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# THESIS

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**By**
ABDELHAMID BENNOUI

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# Particular neutrosophic subsets on a lattice

Defended on 21 / 10 / 2021 before the jury composed of :

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Academic year : 2021/2022

RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ MOHAMED BOUDIAF - M'SILA
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## Sous-ensembles neutrosophiques particuliers dans un treillis

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# Particular neutrosophic subsets on a lattice
# "Sous-ensembles neutrosophiques particuliers sur un treillis"

Abdelhamid Bennoui

Thesis submitted in fulfillment of the requirements for the degree of Doctor of Mathematics

Academic year 2021-2022

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Msila, October 2021
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# Introduction

The notions of ideals and filters are well known in many algebraic structures (e.g., semi-groups, rings, MV-algebras, lattices, et cetera). They have been applied in different subjects of mathematics (see e.g., topology spaces [86], metric spaces [19] and congruence relations [33]). They have been used as tools in the Stone representation theorem for Boolean algebra [75] and in the representations of distributive lattice [30, 70]. Also, in the theory of Lukasiewicz and Post algebras [35], as they are the kernels of the homomorphisms into the power set subalgebras. In ring theory [59], ideals generalize certain subsets of the integers, such as the even numbers or the multiples.

Zadeh in 1965 [94] has introduced the theory of fuzzy sets (FSs), and after that several authors have conducted on the generalizations of this theory. For instance, Atanassov [6, 7] in 1983 introduced the theory of intuitionistic fuzzy sets (IFSs) as an extension of FSs theory. In fuzzy setting, the falsity-membership of an element $x$ can be calculated as $\nu_A(x) = 1 - \mu_A(x)$, which $x$ is fixed. While in the intuitionistic fuzzy setting, the falsity-membership is an independent degree and only satisfies the condition $\nu_A(x) \leq 1 - \mu_A(x)$.

Although FSs (resp. IFSs) are useful in handling uncertainties arising from vagueness, imprecise and incomplete information, it cannot model all sorts of indeterminate or inconsistent information that exists in real-life. Inspired by this situation, Smarandache [72] has introduced the theory of neutrosophy to study the nature, origin and neutrality. To that end, he has introduced the theory of neutrosophic sets (NSs) as a new generalization of FSs and IFSs theories, and for the purpose to ensure the best handling of incomplete, indeterminate and inconsistent information. The notion of NSs described by three degrees; truth-degree ($\mu$), indeterminacy-degree ($\sigma$) and falsity-degree ($\nu$). For more practical use of neutrosophic sets, Wang et al. [84] have defined the single valued neutrosophic set (SVNS) as a subclass of NSs, which can independently express the truth, the indeterminacy and the falsity degrees. To cover the general setting, these three components of a SVNS are independent and their values are enclosed in the interval [0,1]. SVNSs have used in very significant research areas such as image processing [39, 40], medical diagnosis [44, 64, 93], decision making [51, 91, 92] and social problems [58]. More details and applications of NSs can be found in [74].

Due to the importance of the notions of ideals and filters in the study of several mathematical structures, as in fuzzy and intuitionistic fuzzy setting, several papers studied different notions of ideals (resp. filters) on different (intuitionistic) fuzzy structures. In [42], Kim and Jun discussed the notion of intuitionistic fuzzy ideal (IF-ideal) in a semigroup, while Banerjee and Basnet [10] defined a similar notion on a ring structure. Recently, Akram and Dudek [1] studied the notion of intuitionistic fuzzy ideal on Lie algebras. In particular, Thomas and Nair [78, 79, 80] introduced the notion of intuitionistic fuzzy ideal using the idea of fuzzification the membership function of elements on the carrier of a crisp lattice, and investigated some of its properties. Milles et al. [57] characterized based on the lattice meet and join operations the notion of intuitionistic fuzzy ideal (resp. filter) introduced by Thomas and Nair.

In the literature of neutrosophic sets, similar studies of the notions of ideals and filters in neutrosophic context have been done by several authors. Salama and Smarandache [66] introduced the notion of filters via neutrosophic crisp set and investigated several relations between different neutrosophic filters and neutrosophic topologies. Salama and Algamy [67] introduced the notion of filters on a neutrosophic set which is considered as a generalization of the notion of fuzzy filters. Recently, Hamidi et al. [41] studied the notion of single valued neutrosophic filters on EQ-algebras and its relationship with filters on these kind algebras. In [61], Öztürk et al. presented neutrosophic ideals on BCK and BCI algebras with respect to neutrosophic points.

The main aim of this thesis is to extend and study in details several fundamental concepts, properties and characterisations of (crisp, fuzzy and intuitionistic fuzzy) ideals and filters to the neutrosophic setting.

This thesis is structured as follows.

- In Chapter 1, we provide generalities on binary relations, partially ordered sets and lattices, fuzzy sets and fuzzy relations.
- In Chapter 2, we also provide generalities on Intuitionistic fuzzy sets (resp. relations), Intuitionistic fuzzy lattices, Intuitionistic fuzzy ideals and filters on a lattice, Intuitionistic fuzzy ordered lattice, as well as on neutrosophic sets, single valued neutrosophic sets and some related concepts that we need throughout this thesis.
- In Chapter 3, we investigate the neutrosophic ideals and filters concepts on a given lattice. We provide their various characterizations and properties. In particular, we pay attention to their characterizations in terms of the

lattice meet and join operations.

- In Chapter 4, we introduce and study the notions of prime, maximal and principal single-valued neutrosophic ideals (resp. filters) on a lattice. Also, we give several properties and characterizations of these types of ideals and filters. In that chapter, we discus relationships between them.
- Finally, general conclusions and future research are drawn.

Most parts of results presented in this thesis have already been published or submitted for publication in peer-reviewed international journals. Results included in Chapter 3 has been described in [12] (Submitted). Results of Chapter 4 have been described in [13](Accepted).

# 1 Generalities on fuzzy sets

The purpose of this first chapter is to provide a basic introduction to the binary relations, partially ordered sets and lattices. Next, we recall some basic notions of fuzzy sets and fuzzy relations. Note that many of the properties of these concepts will be used in the next chapters.

## 1.1. Binary relations, posets and lattices

This section contains the basic definitions and properties of binary relations, partially ordered sets (posets, for short) and lattices.

### 1.1.1. Posets

**Definition 1.1.** [30] *A binary relation on a set $X$ is a subset of the Cartesian product $X \times X$, it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, if $x$ is related to $y$ by $R$ we often write $xRy$ instead of $(x, y) \in R$. Two elements $x$ and $y$ of a set $X$ equipped with a relation $R$ are called comparable elements if it holds that $xRy$ or $yRx$. Otherwise, they are called incomparable elements, denoted by $x \parallel y$.*

By definition the relations $R$ and $S$ are sets of couples $(x, y)$ on $X^2$. Then, we can apply the classical operations of set theory to them, such as:

• The union of $R$ and $S$ is the relation $R \cup S$ on $X$ and defined as follows:

$$R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}.$$

• The intersection of $R$ and $S$ is the relation $R \cap S$ on $X$ and defined as follows:

$$R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}.$$

• The complement of $R$ is the relation denoted by $R^c$ on $X$ and defined as follows:

$$R^c = \{(x, y) \in X^2 \mid (x, y) \notin R\}.$$

• The transpose of $R$ is the relation denoted by $R^t$ on $X$ and defined as follows:

$$R^t = \{(y, x) \in X^2 \mid xRy\}.$$

• The dual of $R$ is the relation denoted by $R^d$ on $X$ and defined as follows:

$$R^d = \{(x, y) \in X^2 \mid yR^c x\}.$$

• The composition of $R$ and $S$ is the relation denoted by $R \circ S$ on $X$ and defined as follows:

$$R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}.$$

A binary relation $R$ on a set $X$ is called:

(i) reflexive, if $xRx$, for all $x \in X$;

(ii) irreflexive, if $xR^c x$, for all $x \in X$;

(iii) symmetric, if $xRy$ implies $yRx$, for all $x, y \in X$;

(iv) antisymmetric, if $xRy$ and $yRx$ implies $x = y$, for all $x, y \in X$;

(v) asymmetric, if $xRy$ implies $yR^c x$, for all $x, y \in X$;

(vi) transitive, if $xRy$ and $yRz$ implies $xRz$, for all $x, y, z \in X$;

(vi) anti-transitive, if $xRy$ and $yRz$ implies $xR^c z$, for all $x, y, z \in X$;

(vii) complete, if $xRy$ or $yRx$, for all $x, y \in X$.

A binary relation $R$ on a set $X$ is called:

(i) a pseudo-order relation if it is reflexive and antisymmetric;

(ii) an order relation if it is reflexive, antisymmetric and transitive;

(iii) a strict order if it is irreflexive and transitive;

(iv) a total order relation if it is reflexive, antisymmetric, transitive and complete;

(v) an equivalence relation if it is reflexive, symmetric and transitive.

For each equivalence relation $R$ on a set $X$, the equivalence class associated to $x \in X$, denoted by $[x]_R$, is the set $\{y \in X \mid xRy\}$. It is always true that $x \in [x]_R$. It is easy to show that for any $x, y \in X$, either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$. An equivalence therefore partitions $X$ into equivalence classes. The set of all these equivalence classes is called the quotient of $X$ for $R$ and is denoted $X/R$.

For more details on binary relations, we refer to [30, 37, 70].

According to Davey and Priestley [30], we have the following definitions:

**Definition 1.2.** *Let $X$ be a non-empty set. A partial order on $X$ is a binary relation $\leq$ on $X$ such that, for all $x, y, z \in X$,*

*(i) $x \leq x$ (reflexivity);*

*(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);*

*(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).*

A set $X$ equipped with an order relation $\leq$ is said to be a partially ordered set (or poset, for short) and we shall write $(X, \leq)$. On any set, $=$ is an order. A relation $\leq$ on a set $X$ which is reflexive and transitive but not necessarily antisymmetric is called a quasi-order or, by some authors, a preorder. Given any ordered set $(X, \leq)$, the dual of $(X, \leq)$ is denoted by $(X^d, \geq)$ and defined as follows: $x \geq y$ in $X^d$ if and only if $x \leq y$ in $X$. A poset $(X, \leq)$ is called chain, if for any $x, y \in X$, either $x \leq y$ or $y \leq x$, i.e., any two elements of $X$ are comparable. Alternative names for a chain are linearly ordered set or totally ordered set. At the opposite extreme from a chain is an antichain. The ordered set $(X, \leq)$ is an antichain $x \leq y$ in $X$ only if $x = y$.

If $\leq$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \text{ if } a \leq b \text{ and } a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leq$ is the reflexive closure given by:

$$a \leq b \text{ if } a < b \text{ or } a = b.$$

Next we introduce some important special elements

**Definition 1.3.** [30] *Let $(X, \leq)$ be an ordered set. We say $X$ has a least element if, there exists $0 \in X$ such that $0 \leq x$, for any $x \in X$. Dually, $X$ has a greatest element if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$. Note that if $0, 1$ exist then they are unique.*

**Definition 1.4.** *Let $(X, \leq)$ be a poset and $S$ be a subset of $X$, an element $x \in X$ is an upper (lower) bound of $S$ if $y \leq x$ ($y \geq x$) for any $y \in S$. The set of all upper (lower) bounds of $S$ is denoted by $S^u$ ($S^l$).*

$$S^u = \{x \in X : y \leq x \textit{ for any } y \in S\};$$

$$S^l = \{x \in X : y \geq x \textit{ for any } y \in S\}.$$

*If $S^u(S^l)$ has a least (greatest) element $x$, then $x$ is called the least upper bound or supremum (greatest lower bound or infimum) of $S$, denoted by $\sup S$ $(\inf S)$. The supremum (infimum) is unique in $S^u$ $(S^l)$ if it exists.*

*We use the following notation: $x \vee y$ in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ in place of $\inf\{x, y\}$ when it exists. Similarly, $\bigvee S$ and $\bigwedge S$ instead of $\sup S$ and $\inf S$ when these exist.*

One of the most useful and attractive features of ordered sets is that, in the finite case at least, they can be 'drawn'. To describe how to represent ordered sets diagrammatically, we need the idea of covering.

**Definition 1.5.** *Let $X$ be an ordered set and let $x, y \in X$. We say $x$ is covered by $y$ (or $y$ covers $x$), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there be no element $z$ of $X$ with $x < z < y$.*

Since an order is an example of a relation, we can draw it as a directed graph. But there is a more concise and attractive way to draw partial orders and linear orders, in which the reflexivity and transitivity of the order are implicit. A **Hasse diagram** for a partial order $\leq$ on a set $X$ is a graph drawn in the plane, with vertices corresponding to the elements of $X$ and edge going up from $x$ to $y$ if $x < y$ (so excluding $x = y$) and there is no element $z$ with $x < z < y$. In general a poset can be conveniently represented by Hasse diagram, displaying the covering relation $<$. Not that $x < y$ if there is a sequence of connected lines upwards from $x$ to $y$. For $X$ finite, we obtain a diagram for $X^d$ simply by " turning upside down " a diagram for $X$.

Associated with any ordered set by two important families of sets which are down-set and up-set, they play a central role in the representation theory developed in later chapters.

**Definition 1.6.** [30] *Let $X$ be an ordered set and $S$ be a subset on $X$. $S$ is called a down-set (alternative terms include lower-set) if $y \in S$ implies $x \in S$ for all $x \leq y$. Dually, $S$ is called an up-set (alternative terms include upper-set) if $y \in S$ implies $x \in S$ for all $y \leq x$. For a given subset $S$ on $X$, we denote by $\downarrow S$ the set of all elements smaller than or equal to some element of $S$, i.e.,*

$$\downarrow S = \{x \in X \mid x \leq y, \text{ for some } y \in S\},$$

*and $\uparrow S$ the set of all elements bigger than or equal to some element of $S$, i.e.,*

$$\uparrow S = \{x \in X \mid y \leq x, \text{ for some } y \in S\}.$$

*It is easily checked that* $\downarrow S$ *(resp.* $\uparrow S$*) is the smallest down-set (resp. the smallest up-set) containing* $S$*.* $\downarrow S$ *(resp.* $\uparrow S$*) is called the down-set (resp. the up-set) of* $S$*. Similarly, for a given element* $x$ *on a lattice* $X$*, the down-set* $\downarrow \{x\}$ *(*$\downarrow x$*, for short) and the up-set* $\uparrow \{x\}$ *(*$\uparrow x$*, for short) are defined as*

$$\downarrow x = \{y \in X \mid y \leq x\} \text{ (resp. } \uparrow x = \{y \in X \mid x \leq y\}).$$

*Note that if* $S$ *is a down-set (resp. an up-set), then* $\downarrow S$ *(resp.* $\uparrow S$*) coincides with* $S$*.*

**Lemma 1.1.** [30] *Let* $X$ *be an ordered set and* $x, y \in X$*. Then the following statements are equivalent:*

*(i)* $x \leq y$*;*

*(ii)* $\downarrow x \subseteq \downarrow y$*;*

*(iii) For any* $S \subseteq X$*, if* $y \in \downarrow S$ *implies* $x \in \downarrow S$*.*

*We have already made use of maps of an every special type between ordered sets, namely order-isomorphisms.*

**Definition 1.7.** *Let* $X$ *and* $Y$ *be two ordered sets. A mapping* $f : X \to Y$ *is said to be*

*(i) order-preserving (or monotone) if* $x \leq y \in X$ *implies* $f(x) \leq f(y) \in Y$*;*

*(ii) order-embedding (and we write* $f : X \to Y$*) if* $x \leq y \in X$ *if and only if* $f(x) \leq f(y) \in Y$*;*

*(iii) order-isomorphism if it is an order-embedding which mappings* $X$ *onto* $Y$*.*

**Definition 1.8.** *A poset* $(X, \leq)$ *is called a subposet of a poset* $(Y, \grave{\leq})$ *if*

*(i)* $X \subseteq Y$*;*

*(ii) For any* $x, y \in X$*:* $x \leq y$ *if and only if* $x \grave{\leq} y$*.*

### 1.1.2. Lattices

*Many important properties of an ordered set* $(L, \leq)$ *are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of* $L$*. Lattice and complete lattices are two of the most important classes of ordered sets. We shall be particularly interested in ordered sets in which* $x \vee y$ *and* $x \wedge y$ *exist for all*

*$x, y \in L$. Similarly, we write $\bigvee S$ (the join of $S$) and $\bigwedge S$ (the meet of $S$) instead of $\sup S$ and $\inf S$ when they exist.*

**Definition 1.9.** *Let $(L, \leq)$ be a non-empty ordered set.*

*(i)* *$(L, \leq)$ is called a $\vee$-semi-lattice if $x \vee y$ exists for all $x, y \in L$.*

*(ii)* *$(L, \leq)$ is called a $\wedge$-semi-lattice if $x \wedge y$ exists for all $x, y \in L$.*

*(iii)* *$(L, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice.*

*(iv)* *$(L, \leq)$ is called a complete lattice if $\bigvee S$ and $\bigwedge S$ exist for any $S \subseteq L$.*

*For a given lattice $(L, \leq, \wedge, \vee)$, the operations $\wedge$ and $\vee$ defined for any $x, y \in L$ by:*

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\},$$

*satisfy the following algebraic properties:*

*(i)* *$x \wedge x = x \vee x = x$ (idempotency);*

*(ii)* *$x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutativity);*

*(iii)* *$x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity);*

*(iv)* *$x \wedge (y \vee x) = x = x \vee (y \wedge x)$ (absorption-laws);*

*(v)* *the order relation $\leq$ and $\wedge$, $\vee$ are connected as:*

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y, \text{ for any } x, y \in L.$$

*A bounded lattice is a lattice that additionally has a greatest element $1$ and a smallest element $0$, which satisfy $0 \leq x \leq 1$, for all $x \in L$. A finite lattice is automatically bounded with $1 = \bigvee L$ and $0 = \bigwedge L$. A lattice $(L, \leq, \wedge, \vee)$ is distributive if, for all $x, y, z \in L$, the following additional conditions hold*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

*A lattice $(L, \leq, \wedge, \vee)$ is modular if the following condition holds*

$$x \leq z, \text{ implies that } x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ for all } x, y, z \in L.$$

*This condition is also equivalent to its dual*

$$z \leq x, \text{ implies that } x \wedge (y \vee z) = (x \wedge y) \vee z, \text{ for all } x, y, z \in L.$$

*A complemented lattice is a bounded lattice* $(L, \leq, \wedge, \vee, 0, 1)$*, in which each element* $x$ *has a complement, i.e., there exists an element* $y \in L$ *such that*

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

*An element may have more than one complement in general. However, if* $(L, \leq, \wedge, \vee, 0, 1)$ *is distributive then every element will have at most one complement.*

**Definition 1.10.** [30] *A non-empty subset* $M$ *of a lattice* $L$ *is called sub-lattice of* $L$ *if, for all* $x, y \in M$*, it holds that* $x \wedge y \in M$ *and* $x \vee y \in M$*. A sub-lattice* $M$ *of a lattice* $L$ *is called a convex sub-lattice of* $L$*, if* $x \leq z \leq y$ *and* $x, y \in M$ *implies that* $z \in M$*, for all* $x, y, z \in L$*.*

## 1.2. Ideals and filters on a crisp lattice

*Ideals are of fundamental importance in algebra. Filters, the order dual of lattice ideals, have a variety of applications in logic and topology.*

**Definition 1.11.** [30] *A non-empty subset* $I$ *on a lattice* $L$ *is called an ideal if, for all* $x, y \in L$*, the following conditions are satisfied:*

1. *if* $y \in I$ *and* $x \leq y$*, then* $x \in I$*,*
2. *if* $x, y \in I$ *implies* $x \vee y \in I$*.*

*The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.*

*A dual ideal is called a filter determined by the following definition.*

**Definition 1.12.** [30] *A non-empty subset* $F$ *on a lattice* $L$ *is called a filter if, for all* $x, y \in L$*, the following conditions are satisfied:*

1. *if* $y \in F$ *and* $y \leq x$*, then* $x \in F$*,*
2. *if* $x, y \in F$ *implies* $x \wedge y \in F$*.*

*The set of all ideals (resp. filters) of* $L$ *is denoted by* $\mathcal{I}(L)$ *(resp.* $\mathcal{F}(L)$*), and carries the usual inclusion order.*

*An ideal or filter is called proper if it does not coincide with* $L$*. More precisely, an ideal* $I$ *of a lattice with 1 is proper if and only if* $1 \notin I$*, and dually, a filter* $F$ *of a lattice with 0 is proper if and only if* $0 \notin F$*. For each* $x \in L$*, the set* $\downarrow x$ *is an ideal (also known as the principal ideal generated by x). Dually,* $\uparrow x$ *is a principal filter.*

**Definition 1.13.** [30] *An ideal $I$ on a lattice $L$ is called a prime ideal if, for all $x, y \in L$, $x \wedge y \in I$, then $x \in I$ or $y \in I$.*

**Definition 1.14.** [30] *A filter $F$ on a lattice $L$ is called a prime filter if, for all $x, y \in L$, $x \vee y \in F$, then $x \in F$ or $y \in F$.*

**Definition 1.15.** [30] *Let $L$ be a lattice. A proper ideal (resp. filter) $A$ is said to be a maximal ideal (resp. maximal filter or more usually known as an ultrafilter) if the only ideal (resp. filter) properly containing $A$ is $L$.*

**Example 1.1.** *(i) The following are ideals in $\mathcal{P}(X)$*

*(a) all subsets not containing a fixed element of $X$,*

*(b) all finite subsets (this ideal is non-principal if $X$ is infinite).*

*(ii) Let $(X, \mathfrak{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in \mathfrak{T}) x \in U \subseteq V\}$ is a filter in $(X, \mathfrak{T})$.*

*(iii) Let $X = (D(60), |)$ the lattice (i.e., $D(60)$ represents the denominators of the number 60) given in Figure 1.1. Note that any ideal and filter on $X$ is principal. In the following figure we present the principal ideal $I_6 = \{1, 2, 3, 6\}$ and the principal filter $F_6 = \{6, 12, 30, 60\}$ by dash lines.*

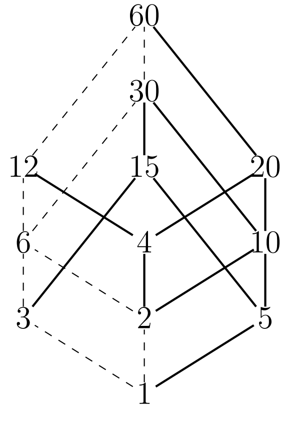


Figure 1.1: Representation of lattice $(D(60), |)$

## 1.3. Fuzzy sets and fuzzy relations

### 1.3.1. t-norms and t-conorms

*The history of triangular-norms (*t-norms*) started with Menger [54]. His main idea was to construct metric spaces where probability distributions are used to* *describe the distance between two elements. Schweizer and Sklar [69] provided the axioms of* t-norms, *as they are used today.*

**Definition 1.16.** [60] *A t-norm $T$ on $[0,1]$ is a function $T:[0,1]^2 \to [0,1]$ satisfies the following four axioms:*

(T1) *Commutativity:* $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$;

(T2) *Associativity:* $(\forall x, y, z \in [0,1])(T(x, T(y,z)) = T(T(x,y), z)))$;

(T3) *Monotonicity:* $(\forall x, y, z \in [0,1)(x \leq y \Rightarrow T(x,z) \leq T(y,z))$;

(T4) *Boundary condition:* $(\forall x \in [0,1])(T(x,1) = x)$.

*Conditions (T4) and (T3) imply that for any t-norm $T$ it holds that $T(x,y) \leq x$, $T(x,y) \leq y$, $T(x,y) \leq Min(x,y)$ and $T(x,0) = 0$.*

**Example 1.2.** *The following four operations are the most common t-norms:*

(T5) *Minimum:* $T_M(x,y) = \min\{x,y\}$

(T6) *Product:* $T_P(x,y) = x.y$

(T7) *Lukasiewicz:* $T_L(x,y) = \max\{x+y-1, 0\}$

(T8) *Drastic product:*

$$T_D(x,y) = \begin{cases} x & \textit{if} \quad y = 1 \\ y & \textit{if} \quad x = 1 \\ 0 & \textit{if} \quad x, y < 1. \end{cases}$$

*Let $T$ be a t-norm on $[0,1]$.*
*An element $a \in ]0,1[$ is called a zero divisor of $T$ if there exists some $b > 0$ such that $T(a,b) = 0$.*
*An element $a \in [0,1]$ is called an idempotent element of $T$ if $T(a,a) = a$.*
*$T$ is called Archimedean if $T(x,x) < x$, for any $x \in [0,1]$.*
*Each $a \in [a,b]$ is an idempotent element of the Minimum t-norm $T_M$ (Actually $T_M$ is the only t-norm whose set of idempotent is equal $[0,1]$), $T_M$ has no zero divisor. Each $a \in ]0,1[$ is a zero divisor of the Lukasiewicz t-norm $T_L$ as well of the Drastic product t-norm $T_D$.*

*For two t-norms $T_1$ and $T_2$ on $[0,1]$, we define:*

$$T_1 \leq T_2 \Leftrightarrow (\forall x, y \in [0,1])(T_1(x,y) \leq T_2(x,y)).$$

*Let be $T_1$ and $T_2$ two t-norms. If $T_1 \leq T_2$, then $T_1$ is called weaker than $T_2$ (or, equivalently, $T_2$ is called stronger than $T_1$). Note that $T_D$ is the weakest t-norm,* *and $T_M$ is the strongest t-norm, i.e. for any t-norm it holds:* (T9) $\mathrm{T_D} \leq \mathrm{T} \leq \mathrm{T_M}$. *Since $T_L \leq T_P$, it obviously holds:* (T10) $\mathrm{T_D} \leq \mathrm{T_L} \leq \mathrm{T_P} \leq \mathrm{T_M}$.

*Triangular conorms (t-conorms) are dual operations of t-norms, we recall the following definition of conorms.*

**Definition 1.17.** [60] *A t-conorm is a function $S : [0,1]^2 \to [0,1]$ that for any $x, y, z \in [0,1]$ satisfies* (T1)-(T3) and the following boundary condition $S(x,0) = S(0,x) = x$, $S(x,1) = S(1,x) = 0$

**Remark 1.1.** *Given a t-norm $T$, we find the associated dual t-conorm $S$ by $S(x,y) = 1 - T(1-x, 1-y)$.*

*The dual t-conorms w.r.t. $T_M, T_P, T_L$ and $T_D$ are given by:*

(S1) *Maximum:* $S_M(x,y) = \max\{x,y\}$

(S2) *Probabilistic sum:* $S_P(x,y) = x + y - s.y$

(T7) *Lukasiewicz:* $S_L(x,y) = \min\{x+y, 1\}$

(T8) *Drastic sum:*

$$S_D(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [0,1)^2 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

### 1.3.2. Fuzzy sets

*This subsection contains the basic definitions and properties of fuzzy sets. The notion of fuzzy set was introduced in 1965 by Lotfi A. Zadeh in the paper [94].*

**Definition 1.18.** [94] *Let $X$ be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element $x$ in the fuzzy subset $A$, for $x \in X$.*

*For two fuzzy sets $A$ and $B$ on a set $X$, several operations are defined in the following way (see [94]).*

*(i)* $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$, *for any* $x \in X$;

*(ii)* $A = B$ *if* $\mu_A(x) = \mu_B(x)$, *for any* $x \in X$;

*(iii)* $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\}$;

*(iv)* $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x)\rangle \mid x \in X\}$;

*(v)* $\overline{A} = \{\langle x, 1 - \mu_A(x)\rangle \mid x \in X\}$.

**Definition 1.19.** [62] *Let $A$ be a fuzzy set on a set $X$. The support of $A$ is the crisp subset on $X$ given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

**Definition 1.20.** [62] *Let $A$ be a fuzzy set on a set $X$. The kernel of $A$ is the crisp subset on $X$ given by*

$$Ker(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

**Definition 1.21.** [62] *Let $A$ be a fuzzy set on a set $X$. The height of $A$ is the highest value taken by its membership function given by*

$$H(A) = \sup_{x \in X} \mu_A(x)$$

**Example 1.3.** *Let $X = [0, 1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We define the fuzzy set $A$ on $X$ by*

$$\mu_A(x) = \begin{cases} 0 \ , \ if \ x < a - \alpha \ or \ b + \beta < x, \\ 1 \ , \ if \ \ a < x < b, \\ 1 + (\frac{x-a}{\alpha}) \ , \ if \ \ a - \alpha < x < a \, , \\ 1 - (\frac{b-x}{\beta}) \ , \ if \ \ b < x < b + \beta. \end{cases}$$

*Then $Ker(A) = [0, 1]$, $Supp(A) = [a - \alpha, b + \beta]$ and $H(A) = 1$.*

*In the sequel, we give the following definition of level sets (which is also often called $\alpha$-cuts) of a fuzzy set.*

**Definition 1.22.** [62] *Let $A$ be a fuzzy set on a set $X$. The $\alpha$-cut of $A$ is a crisp subset*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\},$$

*where $\alpha \in [0, 1]$, $A_0 = X$ and $A_1 = Ker(A)$.*

*More information on L-set can be found in [14, 36, 62, 94].*

### 1.3.3. Fuzzy relations

*The notion of fuzzy relations was first introduced by Zadeh [95] as a natural extension to a fuzzy set and plays an important role in the theory of such sets and their applications.*

**Definition 1.23.** [95] *Let $X$ and $Y$ be two nonempty sets. A binary fuzzy relation from $X$ to $Y$, is a fuzzy subset of $X \times Y$ characterized by a membership function $\mu_R$ which associates with each pair $(x, y)$ its grade of membership $\mu_R(x, y)$ in the interval* $[0, 1]$.

*A binary L-relation $S$ is said to be included in a binary L-relation $R$, denoted by: $S \subseteq R$, if $S(x, y) \leq R(x, y)$, for any $x, y \in X$. The intersection of two binary L-relations $S$ and $R$ on $X$ is the binary L-relation $S \cap R$ on $X$ defined by $(S \cap R)(x, y) = S(x, y) \wedge R(x, y)$. Similarly, the union of two binary L-relations $S$ and $R$ on $X$ is the binary L-relation $S \cup R$ on $X$ defined by: $(S \cup R)(x, y) = S(x, y) \vee R(x, y)$, for any $x, y \in X$. The composition of two binary L-relations $S$ and $R$ on $X$ is the binary L-relation $S \circ R$ on $X$ defined by: $S \circ R(x, z) = \sup_{y \in X}(S(x, y) * R(y, z))$. The transpose $R^t$ of a binary L-relation $R$ is the binary L-relation defined by: $R^t(x, y) = R(y, x)$. As an L-relation is an L-set, then the $\alpha$-level set and Supp of $R$ are defined as in a fuzzy set, i.e., $R_\alpha = \{(x, y) \in X \times X \mid R(x, y) \geq \alpha\}$, for any $x, y \in X$. In the same way we define the support of an L-relation Supp$(R)$ as Supp$(R) = \{(x, y) \in X \times X \mid R(x, y) > 0\}$. A binary L-relation $R$ on a set $X$ is called:*

1. *reflexive, if $R(x, x) = 1$, for all $x \in X$;*
2. *symmetric, if $R^t = R$;*
3. *$*$-transitive, if $R(x, y) * R(y, z) \leq R(x, z)$, for all $x, y, z \in X$;*
4. *$*$-asymmetric, if $R(x, y) * R(y, x) = 0$, for all $x, y \in X$;*
5. *$*$-antisymmetric, if $R(x, y) * R(y, x) \neq 0$ implies $x = y$, for all $x, y \in X$;*
6. *separability, if $R(x, y) = 1$ implies that $x = y$, for all $x, y \in X$.*

*For more details on binary L-relations and their properties, we refer to [14, 20, 77, 85, 95].*

# 2 Generalities on intuitionistic fuzzy sets and neutrosophic sets

*In this chapter, we recall basic definitions and properties of Intuitionistic fuzzy (sets/relations), Intuitionistic fuzzy lattices, Intuitionistic fuzzy ideals and filters on a lattice, Intuitionistic fuzzy ordered lattice. Next, we recall some basic notions of the theory of Neutrosophic sets ( particularly, Single valued neutrosophic sets) and some related concepts that will be needed throughout this thesis.*

## 2.1. Intuitionistic fuzzy sets and intuitionistic fuzzy relations

### 2.1.1. Intuitionistic fuzzy sets

*This subsection contains the basic definitions and properties of intuitionistic fuzzy sets with several operations on them. In 1983, Atanassov [6] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.*

**Definition 2.1.** [6] *Let $X$ be a non-empty set. An intuitionistic fuzzy set (IFS, for short) $A$ on $X$ is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfy the condition:*

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \text{ for each } x \in X.$$

*For any $x \in X$ the number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree or the intuitionistic index of $x$ to $A$.*

*The class of intuitionistic fuzzy sets on $X$ is denoted by $IFS(X)$.*

*Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.*

**Example 2.1.** *Let $X$ be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate that have voted for the corresponding government. Denote it by $M(x)$ and let $\mu(x) = \frac{M(x)}{100}$ (degree of membership, validity, etc.). Let $\nu(x) = 1 - \mu(x)$. This number* *corresponds to the part of electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define* $\nu(x)$ *(degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be* $\pi(x) = 1 - \mu(x) - \nu(x)$ *(degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set* $\{\langle x, \mu(x), \nu(x)\rangle \mid x \in X\}$ *and obviously,*

$$0 \le \mu(x) + \nu(x) \le 1.$$

*For two intuitionistic fuzzy sets* $A$ *and* $B$ *on a set* $X$*, several operations are defined as follows (see, e.g., Atansssov [7, 8, 9], Biswas [16] and Gy [83]). Here we will present only those which are related to the present work.*

(i) $A \subseteq B$ *if* $\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$, *for any* $x \in X$;

(ii) $A = B$ *if* $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$, *for any* $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$;

(v) $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in X\}$;

(vi) $A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in X\}$;

(vii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(viii) $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$;

(ix) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

*In the sequel, we need the following definition of level sets (which is also often called* $(\alpha, \beta)$*-cuts) of an intuitionistic fuzzy set.*

**Definition 2.2.** [89] *Let* $A$ *be an intuitionistic fuzzy set on a set* $X$*. The* $(\alpha, \beta)$*-cut of* $A$ *is a crisp subset*

$$A_{\alpha,\beta} = \{x \in X \mid \mu_A(x) \ge \alpha \text{ and } \nu_A(x) \le \beta\}, \tag{2.1}$$

*where* $\alpha, \beta \in ]0, 1]$ *with* $\alpha + \beta \le 1$.

**Definition 2.3.** [8] *Let* $A$ *be an intuitionistic fuzzy set on a set* $X$*. The support of* $A$ *is the crisp subset on* $X$ *given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}. \tag{2.2}$$

### 2.1.2. Intuitionistic fuzzy relations

*This subsection contains the basic definitions and properties of intuitionistic fuzzy relations, intuitionistic fuzzy order relations and several operations on intuitionistic fuzzy relations.*

*Burillo and Bustince [22, 23] introduced the concept of intuitionistic fuzzy relation as a natural generalization of fuzzy relation.*

**Definition 2.4.** [22, 23] *Let $X$ and $Y$ be two nonempty sets. An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from $X$ to $Y$ is an intuitionistic fuzzy subset of $X \times Y$, i.e., is an expression $R$ given by*

$$R = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle \mid (x,y) \in X \times Y\},$$

*where*

$$\mu_R : X \times Y \to [0,1], \text{ and } \nu_A : X \times Y \to [0,1]$$

*satisfy the condition*

$$0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1, \tag{2.3}$$

*for any $(x,y) \in X \times Y$. The value $\mu_R(x,y)$ is called the degree of membership of $(x,y)$ in $R$ and $\nu_R(x,y)$ is called the degree of non-membership of $(x,y)$ in $R$.*

*Several operations on intuitionistic fuzzy relations are defined (see, e.g., Atansssov [7, 8, 9], Biswas [16] and Gy [83]). Here we will present only those which are related to the present work.*

*Let $R, P$ be two intuitionistic fuzzy relations. $R$ is said to be contained in $P$ (or we say that $P$ contains $R$), denoted by $R \subseteq P$, if for any $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$. The* transpose *(or the* inverse*) $R^t$ of $R$ is the intuitionistic fuzzy relation from $Y$ to $X$ defined by $R^t = \{\langle (x,y), \mu_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\}$, where $\mu_{R^t}(x,y) = \mu_R(y,x)$ and $\nu_{R^t}(x,y) = \nu_R(y,x)$ for any $(x,y) \in X \times Y$.*

*The* intersection *of two intuitionistic fuzzy relations $R$ and $P$ is defined as*

$$R \cap P = \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}$$

*The* union *of two intuitionistic fuzzy relations $R$ and $P$ is defined as*

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \mu_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}.$$

*Let $R, P$ and $Q$ be three intuitionistic fuzzy relations from a universe $X$ to a universe $Y$.*

*(i) if $R \subseteq P$, then $R^t \subseteq P^t$;*

*(ii) $(R \cup P)^t = R^t \cup P^t$;*

*(iii) $(R \cap P)^t = R^t \cap P^t$;*

*(iv) $(R^t)^t = R$;*

*(v) $R \cap (P \cup Q) = (R \cap P) \cup (R \cap Q)$ and $R \cup (P \cap Q) = (R \cup P) \cap (R \cup Q)$;*

*(vi) $R \cup P \supseteq R$, $R \cup P \supseteq P$, $R \cap P \subseteq R$, $R \cap P \subseteq P$;*

*(vii) if $R \supseteq P$ and $R \supseteq Q$, then $R \supseteq P \cup Q$;*

*(viii) if $R \subseteq P$ and $R \subseteq Q$, then $R \subseteq P \cap Q$.*

*Let $R$ be an intuitionistic fuzzy relation (intuitionistic fuzzy relation on $X$, for short). The following properties are crucial in (see e.g., [22, 23, 82, 97]):*

(i) Reflexivity*: $\mu_R(x,x) = 1$, for any $x \in X$. In this case we note that $\nu_R(x,x) = 0$, for any $x \in X$.*

(ii) Antisymmetry*: for any $x, y \in X$, $x \neq y$ then*

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases}, \tag{2.4}$$

*where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.*

(iii) Perfect antisymmetry*: for any $x, y \in X$ with $x \neq y$ and*

$$\begin{cases} \mu_R(x,y) > 0 \\ \text{or} \\ \mu_R(x,y) = 0 \quad \text{and } \nu_R(x,y) < 1\,, \end{cases}$$

*then*

$$\begin{cases} \mu_R(y,x) = 0 \\ \text{and} \\ \nu_R(y,x) = 1\,. \end{cases} \tag{2.5}$$

(iv) Transitivity*:*

$$R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R. \tag{2.6}$$

*In the above definition, the composition* $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ *used in the transitivity means that* $R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\},\ \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\}\rangle \mid x,z \in X\}$, *where* $\alpha$, $\beta$, $\lambda$ *and* $\rho$ *are t-norms or t-conorms taken under the intuitionistic fuzzy condition*

$$0 \le \alpha_{y\in X}\{\beta[\mu_R(x,y),\mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y),\nu_R(y,z)]\} \le 1\,,$$

*for any* $x,z \in X$*. The properties of this composition and the choice of* $\alpha$, $\beta$, $\lambda$ *and* $\rho$*, for which this composition fulfills a maximal number of properties, are investigated in [22]-[26], [32]. If no other conditions are imposed, in the following we will take* $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ *and* $\rho = \max$.

*Note that Bustince and Burillo in [25], mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh [95] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order.*

**Definition 2.5.** [22, 23] *Let* $X$ *be a nonempty crisp set and* $R$ *be an intuitionistic fuzzy relation on* $X$*.* $R$ *is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

*A nonempty set* $X$ *with an intuitionistic fuzzy order* $R$ *defined on it is called an intuitionistic fuzzy ordered set and is denoted by* $(X,\mu_R,\nu_R)$*. It easily follows that each partially ordered set* $(X,\le)$ *and each fuzzy ordered set* $(X,R)$ *can be viewed as intuitionistic fuzzy ordered sets.*

**Example 2.2.** *Let* $X = \{a,b,c,d,e\}$*. Then the intuitionistic fuzzy relation* $R$ *defined on* $X$ *by*

$$R = \{\langle (x,y),\mu_R(x,y),\nu_R(x,y)\rangle \mid x,y \in X\},$$

*where* $\mu_R$ *and* $\nu_R$ *are given by the following tables:*

| $\mu_R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0.55 | 0.40 |
| $b$ | 0 | 1 | 0 | 0.35 | 0.45 |
| $c$ | 0 | 0 | 1 | 0 | 0.70 |
| $d$ | 0 | 0 | 0 | 1 | 0 |
| $e$ | 0 | 0 | 0 | 0 | 1 |

| $\nu_R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | 0.40 | 0.45 | 0.25 |
| $b$ | 0.30 | 0 | 0.20 | 0.35 | 0.10 |
| $c$ | 1 | 1 | 0 | 0.85 | 0.15 |
| $d$ | 1 | 1 | 1 | 0 | 1 |
| $e$ | 1 | 1 | 1 | 0.90 | 0 |

*is an intuitionistic fuzzy order on $X$.*

**Example 2.3.** *Let $m, n \in \mathbb{N}$. Then the following intuitionistic fuzzy relation $R$ on $\mathbb{N}$ is an intuitionistic fuzzy order, where*

$$\mu_R(m,n) = \begin{cases} 1\ , & if\ m = n \\ 1 - \frac{m}{n}\ , & if \quad m < n \\ 0\ , & if \quad m > n\ , \end{cases} \quad and\ \nu_R(m,n) = \begin{cases} 0\ , & if \quad m = n \\ \frac{m}{2n}\ , & if \quad m < n \\ 1\ , & if \quad m > n \end{cases}$$

*On the basis of the above definition of perfect antisymmetry we define a complete (or total) intuitionistic fuzzy order as follows*

**Definition 2.6.** [97] *An intuitionistic fuzzy order $R$ on a universe $X$ is called complete (or total) if for any $x, y \in X$ it holds that*

$$[\mu_R(x,y) > 0 \text{ or } (\mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1)]$$

*or*

$$[\mu_R(y,x) > 0 \text{ or } (\mu_R(y,x) = 0 \text{ and } \nu_R(y,x) < 1)].$$

**Definition 2.7.** [97] *An intuitionistic fuzzy ordered set $(X, \mu_R, \nu_R)$ in which $R$ is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

## 2.2. Intuitionistic fuzzy lattices

*In this section, we recall the basic definitions and properties of intuitionistic fuzzy lattices and some related notions that will be needed throughout the next chapters. The concept of an intuitionistic fuzzy lattice was introduced by Thomas and Nair [79] as an intuitionistic fuzzy set on a crisp lattice stable by the supremum and the infimum of the binary operations $\sqcap$ and $\sqcup$.*

*To avoid any confusion or misunderstanding in some formulas, we use the notation $(\leqslant, \sqcap, \sqcup)$ to refer the (order,* min*,* max*) on the lattice $L$ and $(\leq, \wedge, \vee)$ to refer the (usual order,* min*,* max*) on the real interval $[0,1]$.*

**Definition 2.8.** [79] *Let $L$ be a lattice and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFS on $L$. Then $A$ is called an intuitionistic fuzzy sub-lattice (fuzzy lattice, for short) if for any $x, y \in L$, the following conditions are satisfied:*

*(i)* $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$;

*(ii)* $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$;

*(iii)* $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$;

*(iv)* $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$.

**Example 2.4.** *Figure 1.1 shows the Hasse diagram of a lattice $L = \{0, a, b, 1\}$. The intuitionistic fuzzy set $A$ on $L$ given by $A = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$ is an intuitionistic fuzzy lattice.*

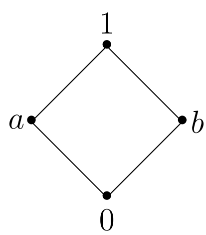


Figure 2.1: Hasse diagram of a lattice $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

*For further details on intuitionistic fuzzy lattices, we refer to [56, 78, 79, 99].*

## 2.3. Intuitionistic fuzzy ideals and filters on a lattice

*The notion of intuitionistic fuzzy ideal (resp. filter) on a lattice $(L, \leqslant, \sqcap, \sqcup)$ was first introduced by Thomas and Nair [79].*

**Definition 2.9.** [79] *Let $L$ be a lattice and $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ be an IFS on $L$. Then $I$ is called an intuitionistic fuzzy ideal on $L$ (IF-ideal, for short) if for all $x, y \in L$ the following conditions are satisfied:*

*(i)* $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$;

*(ii)* $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$;

*(iii)* $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$;

*(iv)* $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

**Example 2.5.** *Let $L$ be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set $I$ on $L$ defined by $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < 1, 0.1, 0.3 >\}$ is an IF-ideal.*

**Definition 2.10.** [79] *Let $L$ be a lattice and $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ be an IFS on $L$. Then $F$ is called an intuitionistic fuzzy filter on $L$ (IF-filter, for short) if for any $x, y \in L$, the following conditions are satisfied:*

*(i)* $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$;

*(ii)* $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$;

*(iii)* $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$;

*(iv)* $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

**Example 2.6.** *Let $L$ be the lattice given by the Hasse diagram in Figure 2.1. The intuitionistic fuzzy set $F$ on $L$ defined by $F = \{< 0, 0.1, 0.6 >, < a, 0.2, 0.6 >, < b, 0.1, 0.5 >, < 1, 0.4, 0.3 >\}$ is an IF-filter.*

**Remark 2.1.** *Notice that every IF-ideal on $L$ is an $L$-IF-lattice, but the converse is not true in general. Indeed, let $L$ be the lattice given by the Hasse diagram in Figure 2.1 and $A \in IFS(L)$ defined by $A = \{< 0, 0.3, 0.1 >, < a, 0.4, 0.5 >, < b, 0.4, 0.3 >, < 1, 0.7, 0.3 >\}$. Then $A$ is an $L$-IF-lattice, but since $\mu_A(a) = \mu_A(a \sqcap 1) = 0.4 \ngeq \max\{0.4; 0.7\}$, then it holds that $A$ is not an IF-ideal on $L$. As well since $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \ngeq \min\{0.4; 0.4\}$, then it holds that $A$ is not an IF-filter on $L$.*

*The following results will be needed throughout this chapter.*

**Proposition 2.1.** *Let $L$ be a lattice, $L^d$ be its order-dual lattice and $A \in IFS(L)$. Then it holds that $A$ is an IF-ideal on $L$ if and only if $A$ is an IF-filter on $L^d$ and conversely.*

**Proposition 2.2.** [79] *Let $L$ be a lattice, $A$ and $B$ are two intuitionistic fuzzy sets on $L$. Then it holds that*

*(i) if $A$ and $B$ are two IF-ideals on $L$, then $A \cap B$ is an IF-ideal on $L$;*

*(ii) if $A$ and $B$ are two IF-filters on $L$, then $A \cap B$ is an IF-filter on $L$.*

## 2.4. Intuitionistic fuzzy ordered lattice

*In this section, we recall some basic concepts related to the intuitionistic fuzzy ordered lattices. Further information can be found in [82].*

**Definition 2.11.** [82] *For an intuitionistic fuzzy ordered set* $(X, \mu_R, \nu_R)$ *and* $x \in X$*. The intuitionistic fuzzy sets* $R_{\geq[x]}$ *and* $R_{\leq[x]}$ *defined in* $X$ *by*

$$R_{\geq[x]} = \{\langle y, \mu_{R_{\geq[x]}}(y), \nu_{R_{\geq[x]}}(y)\rangle \mid y \in X\}, \quad \text{where}$$

$$\mu_{R_{\geq[x]}}(y) = \mu_R(x, y) \text{ and } \nu_{R_{\geq[x]}}(y) = \nu_R(x, y),$$

$$R_{\leq[x]} = \{\langle y, \mu_{R_{\leq[x]}}(y), \nu_{R_{\leq[x]}}(y)\rangle \mid y \in X\}, \text{ where}$$

$$\mu_{R_{\leq[x]}}(y) = \mu_R(y, x) \text{ and } \nu_{R_{\leq[x]}}(y) = \nu_R(y, x).$$

*are called, respectively the dominating class of* $x$ *and the class dominated by* $x$*.*

**Remark 2.2.** *The notions of the dominating class of* $x$ *and the class dominated by* $x$ *are generalizations of the classical notions* $\uparrow x$ *and* $\downarrow x$ *in a usual poset.*

*Next, we recall the definition of the upper bounds, the lower bounds, the supremum and the infimum on intuitionistic fuzzy ordered sets.*

**Definition 2.12.** [82] *Let* $(X, \mu_R, \nu_R)$ *be an intuitionistic fuzzy ordered set and* $A$ *be a subset of* $X$*.*

*(i) The set of upper bounds of* $A$ *with respect to* $R$ *is the intuitionistic fuzzy subset of* $X$ *defined by*

$$U(R, A)(y) = \bigcap_{x \in A} R_{\geq[x]}(y), \tag{2.7}$$

*for* $y \in X$

*(ii) The set of lower bounds of* $A$ *with respect to* $R$ *is the intuitionistic fuzzy subset of* $X$ *defined by*

$$L(R, A)(y) = \bigcap_{x \in A} R_{\leq[x]}(y), \tag{2.8}$$

*for* $y \in X$*.*

**Definition 2.13.** [82] *Let* $(X, \mu_R, \nu_R)$ *be an intuitionistic fuzzy ordered set and* $A$ *be a subset of* $X$*. An element* $x \in X$ *is called the least upper bound (or a supremum) of* $A$ *with respect to* $R$ *if:*

*(i)* $x \in Supp(U(R, A))$ *and*

*(ii) for all other* $y \in Supp(U(R, A))$, $\mu_R(x, y) > 0$ *or* ($\mu_R(x, y) = 0$ *and* $\nu_R(x, y) < 1$)*.*

*An element $x \in X$ is called the greatest lower bound (or an infimum) of $A$ with respect to $R$ if:*

*(i) $x \in Supp(L(R, A))$ and*

*(ii) for all other $y \in Supp(L(R, A))$, $\mu_R(y, x) > 0$ or ($\mu_R(y, x) = 0$ and $\nu_R(y, x) < 1$).*

**Remark 2.3.** *Let $(X, \mu_R, \nu_R)$ be an intuitionistic fuzzy ordered set and $A$ be a subset of $X$. If the supremum and the infimum of $A$ with respect to $R$ exist, then from the perfect antisymmetry of $R$ they are unique and denoted by $\sup_R(A)$ and $\inf_R(A)$, respectively.*

**Definition 2.14.** [82] *An intuitionistic fuzzy ordered set $(X, \mu_R, \nu_R)$ is called an intuitionistic fuzzy lattice with respect to the intuitionistic fuzzy order $R$ (or simply, intuitionistic fuzzy lattice) if each pair of elements $\{x, y\}$ of $X$ has a supremum and an infimum.*

**Example 2.7.** *Let $R$ be an intuitionistic fuzzy relation on $X = \{a, b, c, d, e\}$ defined by the following tables:*

| $\mu_R(.,.)$ | a | b | c | d | e |
|---|---|---|---|---|---|
| a | 1 | 0.7 | 0 | 0 | 0.1 |
| b | 0 | 1 | 0 | 0 | 0.1 |
| c | 0.5 | 0.7 | 1 | 1 | 0.8 |
| d | 0 | 0 | 0 | 1 | 0.5 |
| e | 0 | 0 | 0 | 0 | 1 |

| $\nu_R(.,.)$ | a | b | c | d | e |
|---|---|---|---|---|---|
| a | 0 | 0.2 | 1 | 1 | 0.7 |
| b | 0.8 | 0 | 1 | 0.1 | 0.8 |
| c | 0.3 | 0.2 | 0 | 0 | 0.1 |
| d | 0.8 | 1 | 1 | 0 | 0.4 |
| e | 0.7 | 0.8 | 0.7 | 0.6 | 0 |

*It is easy to verify that $R$ is reflexive, perfect antisymmetric and min-max transitive. This implies that $(X, \mu_R, \nu_R)$ is an intuitionistic fuzzy ordered set. The following table describes the supremum and the infimum of each subset of two elements $\{x, y\}$ of $X$.*

| $\{x,y\}$ | $\sup_R\{x,y\}$ | $\inf_R\{x,y\}$ |
|---|---|---|
| $\{a,b\}$ | $b$ | $a$ |
| $\{a,c\}$ | $a$ | $c$ |
| $\{a,d\}$ | $d$ | $a$ |
| $\{a,e\}$ | $e$ | $a$ |
| $\{b,c\}$ | $b$ | $c$ |
| $\{b,d\}$ | $e$ | $a$ |
| $\{b,e\}$ | $e$ | $b$ |
| $\{c,d\}$ | $d$ | $c$ |
| $\{c,e\}$ | $e$ | $c$ |
| $\{d,e\}$ | $e$ | $d$ |

*So, $(X, \mu_R, \nu_R)$ is an intuitionistic fuzzy lattice.*

**Definition 2.15.** [82] *Let $(X, \mu_R, \nu_R)$ be an intuitionistic fuzzy ordered set.*

*(i) An element $\top \in X$ is called the greatest element (the maximum) of $X$ with respect to $R$ or the intuitionistic fuzzy maximum of $X$ if*

$$\begin{cases} \mu_R(x,\top) > 0 \\ \text{or} \\ \mu_R(x,\top) = 0 \ \text{ and } \nu_R(x,\top) < 1\,, \end{cases}$$

*for any $x \in X$ .*

*(ii) An element $\bot \in X$ is called the smallest element (the minimum) of $X$ with respect to $R$ or the intuitionistic fuzzy minimum if*

$$\begin{cases} \mu_R(\bot,x) > 0 \\ \text{or} \\ \mu_R(\bot,x) = 0 \ \text{ and } \nu_R(\bot,x) < 1\,, \end{cases}$$

*for any $x \in X$ .*

*Next, we provide an overview of the Neutrosophic sets and related concepts.*

## 2.5. Neutrosophic sets

*Neutrosophic set has been derived from a new branch of philosophy, namely Neutrosophy. This theory is capable of dealing with uncertainty, indeterminacy and inconsistent information. Neutrosophic set approaches are suitable to modeling*

*problems with uncertainty, indeterminacy and inconsistent information in which human knowledge is necessary, and human evaluation is needed. In this regard, the theory of neutrosophic and its applications have been expanding in all directions at an astonishing rate especially after the introduction of the journal entitled "Neutrosophic Sets and Systems" (NSS).*
*In 1998, Smarandache [72] introduced the notion of neutrosophic sets (NSs) as a generalization of Atanassov's intuitionistic fuzzy sets.*

**Definition 2.16.** [72] *Let $X$ be a nonempty set. A neutrosophic set (NS, for short) $A$ on $X$ is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to ]^-0, 1^+[$ and an indeterminacy function $\sigma_A : X \to ]^-0, 1^+[$ and a non-membership function $\nu_A : X \to ]^-0, 1^+[$ which satisfy the condition:*

$$^-0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+, \text{ for any } x \in X.$$

*Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting*

$$\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

**Definition 2.17.** [73] *Let $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ be a neutrosophic set on $X$ (nonempty set). Then, the complement of the set $A$ ($\overline{A}$ for short), maybe defined as three kinds of complements:*

(1) $\overline{A} = \{\langle x, \sigma_A(x), 1 - \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$,

(2) $\overline{A} = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x)\rangle \mid x \in X\}$,

(3) $\overline{A} = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle \mid x \in X\}$.

**Remark 2.4.** *Let $A$ and $B$ two neutrosophic sets. Then*

(i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$,

(ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

*One can define several relations and operations between two neutrosophic sets $A$ and $B$ as follows:*

**Definition 2.18.** [73] *Assume that $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle \mid x \in X\}$. Then, we may consider two possible definitions for subsets ($A \subseteq B$).*
*($A \subseteq B$)may be defined as.*

*Type 1:* $A \subseteq B \Leftrightarrow (\forall x \in X; \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \nu_A(x) \geq \nu_B(x)$.
*Type 2:* $A \subseteq B \Leftrightarrow (\forall x \in X; \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x), \nu_A(x) \geq \nu_B(x)$.
*Likewise for the intersection, we may have:*
*Type 1:* $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) \rangle$.
*Type 2:* $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \vee \nu_B(x) \rangle$,
*and for the union, we may have:*
*Type 1:* $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x) \rangle$.
*Type 2:* $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x) \rangle$.

## 2.6. Single valued neutrosophic sets

*From a philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of* $]^-0, 1^+[$. *So instead of* $]^-0, 1^+[$ *we need to take the interval* $[0, 1]$ *for technical applications, because* $]^-0, 1^+[$ *will be difficult to apply in the real applications such as in scientific and engineering problems. In other word, for practical use of neutrosophic sets, Wang et al. [84] introduced the notion of single valued neutrosophic set (SVNS) as a subclass of NSs. In this context, we can say that single valued neutrosophic set has been developing rapidly due to its wide range of theoretical elegance and application areas (see, e.g., [4, 74, 84, 90, 91, 92]).*

**Definition 2.19.** [84] *Let* $X$ *be a nonempty set. A single-valued neutrosophic set (SVNS, for short)* $A$ *on* $X$ *is an object of the form* $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle \mid x \in X\}$ *characterized by a truth-membership function* $\mu_A : X \to [0, 1]$, *an indeterminacy-membership function* $\sigma_A : X \to [0, 1]$ *and a falsity-membership function* $\nu_A : X \to [0, 1]$.

*We denote by* $SVNS(X)$ *the set of all single-valued neutrosophic sets on* $X$.
*For any* $A,\ B \in SVNS(X)$, *several operations are defined.*

*(i)* $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$ *and* $\sigma_A(x) \leq \sigma_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$, *for any* $x \in X$,

*(ii)* $A = B$ *if* $\mu_A(x) = \mu_B(x)$ *and* $\sigma_A(x) = \sigma_B(x)$ *and* $\nu_A(x) = \nu_B(x)$, *for any* $x \in X$,

*(iii)* $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) \rangle \mid x \in X\}$,

*(iv)* $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x) \rangle \mid x \in X\}$,

*(v)* $\overline{A} = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x) \rangle \mid x \in X\}$.

*In the sequel, we recall the following definitions.*

**Definition 2.20.** [2] *Let* $A \in SVNS(X)$*, the level sets of* $A$ *(or the* $(\alpha, \beta, \gamma)$*-cuts of* $A$ *such that* $\alpha, \beta, \gamma \in ]0, 1]$*) are the crisp subsets defined by:*

$$A_{\alpha,\beta,\gamma} = \{x \in X \mid \mu_A(x) \geq \alpha \ , \ \sigma_A(x) \geq \beta \textit{ and } \nu_A(x) \leq \gamma\},$$

*for any* $\alpha, \beta, \gamma \in ]0, 1]$.

**Definition 2.21.** [2] *Let* $A$ *be a single-valued neutrosophic set on a set* $X$*. The support of* $A$ *is the crisp subset on* $X$ *given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) \neq 0, \ \sigma_A(x) \neq 0 \textit{ and } \nu_A(x) \neq 0\}.$$

## 2.7. Single valued neutrosophic relations

*Kim et al. [43] introduced the concept of single valued neutrosophic relation as a natural generalization of fuzzy and intuitionistic fuzzy relation.*

**Definition 2.22.** [43] *A single valued neutrosophic binary relation (A single valued neutrosophic relation, for short) from a universe* $X$ *to a universe* $Y$ *is a single valued neutrosophic subset in* $X \times Y$*, i.e., is an expression* $R$ *given by*

$$R = \{\langle (x, y), \mu_R(x, y), \sigma_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

*where* $\mu_R : X \times Y \to [0, 1]$, and $\sigma_R : X \times Y \to [0, 1]$, and $\nu_A : X \times Y \to [0, 1]$. *For any* $(x, y) \in X \times Y$*. The value* $\mu_R(x, y)$ *is called the degree of a membership of* $(x, y)$ *in* $R$*,* $\sigma_R(x, y)$ *is called the degree of indeterminacy of* $(x, y)$ *in* $R$ *and* $\nu_R(x, y)$ *is called the degree of non-membership of* $(x, y)$ *in* $R$*.*

**Example 2.8.** [47]

*Let* $X = \{a, b, c, d, e\}$*. Then the single valued neutrosophic relation* $R$ *defined on* $X$ *by*

$$R = \{\langle (x, y), \mu_R(x, y), \sigma_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\},$$

*where* $\mu_R$*,* $\sigma_R$ *and* $\nu_R$ *are given by the following tables:*

| $\mu_R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0.35 | 0 | 0 | 0.35 | 0.30 |
| $b$ | 0 | 0.40 | 0 | 0.35 | 0.45 |
| $c$ | 0.20 | 0 | 0.65 | 0 | 0.70 |
| $d$ | 0 | 0 | 0 | 1 | 0 |
| $e$ | 0.25 | 0.35 | 0 | 0 | 0.60 |

| $\sigma_R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0.5 | 0.5 | 0.42 | 0.2 | 0 |
| $b$ | 0.60 | 0.12 | 0.40 | 0.80 | 0.10 |
| $c$ | 0 | 1 | 0.02 | 0.75 | 0.15 |
| $d$ | 0.33 | 1 | 0.88 | 0 | 0.10 |
| $e$ | 0.20 | 0.55 | 1 | 0.55 | 0.30 |

| $\nu_R(.,.)$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | 0.40 | 0.25 | 0.25 |
| $b$ | 0.30 | 0.35 | 0.20 | 0.35 | 0.10 |
| $c$ | 0.80 | 1 | 0 | 0.85 | 0.15 |
| $d$ | 1 | 1 | 1 | 0 | 1 |
| $e$ | 0.70 | 0.55 | 1 | 0.90 | 0.30 |

**Definition 2.23.** [68] *Let $R$ and $P$ be two single valued neutrosophic relations from a universe $X$ to a universe $Y$.*

*(i) The transpose (inverse) $R^t$ of $R$ is the single valued neutrosophic relation from the universe $Y$ to the universe $X$ defined by*

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \sigma_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

*where*

$$\begin{cases} \mu_{R^t}(x,y) = \mu_R(y,x) \\ \text{and} \\ \sigma_{R^t}(x,y) = \sigma_R(y,x) \\ \text{and} \\ \nu_{R^t}(x,y) = \nu_R(y,x)\,, \end{cases}$$

*for any $(x,y) \in X \times Y$.*

*(ii) $R$ is said to be contained in $P$ or we say that $P$ contains $R$, denoted by $R \subseteq P$, if for all $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$, $\sigma_R(x,y) \leq \sigma_P(x,y)$*

*and* $\nu_R(x, y) \geq \nu_P(x, y)$.

(iii) *The intersection (resp.* the union*) of two single valued neutrosophic relations* $R$ *and* $P$ *from a universe* $X$ *to a universe* $Y$ *is a single valued neutrosophic relation defined as*
$R \cap P = \{\langle (x, y), \min(\mu_R(x, y), \mu_P(x, y)), \min(\sigma_R(x, y), \sigma_P(x, y)), \max(\nu_R(x, y), \nu_P(x, y))\rangle \mid (x, y) \in X \times Y\}$ *and*
$R \cup P = \{\langle (x, y), \max(\mu_R(x, y), \max(\sigma_R(x, y), \sigma_P(x, y)), \min(\nu_R(x, y), \nu_P(x, y))\rangle \mid (x, y) \in X \times Y\}$.

**Definition 2.24.** [68, 90] *Let* $R$ *be a single valued neutrosophic relation from a universe* $X$ *into itself.*

(i) Reflexivity*:* $\mu_R(x, x) = \sigma_R(x, x) = 1$ *and* $\nu_R(x, x) = 0$*, for any* $x \in X$.

(ii) Symmetry*: for any* $x, y \in X$ *then*

$$\begin{cases} \mu_R(x, y) = \mu_R(y, x) \\ \sigma_R(x, y) = \sigma_R(y, x) \\ \nu_R(x, y) = \nu_R(y, x) \end{cases},$$

(iii) Antisymmetry*: for any* $x, y \in X$*,* $x \neq y$ *then*

$$\begin{cases} \mu_R(x, y) \neq \mu_R(y, x) \\ \sigma_R(x, y) \neq \sigma_R(y, x) \\ \nu_R(x, y) \neq \nu_R(y, x) \end{cases},$$

(iv) Transitivity*:* $R \circ R \subset R$ *i.e.,* $R^2 \subset R$.

## 2.8. SVN lattices, SVN ideals and filters

*The notion of single valued neutrosophic lattice or fuzzy neutrosophic lattice as introduced by Arockiarani and Antony Crisp in Sweety [5] is a SVNS on a given crisp lattice stable by their meet and join operations. First, we need to fix the following notations.*

**Notation 2.1.** *Here,* $L$ *always denotes a lattice* $(L, \leqslant, \sqcap, \sqcup)$ *and* $L^d$ *its dual-order lattice* $(L, \geqslant, \sqcup, \sqcap)$*. To avoid any confusion or misunderstanding in some formulas, we use the notation* $(\leqslant, \sqcap, \sqcup)$ *to refer the (order,* min, max*) on the lattice* $L$ *and* $(\leq, \wedge, \vee)$ *to refer the (usual order,* min, max*) on the real interval* $[0, 1]$*. For more details on lattices, we refer to [30].*

**Definition 2.25.** [5] *Let $L$ be a lattice and $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in L\}$ be a SVNS on $L$. Then $A$ is called a single-valued neutrosophic lattice (SVN-lattice, for short) if for any $x, y \in L$, the following conditions are satisfied:*

*(i)* $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$,

*(ii)* $\mu_A(x \sqcap y) \geq \mu_A(x) \wedge \mu_A(y)$,

*(iii)* $\sigma_A(x \sqcup y) \geq \sigma_A(x) \wedge \sigma_A(y)$,

*(iv)* $\sigma_A(x \sqcap y) \geq \sigma_A(x) \wedge \sigma_A(y)$,

*(v)* $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$,

*(vi)* $\nu_A(x \sqcap y) \leq \nu_A(x) \vee \nu_A(y)$.

**Example 2.9.** *Let $L = \{0, a, b, 1\}$ be the lattice given by the following Hasse diagram. The SVNS $A = \{< 0, 0.5, 0.4, 0.1 >, < a, 0.4, 0.3, 0.5 >, < b, 0.4, 0.3, 0.3 >, < 1, 0.7, 0.6, 0.3 >\}$ on $L$ is a SVN-lattice.*

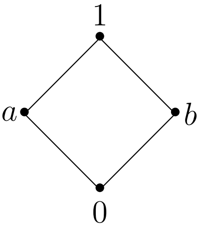


Figure 2.2: Hasse diagram of a lattice $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, 1\}$.

**Definition 2.26.** [5] *Let $L$ be a lattice and $I = \{\langle x, \mu_I(x), \sigma_I(x), \nu_I(x)\rangle \mid x \in L\}$ be a SVNS on $L$. Then $I$ is called a single-valued neutrosophic ideal on $L$ (SVN-ideal, for short) if for all $x, y \in L$, the following conditions are satisfied:*

*(i)* $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$,

*(ii)* $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$,

*(iii)* $\sigma_I(x \sqcup y) \geq \sigma_I(x) \wedge \sigma_I(y)$,

*(iv)* $\sigma_I(x \sqcap y) \geq \sigma_I(x) \vee \sigma_I(y)$,

*(v)* $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$,

*(vi)* $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$.

*Dually, we introduce the notion of a single-valued neutrosophic filter on a lattice.*

**Definition 2.27.** *Let $L$ be a lattice and $F = \{\langle x, \mu_F(x), \sigma_F(x), \nu_F(x)\rangle \mid x \in L\}$ be a SVNS on $L$. Then $F$ is called a single-valued neutrosophic filter on $L$ (SVN-filter, for short) if for all $x, y \in L$, the following conditions are satisfied:*

*(i)* $\mu_F(x \sqcup y) \geq \mu_F(x) \vee \mu_F(y)$,

*(ii)* $\mu_F(x \sqcap y) \geq \mu_F(x) \wedge \mu_F(y)$,

*(iii)* $\sigma_F(x \sqcup y) \geq \sigma_F(x) \vee \sigma_F(y)$,

*(iv)* $\sigma_F(x \sqcap y) \geq \sigma_F(x) \wedge \sigma_F(y)$,

*(v)* $\nu_F(x \sqcup y) \leq \nu_F(x) \wedge \nu_F(y)$,

*(vi)* $\nu_F(x \sqcap y) \leq \nu_F(x) \vee \nu_F(y)$.

*Certainly, intuitionistic fuzzy ideals (resp. filters) are SVN-ideals (resp. SVN-filter).*

**Example 2.10.** *Let $L$ be the lattice represented as in Figure 1. Then*

*(i) the SVNS $I = \{< 0, 0.5, 0.6, 0.1 >, < a, 0.4, 0.5, 0.3 >, < b, 0.1, 0.3, 0.2 >, < 1, 0.1, 0.2, 0.3 >\}$ is a SVN-ideal on $L$,*

*(ii) the SVNS $F = \{< 0, 0.1, 0.2, 0.6 >, < a, 0.2, 0.3, 0.6 >, < b, 0.1, 0.2, 0.5 >, < 1, 0.4, 0.5, 0.3 >\}$ is a SVN-filter on $L$.*

**Remark 2.5.** *As the crisp case, every SVN-ideal on $L$ is a SVN-lattice and not conversely. Indeed, Let $L$ be the lattice represented as in Figure 1 and $A \in SVN(L)$ defined by $A = \{< 0, 0.3, 0.2, 0.1 >, < a, 0.4, 0.3, 0.5 >, < b, 0.4, 0.3, 0.3 >, < 1, 0.7, 0.6, 0.3 >\}$. It is clear that $A$ is a SVN-lattice. But, since $\mu_A(a) = \mu_A(a \sqcap 1) = 0.4 \not\geq \max\{0.4; 0.7\}$, it holds that $A$ is not an SVN-ideal on $L$. Also, since $\mu_A(0) = \mu_A(a \sqcap b) = 0.3 \not\geq \min\{0.4; 0.4\}$, it holds that $A$ is not an SVN-filter on $L$.*

*A SVN-ideal or a SVN-filter is said to be proper if it is not equal to the whole lattice $L$.*

**Example 2.11.** *Let $L$ be the lattice given by the Hasse diagram in Figure 1. Then*

*(i) $I = \{< 0, 0.6, 0.5, 0.2 >, < a, 0.5, 0.6, 0.4 >, < b, 0.2, 0.4, 0.3 >, < 1, 0.2, 0.3, 0.4 > \}$ is a SVN-ideal,*

*(ii) $F = \{< 0, 0.2, 0.3, 0.7 >, < a, 0.3, 0.4, 0.7 >, < b, 0.2, 0.3, 0.6 >, < 1, 0.5, 0.4, 0.4 > \}$ is a SVN-filter.*

*Next, the following proposition is immediate.*

**Proposition 2.3.** *For any SVNS $A$ on a lattice $L$ it holds that $A$ is a SVN-ideal on $L$ if and only if $A$ is a SVN-filter on its dual-order lattice $L^d$.*

*Also, the proof of the following result is obvious.*

**Proposition 2.4.** [5] *For any $A, B \in SVN(L)$ it holds that*

*(i) If $A$ and $B$ are SVN-ideals on $L$, then $A \cap B$ is a SVN-ideal on $L$,*

*(ii) If $A$ and $B$ are SVN-filters on $L$, then $A \cap B$ is a SVN-filter on $L$.*

**Remark 2.6.** *The union of SVN-ideals (resp. SVN-filers) does not necessarily be a SVN-ideal (resp. SVN-filter). Indeed, let $A = \{< 0, 0.5, 0.4, 0.4 >, < a, 0.4, 0.3, 0.5 >, < b, 0.6, 0.5, 0.3 >, < 1, 0.3, 0.2, 0.6 >\}$ and $B = \{< 0, 0.6, 0.5, 0.4 >, < a, 0.5, 0.4, 0.5 >, < b, 0.7, 0.6, 0.3 >, < 1, 0.4, 0.3, 0.4 >\}$ be SVNSs on the lattice $L$ given in Example 2.9. We easily verify that $A$ and $B$ are SVN-ideals on $L$. But since $A \cup B = \{< 0, 0.6, 0.5, 0.4 >, < a, 0.5, 0.4, 0.5 >, < b, 0.7, 0.6, 0.3 >, < 1, 0.4, 0.3, 0.4 >\}$ and $\sigma_{A\cup B}(1) = \sigma_{A\cup B}(a \sqcup b) = 0.3 \ngeq \sigma_{A\cup B}(a) \wedge \sigma_{A\cup B}(b) = 0.4$, it holds that $A \cup B$ is not a SVN-ideal on $L$.*

# 3 Characterizations of Single Valued Neutrosophic Ideals and Filters on a Lattice

*The aim of the present chapter is to deepen the study of these important notions of neutrosophic ideals and filters on a given lattice. We provide their various characterizations and properties, in particular, we pay attention to their characterizations in terms of the lattice meet and join operations.*

## 3.1. Characterizations of SVN-ideals and SVN-filters in terms of their level sets

*The following Proposition shows that the support of a SVN-ideal (resp. SVN-filter) on a given lattice $L$ is also an ideal (resp. a filter) on $L$. The proof can be obtained by a direct application of the definition of an ideal (resp. a filter) on a lattice.*

**Proposition 3.1.** *Let $L$ be a lattice and $A \in SVNS(L)$. The following statements hold:*

*(i) If $A$ is a SVN-ideal, then its support $Supp(A)$ is an ideal on $L$.*

*(ii) If $A$ is a SVN-filter, then its support $Supp(A)$ is a filter on $L$.*

*Proof.* We only show (i), as (ii) can be proved by using Proposition 2.1 and (i). Suppose that $A$ is a SVN-ideal on $L$ and we show that $Supp(A)$ is an ideal on $L$.

(a) Let $x \in Supp(A)$ and $y \leqslant x$, then it holds that

$$\mu_A(x) \neq 0,\ \sigma_A(x) \neq 0 \text{ and } \nu_A(x) \neq 0\,.$$

Since $y \leqslant x$, then it holds that $x \sqcap y = y$. This implies that $\mu_A(y) = \mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$. Hence, $\mu_A(y) \geq \mu_I(x) \neq 0$. In similar way, we obtain that $\sigma_A(x) \neq 0$ and $\nu_A(x) \neq 0$. Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$, we need to show that $x \sqcup y \in Supp(A)$. Since $A$ is a SVN-ideal, then it follows from Definition 2.26 (i) that $\mu_A(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y) \neq 0$. Similarly, we show that $\sigma_A(x \sqcup y) \neq 0$ and $\nu_A(x \sqcup y) \neq 0$. Thus, $x \sqcup y \in Supp(A)$.

Therefore, $Supp(A)$ is an ideal on $L$. □

*The following result provides a characterization of a SVN-ideal (resp. SVN-filter) on a given lattice in terms of its level sets.*

**Proposition 3.2.** *Let $L$ be a lattice and $A \in SVNS(L)$. The following statements hold:*

*(i) $A$ is a SVN-ideal if and only if its level sets are ideals on $L$,*

*(ii) $A$ is a SVN-filter if and only if its level sets are filters on $L$.*

*Proof.* We only show (i), as (ii) can be proved by using Proposition 2.1 and (i). Let $A$ be a SVN-ideal on $L$ and $A_{\alpha,\beta,\gamma}$ such that $\alpha, \beta, \gamma \in ]0, 1]$ their level sets.

(a) Let $x \in A_{\alpha,\beta,\gamma}$ and $y \leqslant x$. From Definition 2.26 of a SVN-ideal, it follows that $\mu_A(y) \geq \mu_A(x)$, $\sigma_A(y) \geq \sigma_A(x)$ and $\nu_A(y) \leq \nu_A(x)$. Now, the fact that $\mu_A(x) \geq \alpha$, $\sigma_A(x) \geq \beta$ and $\nu_A(x) \leq \gamma$ imply that $\mu_A(y) \geq \alpha$, $\sigma_A(y) \geq \beta$ and $\nu_A(y) \leq \gamma$. Hence, $y \in A_{\alpha,\beta,\gamma}$.

(b) Let $x, y \in A_{\alpha,\beta,\gamma}$, then it holds that ($\mu_A(x) \geq \alpha$, $\sigma_A(x) \geq \beta$ and $\nu_A(x) \leq \gamma$) and ($\mu_A(y) \geq \alpha$, $\sigma_A(y) \geq \beta$ and $\nu_A(y) \leq \gamma$). From Definition 2.26 of a SVN-ideal, it follows that $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$, $\sigma_A(x \sqcup y) \geq \sigma_A(x) \wedge \sigma_A(y) \geq \beta$ and $\nu_A(x \sqcup y) = \nu_A(x) \vee \nu_A(y) \leq \gamma$. Hence, $x \sqcup y \in A_{\alpha,\beta,\gamma}$.

Thus, $A_{\alpha,\beta,\gamma}$ is an ideal on $L$, for any $\alpha, \beta, \gamma \in ]0, 1]$.

Conversely, suppose that all level sets of $A$ are ideals on $L$ and we show that $A$ is a SVN-ideal on $L$. Let $x, y \in L$ and assume that $\alpha = \mu_A(x) \wedge \mu_A(y)$, $\beta = \sigma_A(x) \wedge \sigma_A(y)$ and $\gamma = \nu_A(x) \vee \nu_A(y)$. Since $A_{\alpha,\beta,\gamma}$ is an ideal on $L$, then it holds that $x \sqcup y \in A_{\alpha,\beta,\gamma}$, for any $\alpha, \beta, \gamma \in ]0, 1]$. This implies that $\mu_A(x \sqcup y) \geq \alpha$, $\sigma_A(x \sqcup y) \geq \beta$ and $\nu_A(x \sqcup y) \leq \gamma$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \wedge \mu_A(y)$, $\sigma_A(x \sqcup y) \geq \sigma_A(x) \wedge \sigma_A(y)$ and $\nu_A(x \sqcup y) \leq \nu_A(x) \vee \nu_A(y)$. The other three conditions (ii), (iv) and (vi) of Definition 2.26 can be proved similarly. Thus, $A$ is a SVN-ideal on $L$. □

## 3.2. Basic characterization of SVN-ideals and filters on a lattice

*In this section, we characterize the notion of single valued neutrosophic ideals and filters on a lattice in terms of the lattice meet and join operations. We start with the key results.*

**Proposition 3.3.** *Let $L$ be a lattice and $A \in SVN(L)$. Then for all $x, y \in L$, the following equivalences hold:*

*(i)* $(\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$,

*(ii)* $(\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$,

*(iii)* $(\sigma_A(x \sqcap y) \geq \sigma_A(x) \vee \sigma_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \sigma_A(x) \geq \sigma_A(y))$,

*(iv)* $(\sigma_A(x \sqcup y) \geq \sigma_A(x) \vee \sigma_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \sigma_A(x) \leq \sigma_A(y))$,

*(v)* $(\nu_A(x \sqcap y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \leq \nu_A(y))$,

*(vi)* $(\nu_A(x \sqcup y) \leq \nu_A(x) \wedge \nu_A(y))$ *if and only if* $(x \leqslant y \Rightarrow \nu_A(x) \geq \nu_A(y))$.

*Proof.* Let $x, y \in L$.

(i) Suppose that $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$, then $x \sqcap y = x$. Since $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(x) = \mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \geq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \geq \mu_A(y))$. Then it follows that $\mu_A(x \sqcap y) \geq \mu_A(x)$ and $\mu_A(x \sqcap y) \geq \mu_A(y)$. Hence, $\mu_A(x \sqcap y) \geq \mu_A(x) \vee \mu_A(y)$.

(ii) Suppose that $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. If $x \leqslant y$, then $x \sqcup y = y$. Since $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$, it follows that $\mu_A(y) = \mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$. Hence, $\mu_A(x) \leq \mu_A(y)$.

Conversely, suppose that $(x \leqslant y \Rightarrow \mu_A(x) \leq \mu_A(y))$. Then it follows that $\mu_A(x) \leq \mu_A(x \sqcup y)$ and $\mu_A(y) \leq \mu_A(x \sqcup y)$. Hence, $\mu_A(x \sqcup y) \geq \mu_A(x) \vee \mu_A(y)$.

Similarly, we prove (iii) and (v) as (i). Also, we prove (iv) and (vi) as (ii). □

*As corollaries, we obtain the following interesting properties of single valued neutrosophic ideals and filters.*

**Corollary 3.1.** *Let $L$ be a lattice and $I$ be a SVN-ideal on $L$. Then for any $x, y \in L$ it holds that*

*(i) If $x \leqslant y$, then $\mu_I(x) \geq \mu_I(y)$, (i.e., the map $\mu_I : L \to [0, 1]$ is antitone),*

*(ii) If $x \leqslant y$, then $\sigma_I(x) \geq \sigma_I(y)$, (i.e., the map $\sigma_I : L \to [0, 1]$ is antitone),*

*(iii) If $x \leqslant y$, then $\nu_I(x) \leq \nu_I(y)$, (i.e., the map $\nu_I : L \to [0, 1]$ is monotone).*

**Corollary 3.2.** *Let $L$ be a lattice and $F$ be a SVN-filter on $L$. Then for any $x, y \in L$ it holds that*

*(i) If $x \leqslant y$, then $\mu_F(x) \leq \mu_F(y)$, (i.e., the map $\mu_F : L \to [0, 1]$ is monotone),*

*(ii)* *If* $x \leqslant y$, *then* $\sigma_F(x) \leq \sigma_F(y)$, *(i.e., the map* $\sigma_F : L \to [0,1]$ *is monotone),*

*(iii)* *If* $x \leqslant y$, *then* $\nu_F(x) \geq \nu_F(y)$, *(i.e., the map* $\nu_F : L \to [0,1]$ *is antitone).*

**Corollary 3.3.** *Let* $L$ *be a lattice with smallest element* $\perp$ *and greatest element* $\top$, *and* $I$ *be a SVN-ideal on* $L$. *Then it holds that*

*(i)* $\mu_I(\perp) = \max \mu_I(L)$ *and* $\mu_I(\top) = \min \mu_I(L)$, *where* $\mu_I(L) = \{\mu_I(x) \mid x \in L\}$,

*(ii)* $\sigma_I(\perp) = \max \sigma_I(L)$ *and* $\sigma_I(\top) = \min \sigma_I(L)$, *where* $\sigma_I(L) = \{\sigma_I(x) \mid x \in L\}$,

*(iii)* $\nu_I(\perp) = \min \nu_I(L)$ *and* $\nu_I(\top) = \max \nu_I(L)$, *where* $\nu_I(L) = \{\nu_I(x) \mid x \in L\}$.

**Corollary 3.4.** *Let* $L$ *be a lattice with smallest element* $\perp$ *and greatest element* $\top$, *and* $F$ *be a SVN-filter on* $L$. *Then it holds that*

*(i)* $\mu_F(\perp) = \min \mu_I(L)$ *and* $\mu_F(\top) = \max \mu_F(L)$, *where* $\mu_F(L) = \{\mu_F(x) \mid x \in L\}$,

*(ii)* $\sigma_F(\perp) = \min \sigma_I(L)$ *and* $\sigma_F(\top) = \max \sigma_F(L)$, *where* $\sigma_F(L) = \{\sigma_F(x) \mid x \in L\}$,

*(ii)* $\nu_F(\perp) = \max \nu_I(L)$ *and* $\nu_F(\top) = \min \nu_F(L)$, *where* $\nu_F(L) = \{\nu_F(x) \mid x \in L\}$.

*In the following theorem, we provide a basic characterization of single valued neutrosophic ideal on a lattice.*

**Theorem 3.1.** *Let* $L$ *be a lattice and* $I \in SVN(L)$. *Then it holds that* $I$ *is a SVN-ideal on* $L$ *if and only if the following three conditions are satisfied:*

*(i)* $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, *for any* $x, y \in L$,

*(ii)* $\sigma_I(x \sqcup y) = \sigma_I(x) \wedge \sigma_I(y)$, *for any* $x, y \in L$,

*(iii)* $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, *for any* $x, y \in L$.

*Proof.* Suppose that $I$ is a SVN-ideal on $L$, then for any $x, y \in L$ it holds that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$, $\sigma_I(x \sqcup y) \geq \sigma_I(x) \wedge \sigma_I(y)$ and $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$. Since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, it follows from Corollary 3.1 that ($\mu_I(x) \geq \mu_I(x \sqcup y)$, $\sigma_I(x) \geq \sigma_I(x \sqcup y)$) and ($\mu_I(y) \geq \mu_I(x \sqcup y)$, $\sigma_I(y) \geq \sigma_I(x \sqcup y)$). Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$ and $\sigma_I(x) \wedge \sigma_I(y) \geq \sigma_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$ and $\sigma_I(x \sqcup y) = \sigma_I(x) \wedge \sigma_I(y)$.

Also, since $x \leqslant x \sqcup y$ and $y \leqslant x \sqcup y$, we obtain from Corollary 3.1 that $\nu_I(x) \leq \nu_I(x \sqcup y)$ and $\nu_I(y) \leq \nu_I(x \sqcup y)$. Hence, $\nu_I(x) \vee \nu_I(y) \leq \nu_I(x \sqcup y)$. Thus,

$\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$.

Conversely, suppose that $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, $\sigma_I(x \sqcup y) = \sigma_I(x) \wedge \sigma_I(y)$ and $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, for any $x, y \in L$. It is obvious to see that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$, $\sigma_I(x \sqcup y) \geq \sigma_I(x) \wedge \sigma_I(y)$ and $\nu_I(x \sqcup y) \leq \nu_I(x) \vee \nu_I(y)$, for any $x, y \in L$. Next, we will show that $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, $\sigma_I(x \sqcap y) \geq \sigma_I(x) \vee \sigma_I(y)$ and $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$ , for any $x, y \in L$. Let $x, y \in L$ , since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$, then it holds that $\mu_I(x \sqcup (x \sqcap y)) = \mu_I(x)$ and $\mu_I(y \sqcup (x \sqcap y)) = \mu_I(y)$ . From hypothesis (i) and (ii), it follows that $\mu_I(x) \wedge \mu_I(x \sqcap y) = \mu_I(x)$ and $\mu_I(y) \wedge \mu_I(x \sqcap y) = \mu_I(y)$ . Hence, $\mu_I(x \sqcap y) \geq \mu_I(x)$ and $\mu_I(x \sqcap y) \geq \mu_I(y)$. Thus, $\mu_I(x \sqcap y) \geq \mu_I(x) \vee \mu_I(y)$, for any $x, y \in L$. In the same way, we obtain that $\sigma_I(x \sqcap y) \geq \sigma_I(x) \vee \sigma_I(y)$ and $\nu_I(x \sqcap y) \leq \nu_I(x) \wedge \nu_I(y)$ , for any $x, y \in L$. Therefore, $I$ is a SVN-ideal on $L$. □

*In the same manner, the following theorem provides a basic characterization of single valued neutrosophic filters on a lattice.*

**Theorem 3.2.** *Let $L$ be a lattice and $F \in SVN(L)$. Then it holds that $F$ is a SVN-filter on $L$ if and only if the following three conditions are satisfied:*

*(i)* $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, *for any* $x, y \in L$,

*(ii)* $\sigma_F(x \sqcap y) = \sigma_F(x) \wedge \sigma_F(y)$, *for any* $x, y \in L$,

*(ii)* $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$, *for any* $x, y \in L$.

*Proof.* The proof is a direct application of Proposition 2.1 and Theorem 3.1. □

*As corollaries, we obtain the following characterizations of crisp (fuzzy) ideals and intuitionstic fuzzy ideals (resp. crisp (fuzzy) filters and intuitionstic fuzzy filters) on a given lattice.*

**Corollary 3.5.** *Let $L$ be a lattice and $I, F$ be crisp (fuzzy) sets on $L$. Then it holds that*

*(i) $I$ is a crisp (fuzzy) ideal on $L$ if and only if $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, for any $x, y \in L$,*

*(ii) $F$ is a crisp (fuzzy) filter on $L$ if and only if $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, for any $x, y \in L$.*

**Corollary 3.6.** *Let $L$ be a lattice and $I, F$ be intuitionstic fuzzy sets on $L$. Then it holds that* *(i) I is an ituitionstic fuzzy ideal on L if and only if the following two conditions are satisfied:*

*(a)* $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, *for any* $x, y \in L$,

*(b)* $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, *for any* $x, y \in L$.

*(ii) F is an ituitionstic fuzzy filter on L if and only if the following two conditions are satisfied:*

*(a)* $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, *for any* $x, y \in L$,

*(b)* $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$, *for any* $x, y \in L$.

## 3.3. Interaction of SVN-ideals (resp. SVN-filters) with a lattices homomorphism

*In this section, we study the interaction of single valued neutrosophic ideals (resp. filters) with a lattices-homomorphism. First, we introduce the notion of direct image (resp. inverse image) of a single valued neutrosophic set.*

**Definition 3.1.** *Let* $f : L \to L'$ *be a mapping from a lattice* $L$ *into another lattice* $L'$ *and* $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in L\}$ *be an SVNS on* $L$. *The image of* $A$ *by* $f$ *is defined by* $f(A) = \{\langle y, f(\mu_A)(y), f(\sigma_A)(y), f(\nu_A)(y)\rangle \mid y \in L'\}$, *where*

$$f(\mu_A)(y) = \begin{cases} sup\{\mu_A(x) \mid x \in f^{-1}(y)\}, \ if\ f^{-1}(y) \neq \phi \\ 0\ ,\ if\ f^{-1}(y) = \phi \end{cases} ,$$

$$f(\sigma_A)(y) = \begin{cases} sup\{\sigma_A(x) \mid x \in f^{-1}(y)\}, \ if\ f^{-1}(y) \neq \phi \\ 0\ ,\ if\ f^{-1}(y) = \phi \end{cases}$$

*and*

$$f(\nu_A)(y) = \begin{cases} inf\{\nu_A(x) \mid x \in f^{-1}(y)\}, \ if\ f^{-1}(y) \neq \phi \\ 1\ ,\ if\ f^{-1}(y) = \phi. \end{cases}$$

*Similarly, if* $A' = \{\langle y, \mu_{A'}(y), \sigma_{A'}(y), \nu_{A'}(y)\rangle \mid y \in L'\}$ *is an SVNS on* $L'$, *then*

$$f^{-1}(A') = \{\langle x, f^{-1}(\mu_{A'})(x), f^{-1}(\sigma_{A'})(x), f^{-1}(\nu_{A'})(x)\rangle \mid x \in L\},$$

*where* $f^{-1}(\mu_{A'})(x) = \mu_{A'}(f(x))$, $f^{-1}(\sigma_{A'})(x) = \sigma_{A'}(f(x))$ *and* $f^{-1}(\nu_{A'})(x) = \nu_{A'}(f(x))$.

*In the following theorem, we will show that the image of a SVN-ideal (resp. SVN-filter) is a SVN-ideal (resp. SVN-filter).*

**Theorem 3.3.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A$ is an SVN-ideal on $L$, then $f(A)$ is an SVN-ideal on $L'$,*

*(ii) If $A$ is an SVN-filter on $L$, then $f(A)$ is an SVN-filter on $L'$.*

*Proof.* (i) Let $A$ be a SVN-ideal on $L$. For any $y, z \in L'$, it holds that

$$\begin{aligned} f(\mu_A)(y \sqcup z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \sup\{\mu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \wedge f(\mu_A)(z) \end{aligned}$$

and

$$\begin{aligned} f(\sigma_A)(y \sqcup z) &= \sup\{\sigma_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \sup\{\sigma_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\sigma_A(u) \wedge \sigma_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\sigma_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\sigma_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\sigma_A)(y) \wedge f(\sigma_A)(z). \end{aligned}$$

Similarly, for any $y, z \in L'$, it holds that

$$\begin{aligned} f(\nu_A)(y \sqcup z) &= \inf\{\nu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \inf\{\nu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{\nu_A(u) \mid u \in f^{-1}(y)\} \vee \inf\{\nu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\nu_A)(y) \vee f(\nu_A)(z). \end{aligned}$$

Thus, we can conclude that $f(A)$ is an SVN-ideal on $L'$.

(ii) Follows from Proposition 2.3 and (i). □

*In the following theorem, we will show that the inverse image of a SVN-ideal (resp. SVN-filter) is a SVN-ideal (resp. SVN-filter).*

**Theorem 3.4.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A'$ is an SVN-ideal on $L'$, then $f^{-1}(A')$ is an SVN-ideal on $L$,*

*(ii) If $A'$ is an SVN-filter on $L'$, then $f^{-1}(A')$ is an SVN-filter on $L$.*

*Proof.* (i) Let $A'$ be an SVN-ideal on $L'$. For any $x, y \in L$, it holds that

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcup y) &= \mu_{A'}(f(x \sqcup y)) \\ &= \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \wedge f^{-1}(\mu_{A'})(y) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\sigma_{A'})(x \sqcup y) &= \sigma_{A'}(f(x \sqcup y)) \\ &= \sigma_{A'}(f(x)) \wedge \sigma_{A'}(f(y)) \\ &= f^{-1}(\sigma_{A'})(x) \wedge f^{-1}(\sigma_{A'})(y). \end{aligned}$$

Similarly, for any $x, y \in L$, it holds that

$$\begin{aligned} f^{-1}(\nu_{A'})(x \sqcup y) &= \nu_{A'}(f(x \sqcup y)) \\ &= \nu_{A'}(f(x)) \vee \nu_{A'}(f(y)) \\ &= f^{-1}(\nu_{A'})(x) \vee f^{-1}(\nu_{A'})(y). \end{aligned}$$

Therefore, $f^{-1}(A')$ is an SVN-ideal on $L$.

(ii) Follows from Proposition 2.3 and (i). □

# 4 Several Types of Single-Valued Neutrosophic Ideals and Filters on a Lattice

*In this chapter, we introduce and study the notions of prime, maximal and principal single-valued neutrosophic ideals (resp. filters) on a lattice. Several properties and characterizations of these types of ideals and filters are given, and relationships between them are discussed.*

## 4.1. Prime SVN-ideals and filters on a lattice

*In this section, we introduce and characterize the notion of prime SVN-ideals (resp. SVN-filters) on a given lattice.*

### 4.1.1. Basic characterization of prime SVN-ideals and filters on a lattice

*In this section, we characterize the notion of prime single valued neutrosophic ideals and filters on a lattice in terms of the lattice meet and join operations.*

**Definition 4.1.** *A SVN-ideal $I$ on a lattice $L$ is called a prime SVN-ideal if, for any $x, y \in L$, the following conditions hold:*

*(i)* $\mu_I(x \sqcap y) \leq \mu_I(x) \vee \mu_I(y)$,

*(ii)* $\sigma_I(x \sqcap y) \leq \sigma_I(x) \vee \sigma_I(y)$,

*(iii)* $\nu_I(x \sqcap y) \geq \nu_I(x) \wedge \nu_I(y)$.

**Definition 4.2.** *A SVN-filter $F$ on a lattice $L$ is called a prime SVN-filter if, for any $x, y \in L$, the following conditions hold:*

*(i)* $\mu_F(x \sqcup y) \leq \mu_F(x) \vee \mu_F(y)$,

*(ii)* $\sigma_F(x \sqcup y) \leq \sigma_F(x) \vee \sigma_F(y)$,

*(iii)* $\nu_F(x \sqcup y) \geq \nu_F(x) \wedge \nu_F(y)$.

*The following result provides a characterization of prime SVN-ideal on a given lattice.*

**Theorem 4.1.** *Let $L$ be a lattice and $I \in SVNS(L)$. Then it holds that $I$ is a prime SVN-ideal on $L$ if and only if the following conditions hold:*

*(i)* $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$, *for any* $x, y \in L$,

*(ii)* $\mu_I(x \sqcap y) = \mu_I(x) \vee \mu_I(y)$, *for any* $x, y \in L$,

*(iii)* $\sigma_I(x \sqcup y) = \sigma_I(x) \wedge \sigma_I(y)$, *for any* $x, y \in L$,

*(iv)* $\sigma_I(x \sqcap y) = \sigma_I(x) \vee \sigma_I(y)$, *for any* $x, y \in L$,

*(v)* $\nu_I(x \sqcup y) = \nu_I(x) \vee \nu_I(y)$, *for any* $x, y \in L$,

*(vi)* $\nu_I(x \sqcap y) = \nu_I(x) \wedge \nu_I(y)$, *for any* $x, y \in L$.

*Proof.* Suppose that $I$ is a prime SVN-ideal on $L$. We only give the proof of the condition (i), as the others can be proved analogously. By hypothesis we have that $\mu_I(x \sqcup y) \geq \mu_I(x) \wedge \mu_I(y)$. Further, for any $x, y \in L$ it holds from Definition 2.26 (ii) that $\mu_I(x) = \mu_I(x \sqcap (x \sqcup y)) \geq \mu_I(x \sqcup y)$ and $\mu_I(y) = \mu_I(y \sqcap (x \sqcup y)) \geq \mu_I(x \sqcup y)$. Hence, $\mu_I(x) \wedge \mu_I(y) \geq \mu_I(x \sqcup y)$. Thus, $\mu_I(x \sqcup y) = \mu_I(x) \wedge \mu_I(y)$.

Conversely, suppose that $\mu_I$, $\sigma_I$ and $\nu_I$ satisfying the above six conditions. Then it is obvious to see that $I$ is a prime SVN-ideal on $L$. □

*Similarly, combining Theorem 4.1 and Proposition 2.1 leads to the following characterization of prime SVN-filters.*

**Theorem 4.2.** *Let $L$ be a lattice and $F \in SVNS(L)$. Then it holds that $F$ is a prime SVN-filter on $L$ if and only if the following conditions hold:*

*(i)* $\mu_F(x \sqcup y) = \mu_F(x) \vee \mu_F(y)$, *for any* $x, y \in L$,

*(ii)* $\mu_F(x \sqcap y) = \mu_F(x) \wedge \mu_F(y)$, *for any* $x, y \in L$,

*(iii)* $\sigma_F(x \sqcup y) = \sigma_F(x) \vee \sigma_F(y)$, *for any* $x, y \in L$,

*(iv)* $\sigma_F(x \sqcap y) = \sigma_F(x) \wedge \sigma_F(y)$, *for any* $x, y \in L$,

*(v)* $\nu_F(x \sqcup y) = \nu_F(x) \wedge \nu_F(y)$, *for any* $x, y \in L$,

*(vi)* $\nu_F(x \sqcap y) = \nu_F(x) \vee \nu_F(y)$, *for any* $x, y \in L$.

### 4.1.2. Characterizations of prime SVN-ideals and filters in terms of their level sets

*The following proposition shows that the support of a prime SVN-ideal (resp. prime SVN-filter) on a lattice is a prime ideal (resp. prime filter) on that lattice.*

**Proposition 4.1.** *Let $L$ be a lattice and $A \in SVNS(L)$. Then it holds that*

*(i) If $A$ is a prime SVN-ideal, then its support $Supp(A)$ is a prime ideal on $L$.*

*(ii) If $A$ is a prime SVN-filter, then its support $Supp(A)$ is a prime filter on $L$.*

*Proof.* (i) Suppose that $A$ is a prime SVN-ideal on a lattice $L$. From Proposition 3.1, it holds that $Supp(A)$ is an ideal on $L$. Next, we prove that $Supp(A)$ is prime. Let $x, y \in L$ such that $x \sqcap y \in Supp(A)$. It then holds that $\mu_A(x \sqcap y) \neq 0$, $\sigma_A(x \sqcap y) \neq 0$ and $\nu_A(x \sqcap y) \neq 0$. Then the fact that $A$ is a prime SVN-ideal on $L$ implies that $\mu_A(x) \vee \mu_A(y) = \mu_A(x \sqcap y) \neq 0$, $\sigma_A(x) \vee \sigma_A(y) = \sigma_A(x \sqcap y) \neq 0$ and $\nu_A(x) \wedge \nu_A(y) = \nu_A(x \sqcap y) \neq 0$. This implies that either ($\mu_A(x) \neq 0$, $\sigma_A(x) \neq 0$ and $\nu_A(x) \neq 0$) or ($\mu_A(y) \neq 0$, $\sigma_A(y) \neq 0$ and $\nu_A(y) \neq 0$). Hence, either $x \in Supp(A)$ or $y \in Supp(A)$. Therefore, $Supp(A)$ is a prime ideal on $L$.

(ii) Follows in the same way by using Proposition 2.1 and (i).

□

*In the same direction, we get the following theorem which provides a characterization of prime SVN-ideals (resp. prime SVN-filters) in terms of their level sets.*

**Theorem 4.3.** *Let $L$ be a lattice and $A \in SVNS(L)$. Then it holds that*

*(i) $A$ is a prime SVN-ideal if and only if its level sets are prime ideals.*

*(ii) $A$ is a prime SVN-filter if and only if its level sets are prime filters.*

*Proof.* (i) From Proposition 3.2, $A$ is an SVN-ideal on $L$ if and only if $A_{\alpha,\beta,\gamma}$ are an ideals on $L$, for any $\alpha, \beta, \gamma \in ]0, 1]$. It remains to show the primality conditions. Suppose that $A$ is a prime SVN-ideal on $L$, and let $x, y \in L$ such that $x \sqcap y \in A_{\alpha,\beta,\gamma}$. Then from Theorem 4.1 it follows that ($\mu_A(x \sqcap y) = \mu_A(x) \vee \mu_A(y) \geq \alpha$, $\sigma_A(x \sqcap y) = \sigma_A(x) \vee \sigma_A(y) \geq \beta$ and $\nu_A(x \sqcap y) = \nu_A(x) \wedge \nu_A(y) \leq \gamma$). These imply that either ($\mu_A(x) \geq \alpha$, $\sigma_A(x) \geq \beta$ and $\nu_A(x) \leq \gamma$) or ($\mu_A(y) \geq \alpha$, $\sigma_A(y) \geq \beta$ and $\nu_A(y) \leq \gamma$). Hence, either $x \in A_{\alpha,\beta,\gamma}$ or $y \in A_{\alpha,\beta,\gamma}$. Thus, $A_{\alpha,\beta,\gamma}$ are a prime ideals, for any $\alpha, \beta, \gamma \in ]0, 1]$. Conversely, suppose that $A_{\alpha,\beta,\gamma}$ are a prime ideals for any $\alpha, \beta, \gamma \in ]0, 1]$ and $A$ is not a prime SVN-ideal on $L$. Then it holds that there exist $x, y \in L$ such that $\mu_A(x \sqcap y) > \mu_A(x) \vee \mu_A(y)$, $\sigma_A(x \sqcap y) > \sigma_A(x) \vee \sigma_A(y)$ and $\nu_A(x \sqcap y) < \nu_A(x) \wedge \nu_A(y)$. These imply that ($\mu_A(x \sqcap y) > \mu_A(x)$ and $\mu_A(x \sqcap y) > \mu_A(y)$), ($\sigma_A(x \sqcap y) > \sigma_A(x)$ and $\sigma_A(x \sqcap y) > \sigma_A(y)$) and

($\nu_A(x \sqcap y) < \nu_A(x)$ and $\nu_A(x \sqcap y) < \nu_A(y)$). If we put $\mu_A(x \sqcap y) = \alpha$, $\sigma_A(x \sqcap y) = \beta$ and $\nu_A(x \sqcap y) = \gamma$, it follows that ($\mu_A(x) < \alpha$, $\sigma_A(x) < \beta$ and $\nu_A(x) > \gamma$) and ($\mu_A(y) < \alpha$, $\sigma_A(y) < \beta$ and $\nu_A(y) > \gamma$). Hence, $x \sqcap y \in A_{\alpha,\beta,\gamma}$ and $x, y \notin A_{\alpha,\beta,\gamma}$. That is a contradiction with the fact that $A_{\alpha,\beta,\gamma}$ are a prime ideals on $L$ for any $\alpha, \beta, \gamma \in ]0,1]$. Hence, $A$ is a prime SVN-ideal.

(ii) Follows from Proposition 2.1 and (i).

□

### 4.1.3. Interaction of SVN-ideals (resp. filters) with the basic set-operations

*In this subsection, we discuss the interaction of single valued neutrosophic ideals (resp. filters) with intersection, union, complement and some associated single valued neutrosophic sets.*

**Proposition 4.2.** *Let $(A_i)_{i\in I}$ be a family of SVNSs on a lattice $L$. Then it holds that*

*(i) If $A_i$ is a prime SVN-ideal on $L$, for any $i \in I$, then $\bigcap\limits_{i\in I} A_i$ is a prime SVN-ideal on $L$,*

*(ii) If $A_i$ is a prime SVN-filter on $L$, for any $i \in I$, then $\bigcap\limits_{i\in I} A_i$ is a prime SVN-filter on $L$.*

*Proof.* (i) Suppose that $A_i$ is a prime SVN-ideal on $L$, for any $i \in I$. From Proposition 2.4, it follows that $\bigcap\limits_{i\in I} A_i$ is a SVN-ideal on $L$. It remains to show that $\bigcap\limits_{i\in I} A_i$ is prime. Let $x, y \in L$ such that $x \sqcap y \in \bigcap\limits_{i\in I} A_i$. Then it follows that $x \sqcap y \in A_i$, for any $i \in I$. Since for any $i \in I$, $A_i$ is a prime SVN-ideal, it follows that $\mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$, $\sigma_{A_i}(x \sqcap y) \leq \sigma_{A_i}(x) \vee \sigma_{A_i}(y)$ and $\nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y)$, for any $i \in I$. This implies that $\mu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \mu_{A_i}(x \sqcap y) \leq \mu_{A_i}(x) \vee \mu_{A_i}(y)$, $\sigma_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \sigma_{A_i}(x \sqcap y) \leq \sigma_{A_i}(x) \vee \sigma_{A_i}(y)$ and $\nu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \geq \nu_{A_i}(x \sqcap y) \geq \nu_{A_i}(x) \wedge \nu_I(y)$, for any $i \in I$. Hence, $\mu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \bigwedge_{i\in I}(\mu_{A_i}(x) \vee \mu_{A_i}(y))$, $\sigma_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \bigwedge_{i\in I}(\sigma_{A_i}(x) \vee \sigma_{A_i}(y))$ and $\nu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \geq \bigvee_{i\in I}(\nu_{A_i}(x) \wedge \nu_I(y))$. Thus, $\mu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \mu_{\bigcap\limits_{i\in I} A_i}(x) \vee \mu_{\bigcap\limits_{i\in I} A_i}(y)$, $\sigma_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \leq \sigma_{\bigcap\limits_{i\in I} A_i}(x) \vee \sigma_{\bigcap\limits_{i\in I} A_i}(y)$ and $\nu_{\bigcap\limits_{i\in I} A_i}(x \sqcap y) \geq \nu_{\bigcap\limits_{i\in I} A_i}(x) \wedge \nu_{\bigcap\limits_{i\in I} A_i}(y)$. Therefore, $\bigcap\limits_{i\in I} A_i$ is a prime SVN-ideal on $L$.

(ii) Follows from Proposition 2.3 and (i). □

*The following proposition discusses the relationship between a single valued neutrosophic ideal (resp. filter) and its complement.*

**Proposition 4.3.** *Let $L$ be a lattice and $A \in SVN(L)$. Then it holds that*

*(i) $A$ is a prime SVN-ideal if and only if $\overline{A}$ is a prime SVN-filter on $L$,*

*(ii) $A$ is a prime SVN-filter if and only if $\overline{A}$ is a prime SVN-ideal on $L$.*

*Proof.* (i) Suppose that $A$ is a prime SVN-ideal, for any $x, y \in L$ it follows from Proposition 4.1 that

$$\begin{aligned}\mu_{\overline{A}}(x \sqcup y) &= \nu_A(x \sqcup y)\\ &= \nu_A(x) \vee \nu_A(y)\\ &= \mu_{\overline{A}}(x) \vee \mu_{\overline{A}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\overline{A}}(x \sqcap y) &= \nu_A(x \sqcap y)\\ &= \nu_A(x) \wedge \nu_A(y)\\ &= \mu_{\overline{A}}(x) \wedge \mu_{\overline{A}}(y)\end{aligned}$$

In a similar way, we prove that $\sigma_{\overline{A}}(x \sqcup y) = \sigma_{\overline{A}}(x) \vee \sigma_{\overline{A}}(y)$, $\sigma_{\overline{A}}(x \sqcap y) = \sigma_{\overline{A}}(x) \wedge \sigma_{\overline{A}}(y)$, $\nu_{\overline{A}}(x \sqcup y) = \nu_{\overline{A}}(x) \wedge \nu_{\overline{A}}(y)$ and $\nu_{\overline{A}}(x \sqcap y) = \nu_{\overline{A}}(x) \vee \nu_{\overline{A}}(y)$. Applying Proposition 4.2 guarantees that $\overline{A}$ is a prime SVN-filter on $L$. The converse follows from Proposition 2.3 and the first implication.

(ii) Follows from the fact that $A = \overline{\overline{A}}$ and (i). □

**Proposition 4.4.** *Let $L$ be a lattice and $A \in SVN(L)$. Then it holds that*

*(i) $A$ is a prime SVN-ideal if and only if $[A]$ is a prime SVN-ideal on $L$,*

*(ii) $A$ is a prime SVN-filter if and only if $[A]$ is a prime SVN-filter on $L$.*

*Proof.* (i) Suppose that $A$ is a prime SVN-ideal on a lattice $L$. It is easy to show that $[A] = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$ is a SVN-ideal on $L$. Next, we show that $[A]$ is prime. We have that

$$\begin{aligned}\mu_{[A]}(x \sqcap y) &= \mu_A(x \sqcap y)\\ &= \mu_A(x) \vee \mu_A(y)\\ &= \mu_{[A]}(x) \vee \mu_{[A]}(y)\end{aligned}$$

and

$$\begin{aligned}\sigma_{[A]}(x \sqcap y) &= \sigma_A(x \sqcap y)\\ &= \sigma_A(x) \vee \sigma_A(y)\\ &= \sigma_{[A]}(x) \vee \sigma_{[A]}(y).\end{aligned}$$

Also,

$$\begin{aligned}\nu_{[A]}(x \sqcap y) &= 1 - \mu_A(x \sqcap y)\\ &= 1 - (\mu_A(x) \vee \mu_A(y))\\ &= (1 - \mu_A(x)) \wedge (1 - \mu_A(y))\\ &= \nu_{[A]}(x) \wedge \nu_{[A]}(y).\end{aligned}$$

We conclude that $[A]$ is a prime SVN-ideal on $L$. Conversely, suppose that $[A]$ is a prime SVN-ideal. By using the same steps we get that $A$ is a prime SVN-ideal on $L$.

(ii) Follows from Proposition 2.3 and (i). □

**Proposition 4.5.** *Let $L$ be a lattice and $A \in SVNS(L)$. Then it holds that*

*(i) $A$ is a prime SVN-ideal if and only if $\langle A\rangle$ is a prime SVN-ideal on $L$,*

*(ii) $A$ is a prime SVN-filter if and only if $\langle A\rangle$ is a prime SVN-filter on $L$.*

*Proof.* The proof is similar as in Proposition 4.4. □

### 4.1.4. Interaction of SVN-ideals (resp. SVN-filters) with a lattices-homomorphism

*In this section, we study the interaction of (prime) single valued neutrosophic ideals (resp. filters) with a lattices-homomorphism. First, we introduce the notion of direct image (resp. inverse image) of a single valued neutrosophic set.*

**Definition 4.3.** *Let $f : L \to L'$ be a mapping from a lattice $L$ into another lattice $L'$ and $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an SVNS on $L$. The image of $A$ by $f$ is defined by $f(A) = \{\langle y, f(\mu_A)(y), f(\sigma_A)(y), f(\nu_A)(y)\rangle \mid y \in L'\}$, where*

$$f(\mu_A)(y) = \begin{cases} sup\{\mu_A(x) \mid x \in f^{-1}(y)\}, \ \ if \ f^{-1}(y) \neq \phi \\ 0 \ , \ \ if \ f^{-1}(y) = \phi \end{cases} ,$$

$$f(\sigma_A)(y) = \begin{cases} sup\{\sigma_A(x) \mid x \in f^{-1}(y)\}, \ \ if \ f^{-1}(y) \neq \phi \\ 0 \ , \ \ if \ f^{-1}(y) = \phi \end{cases}$$

*and*

$$f(\nu_A)(y) = \begin{cases} inf\{\nu_A(x) \mid x \in f^{-1}(y)\}, \ \ if \ f^{-1}(y) \neq \phi \\ 1 \ , \ \ if \ f^{-1}(y) = \phi. \end{cases}$$

*Similarly, if $A' = \{\langle y, \mu_{A'}(y), \sigma_{A'}(y), \nu_{A'}(y)\rangle \mid y \in L'\}$ is an SVNS on $L'$, then*

$$f^{-1}(A') = \{\langle x, f^{-1}(\mu_{A'})(x), f^{-1}(\sigma_{A'})(x), f^{-1}(\nu_{A'})(x)\rangle \mid x \in L\},$$

*where $f^{-1}(\mu_{A'})(x) = \mu_{A'}(f(x))$, $f^{-1}(\sigma_{A'})(x) = \sigma_{A'}(f(x))$ and $f^{-1}(\nu_{A'})(x) = \nu_{A'}(f(x))$.*

*In the following theorem, we will show that the image of a SVN-ideal (resp. SVN-filter) is a SVN-ideal (resp. SVN-filter).*

**Theorem 4.4.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A$ is an SVN-ideal on $L$, then $f(A)$ is an SVN-ideal on $L'$,*

*(ii) If $A$ is an SVN-filter on $L$, then $f(A)$ is an SVN-filter on $L'$.*

*Proof.* (i) Let $A$ be a SVN-ideal on $L$. For any $y, z \in L'$, it holds that

$$\begin{aligned} f(\mu_A)(y \sqcup z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \sup\{\mu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \wedge f(\mu_A)(z) \end{aligned}$$

and

$$\begin{aligned} f(\sigma_A)(y \sqcup z) &= \sup\{\sigma_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \sup\{\sigma_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\sigma_A(u) \wedge \sigma_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\sigma_A(u) \mid u \in f^{-1}(y)\} \wedge \sup\{\sigma_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\sigma_A)(y) \wedge f(\sigma_A)(z). \end{aligned}$$

Similarly, for any $y, z \in L'$, it holds that

$$\begin{aligned} f(\nu_A)(y \sqcup z) &= \inf\{\nu_A(x) \mid x \in f^{-1}(y \sqcup z)\} \\ &= \inf\{\nu_A(u \vee v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{\nu_A(u) \mid u \in f^{-1}(y)\} \vee \inf\{\nu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\nu_A)(y) \vee f(\nu_A)(z). \end{aligned}$$

Thus, we can conclude that $f(A)$ is an SVN-ideal on $L'$.

(ii) Follows from Proposition 2.3 and (i). □

*In the following theorem, we will show that the inverse image of a SVN-ideal (resp. SVN-filter) is a SVN-ideal (resp. SVN-filter).*

**Theorem 4.5.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A'$ is an SVN-ideal on $L'$, then $f^{-1}(A')$ is an SVN-ideal on $L$,*

*(ii) If $A'$ is an SVN-filter on $L'$, then $f^{-1}(A')$ is an SVN-filter on $L$.*

*Proof.* (i) Let $A'$ be an SVN-ideal on $L'$. For any $x, y \in L$, it holds that

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcup y) &= \mu_{A'}(f(x \sqcup y)) \\ &= \mu_{A'}(f(x)) \wedge \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \wedge f^{-1}(\mu_{A'})(y) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\sigma_{A'})(x \sqcup y) &= \sigma_{A'}(f(x \sqcup y)) \\ &= \sigma_{A'}(f(x)) \wedge \sigma_{A'}(f(y)) \\ &= f^{-1}(\sigma_{A'})(x) \wedge f^{-1}(\sigma_{A'})(y). \end{aligned}$$

Similarly, for any $x, y \in L$, it holds that

$$\begin{aligned} f^{-1}(\nu_{A'})(x \sqcup y) &= \nu_{A'}(f(x \sqcup y)) \\ &= \nu_{A'}(f(x)) \vee \nu_{A'}(f(y)) \\ &= f^{-1}(\nu_{A'})(x) \vee f^{-1}(\nu_{A'})(y). \end{aligned}$$

Therefore, $f^{-1}(A')$ is an SVN-ideal on $L$.

(ii) Follows from Proposition 2.3 and (i). □

*The next results show that the above Theorems 4.4 and 4.5 of the image and the inverse image of an SVN-ideal (resp. SVN-filter) are also relevant to a prime SVN-ideal (resp. prime SVN-filter).*

**Theorem 4.6.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A$ is a prime SVN-ideal on $L$, then $f(A)$ is a prime SVN-ideal on $L'$,*

*(ii) If $A$ is a prime SVN-filter on $L$, then $f(A)$ is a prime SVN-filter on $L'$.*

*Proof.* (i) Let $A$ be a prime SVN-ideal on $L$. Theorem 4.4 guarantees that $f(A)$ is an SVN-ideal on $L'$. Next, we show that $f(A)$ is prime. The fact that $A$ is a prime SVN-ideal implies that for any $y, z \in L$,

$$\begin{aligned} f(\mu_A)(y \sqcap z) &= \sup\{\mu_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \sup\{\mu_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\mu_A(u) \vee \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\mu_A(u) \mid u \in f^{-1}(y)\} \vee \sup\{\mu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\mu_A)(y) \vee f(\mu_A)(z) \end{aligned}$$

and

$$\begin{aligned} f(\sigma_A)(y \sqcap z) &= \sup\{\sigma_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \sup\{\sigma_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{(\sigma_A(u) \vee \sigma_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \sup\{\sigma_A(u) \mid u \in f^{-1}(y)\} \vee \sup\{\sigma_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\sigma_A)(y) \vee f(\sigma_A)(z). \end{aligned}$$

Similarly, for any $y, z \in L'$, it holds that

$$\begin{aligned} f(\nu_A)(y \sqcap z) &= \inf\{\nu_A(x) \mid x \in f^{-1}(y \sqcap z)\} \\ &= \inf\{\nu_A(u \wedge v) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{(\mu_A(u) \wedge \mu_A(v)) \mid u \in f^{-1}(y) \text{ and } v \in f^{-1}(z)\} \\ &= \inf\{\nu_A(u) \mid u \in f^{-1}(y)\} \wedge \inf\{\nu_A(v) \mid v \in f^{-1}(z)\} \\ &= f(\nu_A)(y) \wedge f(\nu_A)(z). \end{aligned}$$

We conclude that $f(A)$ is a prime SVN-ideal on $L'$.

(ii) Follows from Proposition 2.3 and (i). □

*In the following theorem, we will show that the inverse image of a prime SVN-ideal (resp. prime SVN-filter) is a prime SVN-ideal (resp. prime SVN-filter).*

**Theorem 4.7.** *Let $L$, $L'$ be two lattices and $f : L \to L'$ be a lattices-homomorphism. Then it holds that*

*(i) If $A'$ is a prime SVN-ideal on $L'$, then $f^{-1}(A')$ is a prime SVN-ideal on $L$,*

*(ii) If $A'$ is a prime SVN-filter on $L'$, then $f^{-1}(A')$ is a prime SVN-filter on $L$.*

*Proof.* (i) Let $A'$ be a prime SVN-ideal on $L'$. Theorem 4.5 guarantees that $f^{-1}(A')$ is an SVN-ideal on $L$. Next, we show that $f^{-1}(A')$ is prime. Since $A'$ is a prime SVN-ideal, it follows that for any $x, y \in L$,

$$\begin{aligned} f^{-1}(\mu_{A'})(x \sqcap y) &= \mu_{A'}(f(x \sqcap y)) \\ &= \mu_{A'}(f(x)) \vee \mu_{A'}(f(y)) \\ &= f^{-1}(\mu_{A'})(x) \vee f^{-1}(\mu_{A'})(y), \end{aligned}$$

$$\begin{aligned} f^{-1}(\sigma_{A'})(x \sqcap y) &= \sigma_{A'}(f(x \sqcap y)) \\ &= \sigma_{A'}(f(x)) \vee \sigma_{A'}(f(y)) \\ &= f^{-1}(\sigma_{A'})(x) \vee f^{-1}(\sigma_{A'})(y) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\nu_{A'})(x \sqcap y) &= \nu_{A'}(f(x \sqcap y)) \\ &= \nu_{A'}(f(x)) \wedge \nu_{A'}(f(y)) \\ &= f^{-1}(\nu_{A'})(x) \wedge f^{-1}(\nu_{A'})(y). \end{aligned}$$

Therefore, $f^{-1}(A')$ is a prime SVN-ideal on $L$. (ii) Follows from Proposition 2.3 and (i). □

## 4.2. Maximal SVN-ideals and filters on a lattice

*In this section, we introduce and study various properties and characterizations of maximal SVN-ideals (resp. SVN-filters) on a lattice.*

**Definition 4.4.** *Let $L$ be a lattice and let $I \in SVNS(L)$. We say that $I$ is a maximal SVN-ideal if there is no proper SVN-ideal on $L$ containing $I$.*

*The following proposition shows that the support of a maximal SVN-ideal (resp. filter) on a given lattice coincides with that lattice.*

**Proposition 4.6.** *Let $L$ be a lattice and $A \in SVNS(L)$. Then it holds that*

*(i) if $A$ is a maximal SVN-ideal, then its support $Supp(A)$ coincides with $L$,*

*(ii) if $A$ is a maximal SVN-filter, then its support $Supp(A)$ coincides with $L$.*

*Proof.* (i) Suppose that $A$ is a maximal SVN-ideal on a lattice $L$, then it holds from Proposition 3.1 that $Supp(A)$ is an ideal on $L$. Now, the facts that any ideal is a SVN-ideal on $L$ and $A \subseteq Supp(A)$ imply that $Supp(A) = L$.

(ii) Follows from Proposition 2.1 and (i).

□

*The following proposition shows that every proper SVN-ideal (resp. SVN-filter) is contained in a maximal SVN-ideal (resp. maximal SVN-filter). It is a natural generalization of the crisp case.*

**Proposition 4.7.** *Let $L$ be a lattice and $A \in SVNS(L)$. Then it holds that*

*(i) If $A$ is a proper SVN-ideal, then $A$ is contained in a maximal SVN-ideal,*

*(ii) If $A$ is a proper SVN-filter, then $A$ is contained in a maximal SVN-filter.*

*Proof.* The proof is straightforward and similar to that used for the crisp case. □

*In the following theorem, we show the relationship between maximal SVN-ideal (resp. SVN-filter) and prime SVN-ideal (resp. SVN-filter) on a given distributive lattice.*

**Theorem 4.8.** *Let $L$ be a distributive lattice and $A \in SVNS(L)$. Then it holds that*

*(i) If $A$ is a maximal SVN-ideal, then $A$ is a prime SVN-ideal.*

*(ii) If $A$ is a maximal SVN-filter, then $A$ is a prime SVN-filter.*

*Proof.* (i) Suppose that $A$ is a maximal SVN-ideal on a lattice $L$ and it isn't prime. Then it holds that there exist $a, b \in L$ such that $\mu_A(a \sqcap b) > \mu_A(a) \vee \mu_A(b)$, $\sigma_A(a \sqcap b) > \sigma_A(a) \vee \sigma_A(b)$ and $\nu_A(a \sqcap b) < \nu_A(a) \wedge \nu_A(b)$. We define an SVNS on $L$ as follows:

$$B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle \mid x \in L\},$$

where

$$\mu_B(x) = \bigvee_{\{u \in L \ \mid \ x \leqslant u \sqcup a\}} \mu_A(u)\,,$$

$$\sigma_B(x) = \bigvee_{\{u \in L \ \mid \ x \leqslant u \sqcup a\}} \sigma_A(u)$$

and

$$\nu_B(x) = \bigwedge_{\{u \in L \ \mid \ x \leqslant u \sqcup a\}} \nu_A(u)\,.$$

It is clear that $A \subset B$. Now, we show that $B$ is a SVN-ideal on $L$. Let $x, y \in L$, then $\mu_B(x \sqcup y) = \bigvee_{\{u \in L \ \mid \ x \sqcup y \leqslant u \sqcup a\}} \mu_A(u) = (\bigvee_{\{u \in L \ \mid \ x \leqslant u \sqcup a\}} \mu_A(u)) \wedge (\bigvee_{\{u \in L \ \mid \ y \leqslant u \sqcup a\}} \mu_A(u))$. Hence,

$$\mu_B(x \sqcup y) = \mu_B(x) \wedge \mu_B(y)\,.$$

The other conditions (ii)-(vi) of Definition 2.26 can be proved similarly. Thus, $A$ is a SVN-ideal on $L$. Hence, $B$ is a SVN-ideal on $L$. The fact that

$A \subset B$ implies that $A$ isn't a maximal SVN-ideal, a contradiction. Thus, $A$ is a prime SVN-ideal on $L$.

(ii) Follows from Proposition 2.1 and (i).

□

**Remark 4.1.** *The converse of the above implications is not necessarily hold. Indeed, let us consider the distributive lattice $L = \{0, a, b, c, 1\}$ represented by the following Hasse diagram*

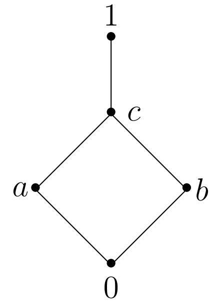


Figure 4.1: Hasse diagram of $(L, \leqslant, \sqcap, \sqcup)$ with $L = \{0, a, b, c, 1\}$.

*Let $I = \{< 0, 1, 0.6, 0 >, < a, 1, 0.6, 0 >, < b, 0.5, 0.3, 0.2 >, < c, 0.5, 0.3, 0.2 >, < 1, 0.5, 0.3, 0.2 >\}$ be a SVNS on $L$. One can easily verifies that $I$ is a prime SVN-ideal on $L$, but it isn't maximal.*

## 4.3. Principal SVN-ideals and filters on a lattice

*In this section, we introduce the notion of principal SVN-ideal (resp. SVN-filter) on a lattice and we characterize these notions in terms of a down-set and an up-set generated by single-valued neutrosophic singletons.*

### 4.3.1. Single-valued neutrosophic down-sets and up-sets

*This subsection is devoted to study the notion of single-valued neutrosophic down-set (resp. single-valued neutrosophic up-set) on a lattice analogously to the crisp down-set (resp. up-set), and then we show their interesting properties.*

#### Definitions

*Analogously to a crisp down-set and up-set on a lattice $L$, we introduce the notions of a single-valued neutrosophic down-set and a single-valued neutrosophic up-set.*

*Also, for a given SVNS $S$ on $L$, we define two associated SVNSs denoted by $\Downarrow S$ and $\Uparrow S$ analogous to those for crisp sets.*

**Definition 4.5.** *Let $L$ be a lattice and $S \in SVNS(L)$.*

*(i) $S$ is called a single-valued neutrosophic down-set (SVN-down-set, for short) if $\mu_S(x) \geq \mu_S(y)$, $\sigma_S(x) \geq \sigma_S(y)$ and $\nu_S(x) \leq \nu_S(y)$, for all $x \leqslant y$.*

*(ii) Dually, $S$ is called a single-valued neutrosophic up-set (SVN-up-set, for short) if $\mu_S(x) \leq \mu_S(y)$, $\sigma_S(x) \leq \sigma_S(y)$ and $\nu_S(x) \geq \nu_S(y)$, for all $x \leqslant y$.*

**Definition 4.6.** *Let $S \in SVNS(L)$ we denote by:*

*(i) $\Downarrow S$ the SVNS associated to $S$ defined as*

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y),$$

$$\sigma_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \sigma_S(y),$$

$$\nu_{\Downarrow S}(x) = \inf_{y \in \uparrow x} \nu_S(y).$$

*(ii) $\Uparrow S$ the SVNS associated to $S$ defined as*

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y),$$

$$\sigma_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \sigma_S(y),$$

$$\nu_{\Uparrow S}(x) = \inf_{y \in \downarrow x} \nu_S(y).$$

**Remark 4.2.** *For any crisp set $S$ on a given lattice $L$, it holds that*

*(i) $\Downarrow S = \downarrow S$;*

*(ii) $\Uparrow S = \uparrow S$.*

*For a given lattice $L$ and $S \in SVNS(L)$, it is clear that*

*(i) $\mu_{\Downarrow S}$ and $\sigma_{\Downarrow S}$ are an antitone mapping and $\nu_{\Downarrow S}$ is a monotone mapping;*

*(ii) $\mu_{\Uparrow S}$ and $\sigma_{\Uparrow S}$ are a monotone mapping and $\nu_{\Uparrow S}$ is an antitone mapping.*

## Properties of SVN-down sets and SVN-up sets

*In this subsection, we present some interesting properties of SVN-down-sets and up-sets on a lattice. We start with the easier one.*

**Proposition 4.8.** *Let $L$ be a lattice, $L^d$ be its dual-order lattice and $S \in SVNS(L)$. The following statements hold:*

*(i)* $S$ *is a SVN-down-set on* $L$ *if and only if* $S$ *is a SVN-up-set on* $L^d$,

*(ii)* $S$ *is a SVN-up-set on* $L$ *if and only if* $S$ *is a SVN-down-set on* $L^d$,

*(iii)* $\Downarrow S$ *on* $L$ *coincides with* $\Uparrow S$ *on* $L^d$,

*(iv)* $\Uparrow S$ *on* $L$ *coincides with* $\Downarrow S$ *on* $L^d$.

**Proposition 4.9.** *Let* $L$ *be a lattice and* $S \in SVNS(L)$. *It holds that*

*(i)* $\Downarrow S$ *is the smallest SVN-down-set containing* $S$*;*

*(ii)* $\Uparrow S$ *is the smallest SVN-up-set containing* $S$.

*Proof.* (i) At first we show $S \subseteq \Downarrow S$. Let $a \in L$ and $b \in \uparrow a$, then it holds that $\mu_S(a) \leq \sup\limits_{b\in\uparrow a}\mu_S(b) = \mu_{\Downarrow S}(a)$. Similarly, $\sigma_S(a) \leq \sup\limits_{b\in\uparrow a}\sigma_S(b) = \sigma_{\Downarrow S}(a)$ and $\nu_S(a) \geq \inf\limits_{b\in\uparrow a}\nu_S(b) = \nu_{\Downarrow S}(a)$. Hence, $S \subseteq \Downarrow S$. Now, we show that $\Downarrow S$ is a SVN-down-set. Let $x, y \in L$ such that $x \leqslant y$. We have $\mu_{\Downarrow S}(x) = \sup\limits_{t\in\uparrow x}\mu_S(t) \geq \sup\limits_{t\in\uparrow y}\mu_S(t)$. Hence, $\mu_{\Downarrow S}(x) \geq \mu_{\Downarrow S}(y)$. In a similar way, we obtain that $\sigma_{\Downarrow S}(x) \geq \sigma_{\Downarrow S}(y)$ and $\nu_{\Downarrow S}(x) \leq \mu_{\Downarrow S}(y)$. Thus, $\Downarrow S$ is a SVN-down-set. Furthermore, let $R$ be a SVN-down-set containing $S$. This implies that $\mu_S(x) \leq \mu_R(x)$, $\sigma_S(x) \leq \sigma_R(x)$ and $\nu_S(x) \geq \nu_R(x)$, for any $x \in L$. Hence, $\sup\limits_{t\in\uparrow x}\mu_S(t) \leq \sup\limits_{t\in\uparrow x}\mu_R(t)$ and $\sup\limits_{t\in\uparrow x}\sigma_S(t) \leq \sup\limits_{t\in\uparrow x}\sigma_R(t)$ and $\inf\limits_{t\in\uparrow x}\nu_S(t) \geq \inf\limits_{t\in\uparrow x}\nu_R(t)$. Thus, $\Downarrow S \subseteq \Downarrow R$. Finally, we conclude that $\Downarrow S$ is the smallest SVN-down-set containing $S$.

(ii) Follows from Proposition 4.8 and (i).

□

From Proposition 4.9, we obtain the following corollary. It shows a characterization of SVN-down-sets and SVN-up-sets.

**Corollary 4.1.** *Let* $L$ *be a lattice and* $S \in SVNS(L)$. *The following equivalences hold:*

*(i)* $S$ *is a SVN-down-set if and only if* $S = \Downarrow S$,

*(ii)* $S$ *is a SVN-up-set if and only if* $S = \Uparrow S$.

The following propositions list some properties of SVN-down and SVN-up sets.

**Proposition 4.10.** *Let* $L$ *be a lattice and* $R, S \in SVNS(L)$. *The following statements hold:*

*(i)* *If* $S \subseteq R$, *then* $\Downarrow S \subseteq \Downarrow R$,

*(ii)* $\Downarrow (\Downarrow S) = \Downarrow S$,

*(iii)* $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$,

*(iv)* $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

*Proof.* (i) Since $R \subseteq \Downarrow R$, it holds that $S \subseteq \Downarrow R$. From Proposition 4.9, it trivially holds that $\Downarrow S \subseteq \Downarrow R$.

(ii) Follows from Proposition 4.9 and Corollary 4.1.

(iii) On the one hand, we easily verify from (i) that $\Downarrow S \cup \Downarrow R \subseteq \Downarrow (S \cup R)$. On the other hand, since $\Downarrow S \cup \Downarrow R$ is a SVN-down-set and $S \cup R \subseteq \Downarrow S \cup \Downarrow R$, it follows from Proposition 4.9 that $\Downarrow (S \cup R) \subseteq \Downarrow S \cup \Downarrow R$. Thus, $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$.

(iv) Follows from Proposition 4.9 and (i).

□

**Remark 4.3.** *In the same direction, a dual version of Proposition 4.10 can also obtained for SVN-up-sets. Its proof follows from Propositions 4.8 and 4.10.*

In the following result, we show that any SVN-ideal (resp. SVN-filter) on a lattice $L$ is a SVN-down-set (resp. SVN-up-set) on $L$.

**Theorem 4.9.** *Let* $L$ *be a lattice and* $S \in SVNS(L)$. *The following implications hold:*

*(i)* *If* $S$ *is a SVN-ideal, then* $S$ *is a SVN-down-set.*

*(ii)* *If* $S$ *is a SVN-filter, then* $S$ *is a SVN-up-set.*

*Proof.* (i) Let $x, y \in L$ such that $x \leqslant y$. Since $S$ is a SVN-ideal, it follows that $\mu_S(x) = \mu_S(x \sqcap y) \geq \mu_S(x) \vee \mu_S(y)$. Hence, $\mu_S(x) \geq \mu_S(y)$. Similarly, we obtain that $\sigma_S(x) \geq \sigma_S(y)$ and $\nu_S(x) \leq \nu_S(y)$. Thus, $S$ is a SVN-down-set.

(ii) Follows from Proposition 2.1, (i) and Proposition 4.8.

□

Combining Theorem 4.9 and Corollary 4.1 leads to the following corollary.

**Corollary 4.2.** *Let* $L$ *be a lattice and* $S \in SVNS(L)$. *The following implications hold:*

*(i) If $S$ is a SVN-ideal, then $\Downarrow S = S$.*

*(ii) If $S$ is a SVN-filter, then $\Uparrow S = S$.*

**Remark 4.4.** *The converse of the implications in the above Theorem 4.9 and Corollary 4.2 does not necessarily hold. Indeed, consider $L$ the lattice given by the Hasse diagram in Figure 1 and $S \in SVNS(L)$ given by $S = \{< 0, 0.7, 0.6, 0.1 >, < a, 0.4, 0.3, 0.2 >, < b, 0.3, 0.2, 0.1 >, < c, 0.2, 0.1, 0.1 >, < 1, 0.1, 0, 0.3 >\}$. We easily verify that*

| $x$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| $\mu_{\Downarrow S}(x)$ | 0.7 | 0.4 | 0.3 | 0.2 | 0.1 |
| $\sigma_{\Downarrow S}(x)$ | 0.6 | 0.3 | 0.2 | 0.1 | 0 |
| $\nu_{\Downarrow S}(x)$ | 0.1 | 0.2 | 0.1 | 0.1 | 0.3 |

*Then $\Downarrow S = \{< 0, 0.7, 0.6, 0.1 >, < a, 0.4, 0.3, 0.2 >, < b, 0.3, 0.2, 0.1 >, < c, 0.2, 0.1, 0.1 >, < 1, 0.1, 0, 0.3 >\}$. Hence, $\Downarrow S = S$, i.e., $S$ is an SVN-down-set. But, $\mu_S(1) = \mu_S(a \sqcup b) \ngeq \min\{0.4, 0.3\}$, which implies that $S$ is not a SVN-ideal on $L$ .*

### 4.3.2. Principal SVN-ideals and filters on a lattice

In this section, we introduce the notion of principal SVN-ideal (resp. SVN-filter) on a lattice. First, we generalize the notion of crisp singleton to the single-valued neutrosophic setting.

**Definition 4.7.** *Let $L$ be a lattice. For any $x \in L$, a single-valued neutrosophic singleton (SVN- singleton, for short) $\tilde{x}$ is a SVNS on $L$ given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t), \sigma_{\tilde{x}}(t), \nu_{\tilde{x}}(t)\rangle \mid t \in L\}$, where*

$$\mu_{\tilde{x}}(t) = \begin{cases} 1 \ , \ if \ x = t \\ f(t) \ , \ \ otherwise \end{cases} ,$$

$$\sigma_{\tilde{x}}(t) = \begin{cases} 1 \ , \ if \ x = t \\ g(t) \ , \ \ otherwise \end{cases}$$

*and*

$$\nu_{\tilde{x}}(t) = \begin{cases} 0 \ , \ if \ x = t \\ h(t) \ , \ \ otherwise \end{cases} ,$$

*such that $f$ and $g$ are mappings from $L$ into $[0,1[$, and $h$ is a mapping from $L$ into $]0,1]$.*

**Definition 4.8.** *Let $L$ be a lattice, Then*

*(i) the principal SVN-ideal generated by a SVN-singleton $\tilde{x}$ is the smallest SVN-ideal contains $\tilde{x}$,*

*(ii) the principal SVN-filter generated by a SVN-singleton $\tilde{x}$ is the smallest SVN-filter contains $\tilde{x}$.*

The following theorem shows that the SVN-down-set (resp. the SVN-up-set) generated by a SVN-singleton on a lattice $L$ is a SVN-ideal (resp. is a SVN-filter) on $L$.

**Theorem 4.10.** *Let $L$ be a lattice and $x$ be an element on $L$. Then it holds that*

*(i) $\Downarrow \tilde{x}$ is a SVN-ideal on $L$,*

*(ii) $\Uparrow \tilde{x}$ is a SVN-filter on $L$.*

*Proof.* (i) Let $x, y \in L$, we will show that $\mu_{\Downarrow\tilde{x}}(x \sqcup y) = \mu_{\Downarrow\tilde{x}}(x) \wedge \mu_{\Downarrow\tilde{x}}(y)$, $\sigma_{\Downarrow\tilde{x}}(x \sqcup y) = \sigma_{\Downarrow\tilde{x}}(x) \wedge \sigma_{\Downarrow\tilde{x}}(y)$ and $\nu_{\Downarrow\tilde{x}}(x \sqcup y) = \nu_{\Downarrow\tilde{x}}(x) \vee \nu_{\Downarrow\tilde{x}}(y)$. Let $a, b \in L$. On the one hand, by Proposition 4.9, $\Downarrow \tilde{x}$ is a SVN-down-set, which implies that $\mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$ and $\mu_{\Downarrow\tilde{x}}(b) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Hence, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(a) \geq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. On the other hand, since $\mu_{\tilde{x}}$ is a monotone mapping, it holds that $\mu_{\tilde{x}}(a) \leq \mu_{\tilde{x}}(a \sqcup b)$ and $\mu_{\tilde{x}}(b) \leq \mu_{\tilde{x}}(a \sqcup b)$. This implies that $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$ and $\sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Hence, $\sup\limits_{a \leqslant t} \mu_{\tilde{x}}(t) \wedge \sup\limits_{b \leqslant t} \mu_{\tilde{x}}(t) \leq \sup\limits_{a \sqcup b \leqslant t} \mu_{\tilde{x}}(t)$. Thus, $\mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b) \leq \mu_{\Downarrow\tilde{x}}(a \sqcup b)$. Therefore, $\mu_{\Downarrow\tilde{x}}(a \sqcup b) = \mu_{\Downarrow\tilde{x}}(a) \wedge \mu_{\Downarrow\tilde{x}}(b)$, for all $a, b \in L$. In analogous way, we easily prove that $\sigma_{\Downarrow\tilde{x}}(a \sqcup b) = \sigma_{\Downarrow\tilde{x}}(a) \wedge \sigma_{\Downarrow\tilde{x}}(b)$ and $\nu_{\Downarrow\tilde{x}}(a \sqcup b) = \nu_{\Downarrow\tilde{x}}(a) \vee \nu_{\Downarrow\tilde{x}}(b)$. Finally, we conclude that $\Downarrow \tilde{x}$ is a SVN-ideal.

(ii) Follows dually by using Proposition 4.8, (i) and Proposition 2.1.

□

In the following result, we show a characterization of a principal SVN-ideal (resp. SVN-filter) in terms of a down-set (resp. up-set) generated by an SVN-singleton.

**Theorem 4.11.** *Let $L$ be a lattice and $I$ (resp. $F$) be a SVN-ideal (resp. SVN-filter) on $L$. Then it holds that* *(i)* $I$ *is a principal SVN-ideal on* $L$ *if and only if there exists* $x \in L$ *such that* $I = \Downarrow \tilde{x}$,

*(ii)* $F$ *is a principal SVN-filter on* $L$ *if and only if there exists* $x \in L$ *such that* $F = \Uparrow \tilde{x}$.

*Proof.* We only prove (i), as (ii) can be proved analogously by using Proposition 4.8 and Proposition 2.1. Suppose that $I$ is a principal SVN-ideal on $L$. Then there exists a SVN-singleton $\tilde{x}$ such that $I$ is the smallest SVN-ideal contains $\tilde{x}$. Since $\tilde{x} \subseteq I$, it follows from Proposition 4.10 that $\Downarrow \tilde{x} \subseteq \Downarrow I = I$. On the other hand, Theorem 4.10 guarantees that $\Downarrow \tilde{x}$ is an ideal. Then the fact that $I$ is the smallest ideal contains $\tilde{x}$ implies that $I \subseteq \Downarrow \tilde{x}$. Thus, $I = \Downarrow \tilde{x}$. Conversely, $I = \Downarrow \tilde{x}$ is a SVN-ideal contains $\tilde{x}$. Now, suppose that $J$ is an other SVN-ideal contains $\tilde{x}$. From Proposition 4.10, it holds that $\Downarrow \tilde{x} \subseteq \Downarrow J = J$. Hence, $I = \Downarrow \tilde{x}$ is the smallest SVN-ideal contains $\tilde{x}$. Thus, $I$ is a principal SVN-ideal. □

# General conclusions and future research

In this thesis, we have characterized the notions of SVN-ideal (resp. SVN-filter) on a lattice in terms of the lattice meet and join operations. As interesting kinds, we have introduced the notions of prime SVN-ideal and SVN-filter and investigated their various characterizations and properties.

In this work, we have introduced the notion of prime, maximal and principal single-valued neutrosophic ideals (resp. filters) on a lattice. Moreover, we have investigated their various properties and characterizations.

Future work is anticipated in multiple directions. We think it makes sense to study the notions of SVN-ideal and SVN-filter for other types of lattices with respect to a SVN-order relation. Moreover, this work will be focused on the characterizations of these types on a single-valued neutrosophic ordered lattice.

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## ملخّـص

في هذه الأطروحة ، قمنا بتمييز مفاهيم كل من المثالي النتروسوفيكي أحادي القيمة والمرشح النتروسوفيكي أحادي القيمة على شبكة من حيث مجموعات المستوى الخاصة بهم ومن حيث عمليات الشبكة. كأنواع مثيرة للاهتمام، قدمنا مفهوم الأولي، الأعظمي والأساسي لكل من المثالي النتروسوفيكي أحادي القيمة والمرشح النتروسوفيكي أحادي القيمة في شبكة. وفي الأخير ، قمنا بإثبات مختلف خصائص هذه الأنواع ومميزاتها.

الكلمات المفتاحية: مجموعة نتروسوفيكية أحادية القيمة، مثالي، مرشح، أولي، أعظمي، أساسي.

## Abstract

In this thesis, we have characterized the notions of single-valued neutrosophic ideal and single-valued neutrosophic filter on a lattice in terms of their level sets and in terms of the lattice meet and join operations. As interesting kinds, we have introduced the notion of prime, maximal and principal single-valued neutrosophic ideals and single-valued neutrosophic filters on a lattice. Moreover, we have investigated their various properties and characterizations.

Key words: Single-valued neutrosophic set, ideal, filter, prime, maximal, principal.

## Résumé

Dans cette thèse, nous avons caractérisé les notions d'idéal neutrosophique à valeur unique et de filtre neutrosophique à valeur unique sur un treillis en termes de leurs ensembles de niveaux et en termes d'opérations de treillis. En tant que types intéressants, nous avons introduit la notion première, maximale et principale d'idéaux neutrosophiques à valeur unique et de filtres neutrosophiques à valeur unique sur un treillis. De plus, nous avons étudié leurs différentes propriétés et caractérisations.

Mot clés: Ensemble neutrosophique à valeur unique, ideal, filtre, premier, maximal, principal.