightarrow F$, $S \circ T$ is Lipschitz p-summing and*

$$\pi_p^{\mathrm{L}}(ST) \leq M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T).$$

Proof. Let T be in $\Lambda_p^{\mathrm{L}}(X, E)$ and consider $S : E \longrightarrow F$ a Lipschitz p-concave operator. We have for every n in $\mathbb{N}$, all (x_i), $(y_i) \subset X$ and all non negative reals $\lambda_1, ..., \lambda_n$

$$\begin{aligned} & \left\| \left(\lambda_i^{\frac{1}{p}} \left(ST(x_i) - ST(y_i) \right) \right)_{1 \leq i \leq n} \right\|_{\ell_p^n(F)} \\ \leq \quad & M_p^{\mathrm{L}}(S) \left\| \left(\lambda_i^{\frac{1}{p}} \left(T(x_i) - T(y_i) \right) \right)_{1 \leq i \leq n} \right\|_{E(\ell_p^n)} \\ \leq \quad & M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T)\omega_p^{L}\left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i), (y_i) \right). \end{aligned}$$

This implies that $S \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(ST) \leq M_p^{\mathrm{L}}(S)\lambda_p^{\mathrm{L}}(T)$. □

Proposition 2.3.2. *Consider $1 \leq p < \infty$. Let $T : X \longrightarrow E$ be a Lipschitz map. If T is in $\Lambda_p^{\mathrm{L}}(X, E)$, then for all p-concave space F and all positive bounded linear operator $\varphi : E \longrightarrow F$, $\varphi \circ T$ is in*

$\Pi_p^{\mathrm{L}}(X,F)$ *and*

$$\pi_p^{\mathrm{L}}(\varphi T) \leq \|\varphi\| M_p(F) \lambda_p^{\mathrm{L}}(T).$$

Proof. We have for every n in $\mathbb{N}$, all $(x_i),(y_i) \subset X$ and all non negative reals $\lambda_1, ..., \lambda_n$

$$\begin{aligned}
\left\| \left(\lambda_i^{\frac{1}{p}} \left(\varphi T(x_i) - \varphi T(y_i) \right) \right)_{1 \leq i \leq n} \right\|_{\ell_p^n(F)} &\leq M_p(F) \left\| \left(\sum_{i=1}^{n} \lambda_i^{\frac{1}{p}} \left| \varphi T(x_i) - \varphi T(y_i) \right|^p \right)^{\frac{1}{p}} \right\| \\
\text{(Proposition 2.2.5)} &\leq \|\varphi\| M_p(F) \left\| \left(\sum_{i=1}^{n} \lambda_i^{\frac{1}{p}} \left| T(x_i) - T(y_i) \right|^p \right)^{\frac{1}{p}} \right\| \\
&\leq \|\varphi\| M_p(F) \lambda_p^{\mathrm{L}}(T) \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i)_i, (y_i)_i \right).
\end{aligned}$$

This implies that $\varphi \circ T$ is Lipschitz p-summing and $\pi_p^{\mathrm{L}}(\varphi T) \leq \|\varphi\| M_p(F) \lambda_p^{\mathrm{L}}(T)$. □

Corollary 2.3.3. *Let X be a pointed metric space and E be a p-concave Banach lattice. Then, every Lipschitz p-lattice summing operator T from X into E is Lipschitz p-summing and*

$$\pi_p^{\mathrm{L}}(T) \leq M_p(E) \lambda_p^{\mathrm{L}}(T).$$

Theorem 2.3.4. *Let X be a pointed metric space and let E be a Banach lattice. Consider $T : X \longrightarrow E$ a Lipschitz map. If the operator T is in $\Pi_p^{\mathrm{L}}(X,E)$, then for all Lipschitz p-convex $S : E \longrightarrow F$, we have $S \circ T$ is in $\Lambda_p^{\mathrm{L}}(X,F)$ and*

$$\lambda_p^{\mathrm{L}}(S \circ T) \leq C_p^{\mathrm{L}}(S) \pi_p^{\mathrm{L}}(T).$$

Proof. As F is p-convex, we have for every n in $\mathbb{N}$, all $(x_i),(y_i) \subset X$ and all non negative reals $\lambda_1, ..., \lambda_n$

$$\begin{aligned}
\left\| \left(\sum_{i=1}^{n} \lambda_i \left| ST(x_i) - ST(y_i) \right|^p \right)^{\frac{1}{p}} \right\|_F &\leq C_p^{\mathrm{L}}(S) \left(\sum_{i=1}^{n} \lambda_i \left\| T(x_i) - T(y_i) \right\|_F^p \right)^{\frac{1}{p}} \\
\text{(Proposition 2.2.5)} &\leq C_p^{\mathrm{L}}(S) \left(\sum_{i=1}^{n} \lambda_i \left\| T(x_i) - T(y_i) \right\|_E^p \right)^{\frac{1}{p}} \\
&\leq C_p^{\mathrm{L}}(S) \pi_p^{L}(T) \omega_p^{\mathrm{L}} \left(\left(\lambda_i^{\frac{1}{p}} \right), (x_i), (y_i) \right).
\end{aligned}$$

This implies that $S \circ T$ *is* Lipschitz p-lattice summing and $\lambda_p^{\mathrm{L}}(S \circ T) \leq C_p^{\mathrm{L}}(S) \pi_p^{\mathrm{L}}(T)$. □

Corollary 2.3.5. *Let X be a pointed metric space and E be a p-convex Banach lattice. Then, every Lipschitz p-summing operator T from X into E is Lipschitz p-lattice summing and*

$$\lambda_p^{\mathrm{L}}(T) \leq C_p(E)\, \pi_p^{\mathrm{L}}(T).$$

Corollary 2.3.6. *If $E = F = L_p(\Omega, \mu)$ and $S = Id_{L_p(\Omega,\mu)}$ then, $\Pi_p^{\mathrm{L}}(X, L_p(\Omega, \mu)) = \Lambda_p^{\mathrm{L}}(X, L_p(\Omega, \mu))$ and $\pi_p^{\mathrm{L}}(T) = \lambda_p^{\mathrm{L}}(T)$.*

Proof. This comes from the fact that $L_p(\Omega, \mu)$ is p-convex and p-concave with $C_p^{\mathrm{L}}\left(L_p(\Omega, \mu)\right) = M_p^{\mathrm{L}}\left(L_p(\Omega, \mu)\right) = 1$. □

Remark 2.3.7. *Let $T : X \longrightarrow E, S : Z \longrightarrow X$ and $R : E \longrightarrow F$ be Lipschitz maps. Then*

(i) If T is Lipschitz p- lattice summing operator and S is Lipschitz p-summing, then $T \circ S$ is Lipschitz p-lattice summing and

$$\lambda_p^{\mathrm{L}}(T \circ S) \leq \lambda_p^{\mathrm{L}}(T)\pi_p^{\mathrm{L}}(S).$$

(ii) If T is p-regular and R is Lipschitz p-summing, then $R \circ T$ is Lipschitz p-summing and

$$\pi_p^{\mathrm{L}}(R \circ T) \leq \rho_p^{\mathrm{L}}(T)\pi_p^{\mathrm{L}}(R).$$

Chapter 3

Characterization of Lipschitz σ-nuclear operators

In this chapter, we introduce a new category of Lipschitz operators which called *Lipschitz σ-nuclear operators*. At the beginning we give a reminder about the classes of σ-nuclear operators. Then, we extend this concept to Lipschitz case and we characterize this type of operators by introducing an equivalent definition and factorization theorem. Furthermore, we prove that the definition of Lipschitz σ-nuclear operators is a good generalization and this space of operators is Banach Lipschitz operator ideals. Finally, we give another class of Lipschitz operators which called *Lipschitz σ-integral operators* as generalization of σ-integral operators and its factorization theorem.

3.1 Linear σ-nuclear operators

In this section, we recall the notion of σ-nuclear for linear case as σ-nuclear operators which introduced by A. Pietsch [38] and $\sigma(p)$-nuclear operators for linear and multilinear operators which introduced by X. Mujica in her thesis [34].

The following definition is mentioned in [38].

Definition 3.1.1. *Let E, F be a Banach spaces and $T : E \longrightarrow F$ be a bounded linear operator. We say that T is σ-nuclear operator if $T = \sum_{i=1}^{\infty} a_i^* \otimes b_i$ with $(a_i^*)_{i=1}^{\infty} \subset E^*$ and $(b_i)_{i=1}^{\infty} \subset F$ such that the family $(a_i^* \otimes b_i)_{i=1}^{\infty}$ is unconditionally summable in the operator norm.*

The set of all σ-nuclear operators is denoted by $\mathcal{N}_\sigma(E, F)$ and the σ-nuclear norm is denoted by $\eta_\sigma(T)$ and it is defined by

$$\eta_\sigma(T) := \inf \sup \left\{ \sum_{i=1}^{\infty} |a_i^*(a) b^*(b_i)|, \quad a \in B_E, b^* \in B_{F^*} \right\},$$

where the infimum is taken over all σ-nuclear representations.

The following definition is an equivalent definition for σ-nuclear operators given by G. Botelho and X. Mujica in their article [11, Proposition 2.1].

Definition 3.1.2. *We say that a bounded linear operator $T : E \longrightarrow F$ is σ-nuclear operator if there exist $(\lambda_i)_{i=1}^\infty \in \ell_\infty, (a_i^*)_{i=1}^\infty \subset E^*$ and $(b_i)_{i=1}^\infty \subset F$ such that $T = \sum_{i=1}^{\infty} \lambda_i a_i^* \otimes b_i$,*

$$\sup_{\substack{a \in B_E \\ b^* \in B_{F^*}}} \sum_{i=1}^{\infty} |a_i^*(a) b^*(b_i)| < \infty$$

and

$$\lim_{m \to \infty} \sup_{\substack{a \in B_E \\ b^* \in B_{F^*}}} \sum_{i=m}^{\infty} |a_i^*(a) b^*(b_i)| = 0.$$

The σ-nuclear norm defined in this case as follows

$$\eta_\sigma(T) := \inf \sup \left\{ \|(\lambda_i)_{i=1}^\infty\|_\infty \sum_{i=1}^{\infty} |a_i^*(a) b^*(b_i)|, \quad a \in B_E, b^* \in B_{F^*} \right\}.$$

Example 3.1.3. *Let E be a Banach space and $(a_i)_{i=1}^\infty \subset E$. We construct the spaces $\mathcal{U}$ and $\mathcal{V}$ as follows*

$$\mathcal{U} := \left\{ (\alpha_i)_{i=1}^\infty \subset \mathbb{C} : (\alpha_i \beta_i^{-2} a_i)_{i=1}^\infty \in \ell_p^u(E) \right\} \tag{3.1}$$

with the norm

$$\|(\alpha_i)_{i=1}^\infty\|_{\mathcal{U}} := \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \beta_i^{-2p} |\alpha_i a^*(a_i)|^p \right)^{\frac{1}{p}},$$

where $(\beta_i)_{i=1}^\infty$ is decreasing sequence in c_0 such that $0 \leq \beta_i \leq 1$.

$$\mathcal{V} := \left\{ (\gamma_i)_{i=1}^\infty \subset \mathbb{C} : (\gamma_i a_i)_{i=1}^\infty \in \ell_p^u(E) \right\} \tag{3.2}$$

with the norm

$$\|(\gamma_i)_{i=1}^{\infty}\|_{\mathcal{V}} := \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} |\gamma_i a^*(a_i)|^p \right)^{\frac{1}{p}}.$$

Then $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$ *and* $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ *are Banach spaces and the sequences* $(e_i)_{i=1}^{\infty}$ *and* $(v_i)_{i=1}^{\infty}$ *of standard unit vectors form a hyperorthogonal bases in* $\mathcal{U}$ *and* $\mathcal{V}$*, respectively.*

Define a diagonal operator

$$D : \mathcal{U} \longrightarrow \mathcal{V}, \quad D((\alpha_i)_{i=1}^{\infty}) = (\beta_i^{-1}\alpha_i)_{i=1}^{\infty} \tag{3.3}$$

for all $(\alpha_i)_{i=1}^{\infty} \in \mathcal{U}$*. We have*

$$\begin{aligned} \|(\beta_i^{-1}\alpha_i)_{i=1}^{\infty}\|_{\mathcal{V}} &= \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} |\beta_i^{-1}\alpha_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\ &\leq \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \beta_i^{-2p} |\alpha_i a^*(a_i)|^p \right)^{\frac{1}{p}} = \|(\alpha_i)_{i=1}^{\infty}\|_{\mathcal{U}}. \end{aligned}$$

Hence, D is well defined and it has a σ*-nuclear representation*

$$D = \sum_{i=1}^{\infty} \beta_i^{-1} e_i^* \otimes v_i,$$

where each $e_i^* \in \mathcal{U}^*$ *is the i-th coordinate functional.*

For every $(\alpha_k)_{k=1}^{\infty} \in B_{\mathcal{U}}$ *and* $\omega^* \in B_{\mathcal{V}^*}$*, we have*

$$\begin{aligned} \sum_{i=1}^{\infty} \left| \beta_i^{-1} e_i^*((\alpha_k)_{k=1}^{\infty}) \omega^*(v_i) \right| &= \sum_{i=1}^{\infty} \left| \beta_i^{-1} \alpha_i \omega^*(v_i) \right| \\ &= \sum_{i=1}^{\infty} \zeta_i \beta_i^{-1} \alpha_i \omega^*(v_i), (|\zeta| = 1) \\ &\leq \left\| \sum_{i=1}^{\infty} \zeta_i \beta_i^{-1} \alpha_i v_i \right\|_{\mathcal{V}} \\ &= \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \beta_i^{-p} |\alpha_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\ &\leq \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \beta_i^{-2p} |\alpha_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\ &= \|(\alpha_k)_{k=1}^{\infty}\|_{\mathcal{U}} \leq 1. \end{aligned}$$

So,

$$\sup_{\substack{(\alpha_k)_{k=1}^{\infty}\in B_{\mathcal{U}} \\ \omega^*\in B_{W^*}}} \sum_{i=1}^{\infty} \left|\beta_i^{-1} e_i^*((\alpha_k)_{k=1}^{\infty})\omega^*(v_i)\right| < \infty.$$

On the other hand, for all $m \in \mathbb{N}$ and using the same steps as above, we get

$$\sum_{i=m}^{\infty} \left|\beta_i^{-1} e_i^*((\alpha_k)_{k=1}^{\infty})\omega^*(v_i)\right| \leq \beta_m \|(\alpha_k)_{k=1}^{\infty}\|_{\mathcal{U}} \leq \beta_m.$$

Then,

$$\lim_{m\to\infty} \sup_{\substack{(\alpha_k)_{k=1}^{\infty}\in B_{\mathcal{U}} \\ \omega^*\in B_{W^*}}} \sum_{i=m}^{\infty} \left|\beta_i^{-1} e_i^*((\alpha_k)_{k=1}^{\infty})\omega^*(v_i)\right| \leq \lim_{m\to\infty} \beta_m = 0.$$

Consequently, $D \in \mathcal{N}_\sigma(\mathcal{U}, \mathcal{V})$ and $\eta_\sigma(D) \leq 1$.

3.2 Lipschitz σ-nuclear operators

In this section, we introduce the class of Lipschitz σ-nuclear operators as generalization of σ-nuclear operators which mentioned in previous section. Furthermore, we present some results that shows that it is indeed a generalization of σ-nuclear operators which presented by G. Botelho and X. Mujica [11, Proposition 2.1].

3.2.1 Definitions

Now, we introduce the notion of σ-nuclear operators in Lipschitz case.

Definition 3.2.1. *Let X be a pointed metric space and E be a Banach space. Consider $T \in \mathrm{Lip}_0(X, E)$, we say that T is Lipschitz σ-nuclear operator if there are sequences $(f_i)_{i=1}^{\infty}$ in $X^\#$ and $(a_i)_{i=1}^{\infty}$ in E such that $T = \sum_{i=1}^{\infty} f_i \otimes a_i$ and the sequence $(f_i \otimes a_i)_{i=1}^{\infty}$ is unconditionally summable in $\mathrm{Lip}_0(X, E)$.*

The set of all Lipschitz σ-nuclear operators is denoted by $\mathcal{N}_\sigma^L(X, E)$ and the Lipschitz σ-nuclear norm is denoted by $\eta_\sigma^L(T)$ and it is defined by

$$\eta_\sigma^L(T) = \inf\left\{ \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)}\right) a^*(a_i) \right| \right\},$$

where the infimum is taken over all Lipschitz σ-nuclear representations of T.

Recall that for Banach space Z, a subset M of the dual unit ball B_{Z^*} is said to be norming, if

$$\|z\| = \sup_{z^* \in M} |z^*(z)|$$

for all $z \in Z$.

Lemma 3.2.2. *A Lipschitz operator $T \in \mathrm{Lip}_0(X,E)$ is Lipschitz σ-nuclear if and only if there are sequences $(\lambda_i)_{i=1}^{\infty} \in \ell_\infty$, $(f_i)_{i=1}^{\infty} \subset X^{\#}$ and $(a_i)_{i=1}^{\infty} \subset E$ such that $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$,*

$$\sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| < \infty$$

and

$$\lim_{m \longrightarrow \infty} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=m}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| = 0.$$

In this case,

$$\eta_{\sigma}^{L}(T) = \eta_{\sigma(1)}^{L}(T) := \inf \left\{ \left\| (\lambda_i)_{i=1}^{\infty} \right\|_{\infty} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| \right\},$$

where the infimum is taken over all Lipschitz $\sigma(1)$-nuclear representations of T.

Proof. By [20], a sequence $(z_i)_{i=1}^{\infty}$ is unconditionally summable in Banach space Z if and only if

$$\sup_{\varphi \in M} \sum_{i=1}^{\infty} |\varphi(z_i)| < \infty$$

and

$$\lim_{m \to \infty} \sup_{\varphi \in M} \sum_{i=m}^{\infty} |\varphi(z_i)| = 0,$$

for some norming set $M \subseteq B_{Z^*}$.

Let $x, y \in X$ and $a^* \in B_{E^*}$ with $x \neq y$, define the map

$$\varphi_{\delta_{xy},a^*} : \mathrm{Lip}_0(X,E) \to \mathbb{K}, \qquad \varphi_{\delta_{xy},a^*}(T) := a^* \left(\frac{T(x) - T(y)}{d(x,y)} \right),$$

where $\delta_{xy} \in (X^{\#})^*$ defined by, $\delta_{xy}(g) := \dfrac{g(x) - g(y)}{d(x,y)}$, for all $g \in X^{\#}$
We can easily to prove that $\varphi_{\delta_{xy},a^*}$ is a continuous linear form and $\|\varphi_{\delta_{xy},a^*}\| \leq 1$. Then, the set

$$M = \left\{ \varphi_{\delta_{xy},a^*}, \quad \delta_{xy} \in B_{(X^{\#})^*}, a^* \in B_{E^*} \right\}$$

is subset of $B_{\left(\mathrm{Lip}_0(X,E)\right)^*}$.

Let us check that M is norming. Give $\varphi \in M$, we have

$$\begin{aligned}
\sup_{\varphi \in M} |\varphi(T)| &= \sup_{\varphi_{\delta_{xy},a^*} \in M} |\varphi_{\delta_{xy},a^*}(T)| \\
&= \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left| a^* \left(\frac{T(x) - T(y)}{d(x,y)} \right) \right| \\
&= \sup_{x \neq y} \left\| \frac{T(x) - T(y)}{d(x,y)} \right\| \\
&= \mathrm{Lip}(T).
\end{aligned}$$

For $Z = \mathrm{Lip}_0(X,E)$ and $(z_i)_{i=1}^{\infty} = (f_i \otimes a_i)_{i=1}^{\infty} \subset Z$. We take $\varphi \in M$

$$\begin{aligned}
|\varphi(f_i \otimes a_i)| &= \left| \varphi_{\delta_{xy},a^*}(f_i \otimes a_i) \right| \\
&= \left| a^* \left(\frac{f_i(x) - f_i(y)}{d(x,y)} a_i \right) \right| \\
&= \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right|
\end{aligned}$$

for $x, y \in X$ with $x \neq y$, $a^* \in E^*$.

Since $(f_i \otimes a_i)_{i=1}^{\infty}$ is unconditionally summable sequence in $\mathrm{Lip}_0(X,E)$, that equivalently with

$$\sup_{\varphi \in D} \sum_{i=1}^{\infty} |\varphi(f_i \otimes a_i)| = \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| < \infty$$

and

$$\lim_{m \to \infty} \sup_{\varphi \in D} \sum_{i=m}^{\infty} |\varphi(f_i \otimes a_i)| = \lim_{m \to \infty} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=m}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| = 0.$$

Now, consider $T = \sum_{i=1}^{\infty} f_i \otimes a_i = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$ with $\lambda_i = 1$, we have

$$\begin{aligned}
\eta^L_{\sigma(1)}(T) &= \inf \left\{ \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| \right\} \\
&\leq \inf \left\{ \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| \right\} = \eta^L_\sigma(T).
\end{aligned}$$

On the other hand, let $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i = \sum_{i=1}^{\infty} f_i \otimes b_i$ where $b_i = \lambda_i a_i$, we have

$$\begin{aligned}
\eta^L_\sigma(T) &= \inf \left\{ \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(b_i) \right| \right\} \\
&= \inf \left\{ \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} |\lambda_i| \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| \right\} \\
&\leq \inf \left\{ \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| \right\} = \eta^L_{\sigma(1)}(T).
\end{aligned}$$

□

Motivated by Lemma 3.2.2, we characterize the Definition 3.2.1 as follows.

Definition 3.2.3. *We say that a Lipschitz operator T from a pointed metric space X to a Banach space E is Lipschitz σ-nuclear if there are sequences $(\lambda_i)_{i=1}^{\infty} \in \ell_\infty$, $(f_i)_{i=1}^{\infty} \subset X^{\#}$ and $(a_i)_{i=1}^{\infty} \subset E$ such that $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$,*

$$\sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| < \infty$$

and

$$\lim_{m \longrightarrow \infty} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=m}^{\infty} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right| = 0.$$

In this case, the Lipschitz σ-nuclear norm becomes defined as

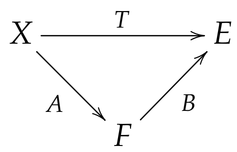
$$\eta_{\sigma}^{L}(T)=\inf\left\{\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}\left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right)a^*(a_i)\right|\right\},$$

where the infimum is taken over all Lipschitz σ-nuclear representations of T.

3.2.2 Factorization theorem and some results

The Lipschitz σ-nuclear operator can be factorized as the following theorem, for the proof, we use the same techniques of [38, Theorem 23.2.5] and [34].

Theorem 3.2.4. *An operator $T\in\mathcal{N}_{\sigma}^{L}(X,E)$ if and only if, the following diagram is commutes*



where, $A\in\overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$, $B\in\overline{\mathcal{L}_f(F,E)}$ and F is a Banach space having a hyperorthogonal basis $(e_i)_{i=1}^{\infty}$. And we have $\eta_{\sigma}^{L}(T)=\inf\mathrm{Lip}(A).\|B\|$ where the infimum is taken over all possible factorizations.

Proof. $\Rightarrow$) Consider $\varepsilon>0$, $T\in\mathcal{N}_{\sigma}^{L}(X,E)$ and let $\sum_{i=1}^{\infty}\lambda_i f_i\otimes a_i$ be a representation of T, such that

$$\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|\leq(1+\varepsilon)\,\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\eta_{\sigma}^{L}(T).$$

We have for all $x,y\in X$ with $x\neq y$ and $a^*\in E^*$,

$$\begin{aligned}\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}\frac{a^*(a_i)}{\|a^*\|}\right| &\leq \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|\\ &\leq (1+\varepsilon)\,\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\eta_{\sigma}^{L}(T),\end{aligned}$$

it follows that

$$\sum_{i=1}^{\infty}|(f_i(x)-f_i(y))\,a^*(a_i)|\leq(1+\varepsilon)\,\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\eta_{\sigma}^{L}(T)\,d(x,y)\,\|a^*\|.$$

We put $|(f_i(x) - f_i(y))\, a^*(a_i)| = |\alpha_i|$, obvious that $(\alpha_i)_{i=1}^{\infty} \in \ell_1$. By Lemma 8.6.4 in [38], there exists a decreasing sequence $(\varrho_i)_{i=1}^{\infty} \in c_0$ with $0 \leq \varrho_i \leq 1$ such that

$$\sum_{i=1}^{\infty} \varrho_i^{-1} |\alpha_i| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |\alpha_i|.$$

We conclude that

$$\begin{aligned} \sum_{i=1}^{\infty} \varrho_i^{-1} |(f_i(x) - f_i(y))\, a^*(a_i)| &\leq (1+\varepsilon) \sum_{i=1}^{\infty} |(f_i(x) - f_i(y))\, a^*(a_i)| \\ &\leq (1+\varepsilon)^2 \, \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \eta_{\sigma}^{L}(T)\, d(x,y) \, \|a^*\| . \end{aligned}$$

We can consider $\varrho_i = \beta_i^2$, where $(\beta_i) \in c_0$ such that $1 \geq \beta_1 \geq \beta_2 \geq ... \geq 0$. Thus,

$$\sum_{i=1}^{\infty} \beta_i^{-2} |(f_i(x) - f_i(y))\, a^*(a_i)| \leq (1+\varepsilon)^2 \, \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \eta_{\sigma}^{L}(T)\, d(x,y) \, \|a^*\| .$$

Let

$$F := \left\{ t = (\tau_i)_{i=1}^{\infty} \in \mathbb{C} : (\tau_i \beta_i^{-1} a_i)_{i=1}^{\infty} \text{ is unconditionally summable} \right\}$$

and

$$\|t\|_F := \sup_{a^* \in B_{E^*}} \sum_{i=1}^{\infty} |\beta_i^{-1} \tau_i a^*(a_i)|.$$

Then $(F, \|\cdot\|_F)$ is a Banach space and the sequence $(e_i)_{i=1}^{\infty}$ of standard unit vectors forms a hyperorthogonal basis in F. Indeed, let $|\gamma_i| \leq |\theta_i|$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} \gamma_i e_i \right\|_F &= \|(\gamma_i)_{i=1}^{n}\|_F \\ &= \sup_{a^* \in B_{E^*}} \sum_{i=1}^{n} \left| \gamma_i \beta_i^{-1} a^*(a_i) \right| \\ &\leq \sup_{a^* \in B_{E^*}} \sum_{i=1}^{n} \left| \theta_i \beta_i^{-1} a^*(a_i) \right| \\ &= \|(\theta_i)_{i=1}^{n}\|_F = \left\| \sum_{i=1}^{n} \theta_i e_i \right\|_F , \end{aligned}$$

for all $n \in \mathbb{N}$.

Define $A \in \mathrm{Lip}_0(X, F)$ and $B \in \mathcal{L}(F, E)$ by

$$A(x) = f_i(x), \qquad \text{for } x \in X, f_i \in X^{\#}$$

and

$$B(t) = B\left((\tau_i)_{i=1}^{\infty}\right) = B\left(\sum_{i=1}^{\infty} \tau_i e_i\right) = \sum_{i=1}^{\infty} \tau_i B(e_i) = \sum_{i=1}^{\infty} \lambda_i \tau_i a_i,$$

for $t \in F$ and $(\lambda_i)_{i=1}^{\infty} \in \ell_\infty$.

Hence, it follows that

$$B \circ A(x) = B((f_i(x))_{i=1}^{\infty}) = \sum_{i=1}^{\infty} \lambda_i f_i(x) a_i = Tx.$$

Usually, we take $P_m : F \to F$ such that $P_m((\xi_i)_{i=1}^{\infty}) := (\xi_i)_{i=1}^{m}$. Then

$$\begin{aligned}
\|(A - P_m A)x - (A - P_m A)y\|_F &= \left\|(f_i(x) - f_i(y))_{i=m+1}^{\infty}\right\|_F \\
&= \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} \beta_i^{-1} |(f_i(x) - f_i(y)) a^*(a_i)| \\
&= \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} \beta_i \beta_i^{-2} |(f_i(x) - f_i(y)) a^*(a_i)| \\
&\leq \beta_{m+1} \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} \beta_i^{-2} |(f_i(x) - f_i(y)) a^*(a_i)| \\
&\leq \beta_{m+1} (1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_\infty^{-1} \eta_\sigma^L(T) d(x,y)
\end{aligned}$$

and

$$\begin{aligned}
\|(B - BP_m)t\| &= \left\|\sum_{i=m+1}^{\infty} \lambda_i \tau_i a_i\right\| \\
&= \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} |\lambda_i \tau_i a^*(a_i)| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} |\tau_i a^*(a_i)| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} \beta_i \beta_i^{-1} |\tau_i a^*(a_i)| \\
&\leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_\infty \sup_{a^* \in B_{E^*}} \sum_{i=m+1}^{\infty} |\tau_i \beta_i^{-1} a^*(a_i)| \\
&\leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_\infty \|t\|_F.
\end{aligned}$$

Hence

$$\text{Lip}(A - P_m A) \leq \beta_{m+1}(1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \eta_{\sigma}^{L}(T)$$

and

$$\|B - BP_m\| \leq \beta_{m+1} \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}.$$

So, $A \in \overline{\text{Lip}_{0\mathcal{F}}(X,F)}$ and $B \in \overline{\mathcal{L}_f(F,E)}$ with

$$\text{Lip}(A) \leq (1+\varepsilon)^2 \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \eta_{\sigma}^{L}(T) \text{ and } \|B\| \leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}.$$

It follows that $\text{Lip}(A)\|B\| \leq (1+\varepsilon)^2 \eta_{\sigma}^{L}(T)$. Then, $\eta_{\sigma}^{L}(T) = \inf \text{Lip}(A)\|B\|$ where the infimum is taken over all possible factorizations.

$\Leftarrow$) Assume that T have the precedent factorization. Consider $A \in \overline{\text{Lip}_{0\mathcal{F}}(X,F)}, B \in \overline{\mathcal{L}_f(F,E)}$ and F is Banach space having a hyperorthogonal basis.

Let $(e_i)_{i=1}^{\infty}$ be a hyperorthogonal basis of F and $(v_i) \in F^*$ be a sequence of corresponding coordinate functionals, we have

$$A(x) = \sum_{i=1}^{\infty} v_i(A(x))\, e_i = \sum_{i=1}^{\infty} A^t v_i(x)\, e_i.$$

By composing with B, we get

$$T(x) = B \circ A(x) = B\left(\sum_{i=1}^{\infty} A^t v_i(x)\, e_i\right) = \sum_{i=1}^{\infty} A^t v_i(x)\, B(e_i).$$

Therefore, considering $f_i = A^t v_i \in X^{\#}$, as there are $(\lambda_i)_{i=1}^{\infty} \in \ell_{\infty} \setminus \{0\}$ and $(a_i)_{i=1}^{\infty} \in E$ such that $B(e_i) = \lambda_i a_i$. Then T has the representation

$$T = \textstyle\sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i.$$

In order to show that T is Lipschitz σ-nuclear operator. We have

$$\begin{aligned}
\sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right| &= \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| \frac{A^t v_i(x) - A^t v_i(y)}{d(x,y)} a^*\left(\frac{B(e_i)}{\lambda_i}\right) \right| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| \frac{v_i(Ax) - v_i(Ay)}{d(x,y)} a^*(B(e_i)) \right| \\
&= \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| v_i\left(\frac{Ax - Ay}{d(x,y)}\right) a^*(B(e_i)) \right| \\
&= \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \sup_{\substack{u \in Im_{\text{Lip}}(A) \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} |v_i(u)\, a^*(B(e_i))|.
\end{aligned}$$

Let $\varepsilon_i = \pm 1$, we have to estimate the latter expression. Observe that

$$\begin{aligned}
\sup_{\substack{u \in Im_{\mathrm{Lip}(A)} \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| v_i(u)\, a^*(B(e_i)) \right| &= \sup_{\substack{u \in Im_{\mathrm{Lip}(A)} \\ a^* \in B_{E^*}}} \left| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, a^*(B(e_i)) \right| \\
&= \sup_{\substack{u \in Im_{\mathrm{Lip}(A)} \\ a^* \in B_{E^*}}} \left| a^* \left(B \left(\sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right) \right) \right| \\
\text{(by Hahn-Banach)} &= \sup_{u \in Im_{\mathrm{Lip}(A)}} \left\| B \left(\sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right) \right\| \\
&\leq \|B\| \sup_{u \in Im_{\mathrm{Lip}(A)}} \left\| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right\|.
\end{aligned}$$

Since $(e_i)_i$ is hyperorthogonal basis and $|\varepsilon_i| = 1$, for all $i \geq 1$, then

$$\sup_{u \in Im_{\mathrm{Lip}(A)}} \left\| \sum_{i=1}^{m} \varepsilon_i v_i(u)\, e_i \right\| \leq \sup_{u \in Im_{\mathrm{Lip}(A)}} \left\| \sum_{i=1}^{m} v_i(u)\, e_i \right\|.$$

Therefore

$$\begin{aligned}
\sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right| &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{u \in Im_{\mathrm{Lip}(A)}} \left\| \sum_{i=1}^{m} v_i(u)\, e_i \right\| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{u \in Im_{\mathrm{Lip}}(A)} \|P_m(u)\|,
\end{aligned}$$

where $(P_m)_m$ the sequence of canonical projections, $P_m(u) = \sum_{i=1}^{m} v_i(u) e_i$.

Thanks to [33, Corollary 26.3 (a)], there is a constant $C \geq 1$ such that $\|P_m(u)\| \leq C\|u\|$. It follows that

$$\begin{aligned}
\sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \sum_{i=1}^{m} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right| &\leq C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{u \in Im_{\mathrm{Lip}}(A)} \|u\| \\
&= C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \sup_{x \neq y} \frac{\|A(x) - A(y)\|}{d(x,y)} \\
&= C\|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1} \|B\| \mathrm{Lip}(A) < \infty.
\end{aligned}$$

Now we will prove that

$$\lim_{m\longrightarrow\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m}^{\infty}\left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right)a^*(a_i)\right|=0.$$

We have

$$\begin{aligned}
\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right| &= \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|\frac{A^t v_i(x)-A^t v_i(y)}{d(x,y)}a^*\left(\frac{B(e_i)}{\lambda_i}\right)\right| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|\frac{v_i(Ax)-v_i(Ay)}{d(x,y)}a^*(B(e_i))\right| \\
&= \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|v_i\left(\frac{Ax-Ay}{d(x,y)}\right)a^*(B(e_i))\right| \\
&= \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\sup_{\substack{u\in Im_{\mathrm{Lip}}(A)\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|v_i(u)\,a^*(B(e_i))\right| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\sup_{\substack{u\in\overline{Im_{\mathrm{Lip}}(A)}\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|v_i(u)\,a^*(B(e_i))\right|.
\end{aligned}$$

Let $\varepsilon_i=\pm1$, we have to estimate the latter expression. Observe that

$$\begin{aligned}
\sup_{\substack{u\in\overline{Im_{\mathrm{Lip}(A)}}\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|v_i(u)\,a^*(B(e_i))\right| &= \sup_{\substack{u\in\overline{Im_{\mathrm{Lip}(A)}}\\ a^*\in B_{E^*}}}\left|\sum_{i=m+1}^{\infty}\varepsilon_i v_i(u)\,a^*(B(e_i))\right| \\
&= \sup_{\substack{u\in\overline{Im_{\mathrm{Lip}(A)}}\\ a^*\in B_{E^*}}}\left|a^*\left(B\left(\sum_{i=m+1}^{\infty}\varepsilon_i v_i(u)\,e_i\right)\right)\right| \\
&= \sup_{u\in\overline{Im_{\mathrm{Lip}(A)}}}\left\|B\left(\sum_{i=m+1}^{\infty}\varepsilon_i v_i(u)\,e_i\right)\right\| \\
&\leq \|B\|\sup_{u\in\overline{Im_{\mathrm{Lip}(A)}}}\left\|\sum_{i=m+1}^{\infty}\varepsilon_i v_i(u)\,e_i\right\|.
\end{aligned}$$

Since $(e_i)_i$ is hyperorthogonal basis and $|\varepsilon_i|=1$, for all $i\geq1$, then

$$\sup_{u\in\overline{Im_{\mathrm{Lip}(A)}}}\left\|\sum_{i=m+1}^{\infty}\varepsilon_i v_i(u)\,e_i\right\|\leq\sup_{u\in\overline{Im_{\mathrm{Lip}(A)}}}\left\|\sum_{i=m+1}^{\infty}v_i(u)\,e_i\right\|.$$

Therefore

$$\begin{aligned}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m+1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right| &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\| \sup_{u\in\overline{Im_{\mathrm{Lip}}(A)}}\left\|\sum_{i=m+1}^{\infty} v_i(u)\, e_i\right\| \\ &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{\infty}^{-1}\|B\| \sup_{u\in\overline{Im_{\mathrm{Lip}}(A)}}\|u-P_m(u)\|,\end{aligned}$$

We know that every Lipschitz approximable operators are Lipschitz compact operators [23], it follows that $\overline{Im_{\mathrm{Lip}}(A)}$ is compact in F. We have by [33, Corollary 26.3 (b)] the sequence (P_m) converges to the identity uniformly on $\overline{Im_{\mathrm{Lip}}(A)}$, i.e.

$$\lim_{m\to\infty}\sup_{u\in\overline{Im_{\mathrm{Lip}}(A)}}\|u-P_m(u)\|=0.$$

Consequently,

$$\lim_{m\longrightarrow\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m}^{\infty}\left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right)a^*(a_i)\right|=0.$$

This proves that T is Lipschitz σ-nuclear operator. □

Proposition 3.2.5. *If X is Banach space and T is bounded linear operator, then T is Lipschitz σ-nuclear if and only if it is σ-nuclear operator, with*

$$\eta_\sigma^L(T)=\eta_\sigma(T).$$

Proof. Let T be a bounded linear operator. If $T\in\mathcal{N}_\sigma^L(X,E)$, then the operator A which mentioned in Theorem 3.2.4 is linear. So, T admits the same factorization of $\mathcal{N}_\sigma(X,E)$.

The other side of the proof is obvious. □

Proposition 3.2.6. *An operator $T\in\mathcal{N}_\sigma^L(X,E)$ if and only if, its linearization $T_L\in\mathcal{N}_\sigma(Æ(X),E)$, with*

$$\eta_\sigma^L(T)=\eta_\sigma(T_L).$$

Proof. Suppose that $T\in\mathcal{N}_\sigma^L(X,E)$, then $T=B\circ A$ such that $A\in\overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$, $B\in\overline{\mathcal{L}_f(F,E)}$ and F is a Banach space having a hyperorthogonal basis.

We have

$$T_L=(B\circ A)_L=B\circ A_L.$$

Since $A\in\overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$ then $A_L\in\overline{\mathcal{L}_f(Æ(X),F)}$, so $T_L\in\mathcal{N}_\sigma(Æ(X),E)$.

Conversely, let $T_L \in \mathcal{N}_\sigma(\text{Æ}(X), E)$ then $T_L = B \circ U$ such that $U \in \overline{\mathcal{L}_f(\text{Æ}(X), F)}$ and $B \in \overline{\mathcal{L}_f(F, E)}$.

We have

$$T = T_L \circ i_X = B \circ U \circ i_X = B \circ A,$$

where $A = U \circ i_X \in \overline{\text{Lip}_{0\mathcal{F}}(X, F)}$. Therefore, $T \in \mathcal{N}_\sigma^L(X, E)$. □

Observe that $\mathcal{N}_\sigma^L$ can be written as the composition of $\mathcal{N}_\sigma$ and Lip_0; $(\mathcal{N}_\sigma^L = \mathcal{N}_\sigma \circ \text{Lip}_0)$. Indeed, let X be a pointed metric space and E be a Banach space. Take $T \in \mathcal{N}_\sigma^L(X, E)$, we have from Proposition 3.2.6 its linearization $T_L \in \mathcal{N}_\sigma(\text{Æ}(X), E)$ and $i_X \in \text{Lip}_0(X, \text{Æ}(X))$, where $T = T_L \circ i_X$. Moreover, $\eta_\sigma^L(T) = \eta_\sigma(T_L)$.

Proposition 3.2.7. *The class of Lipschitz σ-nuclear operators is a Banach Lipschitz operator ideal.*

Proof. From the fact that $(\mathcal{N}_\sigma, \eta_\sigma(\cdot))$ is a Banach operator ideal (see [11]), then $(\mathcal{N}_\sigma^L, \eta_\sigma^L)$ is a Banach Lipschitz operator ideal. □

3.3 Lipschitz σ-integral operators

The *maximal hull* of a Banach Lipschitz operator ideal was introduced by M. Cabrera-Padilla et al [14] by extending that of the linear case (see [20, Definition 17.2]).

Let $\mathcal{I}_{\text{Lip}}$ be a Banach Lipschitz operator ideal. The maximal hull of $\mathcal{I}_{\text{Lip}}$ denoted by $\mathcal{I}_{\text{Lip}}^{max}$ is the biggest Banach Lipschitz operator ideal that satisfies $\mathcal{I}_{\text{Lip}}(X_0, N) = \mathcal{I}_{\text{Lip}}^{max}(X_0, N)$ isometrically, for every finite metric space X_0 and any finite dimensional space N (see [14, Definition 2.5]).

A. Pietsch in [38, 23.3], introduced the ideal of σ-integral operators $\mathfrak{I}_\sigma$, which is the maximal hull of the ideal of σ-nuclear operators $\mathcal{N}_\sigma$ ($\mathfrak{I}_\sigma = \mathcal{N}_\sigma^{max}$). Motivated by Theorem 3.10 in [3], we can define the notion of Lipschitz σ-integral operators as the following definition.

Definition 3.3.1. *Let X be a pointed metric space and E be a Banach space. The class of all Lipschitz σ-integral operators between X and E is the maximal hull of Lipschitz σ-nuclear operators, i.e.,*

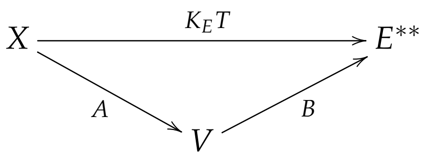
$$\mathfrak{I}_\sigma^L(X, E) := (\mathcal{N}_\sigma^L(X, E))^{max}.$$

Observe that $\mathfrak{I}^L_\sigma$ have the composition ideal. Indeed,

$$\mathfrak{I}^L_\sigma = (\mathcal{N}^L_\sigma)^{max} = (\mathcal{N}_\sigma \circ \mathrm{Lip}_o)^{max} = (\mathcal{N}_\sigma)^{max} \circ \mathrm{Lip}_o = \mathfrak{I}_\sigma \circ \mathrm{Lip}_o.$$

The following theorem is immediate from [38, Theorem 23.3.4] and the composition ideal of $\mathfrak{I}^L_\sigma$.

Theorem 3.3.2. *A Lipschitz operator $T \in \mathrm{Lip}_0(X, E)$ is Lipschitz σ-integral if and only if there is a commutative diagram*



such that $A \in \mathrm{Lip}_0(X, V)$, $B \in \mathcal{L}(V, E^{**})$ and V is some (order complete) Banach lattice.

In this case,

$$\iota^L_\sigma(T) = \inf \|B\| \mathrm{Lip}(A),$$

where the infimum is taken over all possible factorizations.

Chapter 4

Lipschitz $\sigma\,(p)$-nuclear operators

In this chapter, we introduce the notion of Lipschitz $\sigma\,(p)$-nuclear operators as a generalization of $\sigma\,(p)$-nuclear linear operators introduced by X. Mujica and G. Botelho [11] which generalised σ-nuclear linear operators introduced by A. Pietsch in[38] to multilinear case for $p > 1$.

4.1 Definitions

Firstly, we remind definition of $\sigma(p)$-nuclear operators mentioned in [34, 11].

Definition 4.1.1. *Let E, F be a Banach spaces. An operator $T \in \mathcal{L}(E, F)$ is called $\sigma(p)$-nuclear if there are sequences $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$, $(a_i^*)_{i=1}^{\infty} \subset E^*$ and $(b_i)_{i=1}^{\infty} \subset F$, such that $T = \sum\limits_{i=1}^{\infty} \lambda_i a_i^* \otimes b_i$,*

$$\sup_{\substack{a \in B_E \\ b^* \in B_{F^*}}} \left(\sum_{i=1}^{\infty} |a_i^*(a) b^*(b_i)|^p \right)^{\frac{1}{p}} < \infty$$

and

$$\lim_{m \to \infty} \sup_{\substack{a \in B_E \\ b^* \in B_{F^*}}} \left(\sum_{i=m}^{\infty} |a_i^*(a) b^*(b_i)|^p \right)^{\frac{1}{p}} = 0,$$

The set of all $\sigma(p)$-nuclear operators is denoted by $\mathcal{N}_{\sigma(p)}\,(E, F)$ and the $\sigma(p)$-nuclear norm is denoted by $\eta_{\sigma(p)}(T)$ and it is defined by

$$\eta_{\sigma(p)}(T) := \inf\left\{ \|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \sup_{\substack{a\in B_E \\ b^*\in B_{F^*}}} \left(\sum_{i=1}^{\infty} |a_i^*(a)b^*(b_i)|^p\right)^{\frac{1}{p}} \right\},$$

where the infimum is taken over all $\sigma(p)$-nuclear representations.

Let us introduce the class of Lipschitz $\sigma(p)$-nuclear operators.

Definition 4.1.2. *We say that a Lipschitz operator T from a pointed metric space X to a Banach space E is Lipschitz $\sigma(p)$-nuclear if there are sequences $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$, $(f_i)_{i=1}^{\infty} \subset X^{\#}$ and $(a_i)_{i=1}^{\infty} \subset E$ such that $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$,*

$$\sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} < \infty$$

and

$$\lim_{m\longrightarrow\infty} \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=m}^{\infty} \left| \left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} = 0.$$

The set of all Lipschitz $\sigma(p)$-nuclear operators is denoted by $\mathcal{N}_{\sigma(p)}^L(X,E)$ and the Lipschitz $\sigma(p)$-nuclear norm is denoted by $\eta_{\sigma(p)}^L(T)$ and it is defined by

$$\eta_{\sigma(p)}^L(T) = \inf\left\{ \left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*} \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} \right\},$$

where the infimum is taken over all Lipschitz $\sigma(p)$-nuclear representations of T.

Proposition 4.1.3. *Let X be a pointed metric space and E be a Banach space. Then*

$$\mathcal{N}_{\sigma(p)}^L(X,E) \subset \mathcal{N}_{\sigma}^L(X,E).$$

Proof. Give $T \in \mathcal{N}_{\sigma(p)}^L(X,E)$. We have $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i = \sum_{i=1}^{\infty} f_i \otimes b_i$, where $b_i = \lambda_i a_i \in E$, note that T have a Lipschitz σ-nuclear representation. Thanks to Hölder's inequality, we have

$$\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(b_i)\right| = \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}|\lambda_i|\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|$$

$$\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}<\infty$$

and

$$\lim_{m\longrightarrow\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(b_i)\right| = \lim_{m\longrightarrow\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=m}^{\infty}|\lambda_i|\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|$$

$$\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\lim_{m\longrightarrow\infty}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=m}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}=0.$$

So, T is Lipschitz σ-nuclear operator, with

$$\eta_{\sigma}^{L}(T) = \inf\left\{\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(b_i)\right|\right\}$$

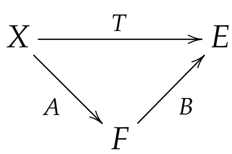
$$\leq \inf\left\{\|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\right\}=\eta_{\sigma(p)}^{L}(T).$$

□

4.2 Factorization theorems

The Lipschitz $\sigma(p)$-nuclear can be factorized as the following theorem, for its proof, we use also the same techniques of [38, Theorem 23.2.5] and [34].

Theorem 4.2.1. *An operator $T\in\mathcal{N}_{\sigma(p)}^{L}(X,E)$ if and only if, the following diagram is commutes*



where, $A \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$*, B diagonal and F is a Banach space having a hyperorthogonal basis* $(e_i)_{i=1}^{\infty}$*, with* $B(e_i) = \lambda_i a_i$*,* $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$ *and* $a_i \in E$ *such that* $(\tau_i a_i)_{i=1}^{\infty} \in \ell_p^u(E)$ *for* $(\tau_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \tau_i e_i \in F$*. And we have*

$$\eta_{\sigma(p)}^{L}(T) = \inf \mathrm{Lip}(A) \|B\|$$

where the infimum is taken over all possible factorizations.

Proof. Consider $\varepsilon > 0$, $T \in \mathcal{N}_{\sigma(p)}^{L}(X,E)$ and let $\sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$ be a representation of T, such that

$$\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \leq (1+\varepsilon)\, \eta_{\sigma(p)}^{L}(T).$$

We have for all $x, y \in X$ with $x \neq y$ and $a^* \in E^*$,

$$\begin{aligned} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} \frac{a^*(a_i)}{\|a^*\|} \right|^p \right)^{\frac{1}{p}} &\leq \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\ &\leq \left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1} (1+\varepsilon)\, \eta_{\sigma(p)}^{L}(T), \end{aligned}$$

it follows that

$$\sum_{i=1}^{\infty} \left| (f_i(x) - f_i(y))\, a^*(a_i) \right|^p \leq \left((1+\varepsilon) \left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1} \eta_{\sigma(p)}^{L}(T)\, d(x,y) \|a^*\| \right)^p.$$

We put $\left| (f_i(x) - f_i(y))\, a^*(a_i) \right|^p = |\alpha_i|$, obvious that $(\alpha_i)_{i=1}^{\infty} \in \ell_1$. By Lemma 8.6.4 in [38], there exists a decreasing sequence $(\varrho_i)_{i=1}^{\infty} \in c_0$ with $0 \leq \varrho_i \leq 1$ such that

$$\sum_{i=1}^{\infty} \varrho_i^{-1} |\alpha_i| \leq (1+\varepsilon) \sum_{i=1}^{\infty} |\alpha_i|.$$

We conclude that

$$\begin{aligned} \sum_{i=1}^{\infty} \varrho_i^{-1} \left| (f_i(x) - f_i(y))\, a^*(a_i) \right|^p &\leq (1+\varepsilon) \sum_{i=1}^{\infty} \left| (f_i(x) - f_i(y))\, a^*(a_i) \right|^p \\ &\leq (1+\varepsilon) \left((1+\varepsilon) \left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1} \eta_{\sigma(p)}^{L}(T)\, d(x,y) \|a^*\| \right)^p. \end{aligned}$$

So,

$$\left(\sum_{i=1}^{\infty}\varrho_i^{-1}\left|\left(f_i(x)-f_i(y)\right)a^*(a_i)\right|^p\right)^{\frac{1}{p}} \leq (1+\varepsilon)^{\frac{1}{p}}(1+\varepsilon)\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1}\eta_{\sigma(p)}^L(T)\,d(x,y)\,\|a^*\|$$
$$\leq (1+\varepsilon)^2\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1}\eta_{\sigma(p)}^L(T)\,d(x,y)\,\|a^*\|.$$

We can consider $\varrho_i=\beta_i^{2p}$, where $(\beta_i)\in c_0$ such that $1\geq\beta_1\geq\beta_2\geq...\geq 0$. Thus,

$$\left(\sum_{i=1}^{\infty}\beta_i^{-2p}\left|\left(f_i(x)-f_i(y)\right)a^*(a_i)\right|^p\right)^{\frac{1}{p}}\leq(1+\varepsilon)^2\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1}\eta_{\sigma(p)}^L(T)\,d(x,y)\,\|a^*\|.$$

Let

$$F:=\left\{t=(\tau_i)_{i=1}^{\infty}\in\mathbb{C}:(\tau_i\beta_i^{-1}a_i)_{i=1}^{\infty}\text{ is unconditionally }p\text{-summable}\right\}$$

and

$$\|t\|_F:=\sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{\infty}|\beta_i^{-1}\tau_i a^*(a_i)|^p\right)^{\frac{1}{p}}.$$

Then $(F,\|\cdot\|_F)$ is a Banach space and the sequence $(e_i)_{i=1}^{\infty}$ of standard unit vectors forms a hyperorthogonal basis in F. Indeed, let $|\gamma_i|\leq|\theta_i|$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}\gamma_i e_i\right\|_F &= \|(\gamma_i)_{i=1}^{n}\|_F\\ &= \sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{n}\left|\gamma_i\beta_i^{-1}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\\ \left(\left|\gamma_i\beta_i^{-1}a^*(a_i)\right|\leq\left|\theta_i\beta_i^{-1}a^*(a_i)\right|\right)\quad &\leq \sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{n}\left|\theta_i\beta_i^{-1}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\\ &= \|(\theta_i)_{i=1}^{n}\|_F=\left\|\sum_{i=1}^{n}\theta_i e_i\right\|_F,\end{aligned}$$

for all $n\in\mathbb{N}$.

Define $A\in\mathrm{Lip}_0(X,F)$ by

$$A(x)=f_i(x),\qquad\text{for } x\in X, f_i\in X^{\#}.$$

Usually, we take $P_m : F \to F$ such that $P_m((\xi_i)_{i=1}^{\infty}) := (\xi_i)_{i=1}^{m}$. Then

$$\begin{aligned}
\|(A-P_mA)x-(A-P_mA)y\|_F &= \left\|(f_i(x)-f_i(y))_{i=m+1}^{\infty}\right\|_F \\
&= \sup_{a^*\in B_{E^*}}\left(\sum_{i=m+1}^{\infty}\beta_i^{-p}|(f_i(x)-f_i(y))a^*(a_i)|^p\right)^{\frac{1}{p}} \\
&= \sup_{a^*\in B_{E^*}}\left(\sum_{i=m+1}^{\infty}\beta_i^{p}\beta_i^{-2p}|(f_i(x)-f_i(y))a^*(a_i)|^p\right)^{\frac{1}{p}} \\
&\leq \beta_{m+1}\sup_{a^*\in B_{E^*}}\left(\sum_{i=m+1}^{\infty}\beta_i^{-2p}|(f_i(x)-f_i(y))a^*(a_i)|^p\right)^{\frac{1}{p}} \\
&\leq \beta_{m+1}(1+\varepsilon)^2\|\,(\lambda_i)_{i=1}^{\infty}\,\|_{p^*}^{-1}\eta_{\sigma(p)}^{L}(T)d(x,y)
\end{aligned}$$

Hence $$\mathrm{Lip}(A-P_mA)\leq \beta_{m+1}(1+\varepsilon)^2\|(\lambda_i)_{i=1}^{\infty}\|_{p^*}^{-1}\eta_{\sigma(p)}^{L}(T).$$

So $A\in\overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$ with

$$\mathrm{Lip}(A)\leq(1+\varepsilon)^2\|(\lambda_i)_{i=1}^{\infty}\|_{p^*}^{-1}\eta_{\sigma(p)}^{L}(T).$$

Now, we define $B\in\mathcal{L}(F,E)$ as follows

$$B(t)=B\left((\tau_i)_{i=1}^{\infty}\right)=B\left(\sum_{i=1}^{\infty}\tau_ie_i\right)=\sum_{i=1}^{\infty}\tau_iB(e_i)=\sum_{i=1}^{\infty}\lambda_i\tau_ia_i.$$

Observe that B is diagonal and we have

$$\begin{aligned}
\|B(t)\| &= \left\|\sum_{i=1}^{\infty}\lambda_i\tau_ia_i\right\| = \sup_{a^*\in B_{E^*}}\sum_{i=1}^{\infty}|\lambda_i\tau_ia^*(a_i)| \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{\infty}|\tau_ia^*(a_i)|^p\right)^{\frac{1}{p}} \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{\infty}\beta_i^{p}\beta_i^{-p}|\tau_ia^*(a_i)|^p\right)^{\frac{1}{p}} \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\beta_1\sup_{a^*\in B_{E^*}}\left(\sum_{i=1}^{\infty}|\tau_i\beta_i^{-1}a^*(a_i)|^p\right)^{\frac{1}{p}} \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\|t\|_F.
\end{aligned}$$

So, $\|B\|\leq\|(\lambda_i)_{i=1}^{\infty}\|_{p^*}$.

Hence, it follows that

$$B \circ A(x) = B(f_i(x)) = \sum_{i=1}^{\infty} \lambda_i f_i(x) a_i = Tx$$

and $\mathrm{Lip}(A)\|B\| \leq (1+\varepsilon)^2 \eta^L_{\sigma(p)}(T)$. Then, $\eta^L_{\sigma(p)}(T) = \inf \mathrm{Lip}(A)\|B\|$ where the infimum is taken over all possible factorizations.

Conversely, assume that T have the precedent factorization. Consider $A \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}, B \in \mathcal{L}(F,E)$ diagonal and F is Banach space having a hyperorthogonal basis.

Let $(e_i)_{i=1}^{\infty}$ be a hyperorthogonal basis of F and $(v_i) \in F^*$ be a sequence of corresponding coordinate functionals, we have

$$A(x) = \sum_{i=1}^{\infty} v_i (A(x)) e_i = \sum_{i=1}^{\infty} A^t v_i (x) e_i.$$

By composing with B, we get

$$T(x) = B \circ A(x) = B\left(\sum_{i=1}^{\infty} A^t v_i (x) e_i\right) = \sum_{i=1}^{\infty} A^t v_i (x) B(e_i).$$

Therefore, considering $f_i = A^t v_i \in X^\#$, as $B(e_i) = \lambda_i a_i$, $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$ and $(a_i)_{i=1}^{\infty} \in E$. So T has the representation

$$T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i.$$

We prove that T is Lipschitz $\sigma(p)$-nuclear operator.

Since $\left(\frac{f_i(x) - f_i(y)}{d(x,y)} a_i\right)_{i=1}^{\infty} \in \ell_p^u(E)$, then

$$\sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \left|\left(\frac{f_i(x) - f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} < \infty$$

and

$$\lim_{m \longrightarrow \infty} \sup_{a^* \in B_{E^*}} \left(\sum_{i=m}^{\infty} \left|\left(\frac{f_i(x) - f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} = 0,$$

for all $x, y \in X$ with $x \neq y$.

This implies

$$\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{\infty}\left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*\left(a_i\right)\right|^p\right)^{\frac{1}{p}} < \infty$$

and

$$\lim_{m\longrightarrow\infty} \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}} \left(\sum_{i=m}^{\infty}\left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*\left(a_i\right)\right|^p\right)^{\frac{1}{p}} = 0.$$

So, $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$. □

The following theorem generalizes Theorem 1 of [27].

Theorem 4.2.2. *Let X be a pointed metric space, E be a Banach space and let $T : X \to E$ be a Lipschitz map. Then the following statements are equivalent.*

(a) $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$.

(b) For the Banach spaces $\mathcal{U}$ and $\mathcal{V}$ defined by (3.1) and (3.2) respectively, and the diagonal operator D defined by (3.3), there exist a Lipschitz operator $R \in \mathcal{N}^L_{\sigma}(X,Z)$ and a linear operator $S \in \mathcal{N}_{\sigma(p)}(W,E)$ with $\eta_{\sigma(p)}(S) \leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ \mathcal{U} & \xrightarrow[D]{} & \mathcal{V} \end{array}$$

In this case,

$$\eta^L_{\sigma(p)}(T) = \inf \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\eta^L_{\sigma}(R),$$

where the infimum is taken over all such factorization of T in (b).

Proof. Consider $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$ and $\varepsilon > 0$ be given. Then, there exist $(f_i)_{i=1}^{\infty} \subset X^{\#}$, $(a_i)_{i=1}^{\infty} \subset E$ and $(\lambda_i)_{i=1}^{\infty} \in \ell_{p^*}$ such that

$$T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$$

and

$$\|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)} a^*\left(a_i\right)\right|^p\right)^{\frac{1}{p}} \leq (1+\varepsilon)\eta^L_{(p)}(T).$$

We have for all $x, y \in X$ with $x \neq y$ and $a^* \in E^*$,

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}\frac{a^*(a_i)}{\|a^*\|}\right|^p\right)^{\frac{1}{p}} &\leq \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\\ &\leq \left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1}(1+\varepsilon)\,\eta_{\sigma(p)}^L(T),\end{aligned}$$

it follows that

$$\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p \leq \left((1+\varepsilon)\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*}^{-1}\eta_{\sigma(p)}^L(T)\,\|a^*\|\right)^p.$$

We put $\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p = |\alpha_i|$, obvious that $(\alpha_i)_{i=1}^{\infty} \in \ell_1$. By [38, Lemma 8.6.4], there exists a decreasing sequence $(\varrho_i)_{i=1}^{\infty} \in c_0$ with $0 \leq \varrho_i \leq 1$ such that

$$\sum_{i=1}^{\infty}\varrho_i^{-1}|\alpha_i| \leq (1+\varepsilon)\sum_{i=1}^{\infty}|\alpha_i|.$$

We conclude that

$$\sum_{i=1}^{\infty}\varrho_i^{-1}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p \leq (1+\varepsilon)\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p.$$

So,

$$\begin{aligned}\left(\sum_{i=1}^{\infty}\varrho_i^{-1}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}} &\leq (1+\varepsilon)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\\ &\leq (1+\varepsilon)\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}\end{aligned}$$

We can consider $\varrho_i = \beta_i^{2p}$, where $(\beta_i) \in c_0$ such that $1 \geq \beta_1 \geq \beta_2 \geq ... \geq 0$. Thus,

$$\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=1}^{\infty}\beta_i^{-2p}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}} \leq (1+\varepsilon)\sup_{\substack{x\neq y\\ a^*\in B_{E^*}}}\left(\sum_{i=1}^{\infty}\left|\frac{f_i(x)-f_i(y)}{d(x,y)}a^*(a_i)\right|^p\right)^{\frac{1}{p}}.$$

Define

$$R : X \to \mathcal{U}, \quad R(x) = (f_i(x))_{i=1}^{\infty} = \sum_{i=1}^{\infty} f_i(x) e_i$$

Observe that R to have a Lipschitz σ-nuclear representation $R = \sum_{i=1}^{\infty} f_i \otimes e_i$, for $x \neq y$ and $z^* \in \mathcal{U}^*$ we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} z^*(e_i) \right| &= \sum_{i=1}^{\infty} \zeta_i \frac{f_i(x) - f_i(y)}{d(x,y)} z^*(e_i) (|\zeta_i| = 1) \\
&\leq \|z^*\| \left\| \sum_{i=1}^{\infty} \zeta_i \frac{f_i(x) - f_i(y)}{d(x,y)} e_i \right\|_{\mathcal{U}} \\
&= \|z^*\| \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \beta_i^{-2p} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\
&\leq (1+\varepsilon) \|z^*\| \sup_{a^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

It follows that

$$\sup_{\substack{x \neq y \\ z^* \in B_{\mathcal{U}^*}}} \sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} z^*(e_i) \right| \leq (1+\varepsilon) \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*\,(a_i) \right|^p \right)^{\frac{1}{p}} < \infty.$$

On the other hand, for all $m \in \mathbb{N}$ and using the same steps as previous, we get

$$\lim_{m \to \infty} \sup_{\substack{x \neq y \\ z^* \in B_{\mathcal{U}^*}}} \sum_{i=m}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} z^*(e_i) \right| \leq (1+\varepsilon) \lim_{m \to \infty} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=m}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*\,(a_i) \right|^p \right)^{\frac{1}{p}} = 0.$$

Hence $R \in \mathcal{N}_{\sigma}^{L}(X, \mathcal{U})$ and

$$\eta_{\sigma}^{L}(R) \leq (1+\varepsilon) \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*\,(a_i) \right|^p \right)^{\frac{1}{p}}.$$

Now, define

$$S : \mathcal{V} \to E, \quad S((\gamma_k)_{k=1}^{\infty}) = \sum_{i=1}^{\infty} \lambda_i \beta_i \gamma_i a_i = \sum_{i=1}^{\infty} \lambda_i \beta_i v_i^*((\gamma_k)_{k=1}^{\infty}) a_i,$$

where each $v_i^* \in \mathcal{V}^*$ is the i-th coordinate functional.

We note that $S = \sum_{i=1}^{\infty} \lambda_i \beta_i v_i^* \otimes a_i$, we have

$$\begin{aligned}
\sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} |\beta_i v_i^*((\gamma_k)_{k=1}^{\infty}) a^*(a_i)|^p \right)^{\frac{1}{p}} &= \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} |\beta_i \gamma_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} |\gamma_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \sup_{\gamma \in B_{\mathcal{V}}} \|(\gamma_i)_{i=1}^{\infty}\|_{\mathcal{V}} \leq 1 < \infty
\end{aligned}$$

and

$$\begin{aligned}
\lim_{m \to \infty} \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=m}^{\infty} |\beta_i v_i^*((\gamma_k)_{k=1}^{\infty}) a^*(a_i)|^p \right)^{\frac{1}{p}} &= \lim_{m \to \infty} \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=m}^{\infty} |\beta_i \gamma_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \lim_{m \to \infty} \beta_m \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=m}^{\infty} |\gamma_i a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \lim_{m \to \infty} \beta_m \sup_{\gamma \in B_{\mathcal{V}}} \|(\gamma_i)_{i=1}^{\infty}\|_{\mathcal{V}} \\
&\leq \lim_{m \to \infty} \beta_m = 0.
\end{aligned}$$

Hence $S \in \mathcal{N}_{\sigma(p)}(\mathcal{V}, E)$ and

$$\begin{aligned}
\eta_{\sigma(p)}(S) &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \sup_{\substack{\gamma \in B_{\mathcal{V}} \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} |\beta_i v_i^*((\gamma_k)_{k=1}^{\infty}) a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}.
\end{aligned}$$

Clearly, $T = S \circ D \circ R$ and

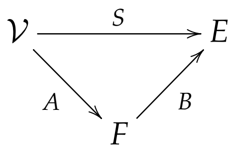
$$\begin{aligned}
\inf \|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \eta_{\sigma}^{L}(R) &\leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \eta_{\sigma}^{L}(R) \\
&\leq (1+\varepsilon) \|(\lambda_i)_{i=1}^{\infty}\|_{p^*} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\
&\leq (1+\varepsilon)^2 \eta_{\sigma(p)}^{L}(T).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, then

$$\inf \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\eta_{\sigma}^{L}(R) \leq \eta_{\sigma(p)}^{L}(T).$$

Conversely, given $\varepsilon > 0$ and let $T = SDR$ such that, $R \in \mathcal{N}_{\sigma}^{L}(X,\mathcal{U})$ and $S \in \mathcal{N}_{\sigma(p)}(\mathcal{V},E)$ with $\eta_{\sigma(p)}(S) \leq \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}$.

Since $S \in \mathcal{N}_{\sigma(p)}(\mathcal{V},E)$ then, by [34, Theorem 1.6.2] it holds that



where, $A \in \overline{\mathcal{L}_f(\mathcal{V},F)}, B \in \mathcal{L}(F,E)$ diagonal and F is a Banach space having a hyperorthogonal basis, with $B(e_i) = \lambda_i a_i$ such that $(\tau_i a_i)_{i=1}^{\infty} \in \ell_p^u(E)$ for $\tau_i = \sum_{i=1}^{\infty} \tau_i e_i \in F$. Moreover,

$$\|A\|\|B\| \leq (1+\varepsilon)\eta_{\sigma(p)}(S).$$

Obviously, $A \circ D \circ R \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$ and

$$\begin{aligned} \mathrm{Lip}(ADR) &\leq \|A\|\|D\|\mathrm{Lip}(R) \\ &\leq \eta_{\sigma}(D)\eta_{\sigma}^{L}(R)\|A\| \leq \eta_{\sigma}^{L}(R)\|A\|. \end{aligned}$$

Observe that T admits a factorization as Theorem 4.2.1, so $T \in \mathcal{N}_{\sigma(p)}^{L}(X,E)$ with

$$\begin{aligned} \eta_{\sigma(p)}^{L}(T) &= \inf \mathrm{Lip}(ADR)\|B\| \\ &\leq \inf \eta_{\sigma}^{L}(R)\|A\|\|B\| \\ &\leq (1+\varepsilon)\inf \eta_{\sigma}^{L}(R)\eta_{\sigma(p)}(S) \\ &\leq (1+\varepsilon)\inf \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\eta_{\sigma}^{L}(R). \end{aligned}$$

Since, ε was arbitrary, then

$$\eta_{\sigma(p)}^{L}(T) \leq \inf \|(\lambda_i)_{i=1}^{\infty}\|_{p^*}\eta_{\sigma}^{L}(R),$$

where the infimum is taken over all such factorization of T. $\square$

4.3 Further results

Proposition 4.3.1. *If X is Banach space and T is bounded linear operator, then T is Lipschitz $\sigma(p)$-nuclear if and only if it is $\sigma(p)$-nuclear operator, with*

$$\eta^L_{\sigma(p)}(T) = \eta_{\sigma(p)}(T).$$

Proof. Let T be a bounded linear operator. If $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$, then the operator A which mentioned in Theorem 4.2.1 is linear. So, T admits the same factorization of $\mathcal{N}_{\sigma(p)}(X,E)$.

The other side of the proof is obvious. $\square$

Proposition 4.3.2. *An operator $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$ if, and only if its linearization $T_L \in \mathcal{N}_{\sigma(p)}(Æ(X),E)$, with*

$$\eta^L_{\sigma(p)}(T) = \eta_{\sigma(p)}(T_L).$$

Proof. Suppose that $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$, then $T = B \circ A$ such that $A \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$ and $B \in \mathcal{L}(E,F)$ diagonal and F is a Banach space having a hyperorthogonal basis, with $B(e_i) = \lambda_i a_i$, $(\lambda_i)_{i=1}^\infty \in \ell_{p^*}$ and $a_i \in E$ such that $(\tau_i a_i)_{i=1}^\infty \in \ell^u_p(E)$ for $(\tau_i)_{i=1}^\infty = \sum_{i=1}^\infty \tau_i e_i \in F$.

We have

$$T_L = (B \circ A)_L = B \circ A_L.$$

Since $A \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$ then $A_L \in \overline{\mathcal{L}_f(Æ(X),F)}$, so $T_L \in \mathcal{N}_{\sigma(p)}(Æ(X),E)$.

Conversely, let $T_L \in \mathcal{N}_{\sigma(p)}(Æ(X),E)$ then $T_L = B \circ U$ such that $U \in \overline{\mathcal{L}_f(Æ(X),F)}$ and $B \in \overline{\mathcal{L}_f(E,F)}$.

We have

$$T = T_L \circ i_X = B \circ U \circ i_X = B \circ A,$$

where $A = U \circ i_X \in \overline{\mathrm{Lip}_{0\mathcal{F}}(X,F)}$. $\square$

Observe that $\mathcal{N}^L_{\sigma(p)}$ can be written as the composition of $\mathcal{N}_{\sigma(p)}$ and Lip_0; $\left(\mathcal{N}^L_{\sigma(p)} = \mathcal{N}_{\sigma(p)} \circ \mathrm{Lip}_0\right)$. Indeed, let X be a pointed metric space and E be a Banach space. Take $T \in \mathcal{N}^L_{\sigma(p)}(X,E)$, we have from Proposition 4.3.2 its linearization $T_L \in \mathcal{N}_{\sigma(p)}(Æ(X),E)$ and $i_X \in \mathrm{Lip}_0(X,Æ(X))$, where $T = T_L \circ i_X$. Moreover, $\eta^L_{\sigma(p)}(T) = \eta_{\sigma(p)}(T_L)$.

Proposition 4.3.3. $\left(\mathcal{N}^L_{\sigma(p)}, \eta^L_{\sigma(p)}\right)$ *is a Banach Lipschitz operator ideal.*

Proof. From the fact that $\left(\mathcal{N}_{\sigma(p)}, \eta_{\sigma(p)}(\cdot)\right)$ is a Banach operator ideal (see [11]), then $\left(\mathcal{N}^L_{\sigma(p)}, \eta^L_{\sigma(p)}\right)$ is a Banach Lipschitz operator ideal. $\square$

4.4 The dual space of $\mathcal{N}^L_{\sigma(p)}$

Since $\left(\mathcal{N}^L_{\sigma(p)}, \eta^L_{\sigma(p)}\right)$ is a Banach Lipschitz operator ideal, we have $\mathrm{Lip}_{0\mathcal{F}}(X,E) \subseteq \mathcal{N}^L_{\sigma(p)}(X,E)$ and

$$\eta^L_{\sigma(p)}(\lambda f \otimes a) = |\lambda| \mathrm{Lip}(f) \|a\|,$$

for all $\lambda \in \mathbb{K}, f \in X^{\#}, a \in E$. Using partial sums of $\sigma(p)$-nuclear representations it is easy to see that $\mathrm{Lip}_{0\mathcal{F}}(X,E)$ is $\eta^L_{\sigma(p)}(\cdot)$-dense in $\mathcal{N}^L_{\sigma(p)}(X,E)$. For $A \in \mathrm{Lip}_{0\mathcal{F}}(X,E)$, define

$$\eta^L_{\sigma(p)f}(A) = \inf\left\{ \left\|(\lambda_i)_{i=1}^m\right\|_{p^*} \sup_{\substack{x\neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^m \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right|^p \right)^{\frac{1}{p}} \right\},$$

where the infimum runs over all finite representations $A = \sum_{i=1}^m \lambda_i f_i \otimes a_i$. It is easy to check that $\eta^L_{\sigma(p)f}(\cdot)$ is a norm on $\mathrm{Lip}_{0\mathcal{F}}(X,E)$ such that $\mathrm{Lip}(\cdot) \leq \eta^L_{\sigma(p)f}(\cdot)$. It follows that $\eta^L_{\sigma(p)f}(\cdot)$ is a complete norm on $\mathrm{Lip}_{0\mathcal{F}}(X,E)$. It is clear that $\eta^L_{\sigma(p)}(\cdot) \leq \eta^L_{\sigma(p)f}(\cdot)$. For further use, we shall establish conditions under which the equality holds.

Lemma 4.4.1. *If the norms $\eta^L_{\sigma(p)}(\cdot)$ and $\eta^L_{\sigma(p)f}(\cdot)$ are equivalent on $\mathrm{Lip}_{0\mathcal{F}}(X,E)$, then they coincide on this space.*

Proof. By assumption there is a constant $c > 0$ such that $\eta^L_{\sigma(p)f}(\cdot) \leq c\eta^L_{\sigma(p)}(\cdot)$ on $\mathrm{Lip}_{0\mathcal{F}}(X,E)$. Given a Lipschitz operator $T \in \mathrm{Lip}_{0\mathcal{F}}(X,E)$ and $\varepsilon > 0$, take an infinite Lipschitz $\sigma(p)$-nuclear representation $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i$, such that

$$\left\|(\lambda_i)_{i=1}^{\infty}\right\|_{p^*} \sup_{\substack{x\neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{\infty} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \leq \left(1 + \frac{\varepsilon}{2}\right) \eta^L_{\sigma(p)}(T).$$

In particular, by the definition of $\eta^L_{\sigma(p)f}(\cdot)$. But $T_{m-1} = \sum_{i=1}^{m-1} \lambda_i f_i \otimes a_i$, we get

$$\begin{aligned} \eta^L_{\sigma(p)f}(T_{m-1}) &\leq \left\|(\lambda_i)_{i=1}^{m-1}\right\|_{p^*} \sup_{\substack{x\neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{m-1} \left| \frac{f_i(x) - f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \eta^L_{\sigma(p)}(T). \end{aligned}$$

Since

$$\lim_{m\to\infty} \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=m}^{\infty} \left| \frac{f_i(x)-f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} = 0,$$

by the definition of $\eta^L_{\sigma(p)}(\cdot)$, for a sufficiently large $m \in \mathbb{N}$ we get

$$\begin{aligned} \eta^L_{\sigma(p)}(T - T_{m-1}) &\leq \left\| (\lambda_i)_{i=m}^{\infty} \right\|_{p^*} \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=m}^{\infty} \left| \frac{f_i(x)-f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\ &\leq \left\| (\lambda_i)_{i=1}^{\infty} \right\|_{p^*} \sup_{\substack{x\neq y \\ a^*\in B_{E^*}}} \left(\sum_{i=m}^{\infty} \left| \frac{f_i(x)-f_i(y)}{d(x,y)} a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\ &\leq \frac{\varepsilon}{2c} \eta^L_{\sigma(p)}(T). \end{aligned}$$

It follows that

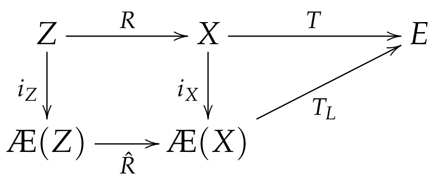
$$\begin{aligned} \eta^L_{\sigma(p)f}(T) &\leq \eta^L_{\sigma(p)f}(T_{m-1}) + \eta^L_{\sigma(p)f}(T - T_{m-1}) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \eta^L_{\sigma(p)}(T) + c\eta^L_{\sigma(p)}(T - T_{m-1}) \\ &\leq \left(1 + \frac{\varepsilon}{2}\right) \eta^L_{\sigma(p)}(T) + \frac{\varepsilon}{2}\eta^L_{\sigma(p)}(T) = (1+\varepsilon)\eta^L_{\sigma(p)}(T). \end{aligned}$$

And as this holds for every $\varepsilon > 0$, the result follows. □

Lemma 4.4.2. *If $T \in \mathcal{N}^L_{\sigma(p)}(X, E)$ and $R \in \mathrm{Lip}_{0\mathcal{F}}(Z, X)$, then*

$$\eta^L_{\sigma(p)f}(T \circ R) \leq \eta^L_{\sigma(p)}(T)\, \mathrm{Lip}(R).$$

Proof. We have the following diagram



From the above diagram we can obvious that $(T \circ R)_L = T_L \circ \hat{R}$. Since $T \in \mathcal{N}^L_{\sigma(p)}(X, E)$ and $R \in \mathrm{Lip}_{0\mathcal{F}}(Z, X)$, then $T_L \in \mathcal{N}_{\sigma(p)}(\text{Æ}(X), E)$, $\hat{R} \in \mathcal{L}_f(\text{Æ}(Z), \text{Æ}(X))$ and $(T \circ R)_L = T_L \circ \hat{R} \in$

$\mathcal{N}_{\sigma(p)}(\text{Æ}(Z), E)$, such that $\eta_{\sigma(p)}(T_L) = \eta^L_{\sigma(p)}(T)$, $\|\hat{R}\| = \text{Lip}(R)$ and we have

$$\begin{aligned} \eta^L_{\sigma(p)f}(T \circ R) &= \eta_{\sigma(p)f}((T \circ R)_L) \\ &= \eta_{\sigma(p)f}(T_L \circ \hat{R}) \\ \text{(By [11, Lemma 2.5])} &\leq \eta_{\sigma(p)}(T_L)\|\hat{R}\| \\ &= \eta^L_{\sigma(p)}(T)\text{Lip}(R). \end{aligned}$$

□

Proposition 4.4.3. *If $X^\#$ have the bounded approximation property, then $\eta^L_{\sigma(p)f}(\cdot) = \eta^L_{\sigma(p)}(\cdot)$ on $\text{Lip}_{0\mathcal{F}}(X, E)$ regardless of the Banach space E.*

Proof. Let $\gamma \geq 1$ be such that $X^\#$ has the γ-bounded approximation property. Given $T \in \text{Lip}_{0\mathcal{F}}(X, E)$ and $\varepsilon > 0$. So, $T_L \in \mathcal{L}_f(\text{Æ}(X), E)$. Since $\text{Æ}(X)^* = X^\#$ have the bounded approximation property, we can apply [38, Lemma 10.2.6] for T_L, then there is an operator $B \in \mathcal{L}_f(\text{Æ}(X), \text{Æ}(X))$ such that $\|B\| \leq (1+\varepsilon)\gamma$ and $T_L \circ B = T_L$.
Obvious that B is linearization of Lipschitz operator $A : X \longrightarrow X$ such that $\text{Lip}(A) = \|B\| \leq (1+\varepsilon)\gamma$, thus $A \in \text{Lip}_{0\mathcal{F}}(X, X)$ and we have

$$(T \circ A)_L = T_L \circ B = T_L \qquad i.e., \quad T \circ A = T.$$

Consequently, By Lemma 4.4.2 we have

$$\eta^L_{\sigma(p)f}(T) = \eta^L_{\sigma(p)f}(T \circ A) \leq \eta^L_{\sigma(p)}(T)\text{Lip}(A) \leq (1+\varepsilon)\gamma\eta^L_{\sigma(p)}(T).$$

Letting $\varepsilon \to 0$ we get $\eta^L_{\sigma(p)f}(T) \leq \gamma\eta^L_{\sigma(p)}(T)$. The result follows from Lemma 4.4.1. □

Our aim is to represent bounded linear functionals on the space $\mathcal{N}^L_{\sigma(p)}(X, E)$ of Lipschitz $\sigma(p)$-nuclear operators. Since this space contains the Lipschitz operators, the Borel transform

$$\mathcal{B} : [\mathcal{N}^L_{\sigma(p)}(X, E), \eta^L_{\sigma(p)}(\cdot)]' \longrightarrow \mathcal{L}(X^\#, E^*),$$

$$\mathcal{B}(\varphi)(f)(a) := \varphi(f \otimes a),$$

is a well defined linear operator. The question is to identify the range of $\mathcal{B}$ and a norm on it that makes $\mathcal{B}$ an isometric isomorphism.

Now, we proceed to show that bounded linear functionals on the space of Lipschitz $\sigma(p)$-nuclear operators are represented by quasi-$\tau(p)$-summing operators via the Borel transform.

Let us justify an equality we shall use soon: iterating the well known equality

$$\sup_{\delta_{xy}\in B_{(X^{\#})^*}} \left(\sum_{i=1}^{m} |\delta_{xy}(f_i)|^p\right)^{\frac{1}{p}} = \sup_{x\neq y} \left(\sum_{i=1}^{m} \left|\frac{f_i(x)-f_i(y)}{d(x,y)}\right|^p\right)^{\frac{1}{p}},$$

for $(f_i)_{i=1}^{\infty} \subset X^{\#}$, it follows that

$$\sup_{\substack{\delta_{xy}\in B_{(X^{\#})^*}\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{m} |\delta_{xy}(f_i)a^*(a_i)|^p\right)^{\frac{1}{p}} = \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{m} \left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}},$$

for $(f_i)_{i=1}^{\infty} \subset X^{\#}$ and $(a_i)_{i=1}^{\infty} \subset E$.

Theorem 4.4.4. *If $X^{\#}$ have the bounded approximation property, then the spaces $[\mathcal{N}^L_{\sigma(p)}(X,E)]'$ and $\Pi_{q\tau(p)}(X^{\#},E^*)$ are isometrically isomorphic via the Borel transform, regardless of the Banach space E.*

Proof. Given $\varphi \in [\mathcal{N}^L_{\sigma(p)}(X,E)]'$, let us denote $\mathcal{B}(\varphi)$ by S_φ. In order to prove that $S_\varphi \in \Pi_{q\tau(p)}(X^{\#},E^*)$, let $m\in\mathbb{N}, f_1,...,f_m \in X^{\#}, a_1,...,a_m \in E$, be given. By duality $(\ell_p^m)^* = \ell_{p^*}^m$ and the Hahn-Banach Theorem, there are scalars $\varepsilon_1,...,\varepsilon_m$ such that $\|(\varepsilon_i)_{i=1}^m\|_{p^*} = 1$ and

$$\Big(\sum_{i=1}^{m} |\varphi(f_i\otimes a_i)|^p\Big)^{\frac{1}{p}} = \Big|\sum_{i=1}^{m} \varepsilon_i\varphi(f_i\otimes a_i)\Big| = \Big|\sum_{i=1}^{m} \varepsilon_i\varphi(f_i\otimes a_i)\Big|.$$

So,

$$\begin{aligned}
\left(\sum_{i=1}^{m} |S_\varphi(f_i)(a_i)|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^{m} |\varphi(f_i\otimes a_i)|^p\right)^{\frac{1}{p}} = \left|\varphi\left(\sum_{i=1}^{m}\varepsilon_i f_i\otimes a_i\right)\right| \\
&\leq \|\varphi\|\cdot\eta^L_{\sigma(p)}\left(\sum_{i=1}^{m}\varepsilon_i f_i\otimes a_i\right) \\
&\leq \|\varphi\|\|(\varepsilon_i)_{i=1}^m\|_{p^*} \sup_{\substack{x\neq y\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{m} \left|\left(\frac{f_i(x)-f_i(y)}{d(x,y)}\right) a^*(a_i)\right|^p\right)^{\frac{1}{p}} \\
&= \|\varphi\| \sup_{\substack{\delta_{xy}\in B_{(X^{\#})^*}\\ a^*\in B_{E^*}}} \left(\sum_{i=1}^{m} |\delta_{xy}(f_i)a^*(a_i)|^p\right)^{\frac{1}{p}},
\end{aligned}$$

proving that S_φ is quasi-$\tau(p)$-summing and $\|S_\varphi\|_{q\tau(p)} \leq \|\varphi\|$.

Conversely, given $S \in \Pi_{q\tau(p)}(X^\#, E^*)$, define

$$T_S : X^\# \times E \longrightarrow \mathbb{K},\ T_S(f,a) := S(f)(a).$$

It is plain that T_S is linear. So, having in mind that $X^\# \otimes E = \mathrm{Lip}_{0\mathcal{F}}(X,E)$, by the universal property of the tensor product there exists a linear operator $\mathcal{T}_S : \mathrm{Lip}_{0\mathcal{F}}(X,E) \longrightarrow \mathbb{K}$ such that

$$\mathcal{T}_S(f \otimes a) = T_S(f,a) = S(f)(a),$$

for all $f \in X^\#, a \in E$. Now we shall prove that $\mathcal{T}_S$ is continuous with respect to the norm $\eta^L_{\sigma(p)}(\cdot)$. Given $\varepsilon > 0$ and $A \in \mathrm{Lip}_{0\mathcal{F}}(X,E)$, by definition of the norm $\eta^L_{\sigma(p)f}(\cdot)$ we can choose a representation $A = \sum_{i=1}^{m} \lambda_i f_i \otimes a_i$, such that

$$\|(\lambda_i)_{i=1}^m\|_{p^*} \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{m} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right|^p \right)^{\frac{1}{p}} \leq (1+\varepsilon)\eta^L_{\sigma(p)f}(A).$$

Therefore,

$$\begin{aligned}
|\mathcal{T}_S(A)| &= \left| \mathcal{T}_S \left(\sum_{i=1}^{m} \lambda_i f_i \otimes a_i \right) \right| \\
&= \left| \sum_{i=1}^{m} \lambda_i S(f_i)(a_i) \right| \\
&\leq \|(\lambda_i)_{i=1}^m\|_{p^*} \cdot \left(\sum_{i=1}^{m} |S(f_i)(a_i)|^p \right)^{\frac{1}{p}} \\
&\leq \|(\lambda_i)_{i=1}^m\|_{p^*} \cdot \|S\|_{q\tau(p)} \cdot \sup_{\substack{\delta_{xy} \in B_{(X^\#)^*} \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{m} |\delta_{xy}(f_i) a^*(a_i)|^p \right)^{\frac{1}{p}} \\
&= \|S\|_{q\tau(p)} \cdot \|(\lambda_i)_{i=1}^m\|_{p^*} \cdot \sup_{\substack{x \neq y \\ a^* \in B_{E^*}}} \left(\sum_{i=1}^{m} \left| \left(\frac{f_i(x) - f_i(y)}{d(x,y)} \right) a^*(a_i) \right|^p \right)^{\frac{1}{p}} \\
&\leq \|S\|_{q\tau(p)} (1+\varepsilon) \eta^L_{\sigma(p)f}(A).
\end{aligned}$$

As this holds for arbitrary $\varepsilon > 0$ and the space $X^\#$ have the bounded approximation property, by Proposition 4.4.3 we conclude that

$$|\mathcal{T}_S(A)| \leq \|S\|_{q\tau(p)} \cdot \eta^L_{\sigma(p)f}(A) = \eta^L_{\sigma(p)}(A).$$

So, $\mathcal{T}_S \in [\mathrm{Lip}_{0\mathcal{F}}(X,E), \eta^L_{\sigma(p)}(\cdot)]'$ and $\|\mathcal{T}_S\| \leq \|S\|_{q\tau(p)}$. As $\mathrm{Lip}_{0\mathcal{F}}(X,E)$ is $\eta^L_{\sigma(p)}(\cdot)$-dense in $\mathcal{N}^L_{\sigma(p)}(X,E)$, there is a unique norm-preserving continuous linear extension φ_S of $\mathcal{T}_S$ to the whole of $\mathcal{N}^L_{\sigma(p)}(X,E)$. In particular, $\|\varphi_S\| \leq \|S\|_{q\tau(p)}$ and for $A = \sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i \in \mathcal{N}^L_{\sigma(p)}(X,E)$,

$$\begin{aligned}
\varphi_S(A) &= \varphi_S\left(\sum_{i=1}^{\infty} \lambda_i f_i \otimes a_i\right) \\
&= \sum_{i=1}^{\infty} \lambda_i \varphi_S(f_i \otimes a_i) \\
&= \sum_{i=1}^{\infty} \lambda_i \mathcal{T}_S(f_i \otimes a_i) \\
&= \sum_{i=1}^{\infty} \lambda_i S(f_i)(a_i).
\end{aligned}$$

From the expression above it follows easily that correspondences $\varphi \mapsto S_\varphi$ and $S \mapsto \varphi_S$ are each other's inverse in the sense that $\varphi_{S_\varphi} = \varphi$ and $S_{\varphi_S} = S$ for $\varphi \in [\mathcal{N}^L_{\sigma(p)}(X,E)]'$ and $S \in \Pi_{q\tau(p)}(X^\#, E^*)$. The equality $\|S_\varphi\|_{q\tau(p)} = \|\varphi\|$ completes the proof. □

Corollary 4.4.5. *If $X^\#$ have the bounded approximation property and E is reflexive, then the spaces $[\mathcal{N}^L_{\sigma(p)}(X,E)]'$ and $\Pi_{\tau(p)}(X^\#, E^*)$ are isometrically isomorphic via the Borel transform.*

4.5 $\sigma(p)$-integral mappings: some possible directions for investigation

Thanks to the idea of extending the maximal hull of Banach Lipschitz operator ideal was introduced by Cabrera-Padilla et al in their paper [14] and developed by Achour et al [3] for the composition Banach Lipschitz operator ideal, we were able to extend σ-integral operators to the nonlinear setting as presented at the end of Chapter 3. Based on the definition of $\sigma(p)$-nuclear operators that Botelho and Mujica gave us in their works [11, 34], as well as the definition of Lipschitz $\sigma(p)$-nuclear operators that we mentioned in this chapter, we inspired another possible generalizations for σ-integral operators as follows.

- Let E and F be a Banach spaces. The ideal of $\sigma(p)$-integral operators between E and F is the maximal hull of $\sigma(p)$-nuclear operators, i.e.,

$$\mathfrak{I}_{\sigma(p)}(E,F) := (\mathcal{N}_{\sigma(p)}(E,F))^{max}.$$

- Let X be a pointed metric space and E be a Banach space. The ideal of Lipschitz $\sigma(p)$-integral operators between X and E is the maximal hull of Lipschitz $\sigma(p)$-nuclear operators, i.e.,

$$\begin{aligned} \mathfrak{I}^L_{\sigma(p)}(X,E) &= (\mathcal{N}^L_{\sigma(p)}(X,E))^{max} = (\mathcal{N}_{\sigma(p)} \circ \mathrm{Lip}_o)^{max} \\ &= (\mathcal{N}_{\sigma(p)})^{max} \circ \mathrm{Lip}_o = \mathfrak{I}_{\sigma(p)} \circ \mathrm{Lip}_o \end{aligned}$$

- For $n \in \mathbb{N}$. Let $E_1, ..., E_n$ and F be a Banach spaces. The ideal of n-linear $\sigma(p)$-integral operators between $E_1 \times \cdots \times E_n$ and F is the maximal hull of n-linear $\sigma(p)$-nuclear operators, i.e.,

$$\mathfrak{I}_{\sigma(p)}(E_1 \times \cdots \times E_n, F) := (\mathcal{N}_{\sigma(p)}(E_1 \times \cdots \times E_n, F))^{max}.$$

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الملخص: في الآونة الأخيرة، تم تطوير و دراسة العديد من فئات المؤثرات الخطية إلى حالة المؤثرات الغير خطية من طرف الكثير من الباحثين. الهدف من رسالتنا هو تعميم و توسيع فئات جديدة من المؤثرات الخطية إلى المؤثرات الغير خطية و بصفة خاصة الليبشيتزية. من بين هذه الفئات، درسنا فئة المؤثرات p−شبكية الجمعية و فئة المؤثرات $\sigma(p)$−نووية في الحالات الليبشيتزية. هذه هي المحاور الرئيسية لهذه الأطروحة.

الكلمات المفتاحية: المؤثرات الليبشيتزية، فضاء آرنس−آلس، المؤثرات الليبشيتزية p−شبكية الجمعية، المؤثرات الليبشيتزية $\sigma(p)$−نووية، المؤثرات شبه −$\tau(p)$−جمعية.

Résumé: Récemment, Beaucoup classes des opérateurs linéaires ont été développées et étudiées dans le cas non-linéaire par plusieurs auteurs. Le but de notre thèse est la généralisation et l'extension des nouvelles classes d'opérateurs linéaires aux opérateurs non linéaires (Lipschitz). Parmi ces classes, nous avons étudié la classe des opérateurs p-lattice sommants et la classe des opérateurs $\sigma(p)$-nucléaires dans le cas de Lipschitz. Tels sont les axes principaux de cette thèse.

Mots-clés: Les opérateurs Lipschitiziens, espace de Arens-Eells, les opérateurs Lipschitz p-lattice sommants, les opérateurs Lipschitz $\sigma(p)$-nucléaires, les opérateurs quasi-$\tau(p)$-sommants.

Abstract: Recently, many classes of linear operators have been developed and studied in the nonlinear setting by several authors. The aim of our thesis is the generalization and extension of new classes of linear operators to nonlinear operators (Lipschitz). Among these classes, we studied the class of p-lattice summing operators and the class of $\sigma(p)$-nuclear operators in the Lipschitz cases. These are the main axes of this thesis.

Keywords: Lipschitz operator, Arens-Eells space, Lipschitz p-lattice summing operators, Lipschitz $\sigma(p)$-nuclear operators, quasi-$\tau(p)$-summing operators.