ÉTUDE NON-CLASSIQUE DE QUELQUES INÉGALITÉS FAISANT INTERVENIR LA FONCTION D'EULER φ

By

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To My Father and Mother

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Résumé

Le cadre de ce travail revient au théorie IST (Internal Set Theory, voir [35]). En vue de [5], rapplons la définition suivante: On dit que les réels ξ_i du système $(\xi_1, \xi_2, \ldots, \xi_k)$ avec $k \geq 1$ sont simultanément approximables au sens infinitésimal, si pour tout nombre réel infiniment petit positif ε, il existe des rationnels $\left(\dfrac{p_i}{q}\right)_{i=1,2,\ldots,k}$ tels que

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon \pounds_i \\ \varepsilon q \cong 0 \end{cases} ; \; i = 1, 2, \ldots k,$$

où $(\pounds_i)_{i=1,2,\ldots,k}$ sont des nombres limités.

Soit $(\xi_0, \xi_1, \ldots, \xi_\omega)$ un système de réels, avec ω illimité. Dans le premier chapitre, nous allons donner une condition nécessaire pour que $(\xi_i)_{i=0,1,\ldots,\omega}$ soient simultanément approximable au sens infinitésimale. L'inverse de cette condition est également discuté.

Soient k un entier positif et W_k l'ensemble des entiers positifs n dont le nombre des facteurs premiers distincts de n est supérieur ou égal à k. Dans le deuxième chapitre, certaines inégalités qui font intervenir plusieurs fonctions arithmétiques sont étudiées sur W_k. Dans le troisième chapitre, nous allons déterminer une fonction arithmétique f pour laquelle l'expression $f^N(n) - \alpha f^N(n + \ell)$ a une infinité de changement de signe sur un sous-ensemble propre infini de W_k, où α, N et ℓ sont des paramètres. Ce résultat sera réalisé en utilisant le théorème de Dirichlet concernant les nombres premiers dans un progression arithmétique.

Dans le quatrième chapitre, certains problèmes ouverts se posent pour d'autres recherches.

Abstract

This study is placed in the framework of Internal Set Theory (see [35]: E. Nelson, *Internal Set Theory*, Bull. Amer. Math. soc. 83 (1977), p. 1165-1198). In view of [5], we recall the following definition: Real numbers $(\xi_i)_{i=1,2,\ldots,k}$ are called simultaneously approximable in the infinitesimal sense, if for every positive infinitesimal ε, there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,\ldots,k}$ such that

$$\left\{\begin{array}{c} \xi_i = \dfrac{p_i}{q} + \varepsilon \pounds_i \\ \varepsilon q \cong 0 \end{array}\right. \quad ; \; i = 1, 2, \ldots k,$$

where $(\pounds_i)_{i=1,2,\ldots,k}$ are limited numbers. Let $(\xi_0, \xi_1, \ldots, \xi_\omega)$ be a system of reals, with ω unlimited. In Chapter 1, we will give a necessary condition for which $(\xi_i)_{i=0,1,\ldots,\omega}$ are simultaneously approximable in the infinitesimal sense. The converse of this condition is also discussed.

Let k be a positive integer, and let W_k denote the set of all positive integers n such that the number of distinct prime factors of n is greater or equal to k. In Chapter 2, we prove the infinity of certain inequalities and equations on the set W_k by using ψ, σ, τ and related functions. In the framework of IST, our working set is an infinite external subset of positive integers. In Chapter 3, we will determine an arithmetic function $f : \mathbb{N} \to \mathbb{N}$ for which $f^N(n) - \alpha f^N(n + \ell)$ has infinitely many sign changes on a proper infinite subset of W_k, where α, N and ℓ are parameters. This result will be realized by using Dirichlet's Theorem about primes in an arithmetic progression.

In Chapter 4, some open problems are posed for further research.

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Bellaouar, Djamel
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Notations

Notation	**Explanation**
$(\xi_1, \xi_2, \ldots, \xi_k)$	A given system of real numbers $(\xi_i)_{i=1,2,\ldots,k}$, usually irrationals.
$\pounds$	A limited number.
$\mathbb{N}^\sigma$	The set of limited positive integers.
ε	A positive infinitesimal real number.
ϕ	An infinitesimal real number (positive or negative).
ω	An unlimited positive integer.
$x \cong y$	Means x, y are infinitely close.
$x \ncong y$	Means x, y are not infinitely close.
$\left(\frac{p_i}{q}\right)_{i=1,2,\ldots,k}$	Rational numbers with common denominator q.
$^\circ(\xi_1, \xi_2, \ldots, \xi_k)$	Is the shadow of $(\xi_1, \xi_2, \ldots, \xi_k)$.
$S_A (\cong 0)$	The set of all systems $(\xi_1, \xi_2, \ldots, \xi_k)$ for which $(\xi_i)_{i=1,2,\ldots,k}$ are simultaneously approximable in the infinitesimal sense.
$\omega(n)$	The number of distinct prime factors of n.
$\gamma(n)$	The Kernel of n given by $\gamma(1) = 1$ and $\gamma(n) = \prod_{p\|n} p$; for $n \geq 2$.
$\tau(n)$	The number of the positive divisors of $n : \tau(n) = \sum_{d\|n} 1$.
$\sigma(n)$	The sum of the positive divisors of $n : \sigma(n) = \sum_{d\|n} d$.
$\sigma_k(n)$	The sum of the k-th positive divisors of $n : \sigma_k(n) = \sum_{d\|n} d^k$; $k \geq 2$.
f^a	Denote the arithmetic function: $f^a(n) = (f(n))^a$; $a \geq 1$.

p_n	The n-th prime number, with $n \geq 1$.
d_n	Denote the gap between p_{n+1} and p_n.
$\mathbb{P}$	The set of prime numbers.
$2 = p_1 < p_2 < \ldots < p_n < \ldots$	The sequence of all primes in increasing order.
$q_1 < q_2 < \ldots < q_n < \ldots$	Denotes an arbitrary sequence of primes.
$p^a \Vert n$	Means, p^a divides n but p^{a+1} does not divide.
$\Omega(n)$	The total number of prime divisors of n: $\Omega(n) = \sum_{p^a \Vert n} a$, and $\Omega(1) = 0$.
$\pi(n)$	The number of primes not exceeding $n : \pi(n) = \sum_{p \leq n} 1$
$\varphi(n) = \varphi_1(n)$	Euler's function : $\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$; $n \geq 2$.
$\varphi_s(n)$	Is given by $\varphi_s(n) = n^s \prod_{p \mid n} \left(1 - \frac{1}{p^s}\right)$; $n \geq 2$.
$\psi_s(n)$	Is given by $\psi_s(n) = n^s \prod_{p \mid n} \left(1 + \frac{1}{p^s}\right)$; $n \geq 2$.
W_k	The set of positive integers n for which the number of distinct prime factors of n is larger or equal to k.
W_∞	The set of positive integers n for which the number of distinct prime factors of n is unlimited and every prime that divides n is also unlimited.
$\overline{W_\infty}$	The set of positive integers n for which the number of distinct prime factors of n is unlimited.
$A_{r,s}$	The set given by: $A_{r,s} = \{n \in \mathbb{N}\ ;\ p_r \varphi_s(n) > p_{r-1} n^s\}$.
$B_{r,s}$	The set given by: $B_{r,s} = \{n \in \mathbb{N}\ ;\ p_{r-1} \psi_s(n) < p_r n^s\}$.
$A_\ell(c)$	The set of positive integers n such that 2^{c-1} divides n or $n + \ell$ and 2^c does not divide both n and $n + \ell$.
$[x]$	The greatest integer less than or equal to x.
$\{x\}$	The fractional part of x.
F_n	Fermat numbers: $F_n = 2^{2^n} + 1\ ; n \geq 1$.
M_p	Mersenne numbers: $M_p = 2^p - 1$, with p prime.
$\frac{P_n}{Q_n}$	The n-th convergent of a simple continued fraction.

Introduction

The theory I.S.T is created by the fact that E. Nelson (1977) added in the usual mathematical language a unary predicate "standard". After this and to govern the use of this predicate, he introduced three axioms which are : Transfer, Idealization and Standardization. The axioms of Z.F.C and the three axioms mentioned above form what is called I.S.T theory. For further information on nonstandard analysis and the theory of I.S.T one can see ([12], [13], [14], [35], [41]). Currently, I.S.T became a framework of researches by a large number of mathematicians, where the field of its applications expanded increasingly.

The objective of this thesis is to introduce the theory of I.S.T in the field of number theory, and especially in the study of the simultaneous rational approximation and the solubility of certain Diophantine inequalities involving several arithmetic functions. This thesis is composed of an introduction, three chapters and a small part devoted to some open problems. First, we give the following definition.

Definition 0.0.1 ([5],[6])**.** The reals of $(\xi_1, \xi_2, \ldots, \xi_n)$ are simultaneously approximable according to the infinitesimal sense if for every positive infinitesimal ε there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,\ldots,n}$ such that

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon.\pounds \\ \varepsilon.q \cong 0, \end{cases} ; 1 \leq i \leq n, \qquad (*)$$

where $\pounds$ is a limited.

This quality of approximation can not be deducted directly from classical results such as Dirichlet's theorem and Kronecker's theorem [21], [45]. Indeed, from classical results if we approximate with a small error we must generally use a large denominator and vice versa, if we want to approximate by using a small denominator the error may be quite large. This quality of approximation with the control of error and of the denominator at a time, was behind the invention and the use of this concept by A. Boudaoud according to the following chronology: unidimensional case (1988), multidimensional case (1993, 1996, 2000) and finally the simultaneous approximation according to the infinitesimal sense of reals of a system whose cardinality is unlimited (2004). For more details, see [4], [5],[6], [7].

In the first chapter, we recall some basic notions of the simultaneous diophantine approximation and the simultaneous diophantine approximation according to the infinitesimal sense. Moreover, we denote by $S_A\,(\cong 0)$ the set of all systems $(\xi_1, \xi_2, \ldots, \xi_n)$ satisfying $(*)$, then we investigate this set by giving some examples. Also, we give a necessary condition for the reals of a system $(\xi_1, \xi_2, \ldots, \xi_n)$ in order to be simultaneously approximable according to the infinitesimal sense, where the converse of this condition was also discussed. The results of this chapter have been the subject of the following publication: "Dj. Bellaouar and A. Boudaoud, *Non-classical Study on the Simultaneous Rational Approximation*, Malaysian Journal of Mathematical Sciences **9**(2), 209 -225 (2015)" [2].

In the second chapter we introduced the following sets: $W_k = \{n \in \mathbb{N} \; ; \omega(n) \geq k\}$ and $W_\infty = \{n \in \mathbb{N} \; ; \omega\,(n) \cong +\infty, \text{ and for all } p|n : p \cong +\infty\}$, where $\omega\,(n)$ denotes the number of distinct prime factors of n. On W_k, we have shown that for the pair of arithmetic functions (f, g), where $f\,(n) = \psi_s\,(n)$ and $g\,(n) = n^s$, the difference

$f(n) - \alpha g(n)$ changes sign infinitely many times, where α is a rational number constructed using two consecutive primes. Here we observe the traces of approximation by rationals because for some n we have $\dfrac{f(n)}{g(n)} > \alpha$ and for others we have $\dfrac{f(n)}{g(n)} < \alpha$. That is, we surround α from the right and from the left by rational numbers. Besides this, we realized other results on W_k involving arithmetic functions such as ω, Ω, γ, τ and σ.

Recall that $W_\infty \subset W_k$ whenever k is limited. On W_∞, as before, we studied by using I.S.T the sign changes of $p_{r-1}\psi_s(n) - p_r n^s$. On the same set, we proved that there is a particular function f that satisfies the inequality $f(n) + \ell > \alpha\, n$ over a finite part of W_∞ whose cardinality is unlimited, where ℓ is a parameter. At the end of this chapter we gave the solubility of some external equations.

In the third chapter we studied, firstly, arithmetic functions f for which the expression $f^N(n) - \alpha f^N(n+\ell)$ has infinitely many sign changes. We noticed that each of the following functions γ , π, d_n and φ_s satisfies the required condition on a suitable subset. Then, by using I.S.T, we studied some inequalities on W_k and others on $\overline{W_\infty}$, where $\overline{W_\infty}$ is the following set: $\overline{W}_\infty = \{n \in \mathbb{N}\ ;\ \omega(n) \cong +\infty\}$.

Finally, we finish our work by a small section devoted to some open problems and perspectives.

We mention that some results of chapters two and three, have been the subject of an article entitled "The Infinity of some Arithmetic Inequalities by using Particular Subsets of Positive Integers" submitted to the Thai Journal of Mathematics.

Chapter 1

Non-classical Study on the Simultaneous Rational Approximation

Dirichlet's approximation theorem ([45]) is a fundamental result in Diophantine approximation, showing that any real number has a sequence of good rational approximations. In 1988, A. Boudaoud [4] introduced a non-classical study to the rational approximation in unidimensional case. Concerning higher dimensional cases, in 1996 [5] he proved that the reals $(\frac{i}{\omega})_{i=1,2,\ldots,\omega}$ are not simultaneously approximable in the infinitesimal sense, where ω is an unlimited positive integer. In 2000 [6], he proved that the reals of any system contains a limited number of reals are simultaneously approximable in the same sense. Moreover, in 2004 [7], he proved that the reals

$$\Big(\frac{q_0}{N!}, \frac{q_0\, q_1}{N!}, \ldots, \frac{q_0\, q_1 \cdots q_\omega}{N!}\Big),$$

are simultaneously approximable in the infinitesimal sense, where N, ω are unlimited and $+\infty \cong q_0 < q_1 < \ldots < q_\omega < N$.

In Chapter 1, our working set will be denoted by $S_A\,(\cong 0)$, is an external subset which contains systems of the form $(\xi_0, \xi_1, \ldots, \xi_k)$, where the reals ξ_i are simultaneously approximable in the infinitesimal sense [5]. Then we prove that $S_A(\cong 0)$ does not contain systems of the form $(\xi_0, \xi_1, \ldots, \xi_\omega)$ whose shadow contains a standard interval $[a, b]$, see [11]. Furthermore, some examples are given in Section 1.3, where we consider only systems which contain an unlimited number of reals.

1.1 Essential notions and preliminaries

1.1.1 Elementary nonstandard notions

We recall some definitions and facts from nonstandard analysis, real numbers, and sets that will be used in the proof of our main result. For more details, see I.P. Van den Berg [13], F. Diener and M. Diener [14], F. Diener and G. Reeb [16], Lutz and Goze [17] and E. Nelson [35].

(a) A real number x is called *unlimited* if its absolute value $|x|$ is larger than any standard integer n. So a nonstandard integer w is also an unlimited real number.

(b) A real number ε is called *infinitesimal* if its absolute value $|\varepsilon|$ is smaller than $\frac{1}{n}$ for any standard n.

(c) A real number r is called *limited* if is not unlimited and *appreciable* if it is neither unlimited nor infinitesimal.

(d) Two real numbers x and y are equivalent (written $x \cong y$) if their difference $x-y$ is infinitesimal.

(f) We distinguish two types of formulas: Formulas which do not contain the symbol '*st*' (for standard) are called *internal*, and formulas which do contain the symbol '*st*' are called *external*.

(g) We call internal any set defined using an internal formula.

(h) We call external any subset of an internal set defined using of an external formula for which a classical theorem at least is in default.

Example 1.1.1. *From F. Diener and M. Diener ([14, p. 06, 19]) and F. Diener and G. Reeb ([16, p. 52]), we have*

1. *Let ε be a positive real number (infinitesimal or not). The following sets are internal.*

$$[1-\varepsilon, 1+\varepsilon]\,, \;\; \{x \in \mathbb{R}\ ;\ \varepsilon x \geq 1\} \;\; \textit{and} \;\; \left\{\frac{n}{\varepsilon}\ ;\ n \in \mathbb{N}\right\}.$$

2. *Let $\mathbb{N}^\sigma$ be the set of limited (standard) positive integers, then $\mathbb{N}^\sigma$ is external. In fact, if it is not we can apply the Principle of Mathematical Induction:*

   - *Since 1 is limited, it follows that $1 \in \mathbb{N}^\sigma$.*
   - *If $s \in \mathbb{N}^\sigma$, then $s+1 \in \mathbb{N}^\sigma$.*

   Therefore $\mathbb{N}^\sigma = \mathbb{N}$, which is impossible because there are unlimited positive integers.

3. *Let ε be an infinitesimal positive real number. The set*

$$\{x \in \mathbb{R}\ ;\ x + \varepsilon \cong x\}$$

   is equal to $\mathbb{R}$. i.e., it is internal. However, the set

$$\{x \in \mathbb{R}\ ;\ x \cong 0\} \tag{1.1.1}$$

is external. In fact, if the set of (1.1.1) is internal, then it has the least upper bound a*; which is neither infinitesimal nor appreciable (If* a *is infinitesimal,* $2a$ *and* $3a$ *are also. If* a *is appreciable,* $\frac{a}{2}$ *is also). That is, the Least Upper Bound Principle "*A nonempty set of reals which is bounded above has the least upper bound*" is in default.*

Lemma 1.1.1 (Robinson's Lemma [28], [41])**.** *If* $(u_n)_{n\geq 0}$ *is a sequence such that* $u_n \cong 0$ *for all standard* n*, there exists an unlimited* N *such that* $u_n \cong 0$ *for all* $n \leq N$*.*

Also, in this chapter, we need to the following notions (see N. Cutland [11], I.P. Van den Berg [13] and F. Diener and M. Diener [14]).

Definition 1.1.1. Let X be a standard set, and let $(A_x)_{x\in X}$ be an internal family of sets.

1. A union of the form $G = \underset{stx\in X}{\cup} A_x$ is called a *pregalaxy*; if it is external G is called a *galaxy.*

2. An intersection of the form $H = \underset{stx\in X}{\cap} A_x$ is called a *prehalo*; if it is external H is called a *halo*.

Example 1.1.2. *We have,*

(a) $\mathbb{N}^{\sigma}$ *is a galaxy.*

(b) $hal\,(0) = \{x \in \mathbb{R}\ ;\ x \cong 0\}$ *is a halo.*

Theorem 1.1.2 (Fehrele's Principle [17])**.** *No halo is a galaxy.*

Definition 1.1.2 (Shadow of a set [14], [17])**.** The shadow of a set A, denoted by $^\circ A$, is the unique standard set whose standard elements are precisely those whose halo intersects A.

Theorem 1.1.3 (Cauchy's Principle [14])**.** *No external set is internal.*

For example, let w be an unlimited positive integer. The shadow of

$$\left(\frac{1}{w}, \frac{2}{w}, \ldots, \frac{w-1}{w}, \frac{w}{w}\right)$$

is equal to $[0, 1]$. Moreover, we see that $e^n < w$ for every standard positive integer n. From Cauchy's Principle there exists an unlimited integer n_0 which satisfies the previous inequality.

In this work limited numbers are denoted by £ and infinitesimal numbers are denoted by ϕ.

1.1.2 Some classical results on the simultaneous rational approximation

We present some well known results on the simultaneous approximation of k numbers $\xi_1, \xi_2, \ldots, \xi_k$ by fractions $\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_k}{q}$. These results were announced by Dirichlet's Theorem [45, page 27], Kronecker's Theorem [21, page 382], and many others.

Theorem 1.1.4 (Dirichlet's Theorem)**.** *Let k be a positive integer, and let $\xi_1, \xi_2, \ldots, \xi_k$ be reals. For any integer $Q > 1$, we can find positive integers q, $p_1, p_2, \ldots, p_k$ such that*

$$1 \leq q < Q^k \text{ and } |q\,\xi_i - p_i| \leq \frac{1}{Q} \;\; ; \text{ for } i = 1, 2, \ldots, k.$$

Theorem 1.1.5 ([21]. page 170)**.** *If* $\xi_1, \xi_2, \ldots, \xi_k$ *are any real numbers, then the system of inequalities*

$$\left|\xi_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+\mu}} \ , \ \mu = \frac{1}{k} \ ; \ \textit{for } i = 1, 2, \ldots, k.$$

has at least one solution. If one ξ_i *at least is irrational, then it has an infinity of solutions.*

Theorem 1.1.6 ([21]. page 170)**.** *Given* $\xi_1, \xi_2, \ldots, \xi_k$ *and any positive* ε*, we can find an integer* q *so that* $q\,\xi_i$ *differs from an integer, for every* i*, by less than* ε*.*

Definition 1.1.3 ([21]. page 170)**.** A set of numbers $\xi_1, \xi_2, \ldots, \xi_r$ is linearly independent if no linear relation:

$$a_1\xi_1 + a_2\xi_2 + \ldots + a_r\xi_r = 0,$$

with integer coefficients, not all zero, holds between them.

Theorem 1.1.7 (Kronecker's Theorem)**.** *If* $\xi_1, \xi_2, \ldots, \xi_k$ *are linearly independent,* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are arbitrary, and* Q *and* ε *are positive, then there are integers*

$$q > Q, \ p_1, p_2, \ldots, p_k$$

such that $|q\,\xi_i - p_i - \alpha_i| < \varepsilon$ *, for i=1,2,...,k.*

1.1.3 Why giving the simultaneous rational approximation in the infinitesimal sense?

In 1990's A. Boudaouad ([5],[6], [7]) proved some results on the simultaneous rational approximation in the infinitesimal sense [4]. He mainly relied on the following definition.

Definition 1.1.4 ([5],[7])**.** Let $(\xi_1, \xi_2, \ldots, \xi_k)$ be a system of reals, with $k \geq 1$. The reals $(\xi_i)_{i=1,2,\ldots,k}$ are said to be simultaneously approximable in the infinitesimal sense, if for every positive infinitesimal ε, there exist rational numbers $\left(\dfrac{p_i}{q}\right)_{i=1,2,\ldots,k}$ such that

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon.\pounds_i \\ \qquad \varepsilon q \cong 0, \end{cases} ; 1 \leq i \leq k, \tag{1.1.2}$$

where $(\pounds_i)_{i=1,2,\ldots,k}$ are limited numbers.

From Definition 1.1.4, the reals $(\xi_i)_{i=1,2,\ldots,k}$ are simultaneously approximable in the infinitesimal sense if for every positive infinitesimal ε, there exist rational numbers $\left(\dfrac{p_i}{q}\right)_{i=1,2,\ldots,k}$ such that

$$\left|\xi_i - \frac{p_i}{q}\right| = \varepsilon.\pounds \text{ and } \varepsilon.q \cong 0.$$

The real ε allows to control the error $\varepsilon.\pounds$ and the denominator q at the same time.

This type of approximation can not be deducted directly from classical results such as Dirichlet's Theorem and Kronecker's theorem. In fact, let ε an infinitesimal positive real number. From Dirichlet's Theorem, for any integer $Q > 1$, the reals $(\xi_i)_{i=1,2,\ldots,k}$ are simultaneously approximable by the rational numbers $\left(\dfrac{p_i}{q}\right)_{i=1,2,\ldots,k}$, with an error less than $\dfrac{1}{qQ}$. That is,

$$\xi_i = \frac{p_i}{q} + e_i, \text{ with } |e_i| \leq \frac{1}{qQ} \text{ and } 1 \leq q < Q^k;\ i = 1, 2, \ldots, k.$$

There are two cases.

- If one can choose Q such that $\varepsilon Q \cong 0$, then the errors e_i may not be all of the form $\varepsilon.\pounds$, because the commun denominator q may be sufficiently small so that $\dfrac{1}{qQ} = \varepsilon.\gamma$, with $\gamma \cong +\infty$.

- If one can choose Q such that $\varepsilon Q \ncong 0$, then the errors e_i are of the form $\varepsilon.\pounds$. However the commun denominator q may be sufficiently large so that $\varepsilon q \ncong 0$, because $1 \leq q \leq Q^k$.

Similarly, from Theorem 1.1.6, for every positive infinitesimal ε we have

$$\xi_i = \frac{p_i}{q} + \varepsilon \pounds_i, \text{ with } |\pounds_i| \leq \frac{1}{q} = \pounds;\ i = 1, 2, \ldots, k.$$

Also, we get the same result by using Theorem 1.1.7 whenever $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. But, we can not have the condition $\varepsilon.q \cong 0$.

Notation 1.1.1. Let $S_A\,(\cong 0)$ denote the set of all systems $(\xi_1, \xi_2, \ldots, \xi_k)$, with $k \geq 1$ for which $(\xi_i)_{i=1,2,\ldots,k}$ satisfy (1.1.2).

In this chapter, we will prove that $S_A\,(\cong 0)$ is a non-empty set. Also, for a given system of reals $(\xi_0, \xi_1, \ldots, \xi_\omega)$, with $\omega \cong +\infty$ we ask if there is a necessary and sufficient condition on the reals $(\xi_i)_{i=0,1,\ldots,\omega}$ for which $(\xi_0, \xi_1, \ldots, \xi_\omega) \in S_A\,(\cong 0)$. We are in a position to give our main results.

1.2 Necessary condition

To prove that $S_A\,(\cong 0)$ is a non-empty set, we need the following lemma.

Lemma 1.2.1. *Let N, ω be two unlimited positive integers. Let ε be an infinitesimal positive real number. If $\varepsilon N^{\omega-1}$ is not infinitesimal, then there exists an integer $i_0 \in \{1, 2, \ldots, \omega - 1\}$ such that $\varepsilon N^{\omega-i_0} \ncong 0$ and $\varepsilon N^{\omega-(i_0+1)} \cong 0$.*

Proof. Let α be an appreciable number strictly less than $\varepsilon N^{\omega-1}$ which we may, because $\varepsilon N^{\omega-1} \ncong 0$. Since

$$0 \simeq \varepsilon N^{\omega-\omega} < \varepsilon N < \varepsilon N^2 < \ldots < \varepsilon N^{\omega-2} < \varepsilon N^{\omega-1} \ncong 0,$$

there exists an integer $s \in \{1, 2, \dots, \omega - 1\}$ such that

$$\varepsilon N^{\omega-(s+1)} \leq \alpha < \varepsilon N^{\omega-s}.$$

There are two cases to consider.

- $\varepsilon N^{\omega-(s+1)} \cong 0$, Lemma 1.2.1 is proved by taking $i_0 = s$.
- $\varepsilon N^{\omega-(s+1)} \not\cong 0$. Since $N \cong +\infty$, we have

$$\varepsilon N^{\omega-(s+2)} = \frac{\varepsilon N^{\omega-(s+1)}}{N} \leq \frac{\alpha}{N} \cong 0.$$

Also, Lemma 1.2.1 is proved by taking $i_0 = s + 1$.

□

Theorem 1.2.2. $S_A\,(\cong 0)$ *is a non-empty set.*

Proof. We will give a system containing an unlimited number of reals that satisfies (1.1.2). That is to say, we prove that $S_A\,(\cong 0)$ contains many systems of the form $(\xi_0, \xi_1, \dots, \xi_k)$, with k is an unlimited. In fact, let N, ω be two unlimited positive integers. We show that

$$\left(\frac{1}{N^{\omega}}, \frac{1}{N^{\omega-1}}, \dots, \frac{1}{N}, 1\right) \in S_A\,(\cong 0)\,. \tag{1.2.1}$$

Let ε be an infinitesimal positive real number. There are two cases.

***A*)** $\varepsilon N^{\omega} \cong 0$. For every $i = 0, 1, \dots, \omega$, we have

$$\begin{cases} \dfrac{1}{N^{\omega-i}} = \dfrac{N^i}{N^{\omega}} + \varepsilon.0 = \dfrac{p_i}{q} + \varepsilon.\pounds_i \\ \varepsilon N^{\omega} = \varepsilon q \cong 0. \end{cases}$$

where $q = N^{\omega}$. In this case, Theorem 1.2.2 is proved.

***B*)** $\varepsilon N^{\omega} \not\cong 0$. Here we distinguish two cases.

***B.1*)** $\varepsilon N^{\omega} = a$, with a is an appreciable. In this case, we can write the system of (1.2.1) as follows:

$$\begin{pmatrix} \dfrac{1}{N^{\omega}} \\ \dfrac{1}{N^{\omega-1}} \\ \dfrac{1}{N^{\omega-2}} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{0}{N^{\omega-1}} + \varepsilon\dfrac{1}{a} \\ \dfrac{1}{N^{\omega-1}} + \varepsilon.0 \\ \dfrac{N}{N^{\omega-1}} + \varepsilon.0 \\ \vdots \\ \dfrac{N^{\omega-1}}{N^{\omega-1}} + \varepsilon.0 \end{pmatrix} = \begin{pmatrix} \dfrac{p_0}{q} + \varepsilon\pounds_0 \\ \dfrac{p_1}{q} + \varepsilon\pounds_1 \\ \dfrac{p_2}{q} + \varepsilon\pounds_2 \\ \vdots \\ \dfrac{p_{\omega}}{q} + \varepsilon\pounds_{\omega} \end{pmatrix},$$

where $q = N^{\omega-1}$ and $\varepsilon q = \varepsilon N^{\omega-1} = \dfrac{\varepsilon N^{\omega}}{N} = \dfrac{a}{N} \cong 0$. So, Theorem 1.2.2 is proved for this case.

B.2) $\varepsilon N^{\omega} \cong +\infty$. In this case we also distinguish two cases.

B.2.1) The real $\varepsilon N^{\omega-1}$ is infinitesimal. Since $\dfrac{1}{\varepsilon N^{\omega}} \cong 0$, it follows that

$$\begin{pmatrix} \dfrac{1}{N^{\omega}} \\ \dfrac{1}{N^{\omega-1}} \\ \dfrac{1}{N^{\omega-2}} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{0}{N^{\omega-1}} + \varepsilon\dfrac{1}{\varepsilon N^{\omega}} \\ \dfrac{1}{N^{\omega-1}} + \varepsilon.0 \\ \dfrac{N}{N^{\omega-1}} + \varepsilon.0 \\ \vdots \\ \dfrac{N^{\omega-1}}{N^{\omega-1}} + \varepsilon.0 \end{pmatrix} = \begin{pmatrix} \dfrac{p_0}{q} + \varepsilon\pounds_0 \\ \dfrac{p_1}{q} + \varepsilon\pounds_1 \\ \dfrac{p_2}{q} + \varepsilon\pounds_2 \\ \vdots \\ \dfrac{p_{\omega}}{q} + \varepsilon\pounds_{\omega} \end{pmatrix},$$

where $q = N^{\omega-1}$ and $\varepsilon q = \varepsilon N^{\omega-1} \cong 0$. Theorem 1.2.2 is proved.

B.2.2) The real $\varepsilon N^{\omega-1}$ is not infinitesimal. Let $i_0 \in \{1, 2, \dots, \omega-1\}$ be the integer constructed in Lemma 1.2.1, then

$$\begin{pmatrix} \dfrac{1}{N^{\omega}} \\ \dfrac{1}{N^{\omega-1}} \\ \vdots \\ \dfrac{1}{N^{\omega-i_0}} \\ \dfrac{1}{N^{\omega-(i_0+1)}} \\ \dfrac{1}{N^{\omega-(i_0+2)}} \\ \vdots \\ \dfrac{1}{N} \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{0}{N^{\omega-(i_0+1)}} + \varepsilon\dfrac{1}{\varepsilon N^{\omega}} \\ \dfrac{0}{N^{\omega-(i_0+1)}} + \varepsilon\dfrac{1}{\varepsilon N^{\omega-1}} \\ \vdots \\ \dfrac{0}{N^{\omega-(i_0+1)}} + \varepsilon\dfrac{1}{\varepsilon N^{\omega-i_0}} \\ \dfrac{1}{N^{\omega-(i_0+1)}} + \varepsilon.0 \\ \dfrac{N}{N^{\omega-(i_0+1)}} + \varepsilon.0 \\ \vdots \\ \dfrac{N^{\omega-i_0-2}}{N^{\omega-(i_0+1)}} + \varepsilon.0 \\ \dfrac{N^{\omega-i_0-1}}{N^{\omega-(i_0+1)}} + \varepsilon.0 \end{pmatrix} = \begin{pmatrix} \dfrac{p_0}{q} + \varepsilon\pounds_0 \\ \dfrac{p_1}{q} + \varepsilon\pounds_1 \\ \vdots \\ \dfrac{p_{i_0}}{q} + \varepsilon\pounds_{i_0} \\ \dfrac{p_{i_0+1}}{q} + \varepsilon\pounds_{i_0+1} \\ \dfrac{p_{i_0+2}}{q} + \varepsilon\pounds_{i_0+2} \\ \vdots \\ \dfrac{p_{\omega-1}}{q} + \varepsilon\pounds_{\omega-1} \\ \dfrac{p_{\omega}}{q} + \varepsilon\pounds_{\omega} \end{pmatrix}$$

with $q = N^{\omega-(i_0+1)}$ and $\varepsilon q = \varepsilon.N^{\omega-(i_0+1)} \cong 0$.

This completes the proof of Theorem 1.2.2. □

In Theorem 1.2.2 we have proved that $S_A\,(\cong 0)$ contains a system where the number of its elements is unlimited. In the following lemma, we will prove that there are many systems which are not belong to $S_A\,(\cong 0)$.

Lemma 1.2.3. *Let ω be an unlimited positive integer, and let $(\xi_0, \xi_1, \ldots, \xi_\omega)$ be a system of reals satisfying the following properties:*

(a) $\xi_0 \cong \xi_1 \cong \ldots \cong \xi_\omega$

(b) $\xi_{i+1} - \xi_i = d_i > 0 \quad$ *for* $i = 0, 1, \ldots, \omega - 1$

(c) $\dfrac{d_i}{d_{i-1}} = a_i \cong 1 \quad$ *for* $i = 1, 2, \ldots, \omega - 1.$

Then $(\xi_0, \xi_1, \ldots, \xi_\omega) \notin S_A\,(\cong 0)$.

Proof. Assume, by way of contradiction, that the reals $(\xi_i)_{i=0,1,\ldots,\omega}$ are simultaneously approximable in the infinitesimal sense. In particular, for $\varepsilon = d_0 \cong 0$ we have

$$\begin{cases} \xi_i = \dfrac{p_i}{q} + \varepsilon \pounds_i \\ \quad \varepsilon q \cong 0, \end{cases} \tag{1.2.2}$$

where $\dfrac{p_i}{q}$ is a rational and $\pounds_i$ is a limited for every $i = 0, 1, \ldots, \omega$.

Let i_0 be an unlimited positive integer strictly less than ω and satisfying

$$i_0 < \frac{1}{N\varepsilon q}, \tag{1.2.3}$$

for a given limited integer $N > 2$ (which we may, because $\dfrac{1}{\varepsilon q} = \dfrac{1}{d_0 q} \cong +\infty$). Since the reals $(a_i)_{i=1,2,\ldots,\omega-1}$ are all appreciable, then for any standard integer $n \geq 1$, the number $S_n = \sum_{i=1}^{n} a_1 a_2 \ldots a_i$ is also an appreciable. Next, consider the set

$$\left\{ n \in \{1, 2, \ldots, \omega\} \ ; \ 1 + \sum_{i=1}^{n} a_1 a_2 \ldots a_i < i_0 \cong +\infty \right\}, \tag{1.2.4}$$

which is internal and contains $\mathbb{N}^\sigma$. According to the Cauchy's Principle there exists an unlimited integer n_0 that satisfies (1.2.4).

On the other hand, since

$$\xi_{n_0} - \xi_0 = d_0 + d_1 + \ldots + d_{n_0 - 1} = \varepsilon \sum_{i=0}^{n_0 - 1} \frac{d_i}{d_0}, \tag{1.2.5}$$

and $\dfrac{d_i}{d_{i-1}} = a_i$, for $i = 1, 2, \ldots, \omega - 1$. From (1.2.2) and (1.2.5), we have

$$\begin{aligned} \xi_{n_0} - \xi_0 &= \varepsilon \left(1 + \sum_{i=1}^{n_0 - 1} a_1 a_2 \ldots a_i \right) \\ &= \frac{p_{n_0} - p_0}{q} + \varepsilon \pounds. \end{aligned} \tag{1.2.6}$$

We use the fact that n_0 satisfies (1.2.4). Then from (1.2.3), (1.2.4), and (1.2.6) we get

$$(p_{n_0} - p_0) + \varepsilon q \pounds < \frac{1}{N}.$$

Since $\varepsilon q \pounds \cong 0$, it follows that $p_{n_0} - p_0 < \dfrac{2}{N}$.

Now we prove that $p_{n_0} > p_0$. First, it suffices to prove that the number $1 + \sum_{i=1}^{n_0-1} a_1 a_2 \ldots a_i$ is unlimited. In fact, consider the following set

$$\left\{ m \in \mathbb{N}\ ; m \leq n_0 - 1 \text{ and } 1 + \sum_{i=1}^{m} a_1 a_2 \ldots a_i > m \right\},$$

which is internal and contains $\mathbb{N}^\sigma$, because for all limited integers s we have

$$1 + \sum_{i=1}^{s} a_1 a_2 \ldots a_i = 1 + s + \phi_s > s,$$

where $\phi_s \cong 0$ (positive or negative). From Cauchy's principle there exists an unlimited integer m_0 (with $m_0 \leq n_0 - 1$) such that

$$1 + \sum_{i=1}^{n_0 - 1} a_1 a_2 \ldots a_i \geq 1 + \sum_{i=1}^{m_0} a_1 a_2 \ldots a_i > m_0 \cong +\infty.$$

We assume that $p_{n_0} = p_0$, by (1.2.6) we get

$$+\infty \cong 1 + \sum_{i=1}^{n_0 - 1} a_1 a_2 \ldots a_i = \pounds,$$

which is a contradiction. Therefore $p_{n_0} \neq p_0$. Moreover, if $p_{n_0} < p_0$, by using (1.2.6) again, we obtain

$$\pounds > \frac{p_0 - p_{n_0}}{\varepsilon q} \cong +\infty,$$

because $\xi_{n_0} > \xi_0$. Which is a contradiction as well, since $\pounds$ is limited. Recall that p_{n_0} and p_0 are positive integers, and since $p_{n_0} > p_0$ it follows that $\dfrac{2}{N} > 1$. Which leads to a contradiction with the hypothesis of $N > 2$. This completes the proof. $\square$

Now we prove the main result of this chapter.

Theorem 1.2.4. *Let ω be an unlimited positive integer, and let $(\xi_0, \xi_1, \ldots, \xi_\omega) \subset [0,1]$ be a system of reals. If $^\circ(\xi_0, \xi_1, \ldots, \xi_\omega)$ contains a standard interval $[a,b]$ with $a < b$, then the reals $(\xi_i)_{i=0,1,\ldots,\omega}$ are not simultaneously approximable in the infinitesimal sense.*

That is, we will prove the necessary condition given by:

$$(\xi_0, \xi_1, \ldots, \xi_\omega) \in S_A(\cong 0) \Rightarrow \forall\, a, b \in \mathbb{N}^\sigma : [a,b] \nsubseteq\ ^\circ(\xi_0, \xi_1, \ldots, \xi_\omega). \tag{N}$$

Proof. Since $^\circ(\xi_0, \xi_1, \ldots, \xi_\omega)$ contains a standard interval $[a,b]$ with $a < b$, there exists a subsystem $(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}) \subset (\xi_0, \xi_1, \ldots, \xi_\omega)$ such that

$$\begin{cases} ^\circ(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}) = [a,b], \\ \xi_{i_0} < \xi_{i_1} < \ldots < \xi_{i_k}, \end{cases}$$

where $k \cong +\infty$. We prove that $a \cong \xi_{i_0} \cong \xi_{i_1} \cong \ldots \cong \xi_{i_k} \cong b$. In fact, suppose the contrary, i.e., there exists $m \in \{1, 2, \ldots, k\}$ such that

$$\xi_{i_{m-1}} \cong \xi_{i_m}.$$

Since $^\circ\xi_{i_{m-1}} \neq {}^\circ\xi_{i_m}$, it follows that

$$\frac{\xi_{i_{m-1}} + \xi_{i_m}}{2} \cong \frac{^\circ\xi_{i_{m-1}} +^\circ \xi_{i_m}}{2} \in [a,b].$$

Which is a contradiction because the number $\dfrac{\xi_{i_{m-1}} + \xi_{i_m}}{2}$ does not belong to the system $(\xi_0, \xi_1, \ldots, \xi_\omega)$.

Put $d_s = \xi_{i_{s+1}} - \xi_{i_s}$ for $s = 0, 1, \ldots, k-1$, then $\max\limits_{0 \le s \le k-1}(d_s) \cong 0$. Let γ be an unlimited real number such that

$$\lambda = \gamma \max_{0 \le s \le k-1}(d_s) \cong 0,$$

and this by using Robinson's Lemma.

Now, we choose a system of N elements $(\theta_r)_{r=0,1,\ldots,N}$ among the numbers $(\xi_{i_s})_{s=0,1,\ldots,k}$ as the following way

$$\begin{cases} \theta_0 \ = \ \xi_{i_0}, \text{ and} \\ \theta_r = \xi_{i_{m_r}} \text{ is the nearest element strictly less than } \theta_0 + r\lambda;\ r = 1, 2, \ldots, N. \end{cases}$$

where $m_r \in \{0, 1, \ldots, k\}$ and N is an unlimited integer, with $N.\lambda \simeq 0$. Then we prove the conditions (**a**), (**b**) and (**c**) of the Lemma 1.2.3 for the new system $(\theta_0, \theta_1, \ldots, \theta_N)$. In fact, from the construction of $(\theta_r)_{0\leq r\leq N}$ we see that

$$\theta_0 \cong \theta_1 \ldots \cong \theta_N \cong \xi_{i_0} \text{ and } \theta_{r+1} - \theta_r = D_r > 0, \text{ for } r = 0, 1, \ldots, N-1.$$

Thus, (**a**) and (**b**) are satisfied. For the proof of (**c**), we put

$$\delta_r = \xi_{i_0} + r\lambda - \theta_r \; ; r = 0, 1, \ldots, N.$$

Then $\delta_r \leq \xi_{i_{m_r+1}} - \xi_{i_{m_r}} = d_{i_{m_r}}$, because $\xi_{i_{m_r}}$ is the nearest element strictly less than $\theta_0 + r\lambda$. Moreover, we have

$$\frac{\delta_r}{\lambda} = \frac{\delta_r}{\gamma \max\limits_{0\leq s\leq k-1} (d_s)} \leq \frac{1}{\gamma}\left(\frac{\delta_r}{d_{i_{m_r}}}\right) \leq \frac{1}{\gamma} \cong 0.$$

Therefore, for every $r = 0, 1, \ldots, N$, there exists an infinitesimal real number ϕ_r such that $\delta_r = \lambda\phi_r$. Hence

$$\theta_{r+1} - \theta_r = \lambda - \delta_{r+1} + \delta_r = \lambda - \lambda\phi_{r+1} + \lambda\phi_r; \text{ for } r = 0, 1, \ldots, N-1.$$

It follows for every $r \in \{1, 2, \ldots, N-1\}$ that

$$\frac{D_r}{D_{r-1}} = \frac{\theta_{r+1} - \theta_r}{\theta_r - \theta_{r-1}} = \frac{1 - \phi_{r+1} + \phi_r}{1 - \phi_r + \phi_{r-1}} \cong 1.$$

Using Lemma 1.2.3 we can also conclude that $(\theta_0, \theta_1, \ldots, \theta_N) \notin S_A\,(\cong 0)$ and therefore $(\xi_0, \xi_1, \ldots, \xi_\omega) \notin S_A\,(\cong 0)$. This completes the proof of Theorem 1.2.4. □

Here we ask the following question: Can we give a criterion by which we can determine whether a given countable system of reals is in $S_A\,(\cong 0)$, or not? The following proposition is given as an example.

Proposition 1.2.5. *Let ω be an unlimited positive integer. Then*

$$S = \left(\frac{1}{\omega}, \frac{1}{\omega+1}, \frac{1}{\omega+2}, \ldots\right) \notin S_A\,(\cong 0)\,.$$

Proof. Suppose that $S \in S_A(\cong 0)$. For $\varepsilon = \dfrac{1}{\omega} \cong 0$, there exist rational numbers $\left(\dfrac{p_i}{q}\right)_{i=0,1,\ldots}$ such that

$$\begin{cases} \dfrac{1}{\omega+i} = \dfrac{p_i}{q} + \varepsilon.\pounds_i \\ \varepsilon.q \cong 0, \end{cases} ; i = 0, 1, \ldots \tag{1.2.7}$$

where $(\pounds_i)_{i=0,1,\ldots}$ are limited numbers. Let i_0 be an unlimited positive integer satisfying

$$+\infty \cong \frac{\omega i_0}{\omega + i_0} < \frac{1}{2\varepsilon.q} \cong +\infty. \tag{1.2.8}$$

Since S is countable, it follows from (1.2.7) that

$$\frac{1}{\omega} - \frac{1}{\omega + i_0} = \varepsilon\left(\frac{\omega i_0}{\omega + i_0}\right) = \frac{p_0 - p_{i_0}}{q} + \varepsilon.\pounds \tag{1.2.9}$$

Similarly, from the proof of Lemma 1.2.3 we get

$$p_0 - p_{i_0} + \varepsilon.q.\pounds < \frac{1}{2},$$

which is impossible because from (1.2.8) and (1.2.9), $p_0 > p_{i_0}$. □

In the following result, for a real number x, let $\{x\}$ and $[x]$ denote the fractional part and the integer part of x, respectively.

Corollary 1.2.6. *Let ω be an unlimited positive integer. If $(\xi_0, \xi_1, \ldots, \xi_\omega) \in S_A(\cong 0)$, then for every limited integer c, $^\circ(\{c\,\xi_0\}, \{c\,\xi_1\}, \ldots, \{c\,\xi_\omega\})$ does not contain any standard interval $[a, b]$, with $a < b$.*

Proof. Suppose that there exists a subset of positive integers:

$$(i_0, i_1, \ldots, i_k) \subset (0, 1, \ldots, \omega), \text{ with } k \cong +\infty,$$

and there is a limited integer c_0 such that $^\circ(\{c_0\xi_{i_0}\}, \{c_0\xi_{i_1}\}, \ldots, \{c_0\xi_{i_k}\}) = [a, b]$ where a and b are standard real numbers ($a < b$), and we prove that $(\xi_0, \xi_1, \ldots, \xi_\omega) \notin S_A(\cong 0)$.

In fact, from Theorem 1.2.4, we get

$$(\{c_0\xi_{i_0}\}, \{c_0\xi_{i_1}\}, \ldots, \{c_0\xi_{i_k}\}) \notin S_A(\cong 0). \tag{1.2.10}$$

It suffices to show that $(c_0\xi_{i_0}, c_0\xi_{i_1}, \ldots, c_0\xi_{i_k}) \notin S_A(\cong 0)$. Suppose the contrary. Then for every positive infinitesimal ε there exist rational numbers $\left(\frac{P_{i_s}}{Q}\right)_{s=0,1,\ldots,k.}$ such that

$$\begin{cases} \{c_0\xi_{i_s}\} = \dfrac{P_{i_s} - [c_0\xi_{i_s}]\, Q}{Q} + \varepsilon \pounds \\ \varepsilon Q \cong 0 \end{cases} ;\ 0 \le s \le k,$$

because $\{c_0\xi_{i_s}\} = c_0\xi_{i_s} - [c_0\xi_{i_s}]$, for $s = 0, 1, \ldots, k$. Thus,

$$(\{c_0\xi_{i_0}\}, \{c_0\xi_{i_1}\}, \ldots, \{c_0\xi_{i_k}\}) \in S_A(\cong 0).$$

Which contradicts the expression (1.2.10).

Finally, since c_0 is a limited integer, we have $(\xi_{i_0}, \xi_{i_1}, \ldots, \xi_{i_k}) \notin S_A(\cong 0)$, and therefore $(\xi_0, \xi_1, \ldots, \xi_\omega) \notin S_A(\cong 0)$. This completes the proof. □

1.3 Remarks and examples

In this section, we give certain remarks and examples about the necessary condition stated in Theorem 1.2.4.

Remark 1.3.1. The converse of (N) is false.

In the following corollary, we give a system of real numbers $(\xi_0, \xi_1, \ldots, \xi_\omega)$ with $\omega \cong +\infty$, whose elements are not simultaneously approximable in the infinitesimal sense but its shadow is different from a standard interval $[a, b]$.

Corollary 1.3.1 (Counterexample)**.** *Let f be the exponential function. For every unlimited positive integer ω, we have*

$$\left(\frac{1}{f(0)}, \frac{1}{f(1)}, \ldots, \frac{1}{f(\omega)}\right) \notin S_A(\cong 0). \tag{1.3.1}$$

Proof. Suppose that the reals of (1.3.1) are simultaneously approximable in the infinitesimal sense. Then for $\varepsilon = \frac{1}{f(\omega)} \cong 0$, there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=0,1,\ldots,\omega}$ such that

$$\begin{cases} \dfrac{1}{f(i)} = \dfrac{p_i}{q} + \varepsilon.\pounds_i \\ \varepsilon q \cong 0, \end{cases}$$

where $\pounds_i$ is a limited number for every $i = 0, 1, \ldots, \omega$. Using Cauchy's principle, there exists an unlimited positive integer i_0 such that

$$f(i_0) < \frac{\gamma}{2}, \tag{1.3.2}$$

where $\gamma = \dfrac{1}{\varepsilon q} \cong +\infty$. Since f is increasing, we have $i_0 < \omega$. In fact, if $i_0 \geq \omega$ it follows that $f(\omega) < \dfrac{f(\omega)}{2q}$. Which is impossible.

Now, we put $s_0 = \omega - i_0$. From the hypothesis, there exist $\dfrac{p_{s_0}}{q}, \dfrac{p_\omega}{q}$ such that

$$\frac{1}{f(s_0)} - \frac{1}{f(\omega)} = \varepsilon(f(i_0) - 1) = \frac{p_{s_0} - p_\omega}{q} + \varepsilon\pounds. \tag{1.3.3}$$

Using (1.3.2) and (1.3.3), we get

$$(p_{s_0} - p_\omega) + \varepsilon q \pounds < \frac{1}{2}. \tag{1.3.4}$$

It follows from (1.3.3) that $p_{s_0} \neq p_\omega$ because $f(i_0) \cong +\infty$. Moreover, if $p_{s_0} < p_\omega$ then

$$\pounds > \frac{p_\omega - p_{s_0}}{\varepsilon . q} \cong +\infty.$$

which we may, because $\dfrac{1}{f(s_0)} > \dfrac{1}{f(\omega)}$. A contradiction, since $\pounds$ is a limited. Thus, $p_{s_0} > p_\omega$. Finally, from (1.3.4) we have $1 \leq p_{s_0} - p_\omega < \dfrac{2}{3}$, since $\varepsilon q \pounds \cong 0$. Which is impossible. □

Remark 1.3.2. Let f be the function of Corollary 1.3.1, we put

$$A = \left(\frac{1}{f(0)}, \frac{1}{f(1)}, \ldots, \frac{1}{f(\omega)} \right).$$

Since f is standard, then $^\circ A = \left(\dfrac{1}{f(0)}, \dfrac{1}{f(1)}, \dfrac{1}{f(2)}, \ldots \right)$ is not an interval.

Corollary 1.3.2 (An example of Theorem 1.2.4). *Let ω be an unlimited positive integer. Then*

$$\left(\frac{1}{\omega}, \frac{2}{\omega}, \ldots, \frac{\omega - 1}{\omega}, \frac{\omega}{\omega} \right) \notin S_A(\cong 0).$$

Proof. It is clear that

$$^{\circ}\left(\frac{1}{\omega}, \frac{2}{\omega}, \ldots, \frac{\omega-1}{\omega}, \frac{\omega}{\omega}\right) = [0,1].$$

Thus we get the result by using Theorem 1.2.4. Moreover, for any standard interval $[a,b]$, with $a < b$, there exists a system of reals $(\xi_0, \xi_1, \ldots, \xi_\omega)$, with $\omega \cong +\infty$ such that $a \cong \xi_0 \cong \xi_1 \cong \ldots \cong \xi_\omega \cong b$, and also from Theorem 1.2.4, $(\xi_0, \xi_1, \ldots, \xi_\omega) \notin S_A(\cong 0)$. □

Remark 1.3.3. Let $(\xi_0, \xi_1, \ldots, \xi_k)$ be an arbitrary system of real numbers. From the proof of Corollary 1.2.6, it is clear that $(\xi_0, \xi_1, \ldots, \xi_k) \in S_A(\cong 0)$, if and only if $(\{\xi_0\}, \{\xi_1\}, \ldots, \{\xi_k\}) \in S_A(\cong 0)$, where $\{x\}$ represent the fractional part of x. We can therefore deduce that if

$$(\xi_0, \xi_1, \ldots, \xi_k) \in S_A(\cong 0),$$

then

$$(\xi_0, \xi_1, \ldots, \xi_k, \{\xi_0\}, \{\xi_1\}, \ldots, \{\xi_k\}) \in S_A(\cong 0).$$

1.4 Conclusion

In this chapter, we aim to give a necessary condition for a system of real numbers $(\xi_0, \xi_1, \ldots, \xi_\omega)$ to be in $S_A(\cong 0)$, where ω is an unlimited positive integer. However, a sufficient condition remains an open problem.

Chapter 2

The the Solubility of certain Arithmetic Inequalities by using Particular Subsets of Positive Integers

Let $k \geq 1$, and let W_k denote the set of positive integers n for which the number of distinct prime factors of n is greater or equal to k. In arithmetic functions, one of important topics is to establish some inequalities (resp. equations) for infinitely many $n \in \mathbb{N}$. Many researchers have obtained their results on the set $\mathbb{N} = W_1 \cup \{1\}$. The purpose of chapters 2 and 3 is to study some inequalities and equations on the set W_k instead of $\mathbb{N}$. In the framework of Internal Set Theory (for more details, see [13],[18],[30],[41]), we denote by W_∞ for integers n for which the number of distinct prime factors of n is unlimited and every prime factor that divides n is also unlimited. In this case, our working set is an infinite external subset of positive integers.

2.1 Preliminaries

Let n be a positive integer, and let $\omega(n)$ denote the number of distinct prime factors of n. Throughout this chapter, k is an arbitrary positive integer. Then

$$W_k = \{n \in \mathbb{N}\ ;\ \omega(n) \geq k\}\,. \qquad (2.1.1)$$

At the beginning, we shall fix the setting in which we work and some definitions.

Definition 2.1.1. A function f is called an arithmetic function or a number theoretic function if it assigns to each positive integer n a unique real or complex number $f(n)$. Typically, an arithmetic function is a real-valued function whose domain is the set of positive integers.

Definition 2.1.2 ([26], [34], [37], [43])**.** In Chapters 2 and 3, we use the following arithmetic functions:

1. Let $s \geq 1$. Then the arithmetic function φ_s is defined as follows

$$\varphi_s\,(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right),\ \varphi_s\,(1) = 1. \qquad (2.1.2)$$

   Euler's totient function, $\varphi(n) = \varphi_1(n)$, is the number of numbers not greater than n and prime to n. For example,

$$\varphi\,(60) = \varphi\left(2^2.3.5\right) = 16 \text{ and } \varphi_2\,(60) = 60^2 \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) = 2304.$$

2. Let $s \geq 1$. We define the arithmetic function ψ_s by

$$\psi_s\,(n) = n^s \prod_{p|n} \left(1 + \frac{1}{p^s}\right),\ \psi_s\,(1) = 1. \qquad (2.1.3)$$

   For $s = 1$, $\psi_1 = \psi$ is the Dedekind's function.

3. Let n be a positive integer. Then the arithmetic functions $\tau(n)$ and $\sigma(n)$ are defined as follows:

$$\tau(n) = \sum_{d|n} 1, \qquad \sigma(n) = \sum_{d|n} d$$

That is, $\tau(n)$ designates the number of all positive divisors of n, and $\sigma(n)$ designates the sum of all positive divisors of n.

4. $\Omega(n)$ denote the total number of prime divisors of n.

5. Let $\gamma(n)$ be the *Kernel* of n given by $\gamma(1) = 1$ and $\gamma(n) = \prod_{p|n} p$, for $n \geq 2$.

Definition 2.1.3 ([49])**.** A real function f defined on the positive integers is said to be multiplicative if

$$f(mn) = f(m) f(n), \ \forall \ m, n \in \mathbb{N} \text{ with } \gcd(m, n) = 1.$$

If

$$f(mn) = f(m) f(n), \ \forall \ m, n \in \mathbb{N},$$

then f is completely multiplicative. Every completely multiplicative function is multiplicative.

It is well-known that the arithmetic functions given previously in Definition 2.1.2 are multiplicative (except $\Omega(n)$).

Definition 2.1.4 ([49])**.** A number of the form $M_n = 2^n - 1$, $n \geq 2$, is said to be a Mersenne number. If M_n is prime, it is called a Mersenne prime. A number of the form $F_n = 2^{2^n} + 1$, $n \geq 0$, is called a Fermat number. If F_n is prime, it is called a Fermat prime.

Fermat hypothesized—he didn't claim to have a proof—that all the numbers

$$F_0, F_1, F_2, \ldots$$

are prime. In fact this is true for

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537.$$

However, Euler showed in 1747 that

$$F_5 = 2^{32} + 1 = 4294967297$$

is composite. In fact, no Fermat prime beyond F_4 has been found. The heuristic argument we used above to suggest that the number of Mersenne primes is probably infinite now suggests that the number of Fermat primes is probably finite.

For by the Prime Number Theorem, the probability of F_n being prime is

$$\approx \frac{2}{\log F_n} \approx \frac{2}{2^n}.$$

Thus the expected number of Fermat primes is

$$2 \approx \sum 2^{-n} = 4 < \infty.$$

This argument assumes that the Fermat numbers are "independent", as far as primality is concerned. It might be argued that our next result shows that this is not so. However, the Fermat numbers are so sparse that this does not really affect our heuristic argument.

Remark 2.1.1. The Fermat numbers are coprime, i.e.,

$$\gcd(F_m, F_n) = 1,$$

if $m \neq n$.

In fact, suppose

$$\gcd(F_m, F_n) > 1.$$

Then we can find a prime p (which must be odd) such that p divides F_m and p divides F_n.

Now the numbers $\{1, 2, \ldots, p-1\}$ form a group $(\frac{\mathbb{Z}}{p})^\times$ under multiplication $\mod p$. Since p divides F_m,

$$2^{2^m} \equiv -1 \mod p.$$

It follows that the order of $2 \mod p$ (i.e., the order of 2 in $(\frac{\mathbb{Z}}{p})^\times$ is exactly 2^{m+1}. For certainly

$$2^{2^{m+1}} = \left(2^{2^m}\right)^2 \equiv 1 \mod p,$$

and so the order of 2 divides 2^{m+1}, i.e., it is 2^e for some $e \leq m+1$. But if $e \leq m$, then

$$2^{2^m} \equiv 1 \mod p.$$

whereas we just saw that the left hand side was $-1 \mod p$. We conclude that the order must be 2^{m+1}. But by the same token, the order is also 2^{n+1}. This is a contradiction, unless $m = n$.

The Fermat number

$$F_n = 2^{2^n} + 1$$

is prime if and only if

$$3^{\frac{F_n - 1}{2}} \equiv -1 \mod F_n.$$

Suppose $P = F_n$ is prime. We have

$$F_n \equiv 5 \mod 12.$$

Evidently

$$F_n \equiv 1 \mod 4,$$

while

$$F_n \equiv (-1)^{2^n} + 1 \mod 3 \equiv 2 \mod 3$$

By the Chinese Remainder Theorem these two congruences determine $F_n \mod 12$; and observation shows that

$$F_n \equiv 5 \mod 12.$$

It follows that

$$\left(\frac{3}{P}\right) = -1.$$

Hence

$$3^{\frac{P-1}{2}} \equiv -1 \mod P$$

Recall that the series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ diverges as n tends to infinity. Euler showed not only that it is approximately $\log n$, but also that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$$

tends to a constant, Euler's constant, denoted by the Greek letter gamma, γ, which he calculated to sixteen decimal places in 1781. It is approximately $0.577215664901532860\ldots$ It is not known whether it is rational. If it is, and $\gamma = \dfrac{a}{b}$, then $b > 10^{244662}$.

This is one of the most mysterious constants in mathematics, which turns up in some unexpected places. For example: if a large integer n is divided by each integer k, $1 \leq k \leq n$, then the average fraction by which the quotients fall short of the next integer is not $\frac{1}{2}$, but γ. It has been conjectured that if $M(n)$ is the number of primes $p \leq n$, such that the Mersenne number $2^p - 1$ is prime, then $\frac{M(n)}{\log n}$ tends to the constant $e^\gamma \log 2 = 2.56954\ldots$as n tends to infinity.

There are many other series for γ. for example

$$\gamma = (\frac{1}{2} - \frac{1}{3}) + 2(\frac{4}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}) + 3(\frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \dots - \frac{1}{15}) + \dots$$

Remark 2.1.2. If when a positive integer is factorized, all its prime factors appear at least squared, then it is powerful. The sequence of powerful numbers starts

$$4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 72, 81, 100, \dots$$

Solomon Golomb showed how to find an infinite number of consecutive pairs of powerful numbers, such as 8 and 9, 288 and 289, and proved that the number of powerful numbers $\leq x$ is approximately $c\sqrt{x}$ where $c = 2.173\dots$He also conjectured that 6 is not the difference between two powerful numbers, but this is false (Narkiewicz: Guy 1994):

$$6 = 5^4 7^3 - 463^2.$$

It has been conjectured that there can not be three consecutive powerful numbers. (Golomb 1970).

Recall that Niven numbers are named after Ivan Niven, author of An Introduction to the Theory of Numbers (1960). (They were also labeled multidigital numbers, or Harshad numbers, by Kaprekar, Harshad meaning “great joy” in Sanskrit.) They are integers divisible by the sum of the digits. Their sequence in base 10 starts

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, \dots$$

Euler proved in 1737 that the sum of the reciprocals of the primes

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

diverges. He also claimed, in modern notation, that it diverged like the function $\log \log n$.

The alternating sum of the prime reciprocals converges, since its value always lies between $\frac{1}{2}$ and 0:

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{11}-\frac{1}{13}+\ldots=0.2696063519\ldots$$

The sum of the squared reciprocals of the primes also converges, to $0.4522474200\ldots$ (Finch 2003).

Theorem 2.1.1 (Arithmetic-Mean-Geometric-Mean Inequality [44])**.** *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers. Then*

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1+a_2+\ldots+a_n}{n}.$$

Theorem 2.1.2 (Bonse's inequality [48])**.** *If p_n is the nth prime, then*

$$p_{n+1}^2 < p_1 p_2 p_3 \ldots p_n$$

provided $n \geq 4$.

Theorem 2.1.3. *[43] Let n be a positive integer. Then*

- *Euler's function is multiplicative, that is, if $\gcd(m,n)=1$, then*

$$\varphi(mn)=\varphi(m)\varphi(n).$$

- *If n is a prime, say p, then*

$$\varphi(p)=p-1.$$

(Conversely, if p is a positive integer with $\varphi(p)=p-1$, then p is prime).

Also, we use the following Theorem which gives in a nonstandard form:

Theorem 2.1.4 (Criterion of convergence of a sequence [16],[29]). *Let $(u_n)_{n\geq 1}$ be a standard sequence of elements of* $\mathbb{R}$. *Then,* $(u_n)_{n\geq 1}$ *converges to l if and only if $u_n \cong l$ for all unlimited n.*

In the study of number theory especially arithmetic functions, many problems are given in the infinity. For example, from the proof of Problem 749 stated in [26, p. 319], $\dfrac{\varphi(n)+\sigma(n)}{\gamma(n)^2}$ is an integer for infinitely many $n \in W_2$ (more precisely, for $n = 2^5.3^{2r+1}, r = 1, 2, \ldots$). In the case $k \neq 2$, we ask if $\dfrac{\varphi(n)+\sigma(n)}{\gamma(n)^2}$ is an integer for infinitely many $n \in W_k$. In Proposition 2.2.5, we will present an example of a fraction of the form $\dfrac{f(g(h(n)))}{h(g(n))}$ which is a positive integer for infinitely many $n \in W_k$, where f, g and h are three arithmetic functions.

Generally, if an arithmetic inequality holds for every positive integers n, then it is very difficult to prove it for infinitely many $n \in W_k$ when we make small changes. For example, J. Sándor [42] proved that $\sigma(n) > n + (\omega(n) - 1)\sqrt{n}$ for every $n \geq 2$, and in Proposition 2.2.11, we will prove that $\sigma(n) > 1 + n + \omega(n)\sqrt{n}$ for infinitely many $n \in W_k$. On the other hand, see [43], for all integers $n \geq 1$, $\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1$. But, when $k \geq 2$, we ask if $\varphi(n)\psi(n)\sigma(n) \geq n^3 + a\,n^2$ holds for infinitely many $n \in W_k$, where $a \in]2, 3[$ is a parameter (it is clear that for $a = 2$, the last inequality holds for every $n = p^\alpha$, with p prime and $\alpha > 1$, i.e., it holds for infinitely many $n \in W_1$).

Moreover, in this chapter:

1. We will determine some pairs (f, g) of arithmetic functions for which the inequality $f(n) > c\,g(n)$ or the equation $f(n) = c\,g(n)$ holds for infinitely many $n \in W_k$, where c is a parameter.

2. In the nonclassical mathematics, we will study certain inequalities (resp. equations) by using external subsets of positive integers. Moreover, it is not known whether $\varphi(n)\psi(n)\sigma(n) \geq n^3 + a\,n^2$ holds for infinitely many $n \in W_k$. We ask if there exists a finite set of positive integers $\{n_1, n_2, \ldots, n_m\}$ such that $\varphi(n_i)\psi(n_i)\sigma(n_i) \geq n_i^3 + a\,n_i^2$, for $i = 1, 2, \ldots, m$, where m is an unlimited integer. Indeed, up to now, this question has not been solved. But, an example of such property is obtained in view of Theorem 2.3.2, where our working set is a subset of positive integers each of whose elements has an unlimited number of distinct prime factors. Finally, we can prove certain classical results by using the techniques of IST as in Theorem 2.4.1.

Also, in the present chapter, $r \geq 2$ is a given positive integer and p_r is the *rth* prime number. We use the fact that $2 = p_1 < p_2 < \ldots < p_n < \ldots$ is the sequence of all primes in increasing order, the sequence $q_1 < q_2 < \ldots < q_n < \ldots$ denotes an arbitrary sequence of primes. Some famous results in mathematics are used, as The Prime Number Theorem, which states that the limit of $\dfrac{p_n}{p_{n-1}}$ as n tends to infinity is 1, and Bertrand's theorem which says that if n is an integer greater than 2, then there is at least one prime between n and $2n-1$. In addition, let d_{n-1} denote the gap between p_n and p_{n-1}, that is, $d_{n-1} = p_n - p_{n-1}$; $n \geq 2$. Then we have

$$\lim_{n\to+\infty} \frac{p_n}{d_n} = +\infty, \tag{2.1.4}$$

or, equivalently (see Theorem 2.1.4), $\dfrac{p_n}{d_n} \cong +\infty$ for every unlimited n.

We can use Cauchy's Principle together with the Prime Number Theorem, we easily get the following:

Example 2.1.1. *Let n be an unlimited positive integer. Using* (2.1.4)*, for every standard positive integer N, we have*

$$1+\frac{1}{N}>\frac{p_n}{p_{n-1}}=1+\varepsilon_n\cong 1,$$

where ε_n is an infinitesimal positive real number. By Cauchy's Principle, there exists an unlimited integer w such that $1+\dfrac{1}{w}>\dfrac{p_n}{p_{n-1}}$.

Our main results in this chapter are as follows.

2.2 Some classical inequalities (resp. equations) on the set W_k

Let A be an infinite set of positive integers. We say that a sequence $(a(n))_{n\geq 1}$ of real numbers has infinitely many sign changes on the set A if there exist infinitely many $n\in A$ such that $a(n)>0$ and there are infinitely $n\in A$ such that $a(n)<0$, see [25].

Let $s\geq 1$. In the following result, from the Prime Number Theorem [46], we assume that p_r and s are chosen so that the following inequality

$$p_k<\left(\frac{p_{r-1}-1}{d_{r-1}+1}\right)^{\frac{1}{s}} \tag{2.2.1}$$

holds, where $k\geq 1$.

Theorem 2.2.1. *Under the same assumption as in* (2.2.1)*, $p_{r-1}\psi_s(n)-p_r n^s$ has infinitely many sign changes on the set W_k.*

Proof. We prove that each of the inequalities $p_{r-1}\psi_s(n)>p_r n^s$ and $p_{r-1}\psi_s(n)<p_r n^s$ holds for infinitely many $n\in W_k$.

First step. We show that there exists a positive integer $n_0 \in W_k$ such that $p_{r-1}\psi_s(n_0) > p_r n_0^s$. Assume, by way of contradiction, that for all $n \in W_k$ we have $p_{r-1}\psi_s(n) \le p_r n^s$. In particular, for $n = 2.3 \ldots p_k \in W_k$, and by using (2.2.1) we get

$$\frac{p_{r-1}}{p_r} \le \frac{n^s}{\psi_s(n)} = \prod_{p|n} \frac{p^s}{p^s+1} \le \frac{p_k^s}{p_k^s+1} < \frac{p_{r-1}-1}{p_r}.$$

It is an impossible case.

Second step. We show that there exists a positive integer $n_1 \in W_k$ such that $p_{r-1}\psi_s(n_1) < p_r n_1^s$. Suppose the opposite, this means that for all $n \in W_k$ we have $p_{r-1}\psi_s(n) \ge p_r n^s$. Let p_N be a prime number satisfying

$$p_N > \left(\frac{1}{\left(\frac{p_r}{p_{r-1}+1}\right)^{\frac{1}{k}} - 1} \right)^{\frac{1}{s}}, \tag{2.2.2}$$

which we may because there are infinitely many primes. In particular, for $n = p_N p_{N+1} \ldots p_{N+k-1} \in W_k$, it follows from (2.2.2) that

$$\frac{p_{r-1}}{p_r} \ge \frac{n^s}{\psi_s(n)} = \prod_{p|n} \frac{p^s}{p^s+1} \ge \left(\frac{p_N^s}{p_N^s+1}\right)^k > \frac{p_{r-1}+1}{p_r}.$$

It is an impossible case as well.

Third step. Now, we can verify the infinity of the above inequalities on the set W_k. Let n_0 and n_1 be two positive integers of W_k such that $p_{r-1}\psi_s(n_0) > p_r n_0^s$ and $p_{r-1}\psi_s(n_1) < p_r n_1^s$. By using the definition of ψ_s (see equation (2.1.3)), the inequality $p_{r-1}\psi_s(n) > p_r n^s$ holds for every integers $n = n_0^m$, with $m \ge 1$, and also the inequality $p_{r-1}\psi_s(n) < p_r n^s$ holds for every integers $n = n_1^m$, with $m \ge 1$. The proof of Theorem 2.2.1 is now complete. $\square$

Remark 2.2.1. Let r, s be positive integers, with $r \geq 2$. We define the set

$$B_{r,s} = \{n \in \mathbb{N}\ ;\ p_{r-1}\psi_s(n) < p_r n^s\}\,. \tag{2.2.3}$$

Let n be an unlimited integer such that for every $s \geq 1$,

$$\frac{n^s}{\psi_s(n)} = 1 - \varepsilon,$$

for a certain infinitesimal positive ε (for example, $n = q_1^{\alpha_1} q_2^{\alpha_2}$, with $q_1 \cong q_2 \cong +\infty$ and $\alpha_1, \alpha_2 \geq 1$). For every limited $r \geq 2$, it follows from Bertrand's theorem that $\frac{p_{r-1}}{p_r} = 1 - a$, where a is an appreciable positive rational number strictly less than 1. Therefore, $n \in B_{r,s}$. That is, $B_{r,s}$ is a nonempty set.

Corollary 2.2.2. *Suppose that $n \notin B_{r,s}$ for some $r \geq 2$ and $s \geq 1$, then there are an infinity of positive integers s' such that $n \in B_{r,s+s'}$.*

Proof. Assume that $n \notin B_{r,s}$ for certain $r \geq 2$ and $s \geq 1$. Because the sequence $\left\{\prod_{p|n}\left(1+\frac{1}{p^m}\right)\right\}_{m\geq 1}$ is decreasing and tends to 1 as m tends to $+\infty$. Then, there exists a unique positive integer s_0 such that

$$\prod_{p|n}\left(1+\frac{1}{p^{s+s_0-1}}\right) \geq \frac{p_r}{p_{r-1}} > \prod_{p|n}\left(1+\frac{1}{p^{s+s_0}}\right) > \prod_{p|n}\left(1+\frac{1}{p^{s+s_0+1}}\right) > \dots$$

That is, for every $s' \geq s_0$, $n \in B_{r,s+s'}$. □

We also have on the set W_k the following result.

Theorem 2.2.3. *There exist two arithmetic functions f, g for which $p_r f(n) - p_{r-1} g(n)$ changes sign infinitely often on the set W_k. Moreover, If $f(1)$ and $g(1)$ are two equal positive integers, then $(p_r, p_{r-1}) = (3, 2)$ and $f(1) = g(1) = 1$.*

Proof. We shall use the *k-th* convergent of an irrational number. (Because, any irrational number can be written uniquely as an infinite simple continued fraction, see [23],[31]). Let $[a_0, a_1, \ldots, a_n, \ldots]$ be an infinite simple continued fraction. We denote the *n-th* convergent by $\dfrac{P_n}{Q_n}$, where

$$\begin{cases} \dfrac{P_0}{Q_0} = \dfrac{a_0}{1} \\ \dfrac{P_1}{Q_1} = \dfrac{a_0 a_1 + 1}{a_1} \\ \ldots \\ \dfrac{P_n}{Q_n} = \dfrac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}}, \text{ for } n \geq 2. \end{cases} \tag{2.2.4}$$

Then, from Chapter 6 on Diophantine approximation stated by Alan Baker in [1], we have

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}, \text{ for } n \geq 1.$$

Now, let $p_r > 2$ be the *r-th* prime number. We put for all positive integer n,

$$f(n) = \frac{P_n Q_{n-1}}{p_r} \text{ and } g(n) = \frac{P_{n-1} Q_n}{p_{r-1}}; n \geq 1.$$

It is clear that $p_r f(n) - p_{r-1} g(n)$ changes sign infinitely often on the sets W_k. Which leads to the interesting result:

$$\text{If } f(1) = g(1) \in \mathbb{N}, \text{ then } p_r = 3,\ f(1) = 1 \text{ and } g(1) = 1.$$

Indeed, if $f(1) = g(1) = m$ for a certain positive integer m, it follows that

$$\frac{P_1 Q_0}{p_r} = \frac{P_0 Q_1}{p_{r-1}} = m.$$

According to equations (2.2.4), we find

$$\frac{a_0 a_1 + 1}{p_r} = \frac{a_0 a_1}{p_{r-1}} = m.$$

And therefore $m(p_r - p_{r-1}) = 1$. So, we must have $m = 1$ and $p_r = 3$. That completes the proof of Theorem 2.2.3. $\square$

Using Problem 639 stated by J-M. De Koninck in [26, p. 83], we have $\prod_{d|n} d = n^3$ if and only if $n = p^5$ or $n = p^2 q$, with p and q are distinct primes. In the following result, we will prove a new similar formula.

Some problems in number theory can be dealt with by applying a general combinatorial theorem about sets called the principle of cross-classification. This is a formula which counts the number of elements of a finite set S which do not belong to a certain prescribed subsets $S_1, S_2, \ldots, S_n$.

Notation 2.2.1. If T is a subset of S we write $N(T)$ for the number of elements of T. We denote by $S - T$ the set of those elements of S which are not in T. Thus,

$$S - \bigcup_{i=1}^{n} S_i$$

consists of those elements of S which are not in any of the subsets $S_1, S_2, \ldots, S_n$. For brevity we write $S_i S_j$, $S_i S_j S_k, \ldots$ of the intersection $S_i \cap S_j$, $S_i \cap S_j \cap S_k, \ldots$, respectively.

Theorem 2.2.4. *Principle of cross-classification. If $S_1, S_2, \ldots, S_n$ are given subsets of a finite set S, then*

$$\begin{aligned} N\left(S - \bigcup_{i=1}^{n} S_i\right) &= N(S) - \sum_{i=1}^{n} N(S_i) + \sum_{1 \le i < j \le n} N(S_i S_j) - \sum_{1 \le i < j < k \le n} N(S_i S_j S_k) + \ldots \\ &\quad + (-1)^n N(S_1 S_2 \ldots S_n). \end{aligned}$$

Proof. If $T \subset S$ let $N_r(T)$ denote the number of elements of T which are not in any of the first r subsets $S_1, S_2, \ldots, S_n$, with $N_0(T)$ being simply $N(T)$. The elements enumerated by $N_{r-1}(T)$ fall into two disjoint sets, those which are note in S_r and those which are in S_r. Therefore we have

$$N_{r-1}(T) = N_r(T) + N_{r-1}(TS_r).$$

Hence

$$N_r(T) = N_{r-1}(T) - N_{r-1}(TS_r) \tag{2.2.5}$$

Now take $T = S$ and use (2.2.5) to express each term on the right in terms of $N_{r-2}(T)$. We obtain

$$\begin{aligned} N_r(S) &= \{N_{r-2}(S) - N_{r-2}(SS_{r-1})\} - \{N_{r-2}(S_r) - N_{r-2}(S_rS_{r-1})\} \\ &= N_{r-2}(S) - N_{r-2}(S_{r-1}) - N_{r-2}(S_r) + N_{r-2}(S_rS_{r-1}). \end{aligned}$$

Applying (2.2.5) repeatedly we finally obtain

$$N_r(S) = N_0(S) - \sum_{i=1}^{r} N_0(S_i) + \sum_{1\le i<j\le n} N_0(S_iS_j) - \ldots + (-1)^n N_0(S_1S_2\ldots S_n).$$

When $r = n$ this gives the required formula.

□

In [26, Problem 706, p. 92], $\tau(\gamma(n)) = \gamma(\tau(n))$ if and only if $n = p^{\alpha}$, with p prime and $\alpha \geq 1$. Next we discuss the above expression for the arithmetic function Ω instead of τ. We have

Proposition 2.2.5. *There are infinitely many $n \in W_k$ such that*

1. $\Omega(\gamma(n)) = \gamma(\Omega(n))$.
2. $\dfrac{\varphi(\Omega(\tau(n)))}{\tau(\Omega(n))}$ *is an integer.*

Proof. Let $(q_1, q_2, \ldots, q_m)$ be an m-tuple of distinct odd primes, where the product $s = q_1 q_2 \ldots q_m$ is greater or equal to k, and let $(l_1, l_2, \ldots, l_s)$ be an s-tuple of distinct primes. For $n = l_1 l_2 \ldots l_s \in W_k$, we establish the statements of our Proposition.

1. From the definitions of Ω and γ, we obtain

$$\Omega(\gamma(n)) = \Omega(l_1 l_2 \ldots l_s) = s = q_1 q_2 \ldots q_m = \gamma(q_1 q_2 \ldots q_m)$$
$$= \gamma(s) = \gamma(\Omega(l_1 l_2 \ldots l_s)) = \gamma(\Omega(n)).$$

2. For each such integer n, the fraction

$$\frac{\varphi(\Omega(\tau(n)))}{\tau(\Omega(n))} = \frac{\varphi(\Omega(2^s))}{\tau(s)} = \frac{\varphi(q_1 q_2 \ldots q_m)}{2^m} = \frac{\prod_{i=1}^{m}(q_i - 1)}{2^m}.$$

   is a positive integer, since $q_i - 1$ is even for every $i = 1, 2, \ldots, m$.

This completes the proof. □

In the next result, we will prove that every square free positive integer $n \in W_k$ is not equal to the sum of their factors with respect to the power k.

Proposition 2.2.6. *Let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes, with $k \geq 2$. Then, $q_1 q_2 \ldots q_k$ is not equal to $q_1^k + q_2^k + \ldots + q_k^k$.*

Proof. Suppose the opposite, this means that for $n = q_1 q_2 \ldots q_k$, we have

$$n = q_1^k + q_2^k + \ldots + q_k^k.$$

Since $q_1 < q_2 < \ldots < q_k < n^{\frac{1}{k}}$, it follows that

$$n^{\frac{1}{k}} > q_k = \frac{n}{q_1 q_2 \ldots q_{k-1}} > \frac{n}{q_1 n^{1-\frac{2}{k}}}.$$

and therefore

$$q_1 > n^{2-\frac{3}{k}} \geq n^{\frac{1}{k}}.$$

A contradiction. □

Remark 2.2.2. Let $n = q_1 q_2 \ldots q_k$ and suppose that $n = q_1^k + q_2^k + \ldots + q_k^k$, where k is ≥ 2. It follows from Arithmetic-Mean-Geometric-Mean Inequality that

$$n = \left(q_1^k q_2^k \ldots q_k^k\right)^{\frac{1}{k}} \leq \frac{q_1^k + q_2^k + \ldots + q_k^k}{k}.$$

Thus, $n \leq \frac{n}{k}$. Which is a contradiction.

Proposition 2.2.7. *Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k} \in W_k$ with $\alpha_i \geq 1$ for every $i = 1, 2, \ldots, k$. Then*

$$\frac{\varphi(n)}{n} \geq \frac{1}{2^k}.$$

Proof. From the definition of φ, we can write

$$\frac{\varphi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right) \geq \prod_{p|n} \left(1 - \frac{1}{2}\right) \geq \frac{1}{2^k}.$$

Which we may because $\omega(n) \geq k$. This completes the proof. □

Proposition 2.2.8. *We put*

$$\delta_k = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

There are infinitely many $n \in W_k$ such that $\frac{\varphi(n)}{n} > \delta_k$.

Proof. Since there are infinitely many prime, there exists an k-tuple of distinct primes $(q_1, q_2, \ldots, q_k)$, with $q_i \geq p_i$ for $i = 1, 2, \ldots, k$. If $n = q_1 q_2 \ldots q_k$, then

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right) \geq \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \delta_k.$$

This completes the proof. □

Proposition 2.2.9. *There exists a positive integer $n \in W_1$ such that*

$$\begin{cases} \dfrac{\ln n}{\ln 2} + 1 \text{ is prime,} \\ \varphi\left(\tau\left(n\right)\right) = \tau\left(\varphi\left(n\right)\right). \end{cases}$$

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k -tuple of distinct primes, where $q_1+q_2+\ldots+q_k-k+1$ is a prime. For $n = 2^{q_1+q_2+\ldots+q_k-k}$, we have

$$\varphi\left(\tau\left(n\right)\right) = \varphi\left(q_1 + q_2 + \ldots + q_k - k + 1\right) = q_1 + q_2 + \ldots + q_k - k.$$

and

$$\tau\left(\varphi\left(n\right)\right) = \tau\left(2^{q_1+q_2+\ldots+q_k-k-1}\right) = q_1 + q_2 + \ldots + q_k - k.$$

This completes the proof. □

Proposition 2.2.10. *Let a be a positive real number. There are infinitely many $n \in W_k$ such that*

$$\frac{n}{\sum_{d|n} \gamma\left(d\right)} > a^k.$$

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k -tuple of distinct primes such that

$$\frac{q_i^{\alpha_i}}{1 + \alpha_i q_i} > a, \text{ for } i = 1, 2, \ldots, k.$$

and let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k} \in W_k$. We have,

$$\frac{n}{\sum_{d|n} \gamma\left(d\right)} = \frac{n}{\prod_{i=1}^{k} \gamma\left(q_i^{\alpha_i}\right)} = \prod_{i=1}^{k} \frac{q_i^{\alpha_i}}{1 + \alpha_i q_i} > a^k.$$

This completes the proof. □

Remark 2.2.3. We ask if there are infinitely many $n \in W_k$ such that

$$\sum_{\substack{1 \le a \le n \\ (a,n)=1}} a - \frac{n\varphi\left(n+1\right)}{2}$$

changes sign infinitely often.

In the rest of this section, we use the paper of J. Sándor [43] where for every $n \geq 2$ he proved that

$$\sigma(n) > n + (\omega(n) - 1)\sqrt{n}.$$

On the set W_k, we prove a new similar result as follows.

Proposition 2.2.11. *There are infinitely many $n \in W_k$ such that*

$$\sigma(n) > 1 + n + \omega(n)\sqrt{n}.$$

Proof. Let s be a positive integer such that $s \geq \max\{k, 4\}$, and let p_s be the *sth* prime number. For $n = p_1 p_2 \ldots p_s \in W_k$, we obtain from Bonse's inequality stated by D. Wells in [48, p. 21] that

$$p_{s+1}^2 < p_1 p_2 \ldots p_s = n.$$

Since $p_i < \sqrt{n}$, for $i = 1, 2, \ldots, s$, it follows that

$$\frac{n}{p_1} > \frac{n}{p_2} > \ldots > \frac{n}{p_s} > \sqrt{n},$$

and therefore

$$\sigma(n) > 1 + n + \sum_{i=1}^{s} \frac{n}{p_i} > 1 + n + s\sqrt{n} = 1 + n + \omega(n)\sqrt{n}.$$

As required. □

2.3 Non-classical notes on certain arithmetic functions

In the previous section we applied our technique only to the set W_k, with $k \geq 1$. Since the number of primes is infinite, we restrict our attention to the non-classical continuation. That is, we deal essentially with using unlimited prime factors.

2.3.1 On integers n for which $\varphi(n)$ is not of the form $2^s p$, with s limited and p unlimited

According to a special case of Dirichlet's famous theorem, (see [38], [39]), given p there exist infinitely many $m \geq 1$ such that $2mp+1$ is a prime. That is, there exist infinitely many n such that $\varphi(n)$ is of the form $2a$, where p divides a. However, it is very difficult to prove that there exist infinitely many primes of the form $2^s p + 1$, where $s \geq 1$ and p is a prime, see the classical unsolved problems stated by L. Moser in [32, p. 73]. In the following Theorem, we will give two kinds of positive integers n for which $\varphi(n)$ is not equal to the product of 2^s and an unlimited prime p, where s is a limited. See [8, p. 107].

Theorem 2.3.1. *We have*

1. *Let p_m be the m-th limited odd prime number, there exists a limited integer k_0 such that*
$$\prod_{i=0}^{k_0} \frac{p_{m+i}}{p_{m+i}-1} > 2. \tag{2.3.1}$$
2. *Let φ be the Euler function. If we put*
$$A = \{a.l; l \in \mathbb{N}; \, a = p_m p_{m+1} \dots p_{m+k_0}\}.$$

Then, $(\varphi, 1_{\mathbb{N}}) \notin F_A$*.That is,* $p_r\varphi(n) - p_{r-1}n$ *changes sign finitely often on the set* A*.*

Proof. First, we suppose the opposite; that means, if there exists a limited prime number p_{n_0} satisfying, for all limited integer $k \geq 1$,

$$\prod_{i=0}^{k} \frac{p_{n_0+i}}{p_{n_0+i} - 1} \leq 2,$$

We get from "Transfer principle [13], the inequality rest valid for all integer $k \in \mathbb{N}$, so we have a contradiction, because

$$\infty \cong \prod_{i=0}^{\infty} \frac{p_{n_0+i}}{p_{n_0+i} - 1} \leq 2.$$

Second, it suffices to prove that if an integer $n \in A_{r,1}$, then n is not divisible by $p_m p_{m+1} \ldots p_{m+k}$. Let $n \in A_{r,1}$, and we suppose that $p_m p_{m+1} \ldots p_{m+k_0}$ is a divisor of n. Where k_0 is the limited integer satisfying (2.3.1), so that

$$n = s \cdot p_m^{a_m} p_{m+1}^{a_{m+1}} \ldots p_{m+k_0}^{a_{m+k_0}} \ ; s \geq 1, a_i > 0$$

Provided that $(s, p_m p_{m+1} \ldots p_{m+k_0}) = 1$, and therefore

$$p_r > \frac{p_{r-1}}{\prod_{p\ |n} \left(1 - \frac{1}{p}\right)}$$

Hence

$$p_r > p_{r-1} \prod_{i=0}^{k_0} \frac{p_{m+i}}{p_{m+i} - 1} \prod_{p|s} \frac{p}{p-1}$$

From Bertrand's postulate, and the previous inequality, we have

$$2 > 2 \prod_{p|s} \frac{p}{p-1}.$$

Which is impossible, and so the assertion follows. □

Example 2.3.1. *Let* p_r *be the r-th prime number, If* $n \in A_{r,1}$, *then* n *is not divisible by* 5.7.11.13.17.19.23. *or* 7.11.13.17.19.23.29.31.37.41.43.47.53.59.61.

2.3.2 Non-classical study of the inequality $f(n) > \alpha\, g(n)$

Notation 2.3.1. We consider the set

$$W_\infty = \{n \in \mathbb{N}\ ;\ \omega(n) \cong +\infty, \text{ and for all } p|n : p \cong +\infty\}.$$

It is clear that W_∞ is an infinite external set. For instance, if N is an unlimited positive integer and if p_N is the Nth prime number, then

$$p_N p_{N+1} \ldots p_{2N} \in W_\infty.$$

As a direct consequence of Theorem 2.2.1, we prove

Let $\mathbb{P}$ be the sequence of primes in ascending order, and let ℓ be a positive integer. In the following result, we assume that $\alpha = \dfrac{p_{r-1}}{p_r}$ is a limited parameter.

Theorem 2.3.2. *Let $p_r > 2$ be the r-th prime number (limited or unlimited) and $A \subseteq N$ be an infinite subset. We put*

$$F_A = \left\{ \begin{array}{c} (f, g)\ ;\ p_r f(n) - p_{r-1} g(n) \text{ changes} \\ \text{sign infinitely often on the set } A. \end{array} \right\}; \qquad (2.3.2)$$

where f and g are two arithmetic functions. The following statements are true:

(a) If we choose $f(n) = d_n$ is the gap between p_{n+1} and p_n, then $p_r f(n) - p_{r-1} f(n+1)$ changes sign infinitely often on the set N and A_∞ respectively.

(b) Let $p_r > 2$ be the r-th prime number, there exist two arithmetic functions f, g for which $p_r f(n) - p_{r-1} g(n)$ changes sign infinitely often on the set N and A_∞. Moreover, If $f(1)$ and $g(1)$ are two equal positive integers, it follows that $(p_r, p_{r-1}) = (3, 2)$ and $f(1) = g(1) = 1$.

(c) For each given $k \geq 1$, there are infinitely many integers $n \in W_\infty$, such that

$$p_r\varphi_k(n) > p_{r-1}n^k;\ p_r > 3.$$

(d) For each given $k \geq 1$ limited, there are infinitely many integers $n \in W_\infty$, such that

$$p_r\varphi_k(n) < p_{r-1}n^k;$$

where p_r is the r-th unlimited prime number.

(e) Let $f(n)$ be a multiplicative arithmetic function increasing on the set P, we suppose further for all $m,\ n,\ i \geq 1$

$$\begin{aligned} 1 &\geq \frac{f(n)}{n} \geq \frac{f(mn)}{mn}, \qquad (2.3.3)\\ \lim_{p^i \longrightarrow \infty} \frac{p^i}{f(p^i)} &= 1. \end{aligned}$$

as p^i runs through the sequence of all prime powers.

And, let s_0 be a fixed limited integer. If

$$\{n\ odd\ ;\ p_r(f(n)+s_0) < p_{r-1}n\} \neq \emptyset \qquad (2.3.4)$$

Then, $p_r(f(n)+s_0) - p_{r-1}n$ changes sign unlimited often on the set W_∞.

2.4 On certain near equations

A prime power is a positive integer power of a single prime number, see [19], [23], [36]. The twenty smallest elements of the sequence of prime powers are:

$$2, 3, 2^2, 5, 7, 2^3, 3^2, 11, 13, 2^4, 17, 19, 23, 5^2, 3^3, 29, 31, 2^5, 37, 41, \ldots.$$

Then we recall a classical theorem about the sequence of prime powers.

Theorem 2.4.1 (see M.B. Nathanson [34], p. 225)**.** *Let $f(n)$ be a multiplicative function. If* $\lim_{q^{\alpha}\to\infty} f(q^{\alpha}) = 0$*, as* q^{α} *runs through the sequence of all prime powers, then* $\lim_{n\to\infty} f(n) = 0$*.*

Remark 2.4.1. Internal Set Theory show us how theorems in standard analysis can be proved using theorems in nonstandard analysis. For instance, we announce the above theorem as follows:

Remark 2.4.2. Since $f(q^{\alpha}) \cong 0$ for every unlimited q^{α}, and since f is standard, it follows that there exist only a limited number of prime powers q^{α} such that $f(q^{\alpha}) \ncong 0$. Thus, there are only a limited number of integers n such that

$$f(q^{\alpha}) \ncong 0$$

for every prime power q^{α} that exactly divides n. Therefore, if n is unlimited, then n is divisible by at least one prime power q^{α} such that $f(q^{\alpha}) \cong 0$, and so n can be written in the form

$$n = \prod_{i=1}^{r} q_i^{\alpha_i} . \prod_{i=r+1}^{r+s} q_i^{\alpha_i}$$

where $q_1, q_2, \ldots, q_{r+s}$ are distinct prime numbers such that

$$\begin{cases} f(q_i^{\alpha_i}) \ncong 0, \text{ for } i = 1, 2, \ldots, r \\ f(q_i^{\alpha_i}) \cong 0, \text{ for } i = r+1, r+2, \ldots, r+s \end{cases}$$

and $s \geq 1$. Since f is multiplicative, and r is limited. It follows that,

$$f(n) = \prod_{i=1}^{r} f(q_i^{\alpha_i}) . \prod_{i=r+1}^{r+s} f(q_i^{\alpha_i}) = \pounds.\varepsilon \cong 0.$$

Proposition 2.4.2. *If k is limited, there are infinitely many $n \in W_k$ such that*

$$\frac{\varphi_s(n)}{n^s} - 1 \cong 0.$$

where k is defined in (2.1.1)*, and $s \geq 1$.*

Proof. Let ε be an infinitesimal positive number, and let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes such that

$$q_i^s \geq \frac{1}{\varepsilon}, \text{ for } i = 1, 2, \ldots, k.$$

For $n = (q_1 q_2 \ldots q_k)^m \in W_k$, with $m \geq 1$. We can write,

$$1 > \frac{\varphi_s(n)}{n^s} = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^s}\right) \geq (1-\varepsilon)^k = 1 - \varepsilon',$$

Since k is limited, then ε' is an infinitesimal positive number. This completes the proof. ☐

Proposition 2.4.3. *For every $n \in W_\infty$, we have*

$$\frac{\sigma(n)}{n^2} \cong 0.$$

Proof. Let $n \in W_\infty$, it is clear that

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s},$$

where $q_1, q_2, \ldots, q_s$ are unlimited distinct prime numbers, with s unlimited and $\alpha_i \geq 1$ for $i = 1, 2, \ldots, s$. Since $\dfrac{\sigma(t)}{t^2}$ is multiplicative, then

$$\begin{aligned} \frac{\sigma(n)}{n^2} &= \prod_{i=1}^{s} \frac{\sigma(q_i^{\alpha_i})}{q_i^{2\alpha_i}} = \prod_{i=1}^{s} \frac{1 + q_i + \ldots + q_i^{\alpha_i}}{q_i^{2\alpha_i}} \\ &= \prod_{i=1}^{s} \frac{1}{q_i^{\alpha_i}} \left(1 + \frac{1}{q_i} + \ldots + \frac{1}{q_i^{\alpha_i}}\right) < \prod_{i=1}^{s} \frac{1}{q_i^{\alpha_i - 1}(q_i - 1)} \cong 0. \end{aligned}$$

This completes the proof. ☐

Remark 2.4.3. We ask if there exist a limited positive integer $a \neq 30$ and n for which $\varphi(an+1) < \varphi(an)$.

Let a, b limited positive integers. Does $\dfrac{\varphi(an+1)}{\varphi(bn+1)} \cong 1$ for infinitely many $n \in W_k$?

Remark 2.4.4. In Proposition 2.4.2, we ask if $\frac{\varphi_s(n+1)}{n^s} - 1 \cong 0$ holds for infinitely many $n \in W_k$ whenever k is limited. Also, in Proposition 2.4.3 we ask if $\frac{\sigma(n+1)}{n^2} \cong 0$ holds for infinitely many $n \in W_\infty$.

Chapter 3

Sign Changes of tow Arithmetic Functions involving γ, π, d_n and φ_s

3.1 Introduction

Let r, N and ℓ be positive integers (parameters) with $r \geq 2$, and let p_r be the rth prime number. Let W_k be the set defined as in (2.1.1). By using Dirichlet's theorem about primes in an arithmetic progression and Bertrand's theorem, we will determine arithmetic functions $f : \mathbb{N} \to \mathbb{N}$ for which $f^N(n) - \alpha f^N(n+\ell)$ has infinitely many sign changes on a proper infinite subset of W_k, where $\alpha = \dfrac{p_{r-1}}{p_r}$. Moreover, we will determine two arithmetic functions (f, g), with $g(n) \neq f(n+\ell)$ and $f^N(n) - \alpha\, g^N(n)$ has infinitely many sign changes on an infinite subset of W_k. In the framework of internal set theory, our working set denoted by $\overline{W_\infty}$, is the set of positive integers n for which the number of distinct prime factors of n is unlimited.

To state main results, we start with a few definitions and introduce notations and terminology that is consistent with the tools and ideas used in this chapter, see [3],[10],[15],[24].

Definition 3.1.1. A positive integer is called square-free if it is the product of distinct prime numbers.

Definition 3.1.2. A positive integer n is multiply perfect if $\sigma(n) = s\,n$, with $s \geq 2$.

It is possible to create higher-order perfect numbers from lower-order, based on the following Theorem.

Theorem 3.1.1 (D. Wells [48], p. 173)**.** *If p is prime, n is p-perfect and $p \nmid n$, then pn is $(p+1)$-perfect.*

Theorem 3.1.2 ([48], p. 254)**.** *If $n = p_1^{r_1} p_2^{r_2} \ldots p_s^{r_s}$ is the standard factorization of n as a product of powers of distinct primes, then*

$$\prod_{i=1}^{r} \frac{p_i}{p_i - 1} > \frac{\sigma(n)}{n}.$$

Definition 3.1.3. Let σ_2 be the arithmetic function given by

$$\sigma_2(n) = \sum_{d|n} d^2.$$

Theorem 3.1.3 (Dirichlet's theorem about primes in arithmetic progression [33])**.** *If a and b are relatively prime integers with $a \geq 1$, then the polynomial $f(n) = an + b$ represents infinitely many primes.*

Can one find $q > 41$ such that $X^2 + X + q$ has prime value for $n = 0, 1, \ldots, q-2$? Are there infinitely many, or only finitely many such primes q? If so, what is the largest possible q?

The same problem should be asked for polynomials of first degree $f(X) = aX + b$, with $a, b \geq 1$. If $f(0)$ is a prime q, then $b = q$. Then $f(q) = aq + q = (a+1)q$ is composite. So, at best, $aX + q$ assumes prime values for X equal to $0, 1, \ldots, q-1$. Can one find such polynomials? Equivalently, can one find arithmetic progressions of q prime numbers, of which the first number is equal to q?

Definition 3.1.4. Let $k \geq 2$. The natural number $n \geq 1$ is said to be a $k-$powerful number when the following property is satisfied: if a prime p divides n, then p^k also divides n.

Definition 3.1.5. The prime numbers p and q are called twin primes if $|p - q| = 2$.

Conjecture 3.1.4 (Twin Prime Conjecture [9])**.** *There are infinitely many twin primes.*

Conjecture 3.1.5 (Goldbach Conjecture [34])**.** *Every even number $n \geq 4$ can be written as the sum of two primes.*

Throughout the first section of this chapter, we use the following notations.

Notation 3.1.1. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be an arithmetic function. For every positive integer a, f^a denote the arithmetic function given by $f^a(n) = (f(n))^a$; $n \geq 1$.

Notation 3.1.2. Let p be prime. Denote by $p^a \parallel n$, when p^a divides n and p^{a+1} does not divide.

There exists a proper infinite subset $W \subset W_k$ such that for all $n \in W$ and for all $i \geq 1$, one of the following inequalities holds :

-$f^N(n) > cg^N(n)$, when n is a square free.

-$f^N(n) < cg^N(n)$, otherewise".

and let $\mathbb{F}$ denote the set of all arithmetic functions (f, g) of such property. In this paper, we will prove the following statements

1) $\mathbb{F}$ is a nonempty set.

2) There exists an infinite sequence of infinite proper subsets of W_k for which $f^i(n) - cg^i(n)$ has infinitely many sign changes, for every $i \geq 1$.

3.2 On the function γ

This work is placed in the framework of Internal set Theory $[1, 2]$. In this paper, we study some inequalities involving the rth prime number and some arithmetic functions. Our working set is a nonclassical subset of $\mathbb{N}$. Let n be a positive integer, we denote by $\omega(n)$ the number of distinct prime factors of n. According to a special case of Dirichlet's famous theorem, we have

Theorem 3.2.1. *Let a and b be relatively prime limited positive integers. There are infinitely many primes of the form $an + b$, where $\omega(n)$ is a limited.*

Example 3.2.1. *There are infinitely many primes of the form $2.3.5n+29$, where $\omega(n)$ is a limited. Let $\{q_1, q_2, \ldots, q_s\}$ be the finite list of primes of the form $2.3.5n-1$, with $\omega(n)$ is limited. Construct the number*

$$N = 2.3.5q_1q_2 \ldots q_s - 1.$$

From [39], N is divisible by a prime q of the form $2.3.5m-1$ not on the list $\{q_1, q_2, \ldots, q_s\}$. Since $\omega(N)$ is limited, then $\omega(m)$ is also a limited. A contradiction.

Let l be an odd positive integer. Let k be a limited positive integer, we put

$$W_k = \{n \in \mathbb{N} \; ; \; \omega(n) \geq k\}. \tag{3.2.1}$$

Theorem 3.2.2. *Let p_r be the r-th prime number, then $p_r \; \gamma(n) - p_{r-1}\gamma(n+1)$ changes sign infinitely often on the set W_1, where*

$$\gamma(n) = \prod_{p \mid n} p.$$

Proof. Because, there are infinitely many primes p such that $p+1$ is of the form $4k$ and also there are infinitely many primes p' such that $p'-1$ is of the form $4k'$; (k and k' are positive integers). If $n = 4k-1$ is prime, then $\gamma(n) > \gamma(n+1)$, and if $n+1 = 4k+1$ is prime, from Bertrand's postulate, it follows that

$$p_r \gamma(n) < p_{r-1} \gamma(n+1).$$

The proof is finished. □

Remark 3.2.1. From the proof of Theorem 3.2.2, we can not easily deduce the sign changes of $\gamma^N(n) - \alpha\,\gamma^N(n+\ell)$ for infinitely many $n \in W_k$, because it is very difficult to prove that there are infinitely many primes of the form $q_1 q_2 \ldots q_k\, m + \ell$, where m is a square-free and there are infinitely many primes of the form $q_1 q_2 \ldots q_k\, m + \ell$, where m is not. For more details, see the proof of Dirichlet's theorem about primes in an arithmetic progression stated by A. Selberg in [47].

The computation techniques of the proof of Theorem 3.2.2 is same as the argument which is used in the proof of the following result.

Theorem 3.2.3. *One of the following inequalities*

$$(a)\ p_r \sigma(\gamma(n)) > p_{r-1} \gamma(n+l),\ \ (b)\ p_r \sigma(\gamma(n)) < p_{r-1} \gamma(n+l).$$

holds infinitely often on a proper subset $W \subset W_k$.

Proof. For every $n = 2q_2 \ldots q_k m$, where $2q_2 \ldots q_k m + l$ is prime, we have

$$\begin{aligned} \frac{\gamma(n+l)}{\sigma(\gamma(n))} &= \frac{2q_2 ... q_k m + l}{\sigma(\gamma(2q_2 ... q_k m))} \\ &= \frac{n}{\sigma(\gamma(n))} + \frac{l}{\sigma(\gamma(n))}. \end{aligned}$$

We put $a_n = \dfrac{n}{\sigma(\gamma(n))}$ and $\varepsilon_n = \dfrac{l}{\sigma(\gamma(n))}$. Since $W^{(2)}$ is infinite, we distinguish two cases.

1. There are infinitely many $n \in W$ such that $n = \prod_{p \| n} p$.

   From the second condition of (2), and since $\gamma(2q_2...q_k s) \geq 2q_2 \ldots q_k$ for every $s \geq 1$, we have

   $$\varepsilon_n = \frac{l}{\sigma(\gamma(n))} \leq \frac{l}{2q_2 \ldots q_k} < \frac{p_r}{p_{r-1}} - 1. \tag{3.2.2}$$

   It follows that

   $$\frac{\gamma(n+l)}{\sigma(\gamma(n))} = \prod_{p \not| n} \frac{p}{1+p} + \varepsilon_n \ < 1 + \varepsilon_n < \frac{p_r}{p_{r-1}}.$$

2. There are infinitely many $n \in W$ such that $n \neq \prod_{p \| m} p$.

   There are two cases :

   (a) In the case, $\omega(m)$ is limited. There are infinitely many m such that every element is divisible by p^α for a certain positive integer α and a prime p, where at least we must have $p \cong \infty$ or $\alpha \cong \infty$. Then,

   $$a_n = \prod_{i=1}^{k} \frac{q_i}{1+q_i} \prod_{p \| m} \frac{p}{1+p} \prod_{\substack{p^\alpha \| m \\ \alpha > 1}} \frac{p^\alpha}{1+p} = \begin{cases} (1-\pounds) \prod\limits_{\substack{p^\alpha \| m \\ \alpha>1}} \dfrac{p^\alpha}{1+p} \cong \infty \\ (1-\varepsilon) \prod\limits_{\substack{p^\alpha \| m \\ \alpha>1}} \dfrac{p^\alpha}{1+p} \cong \infty \end{cases}$$

   and therefore, $a_n > \dfrac{p_r}{p_{r-1}}$.

   (b) In the case : $\omega(m)$ is unlimited. Since m is odd, we also distinguish two cases

- There are infinitely many $m \in W^{(2)}$ such that $card\{p; p \parallel m\}$ is infinite.

$$a_n = \begin{cases} (1-\varepsilon)(1-\pounds) \prod_{\substack{p^\alpha \| m \\ \alpha>1}} \frac{p^\alpha}{1+p} \cong 0 < \frac{p_r}{p_{r-1}} \end{cases}$$

- There are infinitely many $m \in W^{(2)}$ such that card$\{p; p \parallel m\}$ is finite, thus, There are infinitely many $m \in W^{(2)}$ such that

$$a_n = \begin{cases} (1-\varepsilon) \prod_{\substack{p^\alpha \| m \\ \alpha>1}} \frac{p^\alpha}{1+p} \cong \infty > \frac{p_r}{p_{r-1}} \end{cases}$$

This completes the proof.

□

Corollary 3.2.4. *$p_r\sigma(\gamma(n)) - p_{r-1}\gamma(n+l)$ holds infinitely often on the set W_k.*

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes satisfying the first condition of (1.2). There are infinitely many primes of the form $q_1^2 q_2^2 \ldots q_k^2 m + l$, with $\omega(m)$ is limited. So, There are infinitely many m such that $p^\alpha \parallel m$ for a certain positive integer α, where at least $p \cong \infty$ or $\alpha \cong \infty$. For each such integer m, let $n = q_1^2 q_2^2 \ldots q_k^2.m$, then

$$\begin{aligned} \frac{\gamma(n+l)}{\sigma(\gamma(n))} &= \frac{q_1^2 q_2^2 ... q_k^2 m + l}{\sigma(\gamma(q_1^2 q_2^2 \ldots q_k^2 m))} \\ &= \frac{n}{\sigma(\gamma(n))} + \varepsilon_n \ ; \ \varepsilon_n = \frac{l}{\sigma(\gamma(n))} \\ &> \prod_{i=1}^{k} \frac{q_i^2}{1+q_i} \prod_{p \| m} \frac{p}{1+p} \prod_{\substack{p^\alpha \| m \\ \alpha>1}} \frac{p^\alpha}{1+p} + \varepsilon_n \cong \infty. \end{aligned}$$

This completes the proof. □

3.3 On the functions π, d_n and φ_s

Our aim in this section is to extend the previous study of $\gamma(n)$ to the functions $\pi(n)$, d_n and $\varphi_s(n)$; $s \geq 1$. For $\pi(n)$ and d_n, we prove

Theorem 3.3.1. *Let $\pi(n)$ be the number of primes which satisfy $2 \leq p \leq n$, and let ℓ be a positive integer. Then $\pi(n) - \alpha\,\pi(n+\ell)$ has finitely many sign changes on the set $\mathbb{N}$.*

Proof. We suppose that $\pi(n) - \alpha\,\pi(n+\ell)$ has infinitely many sign changes, that is, $\pi(n) < \alpha\,\pi(n+\ell)$ holds for infinitely many n. For each such integer n, we must have

$$\pi(n) < \pi(n+\ell) \leq \pi(n) + \ell. \tag{3.3.1}$$

Noticing that the right hand side in (3.3.1) can be deduced by induction on ℓ. Thus, there are infinitely many integers n such that

$$\pi(n) < \frac{p_{r-1}\ell}{p_r - p_{r-1}}.$$

A contradiction, because $\pi(n)$ is asymptotic to $\dfrac{n}{\ln n}$ which tends to infinity. □

Corollary 3.3.2. *The inequality $\pi(n) > \alpha\pi(n+\ell)$ holds for infinitely many $n \in \mathbb{N}$.*

Proof. In 1849, Polignac conjectured: For every even natural number $2k$ there are infinitely many pairs of consecutive primes p_n, p_{n+1} such that $d_n = p_{n+1} - p_n = 2k$. For more details, see [39, p. 250]. Let ℓ be a positive integer. Then there are infinitely many pairs of consecutive primes p_n, p_{n+1} such that $p_{n+1} - p_n > \ell$. For each such prime p_n, let $n = p_n$. It follows that $\pi(n) = \pi(n+\ell)$, and therefore $\pi(n) > \alpha\pi(n+\ell)$. □

Remark 3.3.1. Let $\pi_{a,b}(n)$ be the number of primes of the form $ak+b$ which are $\leq n$. Does

$$p_r\pi_{a,b}(n) - p_{r-1}\pi_{a,b}(n+1)$$

change sign finitely often?

Theorem 3.3.3. *The inequality* $d_n - \alpha\, d_{n+1}$ *has infinitely many sign changes on the set* W_1.

Proof. For all positive integers n, write $d_n = p_{n+1} - p_n$ so that $d_1 = 1$ and all other d_n are even. Since $(3, 5, 7)$ is the only prime triplet of the form p, $p + 2$, $p + 4$ (see [44, p. 76]), by using Twin Prime Conjecture, there are infinitely many primes $(p_n, p_{n+1}, p_{n+2})_{n \in \mathbb{N}}$ such that

$$\begin{cases} d_n = p_{n+1} - p_n = 2, \\ d_{n+1} = p_{n+2} - p_{n+1} \geq 4. \end{cases} \tag{3.3.2}$$

From Bertrand's theorem and (3.3.2), we have

$$p_r d_n - p_{r-1} d_{n+1} \leq 2\,(p_r - 2p_{r-1}) < 0.$$

Thus, the inequality holds infinitely often.

On the other hand, from [20, p. 26], Erdös and Turán have shown that $d_n > d_{n+1}$ infinitely often. Then, $p_r d_n - p_{r-1} d_{n+1} > 0$ for infinitely many n. This completes the proof. □

The rest of this section is devoted to study the sign changes of $\varphi_s\,(n) - \alpha\, n^s$ on the set W_k. From the Prime Number Theorem: $\lim\limits_{t \longrightarrow \infty} \dfrac{p_{t-1}}{d_{t-1}} = +\infty$, where d_{t-1} is the gap between p_t and p_{t-1}, we can choose the *rth* prime p_r as a sufficiently large number, and the integer s such that

$$p_k < \left(1 + \frac{p_{r-1} - 1}{d_{r-1} + 1}\right)^{\dfrac{1}{s}}, \tag{3.3.3}$$

with k is the integer of (2.1.1), and p_k is the *kth* prime number. Then we have

Theorem 3.3.4. *Under the same assumption as in* (3.3.3)*, we have,* $\varphi_s(n) - \alpha\, n^s$ *has infinitely many sign changes on the set* W_k*.*

Proof. We show that (a) and (b) are each true for infinitely many $n \in W_k$; where

(a) $\varphi_s(n) > \alpha\, n^s$,

(b) $\varphi_s(n) < \alpha\, n^s$.

Firstly, we need the following lemmas.

Lemma 3.3.5. *There exists a positive integer* $n_0 \in W_k$*, such that* $p_r \varphi_s(n_0) > p_{r-1} n_0^s$*.*

Proof. Let p_N be a prime number such that

$$p_N > \frac{1}{\left(\dfrac{p_r}{p_{r-1}+1}\right)^{\frac{1}{k}} - 1} + 1. \tag{3.3.4}$$

Suppose the opposite, this means that for all $n \in W_k$, we have $p_r \varphi_s(n) \leq p_{r-1} n^s$. In particular, for $n = p_N p_{N+1} \ldots p_{N+k-1}$, it follows from (3.3.4) that

$$\frac{p_r}{p_{r-1}} \leq \frac{n^s}{\varphi_s(n)} = \prod_{p|n} \frac{p^s}{p^s - 1} \leq \prod_{p|n} \frac{p}{p-1} < \left(\frac{p_N}{p_N - 1}\right)^k < \frac{p_r}{p_{r-1}+1}. \tag{3.3.5}$$

Which implies that $p_{r-1} > p_{r-1} + 1$. A contradiction. □

Lemma 3.3.6. *Under the same assumption as in* (3.3.3)*, there exists a positive integer* $n_1 \in W_k$ *such that* $p_r \varphi_s(n_1) < p_{r-1} n_1^s$*.*

Proof. Assume, by way of contradiction, that for all $n \in W_k$ we have $p_r \varphi_s(n) \geq p_{r-1} n^s$. In particular, for $n = 2.3 \ldots p_k$ (which is an element of W_k), it follows from (3.3.3) that

$$\frac{p_r}{p_{r-1}} \geq \frac{n^s}{\varphi_s(n)} = \prod_{p|n} \frac{p^s}{p^s - 1} \geq \frac{p_k^s}{p_k^s - 1} > \frac{p_r}{p_{r-1} - 1}. \tag{3.3.6}$$

A contradiction. □

Secondly, we return to prove (a) and (b) for infinitely many n. In fact, from Lemmas 3.3.5, 3.3.6 and the definition of φ_s (see (2.1.2)), there exist positive integers $n_0, n_1 \in W_k$ such that n_0^i, n_1^i satisfy (a) and (b), respectively, for every positive integer i. Thus, we find infinitely many such numbers n_0 and n_1. This completes the proof of Theorem 3.3.4. □

3.4 Some other inequalities on the set W_k

3.4.1 On certain inequalities for σ and σ_2

Proposition 3.4.1. *There are infinitely many $n \in W_k$ such that* $\dfrac{\sigma(n)}{n} > k$.

Proof. Let $\log_2 x$ denote the logarithm of x to the base 2, and let n_0 be a positive integer such that for all $n \geq n_0$,

$$\min\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}, \log_2 \log_2 n\right) > k.$$

For each such integer n, since $\pi(n) > \log_2 \log_2 n$ (see [34, p. 37]), then $n! \in W_k$. It follows that

$$\frac{\sigma(n!)}{n!} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > k.$$

This completes the proof. □

Proposition 3.4.2. *There are infinitely many $n \in\in W_k$ such that*

$$\frac{\sigma_2(n)}{n} > 2^k.$$

Proof. Let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s} \in W_k$, where $\alpha_i \geq 1$ for $i = 1, 2, \ldots, s$ and $s \geq k$. We have,

$$\left(\prod_{d|n} d\right)^2 = \prod_{d|n} d . \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{\tau(n)}.$$

and therefore

$$n = \left(\prod_{d|n} d^2\right)^{\dfrac{1}{\tau(n)}} \leq \frac{1}{\tau(n)} \sum_{d|n} d^2 = \frac{\sigma_2(n)}{\tau(n)}.$$

Thus,

$$\frac{\sigma_2(n)}{n} \geq \tau(n) = \tau\left(q_1^{\alpha_1} q_2^{\alpha_2} \dots q_s^{\alpha_s}\right) = \prod_{i=1}^{s} (1+\alpha_i) \geq 2^k.$$

On the other hand, since $\dfrac{\sigma_2(p)}{p} \neq 2$ and $\dfrac{\sigma_2(t)}{t}$ is multiplicative, then

$$\frac{\sigma_2(n)}{n} > 2^k.$$

This completes the proof of Proposition . □

From [34, p 282], $\limsup\limits_{n\longrightarrow\infty} \dfrac{\sigma(n)}{n} = \infty$. Then we have.

Proposition 3.4.3. *Let a be an unlimited real number and let n be an arbitrary solution of the inequality $\sigma(n) \geq a.n$. Then $\omega(n)$ is unlimited.*

Proof. We suppose the opposite, that is n has a limited number of distinct prime factors $q_1 < q_2 < \dots < q_s$, with s is limited. It follows from [26, Problem 516] that

$$+\infty \cong a \leq \frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1} \cong l,$$

where l is a standard rational number. Which is impossible. □

Remark 3.4.1. Let n be an arbitrary solution of the inequality $\sigma(n) \geq a.n$, with $a \cong +\infty$. Then

$$a \leq \frac{\sigma(n)}{n} = \sum_{d \mid n} \frac{1}{d} = \sum_{p \mid n} \frac{1}{p} + \sum_{\substack{d \mid n \\ d \notin P}} \frac{1}{d} \cong +\infty.$$

It follows that at least one of the numbers $\sum\limits_{p \mid n} \dfrac{1}{p}$ and $\sum\limits_{\substack{d \mid n \\ d \notin P}} \dfrac{1}{d}$ is unlimited. Thus, in each of the cases, n has an unlimited number of distinct prime factors.

Remark 3.4.2. We also prove Proposition 3.4.3 as follows: For every positive integer n, from [26], we have

$$\frac{\sigma(n)}{n} < \begin{cases} \left(\frac{3}{2}\right)^{\omega(n)}; \text{ if } n \text{ is odd,} \\ 2\left(\frac{3}{2}\right)^{\omega(n)-1}; \text{ if } n \text{ is even.} \end{cases}$$

Then, if $\frac{\sigma(n)}{n}$ is unlimited, it follows that $\omega(n)$ is also.

Proposition 3.4.4. *The inequality* $\sigma(n) < \sigma(n-1)$ *holds for infinitely many* $n \in W_2$.

Proof. Since there are infinitely many distinct primes (p, q) such that

$$N = \frac{pq-1}{2} > p + q,$$

then for $n = pq \in W_2$, we have

$$\sigma(n-1) \geq 1 + 2 + N + n - 1 > 1 + p + q + n = \sigma(n).$$

This completes the proof. □

Remark 3.4.3. It may be realized that to prove the inequality of Proposition 3.4.4 for infinitely many $n \in W_k$, with $k \geq 3$.

From [26, problem 489, page 69], we have

$$\sigma_2(n) \geq n\tau(n), \text{ for every } n \geq 1.$$

In the following theorem, we prove a new similar result on the set W_k.

Theorem 3.4.5. *If* $k \geq 3$, *then the inequality*

$$\sigma_2(n) > n\tau(n) + n^2,$$

holds for infinitely many $n \in W_k$.

For the proof, we use the following lemma.

Lemma 3.4.6. *Let p, q and r be distinct primes. If $n = pqr$, then*

$$8n < \sum_{d|n,\ d<n} d^2.$$

Proof. Suppose that $n = pqr$, where $p < q < r$ and $8 < r$. Since $qr \mid n$ and $qr < n$, it follows that $d^2 = q^2r^2 > r(pqr) > 8n$. Then, evidently $8n < \sum_{d|n,\ d<n} d^2$. It therefore remains to verify the triplets $(2,3,5)$, $(2,3,7)$, $(2,5,7)$ and $(3,5,7)$ which are all true. □

Proof of Theorem 3.4.5. Let $n \in W_k$ be a square free integer, with $k \geq 3$. We have,

$$\frac{\sigma_2(n)}{n\tau(n)} = \frac{n^2 + \sum_{d|n,\ d<n} d^2}{n\tau(n)} = \frac{n}{\tau(n)} + \frac{\sum_{d|n,\ d<n} d^2}{n.2^k}.$$

By induction on k, it suffices to prove that $\sum_{d|n,\ d<n} d^2 > n.2^k$. In fact, let $k = 3$ and $n = q_1q_2q_3$. From Lemma 3.4.6, we have

$$n.2^k = q_1q_2q_3.2^3 < \sum_{d|n,\ d<n} d^2 = 1 + q_1^2 + q_2^2 + q_3^2 + (q_1q_2)^2 + (q_2q_3)^2 + (q_1q_3)^2.$$

Let $k \geq 4$, and assume that the result holds for $k-1$. For $n = q_1q_2\ldots q_k$, since $2q_1 \leq q_1^2$, then

$$n.2^k = 2q_1\left(q_2q_3\ldots q_k2^{k-1}\right) < 2q_1 \sum_{\substack{d|q_2q_3\ldots q_k,\\ d<q_2q_3\ldots q_k}} d^2 \leq q_1^2 \sum_{\substack{d|q_2q_3\ldots q_k\\ d<q_1q_2\ldots q_k}} d^2 < \sum_{\substack{d|q_1q_2\ldots q_k,\\ d<q_1q_2\ldots q_k}} d^2.$$

This completes the proof of Theorem 3.4.5. □

Proposition 3.4.7. *Let $n \in W_k$ be an odd integer. Then, $\varphi(n) + \gamma(n) > \sigma(n)$.*

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k -tuple of distinct primes, where $q_1 \geq 3$, and let $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, with $\alpha_i \geq 1$ for $i = 1, 2, \ldots, k$. We see that

$$q_i - 1 > 1 + \frac{\sum\limits_{m=0}^{\alpha_i - 1} q^m}{q_i^{\alpha_i}}, \text{ for } i = 1, 2, \ldots, k.$$

and therefore

$$\frac{\varphi(n)}{n} + \frac{\gamma(n)}{n} = \prod_{i=1}^{k} (q_i - 1) + \frac{1}{\prod\limits_{i=1}^{k} q_i^{\alpha_i - 1}} \prod_{i=1}^{k} \frac{q_i^{\alpha_i + 1} - 1}{q_i^{\alpha_i}(q_i - 1)}.$$

Thus,

$$\frac{\varphi(n)}{n} + \frac{\gamma(n)}{n} > \prod_{i=1}^{k} \left(1 + \frac{\sum\limits_{m=0}^{\alpha_i - 1} q^m}{q_i^{\alpha_i}} \right).$$

This completes the proof. □

Remark 3.4.4. We will give a formula for the n-th non-square. In fact, let T_n be the *n-th* non-square. There is a natural number M such that

$$m^2 < T_n < (m+1)^2.$$

As there are M squares less than T_n and n non-squares up to T_n, we see that

$$T_n = n + m.$$

We have then

$$m^2 < n + m < (m+1)^2 \quad \text{or} \quad m^2 - m < n < m^2 + m + 1.$$

Since $n, m^2 - m, m^2 + m + 1$ are all integers, these inequalities imply

$$m^2 - m + \frac{1}{4} < n < m^2 + m + \frac{1}{4},$$

that is to say, $(m-\frac{1}{2})^2 < n < (m+\frac{1}{2})^2$. But then $m = \left[\sqrt{n}+\frac{1}{2}\right]$. Thus the n-th non-square is $T_n = n + \left[\sqrt{n}+\frac{1}{2}\right]$.

Proposition 3.4.8. *For every $n > 1$, we have*

$$\sum_{\substack{1\le a\le n\\(a,n)=1}} a = \frac{n\varphi(n)}{2}$$

Proof. Clearly if $1 \le a \le n$ and $(a,n) = 1$, $1 \le n-a \le n$ and $(n-a,n) = 1$. Thus

$$S = \sum_{\substack{1\le a\le n\\(a,n)=1}} a = \sum_{\substack{1\le a\le n\\(a,n)=1}} n-a$$

whence

$$2S = \sum_{\substack{1\le a\le n\\(a,n)=1}} n = n\varphi(n).$$

□

Proposition 3.4.9. *Let f be an arithmetic function. For all integer $n \in A_{r,k}$, there exists an integer s such that*

$$f(n) + s \in A_{r,k}.$$

Proof. We suppose for all limited integer s, $f(n)+s \notin A_{r,k}$. From tranfert principle we get $f(n)+s \notin A_{r,k}$ for all integer s. Which is a contradiction because the set $A_{r,k}$ containing infinitely many integers n. For all integer $n \notin A_{r,k}$, there are infinitely many integers s such that

$$n \in A_{r,k+s}.$$

Let n be an integer such that $n \notin A_{r,k}$. Because the sequence $\left\{\prod_{p|n}\left(1-\frac{1}{p^k}\right)\right\}_{k\ge1}$ is increasing, then there exixts a unique positive integer s_0 such that

$$\prod_{p|n}\left(1-\frac{1}{p^{k+s_0-1}}\right) < \frac{p_{r-1}}{p_r} < \prod_{p|n}\left(1-\frac{1}{p^{k+s_0}}\right).$$

□

Proposition 3.4.10. *Let ℓ be a positive integer, then*

$$\lim_{n\longrightarrow\infty}\left(\sigma\left(\ell+p_n\right)-\frac{\sigma\left(p_n\right)}{2}\right)=\infty.$$

Proof. We use the following result. For every positive integers a and b, we have

$$\sigma\left(a+b\right)\geq\frac{\sigma\left(a\right)+\sigma\left(b\right)}{2}.$$

and since $\sigma\left(n\right)>n+\sqrt{n}$ whenever n is composite, we get

$$\begin{aligned}\left(\sigma\left(\ell+p_n\right)-\frac{\sigma\left(p_n\right)}{2}\right)&=\sigma\left(\ell-1+1+p_n\right)-\frac{1+p_n}{2}\\&\geq\frac{\sigma\left(\ell-1\right)+\sigma\left(1+p_n\right)}{2}-\frac{1+p_n}{2}\\&>\frac{\sigma\left(\ell-1\right)}{2}+\frac{1+p_n}{2}+\frac{\sqrt{1+p_n}}{2}-\frac{1+p_n}{2}\\&=\frac{\sigma\left(\ell-1\right)}{2}+\frac{\sqrt{1+p_n}}{2}\longrightarrow\infty.\end{aligned}$$

This completes the proof. □

3.4.2 On certain inequalities for τ

Proposition 3.4.11. *The product of all divisors of a number n is*

$$\prod_{d|n}d=n^{\frac{\tau(n)}{2}}.$$

Proof. Let d denote an arbitrary positive divisor of n, so that

$$n=dd'$$

for some d'. As d ranges over all $\tau\left(n\right)$ positive divisors of n, there are $\tau\left(n\right)$ such equations. Multiplying these together, we get

$$n^{\tau(n)}=\prod_{d|n}d\prod_{d'|n}d'.$$

But as d runs through the divisors of n, so does d', hence

$$\prod_{d|n} d = \prod_{d'|n} d'.$$

So,

$$n^{\tau(n)} = \left(\prod_{d|n} d\right)^2,$$

or equivalently

$$n^{\frac{\tau(n)}{2}} = \prod_{d|n} d.$$

□

Example 3.4.1. *Let* $n = 1371$, *then* $\tau(1371) = 4$. *Therefore*

$$\prod_{d|n} d = 1371^{\frac{4}{2}} = 1879641.$$

It is of course true, since

$$\prod_{d|n} d = 1.3.457.1371 = 1879641.$$

Problem. We ask if

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^3.$$

Proposition 3.4.12. *The inequality*

$$\tau(\gamma(n)) \geq \gamma(\tau(n)) > \frac{p_r}{p_{r-1}},$$

holds for infinitely many $n \in W_k$.

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive integers of the form $2^a - 1$, with $a \geq 1$. For $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have

$$\tau(\gamma(n)) = \tau(q_1 q_2 \ldots q_k) = 2^k,$$

and

$$\gamma(\tau(n)) = \gamma\left(\prod_{i=1}^{k} (1 + \alpha_i)\right) = 2.$$

The result follows from Bertrand's theorem. □

Proposition 3.4.13. *If $k \geq 2$, then $\tau(\gamma(n)) - \gamma(\tau(n))$ has infinitely many sign changes on the set W_k.*

Proof. From Proposition 3.4.12, it suffices to prove that

$$\tau(\gamma(n)) < \gamma(\tau(n)),$$

for infinitely many $n \in W_k$. In fact, let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive integers of the form $l_i^a - 1$, for $i = 1, 2, \ldots, k$, respectively, where $l_1, l_2, \ldots, l_k$ are distinct odd primes and $a \geq 1$. For $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have

$$\gamma(\tau(n)) = \gamma\left(\prod_{i=1}^{k} (1 + \alpha_i)\right) = l_1 l_2 \ldots l_k.$$

and

$$\tau(\gamma(n)) = \tau(q_1 q_2 \ldots q_k) = 2^k.$$

This completes the proof. □

Proposition 3.4.14. *We put*

$$S(n) = \sum_{d|n} \tau(d).$$

There are infinitely many $n \in W_1$ such that

$$\begin{cases} \tau\left(\varphi\left(n\right)\right)+1 \in W_k \\ \varphi\left(\tau\left(n\right)\right) \leq \tau\left(\varphi\left(n\right)\right) \\ \dfrac{\ln n}{\ln 2}+1 \in W_k \\ S\left(n\right) < 2n^2. \end{cases}$$

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k -tuple of distinct primes, for $n = 2^{q_1 q_2 \ldots q_k - 1}$ we have

$$\tau\left(\varphi\left(n\right)\right)+1 = \tau\left(2^{q_1 q_2 \ldots q_k - 2}\right)+1 = q_1 q_2 \ldots q_k \in W_k.$$

and

$$\varphi\left(\tau\left(n\right)\right) = \varphi\left(q_1 q_2 \ldots q_k\right) = \prod_{i=1}^{k}\left(q_i - 1\right) \leq q_1 q_2 \ldots q_k - 1 = \tau\left(2^{q_1 q_2 \ldots q_k - 2}\right) = \tau\left(\varphi\left(n\right)\right).$$

Moreover, we see that

$$s = q_1 q_2 \ldots q_k = \frac{\ln n}{\ln 2} + 1 \in W_k.$$

Also,

$$S\left(n\right) = \sum_{d|n} \tau\left(d\right) = 1 + 2 + \ldots + s = \frac{s\left(s+1\right)}{2} < \frac{4^s}{2} = 2n^2.$$

This completes the proof Proposition 3.4.14. □

3.5 By nonstandard methods

3.5.1 Sign changes using external subsets

In this section, we need to the following notation.

Notation 3.5.1. We put

$$\overline{W}_\infty = \left\{n \in \mathbb{N} \,;\, \omega\left(n\right) \cong +\infty\right\}.$$

That is, $\overline{W_\infty}$ is the set of positive integers n for which the number of distinct prime factors of n is unlimited.

Moreover, for every limited $k \geq 1$, it is clear that

$$W_\infty \subsetneq \overline{W_\infty} \subsetneq W_k.$$

Proposition 3.5.1. *Let $\Lambda(n)$ be the von Mangolt function [20] defined by*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ is a prime power} \\ 0 & \text{; otherwise} \end{cases}$$

We put

$$A = \{\ p = 4k-1;\ k \in \mathbb{N}^*\} \text{ and } B = \{p,\ q \in P;\ \ (p,q) \text{ is twin prime}\},$$

where P is the set of primes. If $\Lambda(n) - \Lambda(n+1) = \Lambda(n)$ finitely often on the set $A \cap B$, then $\Lambda(n) - \Lambda(n+1) = \Lambda(n)$ holds infinitely often on the set B.

Proof. We recall that any prime either equals 2, or is of the form $4k \pm 1$. Let $n = 4k-1 \in A \cap B$, then $\Lambda(n) = \log n$ and $\Lambda(n+1) = \Lambda(4k)$. We suppose that there are finitely many k such that $4k-1 \in A \cap B$ and $\Lambda(4k) = 0$, that means there are finitely many k such that $4k-1 \in A \cap B$ and $k \neq 2^\alpha$ for a certain positive α. So, there exists a positive integer k_0 such that for every $k \geq k_0$, if $4k-1 \in A \cap B$ then k has the form 2^α for some nonnegative integer α. On the other hand, because there are infinitely many twin primes, that means there are infinitely many twin primes (p,q) such that either p or q has the form $4k-1$; $k \geq k_0$. (If $p = 4k+1$ and $q = 4k'+1$, then 4 divides $2(2k+1)$, since $(4, 2k+1) = 1$ we get 4 divides 2. It is a contradiction). Therefore, if $p = 2^\alpha - 1$ then $q = 2^\alpha + 1$, it follows that (From *Fermat* and *Mersenne* primes [37] ,[40], α is an odd prime and a power of 2, which is imposable. Thus, we must have $q = 2^{\alpha} - 1$. Thus, $\alpha = p'$ and $p = 2^{p'} - 3$; where p' is an odd prime. In this case, for $n = p$, $\Lambda(n) = \log p$. Since $2^{p'-1} - 1$ is odd, we get

$$\Lambda(n+1) = \Lambda\left(2^{p'} - 2\right) = \Lambda\left(2\left(2^{p'-1} - 1\right)\right) = 0.$$

This completes the proof. □

3.5.2 Remarks

Remark 3.5.1. Let p_r be the *r-th* unlimited prime number. We prove that there exists at least an integer $n \in W_\infty$ such that $p_r [f(n) + s_0] > p_{r-1} n$. In fact, we suppose the opposite. Let m be an unlimited integer such that

$$\left(1 + \frac{1}{p_{r-1}}\right)^{\frac{1}{m}} > f_1(2)$$

and choose q_l be a prime number such that

$$q_l < f_1^{-1}\left[\left(1 + \frac{1}{p_{r-1}}\right)^{\frac{1}{m}}\right]$$

If we put $n = q_l q_{l+1} \ldots q_{l+m-1}$ we see that $\omega(n) \cong +\infty$. So, we can write

$$\begin{aligned} \frac{p_r}{p_{r-1}} &< \frac{n}{f(n) + s_0} < f_1^m(q_l) \\ &< \frac{p_{r-1} + 1}{p_{r-1}}. \end{aligned}$$

Which is impossible.

Remark 3.5.2. Let p_r be the r-th prime number (limited or unlimited). We prove that there are infinitely many integers $n \in \overline{A}_\infty$, such that

$$p_r \varphi_1(n) < p_{r-1} n. \tag{3.5.1}$$

Firstly, we show the existence of an integer $n \in \overline{W}_\infty$ such that $p_r\varphi_1(n) < p_{r-1}n$.

Because the product $\prod \left(1 - \frac{1}{p^k}\right)^{-1}$ where p takes the set of prime numbers, is divergent if and only if $k \leq 1$; ($\zeta(1) \cong \infty$, [40] . For all unlimited real α, there exists a finite sequence of primes $2 < q_1 < q_2 < ... < q_s \cong \infty$ such that

$$\prod_{i=1}^{s} \frac{q_i}{q_i - 1} > \alpha \frac{p_r}{p_{r-1}}.$$

We suppose for all integer n for which the number of its distinct prime factors is unlimited, $p_r\varphi_1(n) > p_{r-1}n$. It is clear that if $n = q_1.q_2...q_s$ we obtain

$$+\infty > \frac{p_r}{p_{r-1}} > \prod_{i=1}^{s} \frac{q_i}{q_i - 1} \cong +\infty.$$

Which is impossible. Finally, from Remark 3.5.1 the inequality (3.5.1) holds infinitely. By using **(c),** we can deduce that $(\varphi_1, id_{\mathbb{N}}) \in F_{\overline{W}_\infty}$.

Remark 3.5.3. From the proof of Lemma 3.3.5, we see that $A_{r,s}$ has at least an element $n_0 \in W_k$, for every $r \geq 2$ and $s \geq 1$. Then $A_{r,s}$ is a nonempty set.

3.5.3 Some properties of the set $A_{r,s}$, with $r \geq 2$ and $s \geq 1$

Notation 3.5.2. For every $r \geq 2$ and $s \geq 1$, we put

$$A_{r,s} = \{n \in \mathbb{N} \; ; \; p_r\varphi_s(n) > p_{r-1}n^s\}. \tag{3.5.2}$$

In this subsection, we will study the set of positive integers $A_{r,s}$ which is defined in Notation 3.5.2 by using Euler's function φ_s and p_r, with $s \geq 1$ and $r \geq 2$.

Proposition 3.5.2. *$A_{r,s} \cap W_k$ is an infinite set.*

Proof. From the proof of Theorem 3.3.4, there exists a positive integer $n_0 \in W_k$ such that $p_r\varphi_s(n_0) > p_{r-1}n_0^s$. Thus, it follows from the definition of φ_s (see equation (2.1.2)) that $n_0^i \in A_{r,s}$ for every $i \geq 1$. This completes the proof. □

Theorem 3.5.3. *Let p_r be the r-th limited prime number and $n \in A_{r,k}$ be an unlimited integer, If $\omega(n)$ is limited, then $\left(p_r\varphi_k(n) - p_{r-1}n^k\right)$ is always unlimited.*

Proof. Let us suppose N_0 be a limited integer such that

$$N_0 \geq p_r\varphi_k(n) - p_{r-1}n^k.$$

and we put

$$\begin{aligned} \frac{p_r}{p_{r-1}} &= \prod_{p|n}\frac{p^k}{p^k-1} + \lambda\ ;\lambda \in \mathbb{Q}^* \\ &= 1 + a_k + \lambda;\ a_k,\ \lambda \in \mathbb{Q}^*, \end{aligned}$$

where $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k} \in A_{r,k}$ an unlimited integer with $\omega(n) < \infty$, because $\dfrac{p_r}{p_{r-1}} > 1 + a_k$, the number λ can't be infinitesimal. In fact, if $a_k \cong 0$, so that

$$\lambda \cong \frac{p_r - p_{r-1}}{p_{r-1}} \ncong 0$$

and if $a_k \ncong 0$ for all values of $k \geq 1$, since $\omega(n)$ is limited, then there exists the biggest integer $i_0;\ 0 < i_0 \leq k$ such that p_{i_0} is a limited prime number, so, we get

$$\frac{p_r}{p_{r-1}} = \prod_{i=1}^{i_0}\frac{p_i^k}{p_i^k-1} + b_k + \lambda;\ b_k \in \mathbb{Q},$$

where b_k is an infinitesimal rational number. In this case, we obtain

$$\lambda \cong \frac{p_r}{p_{r-1}} - \prod_{i=1}^{i_0}\frac{p_i^k}{p_i^k-1} \ncong 0.$$

A contradiction because $N_0 \geq \lambda p_{r-1}\varphi_k(n) \cong \infty$. □

Proposition 3.5.4. *Let $a \geq 2$ be an almost perfect number (A number such that $\sigma(n) = 2n-1$, see [22]). For all $n \in \{a.m\ ;\ 2 \leq m \leq a \text{ and } (a,m) = 1\}$, we have $n \notin A_{r,1}$.*

Proof. It is clear that for all primes $p \leq a$,

$$\left(2 - \frac{1}{a}\right)\left(\frac{p}{p-1}\right) > 2. \tag{3.5.3}$$

Let $n = am$, with $2 \leq m \leq a$. Suppose the contrary. That is, $n \in A_{r,1}$, it follows that

$$p_r > p_{r-1} \prod_{p\ |n} \frac{p}{p-1}.$$

If $m = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$, where $(q_i)_{1 \leq i \leq s}$ are primes and $(a, m) = 1$. Then

$$p_r > p_{r-1} \prod_{p\ |a} \frac{p}{p-1} \left(\frac{q_s}{q_s - 1}\right)^s.$$

On the other hand, from Theorem 3.1.2 and Bertrand's theorem, we get

$$2 > \left(2 - \frac{1}{a}\right)\left(\frac{q_s}{q_s - 1}\right)^s,$$

because a is an almost perfect number. Which contradicts (3.5.3). □

Proposition 3.5.5. *If n is an unlimited almost perfect number, then $n \notin A_{r,1}$ for every $r \geq 2$.*

Proof. Let n be an unlimited almost perfect number. Suppose the contrary; that is, $n \in A_{r,1}$. It follows that

$$\frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}}.$$

There are two cases:

In the case p_r is unlimited, from the Prime Number Theorem we have

$$2 \cong \frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}} \cong 1.$$

It is an impossible case.

In the case p_r is limited, from Bertrand's theorem, we get

$$2 - \frac{1}{n} = \frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}} \leq 2 - \frac{1}{p_{r-1}}.$$

Because n is unlimited, it is also an impossible case as well. □

Corollary 3.5.6. *Suppose that*

$$\left(\frac{p_r}{d_{r-1}}\right)^{\frac{1}{s}} \geq 3, \tag{3.5.4}$$

and let $n \notin A_{r,s}$ be an odd prime power (for example, by (3.5.4)*, $3^m \notin A_{r,s}$ for every $m \geq 1$). Then there exists a positive integer r_0 such that $n \in A_{r+r_0,s}$ or $n \in A_{r-r_0,s}$.*

Proof. For every integer r', assume that $n = p^m \notin A_{r \pm r',s}$ with $m \geq 1$. Then $p \notin A_{i,s}$ for all $i \geq 2$. Which implies that

$$\frac{p_i}{p_{i-1}} \leq \frac{p^s}{p^s - 1} \leq \frac{p}{p-1} \leq \frac{3}{2}\ ; \text{ for every } i \geq 2.$$

It is a contradiction, since $\dfrac{p_3}{p_2} = \dfrac{5}{3} > \dfrac{3}{2}$. □

Proposition 3.5.7. *Let $l \geq 2$ be a limited positive integer, and let q be an unlimited prime number. If r is limited, then $l \in A_{r,s}$ if and only if $l.q \in A_{r,s}$.*

Proof. From the definition of $A_{r,s}$, we see that $l \in A_{r,s}$ if and only if

$$p_r \prod_{p|l} \left(1 - \frac{1}{p^s}\right) - p_{r-1} > 0, \tag{3.5.5}$$

or, equivalently, for every positive infinitesimal ε we get

$$p_r \prod_{p|l} \left(1 - \frac{1}{p^s}\right) - p_{r-1} - \varepsilon > 0,$$

because l and p_r are limited. In particular, for

$$\varepsilon = \frac{p_r \prod_{p|l} \left(1 - \frac{1}{p^s}\right)}{q^s} \cong 0,$$

we get

$$p_r \prod_{p|l} \left(1 - \frac{1}{p^s}\right) - p_{r-1} - \frac{p_r \prod_{p|l} \left(1 - \frac{1}{p^s}\right)}{q^s} > 0. \tag{3.5.6}$$

Since φ_s is multiplicative, we have $l.q \in A_{r,s}$.

Conversely, if $l.q \in A_{r,s}$, it follows from (3.5.5) and (3.5.6) that $l \in A_{r,s}$. This completes the proof. □

Remark 3.5.4. Let $n \notin A_{r,s}$ for some r, s, where (p_{r-1}, p_r) is a couple of twin primes (for example, if r, s satisfy (3.3.3), then there exists a positive integer $n_1 \in W_k \cap A_{r,s}$). Then for every couple of twin primes (p_{m-1}, p_m) with $m > r$, we have $n \notin A_{m,s}$.

Corollary 3.5.8. *For every $r \geq 3$, $A_{r,s}$ has an infinity of prime numbers.*

Proof. By Bertrand's theorem, for every prime number $p \geq (p_{r-1})^{\frac{1}{s}}$, we have

$$(p_r - p_{r-1})\, p^s - p_r > 0,$$

where $p_r > 3$. It follows that

$$p_r\,(p^s - 1) > p_{r-1} p^s,$$

and therefore $p \in A_{r,s}$. □

Corollary 3.5.9. *If $n \notin A_{r,k}$ and if s_0 is the smallest positive integer such that $n \in A_{r,k+s_0}$. Then*

$$\omega(n) \text{ limited} \Longrightarrow s_0 \text{ limited.}$$

Where $\omega(n)$ is the number of distinct prime factors of n.

Proof. We suppose the opposite; that is s_0 is an unlimited integer. So, if $\omega(n)$ is limited from (13), there exist two infinitesimal positive real numbers ε_1 and ε_2 ($\varepsilon_1 > \varepsilon_2$) such that

$$p_r - \varepsilon_1 < p_{r-1} < p_r - \varepsilon_2.$$

A contradiction. □

Remark 3.5.5. If n is an even positive integer, then $n \notin A_{r,1}$. Indeed, if $n = 2^a N$, with $(2, N) = 1$, it follows from Bertrand's theorem that

$$p_r\varphi(n) = p_r 2^{a-1}\varphi(N) < p_{r-1}2^a\varphi(N) \leq p_{r-1}n,$$

which we may because $\varphi(N) \leq N$.

Example 3.5.1. *For all n, $m \geq 2$, we have $n^m - n \notin A_{r,1}$, because it is an even.*

Proposition 3.5.10. *If $r \geq 3$, then there are no positive integers $n \in W_1$ satisfying*

$$\frac{\varphi(n)}{n} = \frac{p_{r-1}}{p_r}. \tag{3.5.7}$$

Proof. Let n be an odd positive integer. Because $\varphi(n)$ is always an even, then $p_r\varphi(n) \neq p_{r-1}n$. Let n be an even positive integer that satisfies (3.5.7). Since $(p_{r-1}, p_r) = 1$ and $p_r\varphi(n) = p_{r-1}n$, then p_r divides n. Therefore, there exist two positive integers r_1 and β_1 such that

$$n = r_1 p_r^{\beta_1} \ ; \ (r_1, p_r) = 1,$$

where r_1 is even. It follows from equation (3.5.7) that

$$\frac{\varphi(r_1)}{r_1} = \frac{p_{r-1}}{p_r - 1}.$$

Now, for $r_1 = 2^a N$ with $a \geq 1$ and N is odd, we have

$$\frac{\varphi(r_1)}{r_1} = \frac{\varphi(2^a N)}{2^a N} = \frac{\varphi(N)}{2N} = \frac{p_{r-1}}{p_r - 1}.$$

Finally, using Bertrand's theorem we get

$$\frac{\varphi(N)}{N} = \frac{2p_{r-1}}{p_r - 1} > \frac{p_r}{p_r - 1} > 1.$$

Which is impossible, since $\varphi(t) \leq t$. □

Proposition 3.5.11. *If p_r is unlimited and $s \geq 1$ is limited. Then for every limited $n > 1$, we have $n \notin A_{r,s}$. Furthermore, for each such integer n, the integer $(p_{r-1}n^s - p_r\varphi_s(n))$ is always unlimited.*

Proof. For every limited positive integer n, we have

$$1 + \varepsilon_r = \frac{p_r}{p_{r-1}} < \frac{n^s}{\varphi_s(n)} = 1 + a_n(s),$$

where $a_n(s)$ is a positive appreciable rational number, and ε_r is a positive infinitesimal real number. That is $n \notin A_{r,s}$.

Now, let n be a limited integer, with $n > 1$. We assume that there exists a limited integer N_0 satisfying

$$N_0 \geq p_{r-1}n^s - p_r\varphi_s(n).$$

Therefore, we can deduce that $N_0 \geq \frac{p_{r-1}}{2}$, because

$$\begin{aligned} n^s - \frac{p_r}{p_{r-1}}\varphi_s(n) &= n^s - (1+\varepsilon_r)\varphi_s(n) \\ &> \frac{n^s - \varphi_s(n)}{2} \geq \frac{1}{2}. \end{aligned}$$

which contradicts the fact that p_{r-1} is an unlimited and $n^s - \varphi_s(n) \geq 1$. This completes the proof. □

Proposition 3.5.12. *Let p_r be the r-th prime number and Let $n = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$ be an m-perfect number, if*

$$m \left(\frac{p_1 - 1}{p_1} \right)^{\omega(n)} > \frac{p_r}{p_{r-1}},$$

then $n \notin A_{r,k}$.

Proof. We suppose that $n \in A_{r,k}$. Since $\frac{\sigma(n)}{n} < \prod_{p|n} \frac{p}{p-1}$, it follows that

$$\begin{aligned} \frac{p_r}{p_{r-1}} &> m \prod_{p|n} \frac{p^{k-1}}{p^{k-1} + p^{k-2} + \dots + p + 1} \\ &> m \prod_{p|n} \left(\frac{1}{\sum_{i=1}^{\infty} \frac{1}{p^i}} \right) = m \prod_{p|n} \frac{p-1}{p} \\ &> m \left(\frac{p_1 - 1}{p_1} \right)^{\omega(n)} > \frac{p_r}{p_{r-1}}. \end{aligned}$$

A contradiction. □

Proposition 3.5.13. *If n is an e-perfect number, then $n \notin A_{r,1}$; for all $r \geq 2$.*

Proof. Let $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be an e-perfect number, we write

$$\begin{aligned} \frac{\sigma_e(n)}{n} &= \frac{p_1 p_2 \dots p_k + \dots + p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}}{n} \\ &= 1 + \frac{1}{p_1} + \frac{1}{p_1 p_2} + \dots + \frac{1}{p_1 p_2 \dots p_k} + ... \\ &< \left(1 + \sum_{i=1}^{\infty} \frac{1}{p_1^i} \right) \dots \left(1 + \sum_{i=1}^{\infty} \frac{1}{p_k^i} \right) \\ &= \left(\frac{1}{1 - \frac{1}{p_1}} \right) \left(\frac{1}{1 - \frac{1}{p_2}} \right) \dots \left(\frac{1}{1 - \frac{1}{p_k}} \right) \\ &= \left(\frac{p_1}{p_1 - 1} \right) \left(\frac{p_2}{p_2 - 1} \right) \dots \left(\frac{p_k}{p_k - 1} \right). \end{aligned}$$

If $\sigma_e(n) = 2n$, by Proposition 3.5.12, we find $n \notin A_{r,1}$. □

Remark 3.5.6. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ be a composite integer ≥ 4, then for all integer $k \geq 2$,

$$\prod_{p \mid \left[\sqrt{n}\right]} \left(1 - \frac{1}{p^k}\right) > \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

So, if $n \in A_{r,1}$ we have $[\sqrt{n}] \in A_{r,k}$ for all $k \geq 2$. Where $[x]$ represent the biggest integer less than or equal to x. If n is prime, then

$$\prod_{p \mid \left[\sqrt{n}\right]} \left(1 - \frac{1}{p^k}\right) > \left(1 - \frac{1}{p}\right) \text{ for all } k \geq \left[\sqrt{n}\right] + 1.$$

In this case $n \in A_{r,\left[\sqrt{n}\right]+1}$.

Example 3.5.2. *If $n = 5^a 7^b$ with $a, b \geq 0$, we have $[\sqrt{n}] \in A_{3,k}$ for all $k \geq 2$.*

Open problem: Is there an integer $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ such that n is a m-perfect and

$$m \left(\frac{p_1 - 1}{p_1}\right)^{\omega(n)} > \frac{p_r}{p_{r-1}},$$

where $\omega(n)$ denote the number of distinct prime factors of n?

Remark 3.5.7. For all p_s unlimited, we have

$$\prod_{i \geq 0} \frac{p_{s+i}}{p_{s+i} - 1} \geq 2.$$

Finally, we prove the following proposition.

Proposition 3.5.14. *Let $2 = p_1 < p_2 < \dots$ be the sequence of primes in increasing order, and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ with $s \cong +\infty$ and $\alpha_i \geq 1$ for $i = 1, 2, \dots, s$. Let $N \geq 2$ be a limited divisor of n, then we have $\dfrac{n}{N} \notin A_{r,1}$.*

Proof. Let $[x]$ denote the integer part of x. Since $\left[\prod_{p|n} \frac{p}{p-1}\right]$ is unlimited, then for every limited divisor $N \geq 2$ of n, we get

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \leq \frac{n}{\left[\prod_{p|n} \frac{p}{p-1}\right]} < \frac{n}{N}.$$

It follows from Bertrand's theorem that

$$p_r \varphi\left(\frac{n}{N}\right) < \frac{2p_{r-1}\frac{n}{N}}{\left[\prod_{p|\frac{n}{N}} \frac{p}{p-1}\right]} < \frac{2p_{r-1}n}{N^2} \leq p_{r-1}\frac{n}{N}.$$

This completes the proof. □

Conclusion 3.5.1. *It is very important to know some other properties of the set $A_{r,s}$. Namely, it is necessary to know whether $A_{r,s}$ contains Niven numbers, Smidth numbers, Woodall numbers, Cullen numbers,....Etc.*

Chapter 4

Perspective and Open Problems

For further research, we will finish my thesis by stating the following open problems.

4.1 On the simultaneous rational approximation in the infinitesimal sense

Let ω be an unlimited positive integer. Is there a necessary and sufficient condition on the reals $(\xi_0, \xi_1, \ldots, \xi_\omega)$ to be simultaneously approximable in the infinitesimal sense? In Chapter 1 a necessary condition has been solved, while a sufficient condition is still open. Moreover, we have

1. For every $k \geq 1$, we ask if $(\alpha_0, \alpha_1, \ldots, \alpha_k), (\beta_0, \beta_1, \ldots, \beta_k) \in S_A (\cong 0)$, then

   - $(\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k) \in S_A (\cong 0)$.
   - $(\alpha_0\beta_0, \alpha_1\beta_1, \ldots, \alpha_k\beta_k) \in S_A (\cong 0)$.
   - $\left(\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_k}{\beta_k}\right) \in S_A (\cong 0)$.
   - $\left(\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_k}\right) \in S_A (\cong 0)$.

- $(\alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k) \in S_A\,(\cong 0)\,.$
- $(c.\alpha_0, c.\alpha_1, \ldots, c.\alpha_k) \in S_A\,(\cong 0)$, for every integer c.
- $\left(\frac{\alpha_0}{c}, \frac{\alpha_1}{c}, \ldots, \frac{\alpha_k}{c}\right) \in S_A\,(\cong 0)$, for every integer $c \neq 0$.
- $(\alpha_0 + \alpha, \alpha_1 + \alpha, \ldots, \alpha_k + \alpha) \in S_A\,(\cong 0)$, for every real number α.
- $\left(\alpha_0^{\beta_0}, \alpha_0^{\beta_1}, \ldots, \alpha_0^{\beta k}\right) \notin S_A\,(\cong 0)\,.$
- $(\log \alpha_0, \log \alpha_1, \ldots, \log \alpha_k) \notin S_A\,(\cong 0)\,.$
- $(e^{\alpha_0}, e^{\alpha_1}, \ldots, e^{\alpha_k}) \notin S_A\,(\cong 0)\,.$

2. If $(\xi_0, \xi_1, \ldots, \xi_k) \notin S_A\,(\cong 0)$, then there exists a real numbers α such that $(\alpha\xi_0, \alpha\xi_1, \ldots, \alpha\xi_k) \in S_A\,(\cong 0)$.

3. Let $(\xi_0, \xi_1, \ldots, \xi_\omega) \subset [0, 1]$, with $\omega \cong +\infty$. We ask if $(\xi_0, \xi_1, \ldots, \xi_\omega) \in S_A\,(\cong 0)$, then for every real α, $^\circ\left(\{\alpha\xi_0\}, \{\alpha\xi_1\}, \ldots, \{\alpha\xi_\omega\}\right)$ does not contain any standard interval $[a, b]$, with $a < b$.

4. Let n be an unlimited positive integer, and let p_r be the *rth* prime number. Is it true that
$$\left(\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right) \notin S_A\,(\cong 0)?$$

5. Let $\frac{p_n}{q_n}$ be the *n-th* convergent of a continued fraction, with $n \geq 0$. Is it true that
$$\left(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right) \in S_A\,(\cong 0)\,.$$

6. Let ω be an unlimited positive integer. We ask if $S_A\,(\cong 0)$ contains a system of the form $(\xi_0, \xi_1, \ldots, \xi_\omega)$, where $(\xi_i)_{i=1,2,\ldots,\omega}$ are all irrationals.

4.2 On the solubility of certain arithmetic inequalities and equations in integers

1. We ask if Theorem 3.2.2 is true for the functions σ and φ_s, with $s \geq 1$.

2. Let F denote the set of arithmetic functions that satisfy Theorem 3.2.2, and let $f \in F$. Is there a necessary and sufficient condition for which $f^i \in F$ for every $i \geq 1$?

3. Let ℓ, s be positive integers. For a given positive rational number α, we ask if $\psi_s(n+\ell) - \alpha n^s$ has infinitely many sign changes on the set W_k.

4. For every $m \geq 3$, are there infinitely many $n \in W_k$ such that
$$\begin{cases} \dfrac{\ln n}{\ln(\gamma(n))} + 1 \text{ is a prime of the form } x - m + 1 \\ \varphi(\tau(n)) = \tau(\varphi(n)), \end{cases}$$
where x is the sum of m-distinct primes?

5. In the nonstandard analysis, let a and b be relatively prime integers with $a \geq 1$. We ask if there are infinitely many primes of the form $an + b$, with $\omega(n)$ is limited.

6. Woodall numbers are of the form $n.2^n - 1$, with $n \geq 1$. A Woodall number is prime only when $n = 2, 3, 6, 30, 75, 81, 115, 123, 249, 362, 384, \ldots$. The largest known Woodall prime is $3752950.2^{3752950} - 1$ (It was found by J. Thompson in 2007). For more details, see [27], [48, p. 240]. Moreover, it has been conjectured that almost Woodall numbers are composite. Is $\omega.2^\omega - 1$ a composite number for every unlimited ω?

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