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By
Sarra Boudaoud

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Nouredine Benhamidouche	**Prof.**	**University of M'sila**	**President**
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Sarra Boudaoud

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Introduction

One of the most important properties of binary relations is the transitivity property. It is an indispensable requirement for many interesting fields of pure and applied mathematics. In lattice theory, the transitivity is necessary for the associativity of operations of least upper and greatest lower bounds. Nevertheless, the non transitivity relation is also important and ubiquitous in our real life as well as in mathematics. May be, the most common and illustrative example of a non transitive relation, in our real life, is the acquaintance relation: If A knows B, and B knows C, it is not necessary that persons A and C are acquainted. The preference loop or cycle (A is preferred to B, B is preferred to C and C is preferred to A) is non-transitive relation. For instance, in the game theory, the game of rock, paper, scissors provides an immediate example (e.g.; the relation over rock, paper and scissors is "defeats", and the standard rules of the game are such that rock defeats scissors, scissors defeats paper, and paper defeats rock), the non-transitive relation also appear in the football tournament (e.g.; team A beats team B and team B beats team C, but it is not necessarily that team A will beat team C). Furthermore, the non-transitive relations appear in the different fields of pure and applied mathematics. For instance, the relations defined by non-associative binary operations are not transitive. In topology, the closeness relation is an another example (e.g.; in the power set of $\mathbb{R}^2$ the closeness relation defined by: A is close to B if and only if $d(A, B) = 0$ is not transitive relation). In geometry, (e.g.; let $\triangle$ be the set of straight lines in the plane $\mathbb{R}^2$ and we define the binary relation R on $\triangle$ as follows: $d_1 R d_2$ if and only if d_1 and d_2 are orthogonal, for any $d_1, d_2 \in \triangle$. The relation R is not transitive). The theory of graphs is also concerned with non-transitive relations (e.g.; a vertex A is adjacent to B, and vertex B is adjacent to C, but A is not necessarily adjacent to C).

One of the interesting mathematical structure that consider the non-transitive relations is the non-associative lattice. To facilate the meaning, it is a structure like lattice $(L, \leq, \wedge, \vee)$, but without the property of transitivity of $\leq$, which also means, without the associativity of the lattice meet $(\wedge)$ and join $(\vee)$ operations. Fried [34] was among the first authors who discussed the notion of non-associative lattice under the name tournament-lattice (T-lattice), and he investigated its various properties and characterizations. In the same context, Skala [66, 67] discussed the same notion, but she called it trellis. She defined the notion of pseudo order as a reflexive and antisymmetric but not necessarily transitive relation. By stipulation of least upper bounds and greatest lower bounds of finite subsets on a given pseudo ordered set, the result is a structure called trellis (weakly associative lattice as it is called by several authors). In the same purpose, she extended to this structure of trellis several known notions and results on lattices. Since then a lot of studies used these notions of pseudo ordered sets and trellises. For instance, Freid and Grätzer [35] studied the non-associative extension of the class of distributive lattices. Later in [36], Fried and S*óś* studied the connection between trellis and projective planes. In [37], Gladstien characterized the complete trellises of finite length. Bhatta and Shashirekha [14] generalized this characterization in terms of joins of cycles. Chajda [27] studied the notion of ideal on pseudo ordered sets and trellises, and their connections with congruence relations. Rachunek [61, 62], generalized the notion of lattice ordered groups to trellis setting. Some fixed point theorems on lattices was extended to the case of pseudo ordered sets and trellises by Bhatta and George [15], Stouti and Zedam [69]. In fuzzy setting, this notion of trellis was fuzzyfied by Veclar and Patikova [77] under the name L-fuzzy complete propelattice. The Tarski's fixed point theorem for L-fuzzy complete lattice was extended to this non-transitive fuzzy ordered structure.

One of the essential tool in various branches of classical and fuzzy mathematics and its applications is the concept of ideal and its dual (a filter) on ordered structures. For instance, ideals and filters appear in topology to express completeness and compactness in metric spaces [20, 80]. These notions are mainly used to translate connections between properties on algebraic structures and to define congruence relations and quotient algebras [58, 76]. They played a central role in Stone representation theorem for boolean lattice [68] and in the extensive theory of representation of a distributive lattice [31, 40, 64]. In fuzzy setting, for the same purposes, several authors introduced and investigated the concepts of some kinds of fuzzy ideals and fuzzy filters in different ways and on different structures. The first approach considered fuzzy ideal and fuzzy filter as fuzzy sets on crisp structures, as on lattices, residuated lattices [1, 18, 32, 43, 45, 60, 74], BL-algebras [46] and ordered ternary semigroups [3, 9, 28]. The second approach proposed similar notions on fuzzy structures, (see e.g., Mezzomo et al. [49, 50, 51]) for the approach in fuzzy ordered lattices.

Due the huge interest of fuzzy (crisp) ideals and filters in the different algebraic structures, several authors extended and studied these concepts in intuitionistic fuzzy setting. In particular, Kim and Jun [42] introduced the notion of intuitionistic fuzzy interior ideals of semigroups. Banerjee and Basnet in [8] studied the notion of intuitionistic fuzzy ideals on a ring. Akram and Dudek [2] introduced and studied a concept of intuitionistic fuzzy Lie ideals. Qin and Liu [44, 59] introduced and investigated the properties of intuitionistic fuzzy filters on a residuated lattice. Recently, Thomas and Nair [72, 73] considered intuitionistic fuzzy sublattices, intuitionistic fuzzy ideals and intuitionistic fuzzy filters on a lattice.

The notions of ideals and filters are also considered in the pseudo ordered set and trellis structures. Skala [66, 67] introduced the notion of ideal on a trellis as a natural generalization of that ideal on a lattice. Chajda [27], Bhatta and Ramananda [13] studied the notion of ideal on trellis and their connections with congruence relations. $\hat{S}$alounov$\grave{a}$ [63] described and studied the notion of Wal-ideals on weakly associative lattice rings.

Inspired by the above developments of the notions of pseudo ordered sets, trellises and the related notions of ideals and filters, this thesis aims to study these notions in depth in fuzzy setting. First, we study the notion of fuzzy pseudo ordered sets and some of its interesting properties. Also, we introduce the concept of fuzzy trellis as a natural generalization of that of trellis introduced by Skala [66, 67] with respect to the fuzzy pseudo order relation, and we discuss its fundamental properties. Secondly, we study several kinds of fuzzy ideals (resp. filters). We present interesting characterizations of these notions in terms of (crisp) fuzzy trellis operations, and in terms of some of their specific subsets. We pay particular attention to the kind of principal fuzzy ideals (resp. filters) on a given trellis, which is more complicated than the same notions on a lattice. The second approach considered the same characterizations of fuzzy ideals and filters on a lattice in the intuitionistic fuzzy setting. Also, we generalize the notions of fuzzy principal ideal (resp. principal filter) in the same setting.

This thesis is structured as follows.

- In Chapter 1, we provide generalities on pseudo ordered sets, trellises and we recall many properties of these concepts that we need throughout this thesis.

- In Chapter 2, we study the notion of fuzzy pseudo ordered sets and some of its interesting properties. In particular, we show that any fuzzy pseudo ordered set is isomorphic with a contraction set of fuzzy ordered set. Furthermore, we introduce the concept of fuzzy trellis as a natural generalization of the trellis introduced by Skala [66, 67] and we discuss its properties.
- In Chapter 3, we introduce the notions of fuzzy ideal and filter as fuzzy sets on (crisp) fuzzy trellis and we discuss their fundamental properties. Also, we present interesting characterizations of these notions in terms of trellis operations and in terms of their specific subsets.
- In Chapter 4, we generalize the notion of principal ideal (resp. filter) on a lattice to the intuitionistic fuzzy setting, and investigate their various characterizations and properties.
- Finally, general conclusions and future research lines are given.

1 Generalities on pseudo ordered sets, trellises and fuzzy relations

The purpose of this first chapter is to provide some preliminaries on partially ordered sets and lattices. Also, we recall some basic notions of pseudo ordered sets, trellises and fuzzy relations. Many of the properties of these concepts will be used in the next chapters.

1.1. Posets and lattices

This section contains the basic definitions and properties of binary relations, partially ordered sets (posets, for short) and lattices.

1.1.1. Posets

Definition 1.1. [31] *A binary relation on a set X is a subset of the Cartesian product $X \times X$, it is a set of couples $(x, y) \in X^2$. For a relation $R \subseteq X^2$, if x is related to y by R we often write xRy instead of $(x, y) \in R$. Two elements x and y of a set X equipped with a relation R are called comparable elements if it holds that xRy or yRx. Otherwise, they are called incomparable elements, denoted by $x \parallel y$.*

By definition the relations R and S are sets of couples (x, y) on X^2. Then, we can apply the classical operations of set theory to them, such as:

• **The union** of R and S is the relation $R \cup S$ on X and defined as follows:

$$R \cup S = \{(x, y) \in X^2 \mid xRy \vee xSy\}.$$

• **The intersection** of R and S is the relation $R \cap S$ on X and defined as follows:

$$R \cap S = \{(x, y) \in X^2 \mid xRy \wedge xSy\}.$$

• **The complement** of R is the relation denoted by R^c on X and defined as follows:

$$R^c = \{(x, y) \in X^2 \mid (x, y) \notin R\}.$$

• **The transpose** of R is the relation denoted by R^t on X and defined as follows:

$$R^t = \{(y, x) \in X^2 \mid xRy\}.$$

• **The dual** of R is the relation denoted by R^d on X and defined as follows:

$$R^d = \{(x, y) \in X^2 \mid yR^c x\}.$$

• **The composition** of R and S is the relation denoted by $R \circ S$ on X and defined as follows:

$$R \circ S = \{(x, z) \in X^2 \mid (\exists y \in X)(xRy \wedge ySz)\}.$$

A binary relation R on a set X is called:

(i) **reflexive**, if xRx, for all $x \in X$;

(ii) **irreflexive**, if $xR^c x$, for all $x \in X$;

(iii) **symmetric**, if xRy implies yRx, for all $x, y \in X$;

(iv) **antisymmetric**, if xRy and yRx implies $x = y$, for all $x, y \in X$;

(v) **asymmetric**, if xRy implies $yR^c x$, for all $x, y \in X$;

(vi) **transitive**, if xRy and yRz implies xRz, for all $x, y, z \in X$;

(vi) **antitransitive**, if xRy and yRz implies $xR^c z$, for all $x, y, z \in X$;

(vii) **complete**, if xRy or yRx, for all $x, y \in X$.

A binary relation R on a set X is called:

(i) a pseudo-order relation if it is reflexive and antisymmetric;

(ii) an order relation if it is reflexive, antisymmetric and transitive;

(iii) a strict order if it is irreflexive and transitive;

(iv) a total order relation if it is reflexive, antisymmetric, transitive and complete;

(v) an equivalence relation if it is reflexive, symmetric and transitive.

For each equivalence relation R on a set X, the equivalence class defined with $x \in X$, denoted by $[x]_R$, is the set $\{y \in X \mid xRy\}$. It is always true that $x \in [x]_R$. It is easy to show that for any $x, y \in X$, either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$. An equivalence relation therefore partitions X into equivalence classes. The set of all these equivalence classes is called the quotient of X by R and is denoted X/R.

For more details on binary relations, we refer to [16, 31, 64, 40].

According to Davey and Priestley [31], we have the following definitions:

Definition 1.2. *Let X be a non-empty set. A partial order on X is a binary relation $\leq$ on X such that, for all $x, y, z \in X$,*

(i) $x \leq x$ (reflexivity);

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

A set X equipped with an order relation $\leq$ is said to be a partially ordered set (or poset, for short) and we shall write $(X, \leq)$. On any set, $=$ is an order. A relation $\leq$ on a set X which is reflexive and transitive but not necessarily antisymmetric is called a quasi-order or, by some authors, a preorder. Given any ordered set $(X, \leq)$, the dual of $(X, \leq)$ is denoted by $(X^d, \geq)$ and defined as follows: $x \geq y$ in X^d if and only if $x \leq y$ in X. A poset $(X, \leq)$ is called a chain, if for any $x, y \in X$, either $x \leq y$ or $y \leq x$, i.e., any two elements of X are comparable. Alternative names for a chain are linearly ordered set or totally ordered set. At the opposite extreme from a chain is an antichain.

The ordered set $(X, \leq)$ is an antichain if $x \leq y$ in X only if $x = y$.

If $\leq$ is an order, then the corresponding strict order $<$ is the irreflexive kernel given by:

$$a < b \text{ if } a \leq b \text{ and } a \neq b.$$

Conversely, if $<$ is a strict order, then the corresponding order $\leq$ is the reflexive closure given by:

$$a \leq b \text{ if } a < b \text{ or } a = b.$$

Next we introduce some important special elements

Definition 1.3. [31] *Let $(X, \leq)$ be an ordered set. We say X has a least element if, there exists $0 \in X$ such that $0 \leq x$, for any $x \in X$. Dually, X has a greatest element if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$. Note that if $0, 1$ exist then they are unique.*

Definition 1.4. *Let $(X, \leq)$ be a poset and S be a subset of X, an element $x \in X$ is an upper (lower) bound of S if $y \leq x$ $(y \geq x)$ for any $y \in S$. The set of all upper (lower) bounds of S is denoted by S^u (S^l).*

$$S^u = \{x \in X : y \leq x \textit{ for any } y \in S\};$$

$$S^l = \{x \in X : y \geq x \textit{ for any } y \in S\}.$$

If $S^u(S^l)$ has a least (greatest) element x, then x is called the least upper bound or supremum (greatest lower bound or infimum) of S, denoted by $\sup S$ $(\inf S)$. The supremum (infimum) is unique in S^u (S^l) if it exists.

We use the following notation: $x \vee y$ in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ in place of $\inf\{x, y\}$ when it exists. Similarly, $\bigvee S$ and $\bigwedge S$ instead of $\sup S$ and $\inf S$ when these exist.

One of the most useful and attractive features of ordered sets is that, in the finite case at least, they can be 'drawn'. To describe how to represent ordered sets diagrammatically, we need the idea of covering.

Definition 1.5. *Let X be an ordered set and let $x, y \in X$. We say x is covered by y (or y covers x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there is no element z of X with $x < z < y$.*

Since an order is an example of a relation, we can draw it as a directed graph. But there is a more concise and attractive way to draw partial orders and linear orders, in which the reflexivity and transitivity of the order are implicit. A **Hasse diagram** for a partial order $\leq$ on a set X is a graph drawn in the plane, with vertices corresponding to the elements of X and edge going up from x to y if $x < y$ (so excluding $x = y$) and there is no element z with $x < z < y$. In general a poset can be conveniently represented by Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y. For X finite, we obtain a diagram for X^d simply by " turning upside down " a diagram for X.

Associated with any ordered set by two important families of sets which are down-set and up-set, they play a central role in the representation theory developed in later chapters.

Definition 1.6. [31] *Let X be an ordered set and S be a subset on X. S is called a down-set (alternative terms include lower-set) if $y \in S$ implies $x \in S$ for all $x \leq y$. Dually, S is called an up-set (alternative terms include upper-set) if $y \in S$ implies $x \in S$ for all $y \leq x$. For a given subset S on X, we denote by $\downarrow S$ the set of all elements smaller than or equal to some element of S, i.e.,*

$$\downarrow S = \{x \in X \mid x \leq y, \text{ for some } y \in S\},$$

and $\uparrow S$ the set of all elements bigger than or equal to some element of S, i.e.,

$$\uparrow S = \{x \in X \mid y \leq x, \text{ for some } y \in S\}.$$

It is easily checked that $\downarrow S$ (resp. $\uparrow S$) is the smallest down-set (resp. the smallest up-set) containing S. $\downarrow S$ (resp. $\uparrow S$) is called the down-set (resp. the up-set) of S. Similarly, for a given element x on a lattice X, the down-set $\downarrow \{x\}$ ($\downarrow x$, for short) and the up-set $\uparrow \{x\}$ ($\uparrow x$, for short) are defined as

$$\downarrow x = \{y \in X \mid y \leq x\} \text{ (resp. } \uparrow x = \{y \in X \mid x \leq y\}).$$

Note that if S is a down-set (resp. an up-set), then $\downarrow S$ (resp. $\uparrow S$) coincides with S.

Lemma 1.1. [31] *Let X be an ordered set and $x, y \in X$. Then the following statements are equivalent:*

(i) $x \leq y$;

(ii) $\downarrow x \subseteq \downarrow y$;

(iii) For any $S \subseteq X$, if $y \in \downarrow S$ implies $x \in \downarrow S$.

We have already used a special type of mappings between sets, namely order isomorphisms.

Definition 1.7. *Let X and Y be two ordered sets. A mapping $f : X \to Y$ is said to be*

(i) order-preserving (or monotonic) if $x \leq y \in X$ implies $f(x) \leq f(y) \in$

Y;

(ii) order-embedding (and we write $f : X \to Y$) if $x \leq y \in X$ if and only if $f(x) \leq f(y) \in Y$;

(iii) order-isomorphism if it is an order-embedding which maps X onto Y.

1.1.2. Lattices

Many important properties of an ordered set $(L, \leq)$ are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of L. Lattices and complete lattices are two of the most important classes of ordered sets. We shall be particularly interested in ordered sets in which $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$. Similarly, we write $\bigvee S$ (the join of S) and $\bigwedge S$ (the meet of S) instead of $\sup S$ and $\inf S$ when they exist.

Definition 1.8. *Let $(L, \leq)$ be a non-empty ordered set.*

(i) $(L, \leq)$ is called a $\vee$-semi-lattice if $x \vee y$ exists for all $x, y \in L$.

(ii) $(L, \leq)$ is called a $\wedge$-semi-lattice if $x \wedge y$ exists for all $x, y \in L$.

(iii) $(L, \leq)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice.

(iv) $(L, \leq)$ is called a complete lattice if $\bigvee S$ and $\bigwedge S$ exist for any $S \subseteq L$.

For a given lattice $(L, \leq, \wedge, \vee)$, the operations $\wedge$ and $\vee$ defined for any $x, y \in L$ by:

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\},$$

satisfy the following algebraic properties:

(i) $x \wedge x = x \vee x = x$ (idempotency);

(ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutativity);

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity);

(iv) $x \wedge (y \vee x) = x = x \vee (y \wedge x)$ (absorption-laws);

(v) the order relation $\leq$ and $\wedge$, $\vee$ are connected as:

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y, \text{ for any } x, y \in L.$$

A bounded lattice is a lattice that additionally has a greatest element 1 *and a smallest element* 0*, which satisfy* $0 \leq x \leq 1$*, for all* $x \in L$*. A finite lattice is automatically bounded with* $1 = \bigvee L$ *and* $0 = \bigwedge L$*. A lattice* $(L, \leq, \wedge, \vee)$ *is distributive if, for all* $x, y, z \in L$*, the following additional conditions hold*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice $(L, \leq, \wedge, \vee)$ *is modular if the following condition holds*

$$x \leq z, \text{ implies that } x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ for all } x, y, z \in L.$$

This condition is also equivalent to its dual

$$z \leq x, \text{ implies that } x \wedge (y \vee z) = (x \wedge y) \vee z, \text{ for all } x, y, z \in L.$$

A complemented lattice is a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$*, in which each element* x *has a complement, i.e., there exists an element* $y \in L$ *such that*

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

An element may have more than one complement in general. However, if $(L, \leq, \wedge, \vee, 0, 1)$ *is distributive then every element will have at most one complement.*

Definition 1.9. [31] *A non-empty subset* M *of a lattice* L *is called sub-lattice of* L *if, for all* $x, y \in M$*, it holds that* $x \wedge y \in M$ *and* $x \vee y \in M$*. A sub-lattice* M *of a lattice* L *is called a convex sub-lattice of* L*, if* $x \leq z \leq y$ *and* $x, y \in M$ *implies that* $z \in M$*, for all* $x, y, z \in L$*.*

Ideals and filters on a crisp lattice

Ideals are of fundamental importance in algebra. Filters, the order dual of lattice ideals, have a variety of applications in logic and topology.

Definition 1.10. [31] *A non-empty subset* I *on a lattice* L *is called an ideal if, for all* $x, y \in L$*, the following conditions are satisfied:*

1. *if* $y \in I$ *and* $x \leq y$*, then* $x \in I$*,*
2. *if* $x, y \in I$ *implies* $x \vee y \in I$*.*

The definition can be more compactly stated by declaring an ideal to be a non-empty down-set closed under join.

A dual ideal is called a filter determined by the following definition.
Definition 1.11. [31] *A non-empty subset F on a lattice L is called a filter if, for all $x, y \in L$, the following conditions are satisfied:*

1. *if $y \in F$ and $y \leq x$, then $x \in F$,*

2. *if $x, y \in F$ implies $x \wedge y \in F$.*

The set of all ideals (resp. filters) of L is denoted by $\mathcal{I}(L)$ (resp. $\mathcal{F}(L)$), and carries the usual inclusion order.

An ideal or filter is called proper if it does not coincide with L. More precisely, an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$. For each $x \in L$, the set $\downarrow x$ is an ideal (also known as the principal ideal generated by x). Dually, $\uparrow x$ is a principal filter.
Definition 1.12. [31] *An ideal I on a lattice L is called a prime ideal if, for all $x, y \in L$, $x \wedge y \in I$, then $x \in I$ or $y \in I$.*
Definition 1.13. [31] *A filter F on a lattice L is called a prime filter if, for all $x, y \in L$, $x \vee y \in F$, then $x \in F$ or $y \in F$.*
Definition 1.14. [31] *Let L be a lattice. A proper ideal (resp. filter) A is said to be a maximal ideal (resp. maximal filter or more usually known as an ultrafilter) if the only ideal (resp. filter) properly containing A is L.*
Example 1.1. *(i) The following are ideals in $\mathcal{P}(X)$*

  (a) all subsets not containing a fixed element of X,

  (b) all finite subsets (this ideal is non-principal if X is infinite).

(ii) Let $(X, \mathfrak{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X \mid (\exists U \in \mathfrak{T}) x \in U \subseteq V\}$ is a filter in $(X, \mathfrak{T})$.

(iii) Let $X = (D(60), |)$ the lattice (i.e., $D(60)$ represents the divisors of the number 60) given in Figure 1.1. Note that any ideal and filter on X is principal. In the following figure we present the principal ideal $I_6 = \{1, 2, 3, 6\}$ and the principal filter $F_6 = \{6, 12, 30, 60\}$ by dash lines.

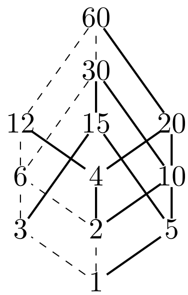

Figure 1.1: Depicted of the lattice $(D(60), |)$

1.1.3. T-norms and T-conorms

*The history of triangular-norms (*t-norms*) started with Menger [48]. His main idea was to construct metric spaces where probability distributions are used to describe the distance between two elements. Schweizer and Sklar [65] provided the axioms of* t-norms*, as they are used today.*

Definition 1.15. [56]

(i) A t-norm T on $[0,1]$ is a function $T : [0,1]^2 \to [0,1]$ satisfing the following axioms:

(T1) *Commutativity:* $(\forall x, y \in [0,1])(T(x,y) = T(y,x))$;

(T2) *Associativity:* $(\forall x, y, z \in [0,1])(T(x, T(y,z)) = T(T(x,y), z))$;

(T3) *Monotonicity:* $(\forall x, y, z \in [0,1])(x \leqslant y \Rightarrow T(x,z) \leqslant T(y,z))$;

(T4) *Boundary condition:* $(\forall x \in [0,1])(T(x,1) = x), (T(x,0) = 0)$.

(ii) A t-conorm S on $[0,1]$ is a function $S : [0,1]^2 \to [0,1]$ satisfing the following axioms:

(S1) *Commutativity:* $(\forall x, y \in [0,1])(S(x,y) = S(y,x))$;

(S2) *Associativity:* $(\forall x, y, z \in [0,1])(S(x, S(y,z)) = S(S(x,y), z)))$;

(S3) *Monotonicity:* $(\forall x, y, z \in [0,1])(x \leqslant y \Rightarrow S(x,z) \leqslant S(y,z))$;

(S4) *Boundary condition:* $(\forall x \in [0,1])(S(x,1) = 1), (S(x,0) = x)$.

Remark 1.1. *Given a t-norm T, we find the associated dual t-conorm S by $S(x,y) = 1 - T(1-x, 1-y)$.*

Example 1.2. *The following four operations are the most common t-norms:*

(T5) *Minimum:* $T_M(x,y) = \min\{x,y\}$

(T6) *Product:* $T_P(x,y) = x.y$

(T7) *Łukasiewicz:* $T_L(x,y) = \max\{x+y-1, 0\}$

(T8) *Drastic product:*

$$T_D(x,y) = \begin{cases} x & \textit{if} \;\; y = 1 \\ y & \textit{if} \;\; x = 1 \\ 0 & \textit{if} \;\; x, y < 1. \end{cases}$$

The dual t-conorms w.r.t. T_M, T_P, T_L *and* T_D *are given by:*

(S1) *Maximum:* $S_M(x,y) = \max\{x,y\}$

(S2) *Probabilistic sum:* $S_P(x,y) = x + y - s.y$

(T7) *Łukasiewicz:* $S_L(x,y) = \min\{x+y, 1\}$

(T8) *Drastic sum:*

$$S_D(x,y) = \begin{cases} 1 & \textit{if} \;\; (x,y) \in [0,1)^2 \\ \max\{x,y\} & \textit{otherwise} \end{cases} \tag{1.1}$$

Now, we give a brief overview of specific properties and important classes of t-norms that will be essential throughout this thesis.

Definition 1.16. [19] *Special properties and classes of triangular norms:*

(1) *For any t-norm* T*, we have* $T(x,y) \leqslant x$, $T(x,y) \leqslant y$ *and* $T(x,y) \leqslant$ *Min* (x,y).

(2) *An element* $x \in]0,1[$ *is called a zero divisor of* T *if there exists some* $y > 0$ *such that* $T(x,y) = 0$.

(3) *An element* $x \in [0,1]$ *is called an idempotent element of* T *if* $T(x,x) = x$.

(4) *A t-norm* T *is called Archimedean if and only if, for all pair* $(x,y) \in]0,1[^2$, *there is an* $n \in \mathbb{N}$ *such that*

$$T(\overbrace{x, ..., x}^{n \; times}) < y.$$

(5) A t-norm T is called left-continuous if the following holds for all $x \in [0,1]$ and all families $(y_i)_{i \in I} \in [0,1]^I$:

$$T(x, \bigvee_{i \in I} y_i) = \bigvee_{i \in I} T(x, y_i).$$

(6) For two t-norms T_1 and T_2 on $[0,1]$, we define:

$$T_1 \leqslant T_2 \Leftrightarrow (\forall x, y \in [0,1])(T_1(x,y) \leqslant T_2(x,y)) \, .$$

Each $x \in [x,y]$ is an idempotent element of the Minimum t-norm T_M (Actually T_M is the only t-norm whose set of idempotent elements is equal $[0,1]$), T_M has no zero divisor. Each $x \in]0,1[$ is a zero divisor of the Łukasiewicz t-norm T_L as well as of the Drastic product t-norm T_D.

Let T_1 and T_2 be two t-norms. If $T_1 \leq T_2$, then T_1 is called weaker than T_2 (or, equivalently, T_2 is called stronger than T_1). Note that T_D is the weakest t-norm, and T_M is the strongest t-norm, i.e.; for each t-norm it holds:

(T9) $T_D \leqslant T \leqslant T_M$.

Since $T_L \leqslant T_P$, it obviously holds:

(T10) $T_D \leqslant T_L \leqslant T_P \leqslant T_M$.

Remark 1.2. *We will mention when a t-norm T on a bounded lattice satisfying sup-preserving property:*

$$T(a, \bigvee_{i \in I} a_i) = \bigvee_{i \in I} T(a, a_i).$$

Note that any t-norm on $[0,1]$, or on a complete residuated lattice, satisfyies the sup-preserving property.

1.2. Pseudo ordered sets and trellises

This section contains the basic definitions and properties of pseudo-ordered sets, trellis structure and some notions that will be needed later. More information can be found in [34, 67].

1.2.1. Pseudo ordered sets

Definition 1.17. [67] *Let X be a non-empty set, a pseudo order on X is a binary relation $\trianglelefteq$ on X such that, for all $x, y \in X$, the conditions hold*

(i) $x \trianglelefteq x$ *(reflexivity);*

(ii) $x \trianglelefteq y$ *and* $y \trianglelefteq x$ *implies* $x = y$ *(antisymmetry).*

A set X equipped with a pseudo order relation $\trianglelefteq$ is called a pseudo ordered set (or psoset, for short). For $x, y \in X$, if $x \trianglelefteq y$ and $x \neq y$, then we write $x \triangleleft y$. For a subset $A \subseteq X$, the notions of a lower bound, an upper bound, greatest lower bound (g.l.b), least upper bound (l.u.b) and other notions are defined analogously to the corresponding notion in a partially ordered set (poset, for short). Now, we will only introduce the new concepts related to the pseudo-order relation.

Remark 1.3. *It is easily seen that any order relation on a set X is a pseudo order relation on X.*

Example 1.3. *(i) Let $X = \mathbb{R}$ be the set of real numbers. The relation on X defined, for any $x, y \in X$*

$$x \trianglelefteq y \text{ if and only if } 0 \leqslant y - x \leqslant a, \text{ where } a \in \mathbb{R}^+,$$

is a pseudo order relation on X.

(ii) Let $X = \{a, b, c, d, e\}$ and the binary relation defined on X as follows

$$\trianglelefteq = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),$$

$$(b,c),(b,e),(c,d),(c,e),(d,e)\},$$

the couple $(X, \trianglelefteq)$ is a pseudo ordered set. We note that $\trianglelefteq$ is not transitive because (b,c) and $(c,d) \in \trianglelefteq$, while $(b,d) \notin \trianglelefteq$.

In the sequel we will represent finite pseudo ordered sets by ***Hasse diagram*** *with the convention that if x is below y and is joined to y by a line if $x \trianglelefteq y$. If x is below y, and x and y are not related, then x and y will be joined by a dashed curve.*

Example 1.4. *Let $A = \{a, b, c, d, e\}$ be a set and $\trianglelefteq$ be a pseudo order relation on A, where $a \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq e$ (i.e.; $a \trianglelefteq x \trianglelefteq e$, for any $x \in A$). The Hasse diagram of A is depicted by the following figure:*

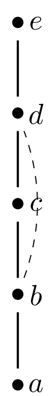

Figure 1.2: Hasse diagram of the psoset A

According to Skala [67], we have the following definitions

Definition 1.18. [67] *Let $(X, \trianglelefteq)$ be a psoset and B be a subset of X. We define a relation $\trianglelefteq_B$ on B as follows: for any $b, \grave{b} \in B$ there is a finite sequence $(b_1, ..., b_n) \in B$ such that $b \trianglelefteq b_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq \grave{b}$. $(B, \trianglelefteq_B)$ is called pseudo chain or (P-chain, for short) if for each pair of elements $b, \grave{b}$ of B either of the relations $b \trianglelefteq_B \grave{b}$ or $\grave{b} \trianglelefteq_B b$ holds, ($\trianglerighteq_B$ can be defined dually). A subset B is called a cycle if, for each pair of elements b and $\grave{b}$ of B both the relations $b \trianglelefteq_B \grave{b}$ and $\grave{b} \trianglelefteq_B b$ hold. Empty set is a cycle, any single element set on psoset is also a cycle. A non-trivial cycle is a cycle having more than one element and it contains at least three elements. A psoset is called acyclic if it does not contain a non-trivial cycle.*

A pseudo chain (P-chain) $C = \{a_i \mid i = 1, 2, ...\}$ of elements of psoset $(X, \trianglelefteq)$ is said to be an infinite ascending p-chain in X if $a_1 \trianglelefteq a_2 \trianglelefteq a_3, ...,$ where C is infinite. A psoset $(X, \trianglelefteq)$ is said to satisfy the ascending p-chain condition if there is no infinite ascending p-chain in X.

Definition 1.19. [67] *A preorder relation (preorder, for short) is a binary relation $\lesssim$ over a set X which is reflexive ($a \lesssim a$, for all $a \in X$), and transitive ($a \lesssim b$ and $b \lesssim c$ implies $a \lesssim c$, for any $a, b, c \in X$) but not necessarily antisymmetric. A set with a preorder relation is called a preordered set.*

Proposition 1.1. [67] *Let $(X, \trianglelefteq)$ be a pseudo-ordered set and $\mathcal{A}$ be a subset on X. A binary relation $\lesssim$ can be defined on $\mathcal{A}$ by setting $x \lesssim y$ for two elements of $\mathcal{A}$ if and only if there exists a finite sequence $(a_1, ..., a_n)$ of elements from X such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq y$, is a preorder relation on $\mathcal{A}$.*

Proof. Let $(X, \trianglelefteq)$ be a pseudo ordered set and $\mathcal{A}$ be a non-empty set on X.

(i) For any $x \in \mathcal{A}$, we have $x \trianglelefteq x$ this implies $x \lesssim x$. Hence, $\lesssim$ is reflexive;

(ii) Let $x, y, z \in \mathcal{A}$. Suppose that $x \lesssim y$ and $y \lesssim z$ and we show that $x \lesssim z$. From the supposition, it follows that there exist two sequences $(a_1, ..., a_n)$ and $(b_1, ..., b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq y$ and $y \trianglelefteq b_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq z$, it is easy to see that, there exists another sequence $(a_1, ..., b_n) \in X$ such that $x \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq b_n \trianglelefteq z$. That is, $x \lesssim z$. Hence, $\lesssim$ is transitive. Therefore, $\lesssim$ is a preorder relation on $\mathcal{A}$.

□

Remark 1.4. *If for each pair of elements x, y of $\mathcal{A}$ at least one of the relations $x \lesssim y$ or $y \lesssim x$ holds then $\mathcal{A}$ will be called a pseudo-chain or (p-chain, for short).*

A pseudo order on a set X is said to be linear if X itself is a pseudo-chain. It can easily shown that any pseudo order can be extended to a linear which leads the following theorem.

Theorem 1.1. [67] *Any pseudo order relation $\trianglelefteq$ on a set X can be extended to a linear pseudo order $\grave{\trianglelefteq}$ on X such that any cycle which respects $\trianglelefteq$ also respects $\grave{\trianglelefteq}$.*

The following theorem is immediately obtained by using Theorem 1.1.

Theorem 1.2. [67] *Any pseudo order is the intersection of all its linear extensions.*

Definition 1.20. [67] *Let $(X, \trianglelefteq)$ be a psoset. We can define the dual psoset of X, $(X^d, \trianglerighteq)$ by: $x \trianglerighteq y$ to holds in X^d if and only if $x \trianglelefteq y$ holds in X. For X finite, we obtain a diagram for X^d simply by "tuning upside down" a diagram of X. Given a statement Φ about pseudo ordered sets which is true* *in all pseudo ordered sets, the dual statement Φ^d is also true in all pseudo ordered sets.*

The notion of consistent family and contraction set of a pseudo ordered set $(X, \trianglelefteq)$ was introduced by Skala [67].

Definition 1.21. [67] *Let $(X, \trianglelefteq)$ be a psoset and $\mathcal{C}$ be a family of subsets of X,(i.e., $\mathcal{C} \subseteq \wp(X)$, where $\wp(X)$ is the power set of X). The family $\mathcal{C}$ is called a consistent family of $(X, \trianglelefteq)$ if it satisfies the following condition: For any A and B two distinct members of $\mathcal{C}$, there do not exist elements $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $x_1 \trianglelefteq y_1$ and $y_2 \trianglelefteq x_2$.*

Definition 1.22. [67] *For any consistent family $\mathcal{C}$ of $(X, \trianglelefteq)$, the pseudo order relation defined as follows, for any $A, B \in \mathcal{C}$, $A \trianglelefteq_{\mathcal{C}} B$ if and only if there exist $x \in A$ and $y \in B$ such that $x \trianglelefteq y$. Any consistent family $\mathcal{C}$ of $(X, \trianglelefteq)$ with the pseudo order $\trianglelefteq_{\mathcal{C}}$ (i.e., $(\mathcal{C}, \trianglelefteq_{\mathcal{C}})$) will be called a contraction set of $(X, \trianglelefteq)$.*

Example 1.5. *The family of all pairs $\{2k-1, 2k\}$, and $k = 1, 2, 3, ...$ is consistent family of $(\mathbb{N}, \leqslant)$.*

Theorem 1.3. [67] *Any pseudo ordered set is isomorphic with a contraction set of partially ordered set.*

Definition 1.23. *Let $(X, \trianglelefteq)$ be a psoset, A be a subset of X. The set A is called a convex subset of $(X, \trianglelefteq)$, if for all $b, c \in A$ and $a \in X$:*

$$b \trianglelefteq a \trianglelefteq c \text{ implies } a \in A.$$

Definition 1.24. [67] *A psoset is called to have the fixed point property if every order-preserving mapping $f : X \to X$ has a fixed point (i.e., there exists an element x of X such that $f(x) = x$).*

1.2.2. Trellises

The notion of trellis was introduced by Fried [34] and Skala [66, 67] as one of the most important algebraic structures in order theory. Which is considered as an extension of the notion of lattice by dropping the property of transitivity. A trellis is defined as a psoset $(T, \trianglelefteq)$ in which pair of elements has a least upper bound and a greatest lower bound. We use the following notation, we write $x \sqcup y$ in place of $sup\{x, y\}$ when it exists and $x \sqcap y$ in place of $inf\{x, y\}$ when it exists. Similarly, we write $\sqcup S$ and $\sqcap S$ instead of $(supS)$ and $(infS)$ when these exist.

Definition 1.25. *Let $(T, \trianglelefteq)$ be a non-empty pseudo ordered set.*

(i) $(T, \trianglelefteq)$ is called a $\sqcap$-semi-trellis if $x \sqcap y$ exists for any $x, y \in T$.

(ii) $(T, \trianglelefteq)$ is called a $\sqcup$-semi-trellis if $x \sqcup y$ exists for any $x, y \in T$.

(iii) $(T, \trianglelefteq)$ is called a trellis if it is both a $\sqcap$-semi-trellis and $\sqcup$-semi-trellis.

In other words, a trellis is an algebraic structure $(T, \sqcap, \sqcup)$ where the binary operations $\sqcap$ and $\sqcup$ satisfy the following properties, for any $x, y, z \in T$.

(i) $x \sqcap x = x \sqcup x = x$ (idempotency);

(ii) $x \sqcap y = y \sqcap x$ and $x \sqcup y = y \sqcup x$ (commutativity);

(iii) $x \sqcap (x \sqcup y) = x = x \sqcup (x \sqcap y)$ (absorption laws);

(iv) $x \sqcap ((x \sqcup y) \sqcap (x \sqcup z)) = x = x \sqcup ((x \sqcap y) \sqcup (x \sqcap z))$ (weak associativity).

For a given trellis $(T, \trianglelefteq, \sqcap, \sqcup)$ and $x, y \in T$, the following statements are equivalent:

(i) $x \trianglelefteq y$;

(ii) $x \sqcup y = y$;

(iii) $x \sqcap y = x$.

The notions of sub-trellis, complete trellis, distributive and modular trellis are defined analogously to the corresponding notions in partially ordered sets.

Remark 1.5. *It is will known that every finite lattice is complete, but this property is not true in the finite trellis. Indeed, let us consider the trellis $(T, \trianglelefteq, \sqcap, \sqcup)$ given in the following table and figure.*

$\sqcap\sqcup$	a	b	c	d	e
a		b	c	d	a
b	a		e	e	e
c	a	a		e	e
d	a	a	a		e
e	e	b	c	d	

We can see that T is not complete, since for example $\sqcup\{a, b, d\}$ does not exist.

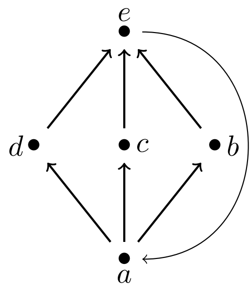

Figure 1.3: Finite trellis not complete.

Definition 1.26. [67] *Let T be a trellis. We say T has a greatest element if there exists $1 \in T$ such that $x \sqcap 1 = x$, for any $x \in T$. Dually, we say T has a smallest element if there exists $0 \in T$ such that $x \sqcup 0 = x$, for any $x \in T$. A trellis $(T, \sqcap, \sqcup)$ possessing 0 and 1 is called bounded trellis.*

Lemma 1.2. [67] *If every non-empty subset of a pseudo ordered set T with the smallest element 0 has a least upper bound, then T is a complete trellis.*

Example 1.6. *Let $(X, \trianglelefteq)$ be a psoset given in Example 1.4 with the l.u.bs and g.l.bs given in the following table (see eg. Figure 1.4)*

$\{x, y\}$	$x \sqcap y$	$x \sqcup y$
$\{a, b\}$	b	a
$\{a, c\}$	a	c
$\{a, d\}$	a	d
$\{a, e\}$	a	e
$\{b, c\}$	b	c
$\{b, d\}$	a	e
$\{b, e\}$	b	e
$\{c, d\}$	c	d
$\{c, e\}$	c	e
$\{d, e\}$	d	e

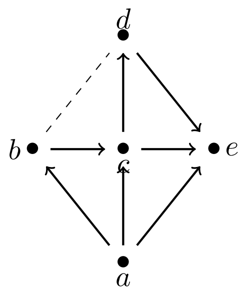

Figure 1.4: Representation of a trellis.

is a trellis.

Definition 1.27. [67] *A mapping f from a trellis $\mathcal{L} = (T, \trianglelefteq)$ into a trellis $\mathcal{M} = (\grave{T}, \grave{\trianglelefteq})$ (i.e., $f : \mathcal{L} \to \mathcal{M}$) is called*

(i) Homomorphism if $f(x \sqcap_{\mathcal{L}} y) = f(x) \sqcap_{\mathcal{M}} f(y)$ and $f(x \sqcup_{\mathcal{L}} y) = f(x) \sqcup_{\mathcal{M}} f(y)$, for any $x, y \in T$;

(ii) Embedding if f is a one to one homomorphism;

(iii) Isomorphism if f is both one to one and onto homomorphism.

A homomorphism f of a trellis T into itself is called an endomorphism.

Next, we introduce some important special elements of a trellis.

Definition 1.28. [67] *An element z of a trellis T is said to be*

(i) right-transitive if $z \trianglelefteq x \trianglelefteq y$ implies $z \trianglelefteq y$;

(ii) left-transitive if $x \trianglelefteq y \trianglelefteq z$ implies $x \trianglelefteq z$;

(iii) middle-transitive if $x \trianglelefteq z \trianglelefteq y$ implies $x \trianglelefteq y$;

(v) transitive if z is right, left and middle transitive.

Example 1.7. *In the trellis T given in Figure 1.4, note that $b \trianglelefteq c \trianglelefteq d$ and $b \ntrianglelefteq d$ then b isn't right-transitive. Otherwise, since if $x \trianglelefteq y \trianglelefteq e$ implies $x \trianglelefteq e$, for any $x, y \in T$, then it holds that e is a left-transitive element.*

Theorem 1.4. [67] *Let T be a trellis and $x, a \in X$. It hold that*

(i) If a is left-transitive element, then

(a) $x \lesssim a$, implies $x \trianglelefteq a$;

(b) $x \trianglelefteq a$, implies that $x \sqcup y \trianglelefteq a \sqcup y$, for any $y \in X$.

(ii) Dually, if a is right-transitive element, then

(a) $a \lesssim x$*, implies* $a \trianglelefteq x$*;*

(b) $a \trianglelefteq x$*, implies that* $a \sqcap y \trianglelefteq x \sqcap y$*, for any* $y \in X$*.*

Definition 1.29. [67] *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis and* (x, y, z) *a sequence of elements, then*

(i) (x, y, z) *is* $\sqcap$*-associative if* $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$*;*

(ii) (x, y, z) *is* $\sqcup$*-associative if* $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$*.*

Definition 1.30. [67] *An element a is called* $\sqcap$*-associative if any sequence of three elements including a is* $\sqcap$*-associative. The* $\sqcup$*-associative of a is defined dually. If an element is both* $\sqcap$*-associative and* $\sqcup$*-associative, it is said to be associative.*

Theorem 1.5. [67] *Let T be a trellis and* $\mathcal{B}$ *be a pseudo chain on T. An element of* $\mathcal{B}$ *is associative if and only if it is transitive.*

Theorem 1.6. *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis, then the following statements are equivalent*

(i) $\trianglelefteq$ *is transitive;*

(ii) the operations $\sqcup$ *and* $\sqcap$ *are associative.*

Theorem 1.7. [67] *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis and a be an element of T. If a is* $\sqcup$*-associative (resp.*$\sqcap$*-associative), then a is transitive.*

Proof. The proof of this theorem is straightforward. □

Remark 1.6. *The converse of the Theorem 1.7 does not hold in general. Indeed, let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *the trellis given by the Hasse diagram in Figure 1.5. Note that d is transitive but not* $\sqcup$*-associative since* $(d \sqcup a) \sqcup c = c$ *but* $d \sqcup (a \sqcup c) = u$*.*

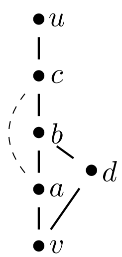

Figure 1.5: Figure 4

2 Fuzzy pseudo ordered sets and fuzzy trellises

In this chapter, we study the notion of fuzzy pseudo ordered sets and some of its interesting properties. In particular, we show that any fuzzy pseudo ordered set is isomorphic to a contraction set of fuzzy ordered set. Furthermore, we introduce the concept of fuzzy trellis as a natural generalization of the trellis introduced by Skala [66, 67] and we discuss its properties. We start by recalling the concepts of fuzzy sets and fuzzy relations.

2.1. Fuzzy sets and fuzzy relations

In this section, we recall the definitions and some basic properties of fuzzy sets and fuzzy relations. These notions are fundamental importance in almost all sub-fields of fuzzy logic and fuzzy set theory. They play an important role in fuzzy modelling, fuzzy control and fuzzy inference. Also, they have important applications in relational databases, approximate reasoning and medical diagnosis. In many cases, fuzzy relations can handle real life problems better than the crisp ones.

2.1.1. Fuzzy sets

A fuzzy subset (L-set, for short) in a universe set X is a mapping $A : X \to L$, where L is bounded lattice, assigning to every element $x \in X$ an element $A(x) \in L$ interpreted as the truth degree to which x belongs to A. Ordinary crisp subset of X are considered as an L-set of X, taking membership values in the set $\{0, 1\} \subseteq L$. For later, L^X denotes the set of all L-sets of X (i.e., the set of all mappings from X to L). Let A and B be two L-sets of X.

The equality of A and B is defined as the usual equality of mappings by:

$$A = B \text{ if and only if } A(x) = B(x), \text{ for any } x \in X.$$

The inclusion is also defined by:

$$A \subseteq B \text{ if and only if } A(x) \leq B(x), \text{ for any } x \in X.$$

The intersection and the union of an arbitrary family $\{A_i\}_{i\in I}$ *of L-sets of* X*, are mappings from* X *into* L *defined by:*

$$(\bigwedge_{i\in I} A_i)(x) = \bigwedge_{i\in I} A_i(x), \ (\bigvee_{i\in I} A_i)(x) = \bigvee_{i\in I} A_i(x).$$

Endowed with this partial order, the set $\pounds(X)$ *of all L-sets of* X *forms a complete lattice, in which the meet is (intersection)* $\bigwedge_{i\in I} A_i$ *and the join is (union)* $\bigvee_{i\in I} A_i$.

The product $A \otimes B$ *is an L-set defined by:*

$$(A \otimes B)(x) = A(x) \otimes B(x), \text{ for every } x \in X.$$

The crisp part of an L-set A *of* X *is a crisp subset* $\hat{A} = \{x \in X \mid A(x) = 1\}$ *of* X*. We will also consider* $\hat{A}$ *as a mapping* $\hat{A} : X \to L$ *defined by* $\hat{A}(x) = 1$*, if* $A(x) = 1$*, and* $\hat{A}(x) = 0$*, if* $A(x) \neq 1$*. By the support of L-set* A *of* X*, denoted by Supp(A), we mean all elements of* X *that belong to* A *to a nonzero degree. That is, Supp(A) is a classical set defined by* $Supp(A) = \{x \in X \mid A(x) > 0\}$*. The* α*-level set of* A*, denoted by* A_α*, is a set consisting of those elements of universe* X *whose membership values are equal or exceed the value of* α*. That is,* $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$*. Note that* $\alpha \in]0,1]$ *is arbitrary and* α*-level set is a crisp set. In other words,* A_α *consists of elements of* X *defined with* A *to a degree of at least* α*. Clearly, the lower level of* α*, the more elements are admitted to the corresponding* α*-level. That is, if* $\alpha_1 > \alpha_2$*, then* $A_{\alpha_1} \subset A_{\alpha_2}$*.*

Example 2.1. *Let* $X = [0,1]$ *with* $\alpha, \beta \in \mathbb{R}$ *and let* $a, b \in \mathbb{R}$*. We define the fuzzy set* A *on* X *by:*

$$\mu_A(x) = \begin{cases} 0 \ , \ if \ x < a - \alpha \ or \ b + \beta < x, \\ 1 \ , \quad if \quad a < x < b, \\ 1 + (\frac{x-a}{\alpha}) \ , \quad if \quad a - \alpha < x < a \, , \\ 1 - (\frac{b-x}{\beta}) \ , \quad if \quad b < x < b + \beta. \end{cases}$$

Then $Ker(A) = [0, 1]$, $Supp(A) = [a - \alpha, b + \beta]$ *and* $H(a) = 1$.

More information on L-set can be found in [10, 38, 41, 57, 83].

2.1.2. Fuzzy relations

The notion of fuzzy relation (an L-relation, for short) on a set X *generalizes the notion of a relation by expressing degrees of relationship in some bounded lattice* $(L, \leq, \vee, \wedge, 0, 1)$. *A binary L-relation* R *on a set* X *is a mapping* $R : X \times X \to L$. *For every* $x, y \in X$ *is called the grade of membership of* (x, y) *in* R *and means haw far* x *and* y *are related under* R. *Obviously, if* $L = \{0, 1\}$, *then binary relations are retrieved, often referred to as a crisp relations.*

A binary L-relation S *is said to be included in a binary L-relation* R, *denoted by:* $S \subseteq R$, *if* $S(x, y) \leq R(x, y)$, *for any* $x, y \in X$. *The intersection of two binary L-relations* S *and* R *on* X *is the binary L-relation* $S \cap R$ *on* X *defined by* $(S \cap R)(x, y) = S(x, y) \wedge R(x, y)$. *Similarly, the union of two binary L-relations* S *and* R *on* X *is the binary L-relation* $S \cup R$ *on* X *defined by:* $(S \cup R)(x, y) = S(x, y) \vee R(x, y)$, *for any* $x, y \in X$. *The composition of two binary L-relations* S *and* R *on* X *is the binary L-relation* $S \circ R$ *on* X *defined by:* $S \circ R(x, z) = \sup_{y \in X}(S(x, y) * R(y, z))$. *The transpose* R^t *of a binary L-relation* R *is the binary L-relation defined by:* $R^t(x, y) = R(y, x)$. *As an L-relation is an L-set, then the* α*-level set and Supp of* R *are defined as in a fuzzy set, i.e.,* $R_\alpha = \{(x, y) \in X \times X \mid R(x, y) \geq \alpha\}$, *for any* $x, y \in X$. *In the same way we define the support of an L-relation* $Supp(R)$ *as* $Supp(R) = \{(x, y) \in X \times X \mid R(x, y) > 0\}$. *A binary L-relation* R *on a set* X *is called:*

1. *reflexive, if* $R(x, x) = 1$, *for all* $x \in X$;
2. *symmetric, if* $R^t = R$;
3. $*$*-transitive, if* $R(x, y) * R(y, z) \leq R(x, z)$, *for all* $x, y, z \in X$;
4. $*$*-asymmetric, if* $R(x, y) * R(y, x) = 0$, *for all* $x, y \in X$;
5. $*$*-antisymmetric, if* $R(x, y) * R(y, x) \neq 0$ *implies* $x = y$, *for all* $x, y \in X$;
6. *separability, if* $R(x, y) = 1$ *implies that* $x = y$, *for all* $x, y \in X$.

For more details on binary L-relations and their properties, we refer to [10, 21, 71, 79, 84].

Notations
For consistent of the notations and to avoid any confusion in the comprehension of the formulas in the next chapters, we will use the following notations.

(i) $(L, \leq, \wedge, \vee)$ *refers to lattice;*

(ii) $(T, \trianglelefteq, \sqcap, \sqcup)$ *refers to trellis;*

(iii) $([0,1], \leqslant, \barwedge, \veebar)$ *refers the real interval.*

2.2. Fuzzy pseudo ordered sets

In this section, we introduce and study the notion of L-fuzzy pseudo order (L-pseudo order, for short). First, we recall the definition of fuzzy order relation. Further information on fuzzy order relations can be found in [11, 21, 39, 84].
Definition 2.1. [21] *A mapping* $\mathcal{R} : X \times X \to L$ *is called a fuzzy order relation (L-order relation, for short) on* X *with respect to a t-norm* $*$, *if and only if the following conditions are satisfied:*

(i) $\mathcal{R}(x, x) = 1$, *for all* $x \in X$ *(reflexivity);*

(ii) $\mathcal{R}(x, y) * \mathcal{R}(y, x) \neq 0$ *implies* $x = y$, *for all* $x, y \in X$ *($*$-antisymmetry);*

(iii) $\mathcal{R}(x, y) * \mathcal{R}(y, z) \leq \mathcal{R}(x, z)$, *for all* $x, y, z \in X$ *($*$-transitivity).*

A set X *equipped with L-order relation* $\mathcal{R}$ *is called an L-ordered set, noted by* $(X, \mathcal{R})$.
Definition 2.2. *An L-relation* $P : X \times X \to L$ *is called an L-pseudo order relation on* X *with respect to a t-norm* $*$, *if it satisfies the following two conditions:*

(i) $P(x, x) = 1$, *for all* $x \in X$ *(reflexivity);*

(ii) $P(x, y) * P(y, x) \neq 0$ *implies* $x = y$, *for all* $x, y \in X$ *($*$-antisymmetry).*

A set X *equipped with L-pseudo order relation* P *is called an L-pseudo ordered set, noted by* (X, P).

Example 2.2. *Let $X = \{a, b, c, d\}$, then the L-relation P on X defined by the following table:*

P	a	b	c	d
a	1	0.5	0.3	0.1
b	0	1	0	0.4
c	0	0	1	0.5
d	0	0	0	1

is an L-pseudo order on X.

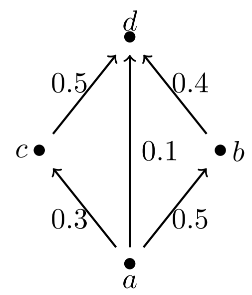

Figure 2.1: Representation of the L-pseudo ordered set $X = \{a, b, c, d\}$.

Definition 2.3. *Let (X, P) be an L-pseudo ordered set and $\mathcal{C}$ be a family of subsets of X. $\mathcal{C}$ is called a fuzzy consistent family of X with respect to the t-norm $*$ and the L-pseudo order P (an $*$-P-consistent family of X, for short) if for any two distinct members $A, B \in \mathcal{C}$, there do not exist $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $P(x_1, y_1) * P(y_2, x_2) \neq 0$.*

Next, we need the following lemma.

Lemma 2.1. *Let L be a complete lattice and $*$ a t-norm on it. Suppose that L satisfyies the sup-preserving property, I and J be two subsets of $\mathbb{N}$ such that $I \cap J = \phi$. Then it holds that*

$$\bigvee_{i \in I} a_i * \bigvee_{j \in J} b_j = \bigvee_{(i,j) \in I \times J} (a_i * b_j), \text{ for any } a_i, b_j \in L\,.$$

Proof. From Definition 1.16, it follows that

$$\begin{aligned}(\bigvee_{i\in I} a_i) * (\bigvee_{j\in J} b_j) &= \bigvee_{i\in I}(a_i * \bigvee_{j\in J} b_j)\\ &= \bigvee_{i\in I}(\bigvee_{j\in J}(a_i * b_j))\\ &= \bigvee_{i\in I}\bigvee_{j\in J}(a_i * b_j).\end{aligned}$$

Since $I \cap J = \phi$, it holds that

$$(\bigvee_{i\in I} a_i) * (\bigvee_{j\in J} b_j) = \bigvee_{(i,j)\in I\times J}(a_i * b_j).$$

□

The following results will be needed in the next section.

Proposition 2.1. *Let $\mathcal{C}$ be an $*$-P-consistent family of X and A, B be two distinct elements of $\mathcal{C}$. Then it holds that $A \cap B = \emptyset$.*

Proof. Suppose that there exist $A, B \in \mathcal{C}$ and $A \cap B \neq \emptyset$, (i.e., there exists $x \in A \cap B$). From the reflexivity of P, it follows that $P(x,x) * P(x,x) = 1 * 1 \neq 0$. Then, there exist $a_1 = a_2 = x \in A$ and $b_1 = b_2 = x \in B$ such that $P(a_1, b_1) * P(b_2, a_2) = 1 \neq 0$. Hence, $\mathcal{C}$ is not an $*$-P consistent family of X. This is a contradiction. Thus, $A \cap B = \emptyset$. □

In the following proposition we show that $(\mathcal{C}, \hat{P})$ is an L-pseudo ordered set.

Proposition 2.2. *Let (X, P) be an L-pseudo ordered set and $\mathcal{C}$ be an $*$-P-consistent family of X. Then $(\mathcal{C}, \hat{P})$ is an L-pseudo ordered set, where $\hat{P}$ is an L-relation defined on $\mathcal{C}$ in terms of P as:*

$$\hat{P}(A,B) = \sup\{P(x,y) \mid x \in A \textit{ and } y \in B\} = \bigvee_{(x,y)\in A\times B} P(x,y),$$

for any $A, B \in \mathcal{C}$.

Proof. Let $A, B \in \mathcal{C}$. First, since P is reflexive, it follows that $\hat{P}(A,A) = \bigvee_{(x,y)\in A^2} P(x,y) = P(x,x) = 1$. Hence, $\hat{P}$ is reflexive. Second, suppose that $(\hat{P}(A,B) * \hat{P}(B,A)) \neq 0$ and $A \neq B$. From Proposition 2.1, it follows that $A \cap B = \phi$. Also from Lemma 2.1 it follows that

$$\begin{aligned} \hat{P}(A,B) * \hat{P}(B,A) &= \bigvee_{(x,y)\in A\times B} P(x,y) * \bigvee_{(\grave{y},\grave{x})\in B\times A} P(\grave{y},\grave{x}) \\ &= \bigvee_{(x,y)\in A\times B}\bigvee_{(\grave{y},\grave{x})\in B\times A} (P(x,y) * P(\grave{y},\grave{x})) \\ &\neq 0. \end{aligned}$$

Hence, there exist $(x,y) \in A \times B$ and $(\grave{y},\grave{x}) \in B \times A$ such that $P(x,y) * P(\grave{y},\grave{x}) \neq 0$. That is a contradiction with the fact that $\mathcal{C}$ is an $*$-P-consistent family. Thus, $\hat{P}$ is an $*$-antisymmetric. Therefore, $(\mathcal{C}, \hat{P})$ is an L-pseudo ordered set. □

2.3. Fuzzy ordered set induced by a fuzzy pseudo ordered set

In this section, we study an L-ordered set created by an L-pseudo ordered set. First, we show the following key result.

Proposition 2.3. *Let (X,P) be an L-pseudo ordered set, $Y = X \times \{0,1\}$ and $\mathcal{R}$ be an L-relation on Y defined by:*

$$\mathcal{R}((x,\alpha),(y,\beta)) = \begin{cases} 1\ , & if \quad x = y \quad and \quad \alpha \leq \beta, \\ P(x,y)\ , & if \quad x \neq y \quad and \quad \alpha < \beta, \\ 0\ , & otherwise. \end{cases}$$

It holds that $(Y,\mathcal{R})$ is an L-ordered set.

Proof. First, since $\mathcal{R}((x,\alpha),(x,\alpha)) = 1$, for any $(x,\alpha) \in Y$, it holds that $\mathcal{R}$ is reflexive. Second, Let $(x,\alpha),(y,\beta) \in Y$, we need to show that, if $(x,\alpha) \neq (y,\beta)$, then $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$. The fact that $(x,\alpha) \neq (y,\beta)$ implies that $x \neq y$ or $\alpha \neq \beta$.

(1) If $x \neq y$, since $\alpha, \beta \in \{0,1\}$, there are three subcases to consider.

  (a) If $\alpha < \beta$, then it follows that $\mathcal{R}((x,\alpha),(y,\beta)) = P(x,y)$ and $\mathcal{R}((y,\beta),(x,\alpha)) = 0$. Hence,

  $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0.$

(b) If $\beta < \alpha$, then it follows that $\mathcal{R}((x,\alpha),(y,\beta)) = 0$ and $\mathcal{R}((y,\beta),(x,\alpha)) = P(y,x)$.
Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$.

(c) If $\alpha = \beta$, then it follows that $\mathcal{R}((x,\alpha),(y,\beta)) = \mathcal{R}((y,\beta),(x,\alpha)) = 0$. Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$.

(2) If $\alpha \neq \beta$, there are three subcases to consider.

(d) If $x = y$ and $\alpha < \beta$, then it follows that $\mathcal{R}((x,\alpha),(y,\beta)) = 1$ and $\mathcal{R}((y,\beta),(x,\alpha)) = 0$.

Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$.

(e) If $x = y$ and $\alpha > \beta$, then it follows that $\mathcal{R}((x,\alpha),(y,\beta)) = 0$ and $\mathcal{R}((y,\beta),(x,\alpha)) = 1$.

Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$.

(f) If $x \neq y$, then it follows that one of the cases (a) or (b) is true.
Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(x,\alpha)) = 0$.

Thus, $\mathcal{R}$ is $*$-antisymmetric.

Third, let $(x,\alpha),(y,\beta),(z,\gamma) \in Y$, we need to show that $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(z,\gamma)) \leq \mathcal{R}((x,\alpha),(z,\gamma)$.

If $(x,\alpha) = (y,\beta)$ or $(y,\beta) = (z,\gamma)$ or $(x,\alpha) = (z,\gamma)$, then it is trivially true that $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(z,\gamma)) \leq \mathcal{R}((x,\alpha),(z,\gamma))$.
If $(x,\alpha) \neq (y,\beta)$,$(y,\beta) \neq (z,\gamma)$ and $(x,\alpha) \neq (z,\gamma)$, then it follows that

$$\mathcal{R}((x,\alpha),(y,\beta)) = \begin{cases} 1\ , & if \quad x = y \quad and \quad \alpha < \beta \\ P(x,y)\ , & if \quad x \neq y \quad and \quad \alpha < \beta \\ 0\ , & otherwise \end{cases}$$

and

$$\mathcal{R}((y,\beta),(z,\gamma)) = \begin{cases} 1\ , & if \quad y = z \quad and \quad \beta < \gamma \\ P(y,z)\ , & if \quad y \neq z \quad and \quad \beta < \gamma \\ 0\ , & otherwise \end{cases} .$$

Since $\alpha < \beta$ and $\beta < \gamma$ with $\alpha, \beta, \gamma \in \{0,1\}$ is an impossible case, it follows that

$\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(z,\gamma)) = 0$. Hence, $\mathcal{R}((x,\alpha),(y,\beta)) * \mathcal{R}((y,\beta),(z,\gamma)) \leq \mathcal{R}((x,\alpha),(z,\gamma))$. Thus, $\mathcal{R}$ is $*$-transitive. Therefore, $(Y,\mathcal{R})$ is an L-order on Y. □

2.4. Fuzzy pseudo ordered set isomorphic to a contraction set of fuzzy ordered set

In this section, we show that any L-pseudo ordered set is isomorphic with a contraction set of an L-ordered set. First, we mention the definition of a contraction set of L-pseudo ordered set.

Definition 2.4. *Let $\mathcal{C}$ be an $*$-P-consistent family of L-pseudo ordered set (X,P). Then $(\mathcal{C},\hat{P})$ is called a contraction set of $(\mathcal{C},P)$, where $\hat{P}$ is the L-pseudo order defined in Proposition 2.2 on $\mathcal{C}$ as follows:*

$$\hat{P}(A,B) = \sup\{P(x,y) \mid x \in A \text{ and } y \in B\} = \bigvee_{(x,y)\in A\times B} P(x,y),$$

for any $A,B \in \mathcal{C}$.

Example 2.3. *Consider the L-pseudo ordered set (X,P) defined in Example 2.2. The family $\mathcal{C}$ =$\{\{a\},\{b,c\},\{d\}\}$ is an $*$-P-consistent family of (X,P). Indeed, for any two distinct elements $A,B \in \mathcal{C}$, there do not exist $a_1,a_2 \in A$ and $b_1,b_2 \in B$ such that $P(a_1,b_1) * P(b_2,a_2) \neq 0$. Furthermore, $(\mathcal{C},\hat{P})$ is a contraction set of (X,P).*

Now, we provide the notion of contraction set of L-ordered set. First, we show that the set $\mathcal{L}$ is an $*$-$\mathcal{R}$-consistent family of Y in the following proposition.

Proposition 2.4. *Let $\mathcal{L} = \{\{x\}\times\{0,1\} \mid x \in X\}$ be a family of subsets of Y, then $\mathcal{L}$ is an $*$-$\mathcal{R}$-consistent family of Y.*

Proof. Let $\mathcal{L} = \{\{x\}\times\{0,1\}\}$ be a family of subsets of Y, $(\mathcal{L} \subseteq \wp(Y))$. Suppose that there exist two distinct members A and B in $\mathcal{L}$, then it follows that $A = \{(x,0);(x,1)\}$ and $B = \{(y,0);(y,1)\}$, such that $\mathcal{R}((x,\alpha_1),(y,\beta_1)) * \mathcal{R}((y,\beta_2),(x,\alpha_2)) \neq 0$ where $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \{0,1\}$, then we have two cases to consider

(a) $\mathcal{R}((x,0),(y,0)) = \mathcal{R}((y,1),(x,1)) = 0$. Hence, $\mathcal{R}((x,0),(y,0)) * \mathcal{R}((y,1),(x,1))$ 0;

(b) $\mathcal{R}((x,0),(y,1)) = P(x,y)$, and $\mathcal{R}((y,0),(x,1)) = P(y,x)$. Then it follows that $\mathcal{R}((x,0),(y,1)) * \mathcal{R}((y,0),(x,1)) = P(x,y) * P(y,x)$.

Since $x \neq y$ and P is $*$-antisymmetry, it holds that $P(x,y) * P(y,x) = 0$.

Hence, for any two distinct members A and B of $\mathcal{L}$, it follows that there do not exist $(x,\alpha_1),(x,\alpha_2) \in R$ and $(y,\beta_1),(y,\beta_2) \in S$ such that $\mathcal{R}((x,\alpha_1),(y,\beta_1)) * \mathcal{R}((y,\beta_2),(x,\alpha_2)) \neq 0$. Thus, $\mathcal{L}$ is an $*$-$\mathcal{R}$-consistent family of Y. □

Combining Proposition 2.2 and Proposition 2.4 easily lead to the following corollary.

Corollary 2.1. *$(\mathcal{L}, \hat{\mathcal{R}})$ is a contraction set of the L-ordered set $(Y, \mathcal{R})$, where*

$$\hat{\mathcal{R}}(A,B) = \sup\{\mathcal{R}((x,\alpha),(y,\beta)) \mid (x,\alpha) \in A \text{ and } (y,\beta) \in B\},$$

for any $A, B \in \mathcal{L}$.

Definition 2.5. *Let (X,P) and $(Y,\grave{P})$ be L-pseudo ordered sets. A mapping $f : X \to Y$ is an isomorphism from X into Y if, the following conditions are fulfilled:*

(i) f is bijective.

(ii) $\grave{P}(f(x), f(y)) \geq P(x,y)$.

for any $x, y \in X$,

In the following theorem, we show that any L-pseudo ordered set is isomorphic with a contraction set of L-ordered set.

Theorem 2.1. *Let (X,P) be an L-pseudo ordered set and $(\mathcal{L}, \hat{\mathcal{R}})$ the contraction set of $(Y, \mathcal{R})$ defined above. Then the mapping $f : (X,P) \to (\mathcal{L}, \hat{\mathcal{R}})$ from X into $\mathcal{L}$ defined by:*

$$f(x) = \{x\} \times \{0,1\},$$

is an isomorphism of L-pseudo ordered sets.

Proof. Let f be a map $f : (X,P) \to (\mathcal{L}, \hat{\mathcal{R}})$ defined by:
$f(x) = \{(x,0),(x,1)\} = \{x\} \times \{0,1\}$, for any $x \in X$.

We show that f is an isomorphism mapping. First, if we take $x_1 \neq x_2$ and suppose that $f(x_1) = f(x_2)$, we get that

$$\begin{aligned} f(x_1) &= \{\{(x_1,0),(x_1,1)\} \mid x_1 \in X\} \\ &\neq \{\{(x_2,0),(x_2,1)\} \mid x_2 \in X\} \\ &\neq f(x_2), \end{aligned}$$

a contradiction. Therefore f is one to one.

Second, let $A \in \mathcal{L}$, it follows that there exists $x \in X$ such that $A = \{(x_1,0),(x_1,1)\}$, hence $A = f(x)$. Therefore f is a surjective mapping.

Third, let $x, y \in X$ and $A, B \in \mathcal{L}$ such that $f(x) = A$ and $f(y) = B$, we show that $\hat{\mathcal{R}}(A,B) \geq P(x,y)$. The fact that $P(x,y) \neq 0$ implies that $x = y$ and $\alpha \leq \beta$ or $x \neq y$ and $\alpha < \beta$.

(1) If $x = y$ and $\alpha \leq \beta$, then it trivially holds that $\hat{\mathcal{R}}(A,B) = \bigvee\{\mathcal{R}((x,\alpha),(y,\beta))\} = 1 \geq P(x,y)$.

(2) If $x \neq y$ and $\alpha < \beta = 1$, then it trivially holds that $\hat{\mathcal{R}}(A,B) = \bigvee\{\mathcal{R}((x,\alpha),(y,\beta))\} = P(x,y) \geq P(x,y)$.

Hence, f is an isomorphism. □

Example 2.4. *Let $X = \{x,y,z\}$ and P be an L-pseudo order relation on X defined by the following table:*

P	x	y	z
x	1	0.5	0.1
y	0	1	0.4
z	0	0	1

and the set $Y = \{\{x\} \times \{0,1\}\}$, we define an L-order relation $\mathcal{R}$ on Y as follows

$\mathcal{R}$	$(x,0)$	$(x,1)$	$(y,0)$	$(y,1)$	$(z,0)$	$(z,1)$
$(x,0)$	1	1	0	0.5	0	0.1
$(x,1)$	0	1	0	0	0	0
$(y,0)$	0	0	1	1	0	0.4
$(y,1)$	0	0	0	1	0	0
$(z,0)$	0	0	0	0	1	1
$(z,1)$	0	0	0	0	0	1

Let $(\mathcal{L}, \hat{\mathcal{R}})$ be the contraction set of $(Y, \mathcal{R})$. Then the mapping $f : X \to \mathcal{L}$ defined by $f(x) = \{x\} \times \{0, 1\}$, is an L-pseudo order isomorphism from X into $\mathcal{L}$ (see, Figure 2.2).

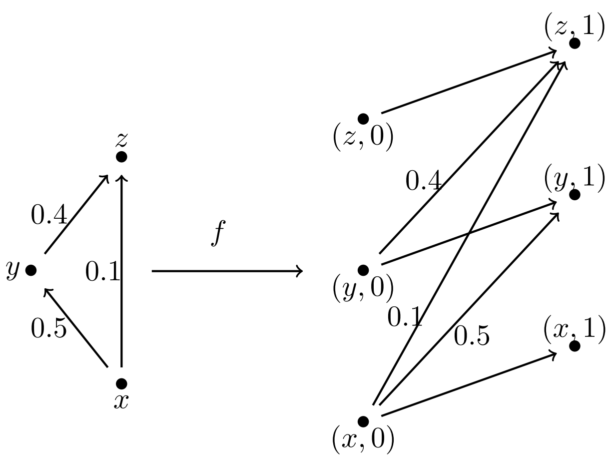

Figure 2.2: Representation of f.

2.5. Fuzzy ordered trellises

In this section, we introduce the notions of the meet and the join of any subset of an L-pseudo ordered set (X, P). Further, we define the notion of a L-fuzzy trellis (L-trellis, for short), and we show that it is isomorphic with a contraction trellis of an L-fuzzy lattice. Some notions and results related to the meet and join are to be found in [29, 30, 49].

Remark 2.1. *To avoid any confusion or misunderstanding in some formulas, we use the notation $(P, \sqcap_P, \sqcup_P)$ to refer to (L-pseudo order, min, max)* *on the trellis X, the notation $(\mathcal{R}, \sqcap_{\mathcal{R}}, \sqcup_{\mathcal{R}})$ to refer to (L-order, min, max) on the lattice Y. Also, we recall that the notation $(\leq, \wedge, \vee)$ is used to refer the (usual order, min, max) on the lattice L.*

Definition 2.6. *Let (X, P) an L-pseudo ordered set and A be a subset of X. The set of lower bounds of A with respect to P is an L-set of X defined, for any $y \in X$, by:*

$$A^l(y) = \bigwedge_{x \in A} P(y, x).$$

The set of upper bounds of A with respect to P is an L-set of X defined, for any $y \in X$, by:

$$A^u(y) = \bigwedge_{x \in A} P(x, y).$$

Remark 2.2. *Let A be a subset of an L-pseudo ordered set (X, P). The infimum of A is denoted by $\sqcap_P A$ (i.e., $\inf(A)$) and the supremum of A is denoted by $\sqcup_P A$ (i.e., $\sup(A)$).*

Definition 2.7. *Let (X, P) ba an L-pseudo ordered set and A be an L-set of X. An element x_0 of X is called the greatest lower bound (or an infimum) of A (i.e., $x_0 = \sqcap_P A$) with respect to P, if:*

(i) $A^l(x_0) = 1$,

(ii) $A^l(y) \leq P(y, x_0)$, *for any* $y \in X$.

An element x_1 of X is called the least upper bound (or a supremum) of A (i.e., $x_1 = \sqcup_P A$) with respect to P, if:

(i) $A^u(x_1) = 1$,

(ii) $A^u(y) \leq P(x_1, y)$, *for any* $y \in X$.

Proposition 2.5. *Let A be a subset of an L-pseudo ordered set (X, P).*

(i) If $\sqcap_P A$ exists, then it is unique;

(ii) If $\sqcup_P A$ exists, then it is unique.

Proof. (i) Let $z, \grave{z}$ be two supremums of A. Then it holds that $A^u(z) = A^u(\grave{z}) = 1$, $A^u(y) \leq P(z, y)$ and $A^u(y) \leq P(z', y)$ for any $y \in X$. Since $z \in A$ and by putting $y = z'$ (resp, $y = z$), we obtain that $P(z, z') = 1$ and $P(z, z') = 1$. Since P is $*$-antisymmetry, it follows that $z = z'$.

(ii) Follows similarly as (i).

□

Remark 2.3. *If A is a pair $\{x, y\}$, then we also write $\sqcup_P A = x \sqcup_P y$ and $\sqcap_P A = x \sqcap_P y$.*

Definition 2.8. *An L-pseudo ordered set (X, P) is called an L-trellis with respect to the L-pseudo order P if each pair of elements $\{x, y\}$ of X has an infimum and a supremum.*

Example 2.5. *Let P be an L-pseudo order relation on $X = \{a, b, c, d\}$ defined by the following tables.*

P	a	b	c	d
a	1	0.5	0.3	0.1
b	0	1	0	0.4
c	0	0	1	0.5
d	0	0	0	1

$\{x, y\}$	$x \sqcup_P y$	$x \sqcap_P y$
$\{a, b\}$	b	a
$\{a, c\}$	a	c
$\{a, d\}$	d	a
$\{b, c\}$	d	a
$\{b, d\}$	d	b
$\{c, d\}$	d	c

It is easy to verify that P is reflexive, $*$-antisymmetric and is not $*$-transitive. Then, P is an L-pseudo order and (X, P) with the supremum and the infimum of each pair of elements $\{x, y\}$ on X given in the second table, is an L-trellis.

Definition 2.9. *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis. An element $x_0 \in T$ is called the least element (the minimum) of T with respect to P, if:*

(i) $P(x_0, x) = 1$*, for any* $x \in T$*;*

An element $x_1 \in T$ is called the greatest element (the maximum) of T with respect to P, if:

(i) $P(x, x_1) = 1$*, for any* $x \in T$*.*

Definition 2.10. *An L-trellis $(T, P, \sqcap_P, \sqcup_P)$, if P is $*$-transitive, then $(T, P, \sqcap_P, \sqcup_P)$ will be called an L-lattice.*

Proposition 2.6. *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis. Then the following statements are equivalent:*

(i) $P(x, y) = 1$;

(ii) $x \sqcap_P y = x$;

(iii) $x \sqcup_P y = y$.

Proof. Let $x, y \in T$ and $A = \{x, y\}$. First, suppose that $P(x, y) = 1$ and we show that $x \sqcap_P y = x$. Since P is a reflexive, it holds that $P(x, x) = 1$ and from the supposition $P(x, y) = 1$ it follows that $A^l(x) = \bigwedge_{t \in A} P(x, t) = P(x, x) \wedge P(x, y) = 1$. Next, we show that $A^l(t) \leq P(t, x)$, for any $t \in T$. $A^l(t) = \bigwedge_{a \in A} P(t, a) = P(t, x) \wedge P(t, y) \leq P(t, x)$, for any $t \in T$. Hence, $x = \sqcap_P A = x \sqcap_P y$. Second, if $x \sqcap_P y = x$, then it trivially holds that $x \sqcup_P y = y$. Third, suppose that $x \sqcup_P y = y$, then it holds that $A^u(y) = 1$. Hence, $\bigwedge_{t \in A} P(t, y) = 1$. Thus, $P(x, y) = 1$. □

Proposition 2.7. *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis. Then it holds that :*

(i) $x \sqcap_P x = x \sqcup_P x = x$*, for any* $x \in T$ *(idempotency);*

(ii) $x \sqcap_P y = y \sqcap_P x$ *and* $x \sqcup_P y = y \sqcup_P x$*, for any* $x, y \in T$ *(commutativity);*

(iii) $x \sqcap_P (x \sqcup_P y) = x = x \sqcup_P (x \sqcap_P y)$*, for any* $x, y \in T$ *(absorption);*

(iv) $x \sqcap_P ((x \sqcup_P y) \sqcap_P (x \sqcup_P z)) = x = x \sqcup_P ((x \sqcap_P y) \sqcup_P (x \sqcap_P z))$*, for any* $x, y, z \in T$ *(weak associativity).*

Proof. Assertion (i), (ii) are trivial.

(iii) We only prove (iii), as (iv) can be proved analogously.
Let $B = \{x, x \sqcup_P y\}$. From Proposition 2.6, it follows that $P(x, x \sqcup_P y) = 1$. Since, $B^l(x) = \bigwedge_{t \in B} P(x, t) = P(x, x) \wedge P(x, x \sqcup_P y) = 1$. Hence, x is a lower bound of B. If z is a lower bound of B, then $B^l(z) = \bigwedge_{t \in B} P(z, t) = P(z, x) \wedge P(z, x \sqcup_P y) \leq P(z, x)$. Hence, x is the greatest lower bound of B. Thus, $x \sqcap_P (x \sqcup_P y) = x$. Similarly, we obtain that $(x \sqcap_P y) \sqcup_P x = x$.

□

Proposition 2.8. *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis. If P is $*$-transitive, then it holds that*

(i) $P(x, y) \leq P(x \sqcap_P z, y)$*, for any* $z \in T$*;*

(ii) $P(x, y) \leq P(x, y \sqcup_P z)$*, for any* $z \in T$*.*

Proof. (i) From Proposition 2.6, it follows that $P(x \sqcap_P z, x) = 1$. Since P is $*$-transitive, it follows that $P(x \sqcap_P z, x) * P(x, y) \leq P(x \sqcap_P z, y)$. Hence, $P(x, y) \leq P(x \sqcap_P z, y)$.

(ii) Follows similarly like (i).

□

Proposition 2.9. [49] *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-lattice and $x, y, z \in T$, then*

(i) $P(z, x) * P(z, y) \leq P(z, x \sqcap_P y)$.

(i) $P(x, z) * P(y, z) \leq P(x \sqcup_P y, z)$.

Proof. (i) From Proposition 2.6, it follows that if $P(z, x) * P(z, y) \neq 0$, then z is a lower bound of $\{x, y\}$. Since $x \sqcap_P y$ is the greatest lower bound of $\{x, y\}$, it holds that $P(z, x \sqcap_P y) = 1$. Hence, $P(z, x) * P(z, y) \leq P(z, x \sqcap_P y)$.

(ii) Follows similarly as (i).

□

The following proposition discusses the associativity of the operations $\sqcap_P$ and $\sqcup_P$ in L-trellis if P is $*$-transitive.

Proposition 2.10. *Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis. The following statements are equivalent:*

(i) P is $*$-transitive;

(ii) $\sqcap_P$ and $\sqcup_P$ are associative.

Proof. Let $(T, P, \sqcap_P, \sqcup_P)$ be an L-trellis and $B = \{x, y, z\}$ be a subset of T. Suppose that P is an $*$-transitive relation and we show that $\sqcap_P$, $\sqcup_P$ are associative binary operations. On the one hand, from Proposition 2.6, it follows that, $P(x, x \sqcup_P (y \sqcup_P z)) = 1$. Since P is $*$-transitive relation, it holds that $P(y, x \sqcup_P (y \sqcup_P z)) \geq P(y, k) * P(k, x \sqcup_P (y \sqcup_P z))$ for any $k \in T$. Putting $k = y \sqcup_P z$ this implies that $P(y, x \sqcup_P (y \sqcup_P z)) \geq P(y, y \sqcup_P z) * P(y \sqcup_P z, x \sqcup_P (y \sqcup_P z)$, that is, $P(y, x \sqcup_P (y \sqcup_P z)) \geq 1 * 1$. Hence, $P(y, x \sqcup_P (y \sqcup_P z)) = 1$.

From Proposition 2.9 and Definition 2.10, it follows that $P(x, x \sqcup_P (y \sqcup_P z) * P(y, x \sqcup_P (y \sqcup_P z) \leq P(x \sqcup_P y, x \sqcup_P (y \sqcup_P z)$. Hence, $1 * 1 \leq P(x \sqcup_P y, x \sqcup_P (y \sqcup_P z)$. Thus, $P(x \sqcup_P y, x \sqcup_P (y \sqcup_P z)) = 1$. On the other hand, from the fact that $P(z, x \sqcup_P (y \sqcup_P z)) \geq P(z, y \sqcup_P z) * P(y \sqcup_P z, x \sqcup_P (y \sqcup_P z))$, then $P(z, x \sqcup_P (y \sqcup_P z)) \geq 1 * 1$. Hence, $P(z, x \sqcup_P (y \sqcup_P z)) = 1$. Thus,

$P((x \sqcup_P y) \sqcup_P z, x \sqcup_P (y \sqcup_P z)) = 1$. Similarly, we obtain that $P(x \sqcup_P (y \sqcup_P z), (x \sqcup_P y) \sqcup_P z) = 1$,since P is $*$-antisymmetric, we conclude that $x \sqcup_P (y \sqcup_P z) = (x \sqcup_P y) \sqcup_P z$. On a same way, we get that $x \sqcap_P (y \sqcap_P z) = (x \sqcap_P y) \sqcap_P z$. Therefore, $\sqcap_P$ and $\sqcup_P$ are associative binary operations on T.

Secondly, suppose that $x \sqcup_P y = y$ and $y \sqcup_P z = z$. Since $P(x, x \sqcup_P (y \sqcup_P z)) = 1$, it follows that $P(x, y) * P(y, z) \leq P(x, x \sqcup_P (y \sqcup_P z))$. Since $\sqcap_P$ and $\sqcup_P$ are associative, it follows that $P(x, y) * P(y, z) \leq P(x, (x \sqcup_P y) \sqcup_P z)$, then we get $P(x, y) * P(y, z) \leq P(x, y \sqcup_P z)$. Hence, $P(x, y) * P(y, z) \leqslant P(x, z)$. Thus P is $*$-transitive. □

The following theorem characterizes the structure of L-lattice $(Y, \mathcal{R}, \sqcap_{\mathcal{R}}, \sqcup_{\mathcal{R}})$.

Theorem 2.2. *Let (T, P) be an L-trellis and $(Y, \mathcal{R})$, then the L-ordered set$(Y, \mathcal{R})$ is an L-lattice, where*

$$(x, \alpha) \sqcap_{\mathcal{R}} (y, \beta) = (x \sqcap_P y, \alpha \wedge \beta) \text{ and } (x, \alpha) \sqcup_{\mathcal{R}} (y, \beta) = (x \sqcup_P y, \alpha \vee \beta),$$

for all $(x, \alpha), (y, \beta) \in Y$.

Proof. Let $(Y, \mathcal{R})$ be an L-ordered set and $B = \{(x, \alpha), (y, \beta)\}$. We have

$$\begin{aligned} B^l(x \sqcap_P y, \alpha \wedge \beta) &= \bigwedge_{(t,\lambda) \in B} \mathcal{R}((x \sqcap_P y, \alpha \wedge \beta); (t, \lambda)) \\ &= \mathcal{R}((x \sqcap_P y, \alpha \wedge \beta); (x, \alpha)) \bigwedge \mathcal{R}((x \sqcap_P y, \alpha \wedge \beta); (y, \beta)). \end{aligned}$$

and

$$\begin{aligned} B^u(x \sqcup_P y, \alpha \vee \beta) &= \bigwedge_{(t,\lambda) \in B} \mathcal{R}((t, \lambda); (x \sqcup_P y, \alpha \vee \beta)) \\ &= \mathcal{R}((x, \alpha); (x \sqcup_P y, \alpha \vee \beta)) \bigwedge \mathcal{R}((y, \beta); (x \sqcap_P y, \alpha \wedge \beta)). \end{aligned}$$

First, since (T, P) is an L-trellis, then the $x \sqcap_P y$ and $x \sqcup_P y$ exist for any $x, y \in T$. On the one hand, from Propositions 2.3 and 2.6, and the fact $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$, then it holds that

$$\begin{aligned} B^l(x \sqcap_P y, \alpha \wedge \beta) &= \mathcal{R}((x \sqcap_P y, \alpha \wedge \beta); (x, \alpha)) \bigwedge \mathcal{R}((x \sqcap_P y, \alpha \wedge \beta); (y, \beta)) \\ &= P(x \sqcap_P y, x) \bigwedge P(x \sqcap_P y, y) \\ &= 1 \bigwedge 1 \\ &= 1. \end{aligned}$$

Thus, $B^l(x \sqcap_P y, \alpha \wedge \beta) = 1$.

On the other hand, from Proposition 2.9, it follows that

$$\begin{aligned} B^l(t,\lambda) &= \bigwedge_{(a,\psi)\in B} \mathcal{R}((t,\lambda);(a,\psi)) \\ &= \mathcal{R}((t,\lambda);(x,\alpha)) \bigwedge \mathcal{R}((t,\lambda);(y,\beta)) \\ &\leq \mathcal{R}((t,\lambda);(x,\alpha) \sqcap_{\mathcal{R}} (y,\beta)) \\ &\leq \mathcal{R}((t,\lambda);(x \sqcap_P y, \alpha\wedge\beta)). \end{aligned}$$

Hence, $(x,\alpha) \sqcap_{\mathcal{R}} (y,\beta) = (x \sqcap_P y, \alpha \wedge \beta)$ is the greatest lower bound of B.

Second, since $\alpha \leq \alpha \vee \beta$ and $\beta \leq \alpha \vee \beta$, it follows from Propositions 2.3 and 2.6 that

$$\begin{aligned} B^u(x \sqcup_P y, \alpha\vee\beta) &= \mathcal{R}((x,\alpha);(x \sqcup_P y, \alpha\vee\beta)) \bigwedge \mathcal{R}((y,\beta);(x \sqcup_P y, \alpha\vee\beta)) \\ &= P(x, x\sqcup_P y) \bigwedge P(y, x \sqcup_P y) \\ &= 1 \bigwedge 1 \\ &= 1. \end{aligned}$$

Thus, $B^u(x \sqcup_P y, \alpha \vee \beta) = 1$.

Furthermore, from Proposition 2.9 it follows that

$$\begin{aligned} B^u(z,\gamma) &= \bigwedge_{(a,\phi)\in B} \mathcal{R}((a,\phi);(z,\gamma)) \\ &= \mathcal{R}((x,\alpha);(z,\gamma)) \bigwedge \mathcal{R}((y,\beta);(z,\gamma)) \\ &\leq \mathcal{R}((x,\alpha) \sqcup_{\mathcal{R}} (y,\beta);(z,\gamma)) \\ &\leq \mathcal{R}((x \sqcup_P y, \alpha\vee\beta);(z,\gamma)). \end{aligned}$$

Therefore, $(x,\alpha) \sqcup_{\mathcal{R}} (y,\beta) = (x \sqcup_P y, \alpha \vee \beta)$ is the smallest upper bound of B. Thus, $(Y, \mathcal{R})$ is an L-lattice. □

Combining Theorem 2.1 and Theorem 2.2 easily leads to the following theorem.

Theorem 2.3. *Any L-trellis is isomorphic with a contraction trellis of some L-lattice.*

3 Fuzzy ideals and filters on a trellis

The purpose of this chapter is to investigate fuzzy ideal and fuzzy filter concepts as fuzzy sets on (crisp) fuzzy trellis and their fundamental properties. We present interesting characterizations of these notions in terms of trellis operations and in terms of their specific subsets. Moreover, we introduce two interesting kinds, prime and principal fuzzy ideals and fuzzy filters with respect the weakly associative meet and join operations of this trellis and investigate their various characterizations and properties. The most results of this chapter in crisp and fuzzy case are discussed in [55].

3.1. Fuzzy ideals and filters on a trellis

In this section, we recall the notions of ideals and filters on a trellis introduced by Skala [67] and their fundamental properties. Next, we generalize these notions as an L-sets on crisp trellis.

3.1.1. Ideals and filters on a trellis

The notions of an ideal and a filter on a trellis was introduced by Skala [67].
Definition 3.1. [67] *Let T be a trellis. An ideal I of T is a non-empty subset of T such that*

(i) If $x \in I$, $y \in T$ and $y \trianglelefteq x$ implies $y \in I$;

(ii) If $x, y \in I$, then $x \sqcup y \in I$.

Let I be an ideal of a trellis T, some common types of ideals are defined as follows:

(i) I is a proper ideal if $I \neq T$ and if T contains 1, $1 \notin I$;

(ii) I is a prime ideal if is proper and $x \sqcap y \in I$ imply that either $x \in I$ or $y \in I$, for all $x, y \in T$;

(iii) I is maximal ideal of T if I is proper and the only ideal properly contains I is T.

Example 3.1. *Let T be the trellis given in Figure 1.4, a is the l.u.b of T. Then, the following sets*

$$M = \{b, a\}, N = \{c, b, a\},$$

are proper ideals of T and N is prime. An example of a subset which is not an ideal is $\{d, c, a\}$.

Definition 3.2. [67] *Let T be a trellis. A filter F of T is a non-empty subset of T such that*

(i) If $x \in F$, $y \in T$ and $x \trianglelefteq y$ implies $y \in F$;

(ii) If $x, y \in F$, then $x \sqcap y \in F$.

Let F be a filter of a trellis T, some common types of filters are defined as follows:

(i) F is a proper filter if $F \neq T$ and if T contains 0, $0 \notin F$;

(ii) F is a prime filter if is proper and $x \sqcup y \in F$ implies that either $x \in F$ or $y \in F$ for all $x, y \in T$;

(iii) F is an ultrafilter (or maximal filter) of T if F is proper and the only filter properly containing F is T.

The following proposition is immediate.

Proposition 3.1. *Let T be a trellis, T^d be its pseudo order-dual trellis and A is a set on T. Then it holds that A is an ideal on T if and only if A is a filter on T^d and conversely.*

The following propositions show some properties of ideals and filters in trellis structure, which are different from those in lattice. For more information (see.eg., [12, 13, 27]).

Proposition 3.2. [27] *Let T be a trellis, $\{I_\gamma, \gamma \in \Gamma\}$ a chain of they ideals (i.e.; for each $\gamma, \delta \in \Gamma, I_\gamma \subseteq I_\delta$ or $I_\delta \subseteq I_\gamma$). Then, $I = \bigcup\limits_{\gamma \in \Gamma} I_\gamma$ is also an ideal on T.*

Proposition 3.3. [27] *Let T be a trellis and I be a prime ideal on T. If $F = T - I$ is non-empty, then F is a prime filter on T.*

Proposition 3.4. [67] *Let T be a trellis and $\mathcal{I}(T)$ the set of all ideals of T. The intersection of any collection of ideals is an ideal. Furthermore, The set $\mathcal{I}(T)$ ordered by the set inclusion forms a complete lattice. The operation meet in $\mathcal{I}(T)$ coincides with the intersection.*

Every convex subset of a partially ordered set is the intersection of an ideal and a filter. It can be easily shown that this result is not true for the pseudo ordered sets in the general case.

Example 3.2. *Let $X = \{d, f, a, b, c, g, e\}$ be a pseudo ordered set given by Hasse diagram in the following Figure 3.1. Let set $C = \{a, b, c, g, f\}$, then C is clearly a convex subset of X, however, $I = X = F$ for each I an ideal and F a filter of X. Thus, $I \cap F = X \neq C$.*

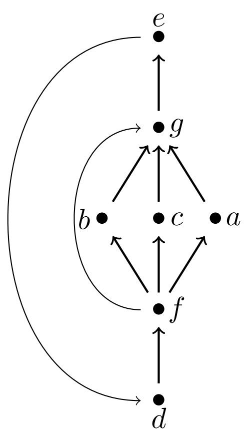

Figure 3.1: Representation of the psoset $X = \{d, f, a, b, c, g, e\}$

Lemma 3.1. [13] *If every convex subtrellis of a trellis T is the intersection of an ideal and a filter of T, then T is a lattice.*

A congruence relation $\equiv$ on a trellis T is an equivalence relation.

Definition 3.3. [67] *Let T be a trellis. A relation $\equiv$ defined on T as follows: $a \equiv b$ and $c \equiv d$, then $a \sqcup c \equiv b \sqcup d$ and $a \sqcap c \equiv b \sqcap d$, for any $a, b, c, d \in T$, is called a congruence relation.*

Proposition 3.5. [27] *Let T be a trellis and $\equiv$ be a congruence relation on T. If 0 is the least element of T, then the set $I = \{x \in T \mid x \equiv 0\}$ is an ideal of T.*

Proposition 3.6. [27] *Let $\mathcal{L} = (T, \trianglelefteq), \mathcal{M} = (\grave{T}, \grave{\trianglelefteq})$ be two trellises and let T have a least element 0. If φ is an homomorphism of T into $\grave{T}$, then $I = \{y \in T \mid \varphi(y) = 0\}$ forms an ideal on T.*

Remark 3.1. *These propositions can be also dualized for a trellis with the greatest element and for a filter.*

3.1.2. Down-sets and up-sets on a trellis

In this subsection, we introduce some important families of sets associated with any finite set on a trellis then, we provide the definition of down-set (resp. up-set) on a trellis.

Definition 3.4. *Let T be a trellis. For a given subset S on T. We define the following special subsets associated with S by:*

(i) $\downarrow S = \{x \in T : x \trianglelefteq y, \text{ for some } y \in S\}$;

(ii) $\uparrow S = \{x \in T : y \trianglelefteq x, \text{ for some } y \in S\}$;

(iii) $\downharpoonleft S = \{x \in T : x \lesssim y, \text{ for some } y \in S\}$;

(iv) $\upharpoonleft S = \{x \in T : y \lesssim x, \text{ for some } y \in S\}$.

Similarly, for a given element x on a trellis T,

(i) $\downarrow x = \{y \in T \mid y \trianglelefteq x\}$;

(ii) $\uparrow x = \{y \in T \mid x \trianglelefteq y\})$;

(iii) $\downharpoonleft x = \{y \in T \mid y \lesssim x\})$;

(iv) $\upharpoonleft x = \{y \in T \mid x \lesssim y\})$.

In the following propositions, we show some interesting properties of down-sets and up-sets on a trellis.

Proposition 3.7. *Let T be a trellis, T^d be its pseudo order-dual trellis and S be a set on T. The following statements hold*

(i) S is a down-set on T if and only if S is an up-set on T^d;

(ii) S is an up-set on T if and only if S is a down-set on T^d;

(iii) $\downarrow S$ on T coincides with $\uparrow S$ on T^d;

(iv) $\uparrow S$ on T coincides with $\downarrow S$ on T^d;

(v) $\downharpoonleft S$ on T coincides with $\upharpoonleft S$ on T^d;

(vi) $\upharpoonleft S$ on T coincides with $\downharpoonleft S$ on T^d.

Remark 3.2. *(i) One can see that if T is a lattice, then $\downarrow S = \downharpoonleft S$ and $\uparrow S = \upharpoonleft S$, for any set S on T;*

(ii) Contrary to the case of trellis, one easily verifies that the elimination of the property of transitivity of $\trianglelefteq$ makes that $\downarrow S$ (resp. $\uparrow S$) would

not necessarily down-set (resp. up-set) on T, for any set S on T.

For a given trellis T, the following proposition shows that the associated set $\downharpoonleft S$ (resp. $\upharpoonright S$) with a given set S on T is a down-set (resp. up-set).

Proposition 3.8. *Let T be a trellis and S be a set on T. It holds that*

(i) $\downharpoonleft S$ is the smallest down-set containing S;

(ii) $\upharpoonright S$ is the smallest up-set containing S.

Proof. Let T be a trellis and S be a subset on T. It is clear that $S \subset \downharpoonleft S$.

(i) Suppose that there exists another down-set $\downharpoonleft R$ containing S such that $S \subset \downharpoonleft R \subset \downharpoonleft S$. then there exists $y \in T$ such that $y \in \downharpoonleft S$ and $y \notin \downharpoonleft R$. From supposition it follows that $y \lesssim x$ for some $x \in S$, so $y \in \downharpoonleft R$. This is a contradiction. Hence, $\downharpoonleft S \subset \downharpoonleft R$. Therefore, $\downharpoonleft S = \downharpoonleft R$. Finally, we conclude that $\downharpoonleft S$ is the smallest down-set contain S.

(ii) Follows from Proposition 3.7 and (i).

□

Corollary 3.1. *let T be a trellis and S be a subset on T. Then, it holds that*

$$S \subseteq \downarrow S \subseteq \downharpoonleft S \text{ and } S \subseteq \uparrow S \subseteq \upharpoonright S.$$

Proposition 3.9. *Let T be a trellis and R, S be two sets on T. The following statements hold:*

(i) If $R \subseteq S$, then $\downarrow R \subseteq \downarrow S$;

(ii) $\downarrow (R \cup S) = \downarrow R \cup \downarrow S$;

(iii) $\downarrow (R \cap S) \subseteq \downarrow R \cap \downarrow S$.

Proof. (i) Since $R \subseteq \downarrow R$, it holds that $S \subseteq \downarrow R$. From Proposition 3.8 it holds that $\downarrow S \subseteq \downarrow R$.

(ii) On the one hand, it is easily verified from (i) that $\downarrow S \cup \downarrow R \subseteq \downarrow (S \cup R)$. On the other hand, let $y \in \downarrow (S \cup R)$, then $y \trianglelefteq x$ for some $x \in S \cup R$. Therefore, $x \in S$ or $x \in R$ this implies that $y \in \downarrow S$ or $y \in \downarrow R$. Hence, $y \in \downarrow S \cup \downarrow R$. Thus, $\downarrow (S \cup R) \subseteq \downarrow S \cup \downarrow R$. Finally we conclude that $\downarrow (R \cup S) = \downarrow R \cup \downarrow S$.

(iii) Follows from Proposition 3.8 and (i).

□

In the same direction, a dual version of the Proposition 3.9 can also be obtained for up-sets. Its proof follows from Propositions 3.7 and 3.9.

Proposition 3.10. *Let T be a trellis and R, S be two sets on T. The following statements hold:*

(i) If $R \subseteq S$, then $\uparrow R \subseteq \uparrow S$;

(ii) $\uparrow (R \cup S) = \uparrow R \cup \uparrow S$;

(iii) $\uparrow (R \cap S) \subseteq \uparrow R \cap \uparrow S$.

Remark 3.3. *These previous propositions remain true for the down-set $\downharpoonleft S$ (resp. up-set $\upharpoonleft S$). Contrary of the lattice, for a subset S of a trellis T, $\downarrow (\downarrow S) \neq \downarrow S$ and $\uparrow (\uparrow S) \neq \uparrow S$ but $\downharpoonleft (\downharpoonleft S) = \downharpoonleft S$ and $\upharpoonleft (\upharpoonleft S) = \upharpoonleft S$. Indeed, let us consider the trellis T given in Figure 1.4 and the set $S = \{d\}$. Then, $\downarrow d = \{d, c, a\}$ and $\downarrow (\downarrow d) = \{d, c, b, a\}$. Note that $\downarrow d \neq \downarrow\downarrow d$. Otherwise, $\downharpoonleft d = \{d, c, b, a\}$ and $\downharpoonleft (\downharpoonleft d) = \{d, c, b, a\}$, then we find that $\downharpoonleft (\downharpoonleft d) = \downharpoonleft d$.*

In the following result, we show that any ideal (resp. filter) on a trellis T is a down-set (resp. up-set) on T.

Theorem 3.1. *Let T be a trellis and S be a set on T. The following implications hold:*

(i) If S is an ideal, then S is a down-set;

(ii) If S is a filter, then S is an up-set.

Proof. We only prove (i), (ii) is proved analogously.

(i) Let S be an ideal on a trellis T. Let $x \in S$. For any element $y \in T$ such that $y \lesssim x$, it holds that $y \in S$. Thus, S is a down-set on T. □

In view of the previous theorem, we obtain the following proposition.

Proposition 3.11. *Let T be a trellis and S be a subset on T. The following implications hold:*

(i) If S is an ideal, then $\downharpoonleft S = S$;

(ii) If S is a filter, then $\upharpoonleft S = S$.

Proof. We only prove (i), as (ii) can be proved analogously.

(i) Suppose that S is an ideal on T. It is easy to check that $S \subseteq \downharpoonleft S$. Let $y \in \downharpoonleft S$, then $y \lesssim x$ for some $x \in S$. Since S is an ideal, then $y \in S$. Hence, $\downharpoonleft S \subseteq S$. Thus, $S = \downharpoonleft S$.

(ii) Follows from Proposition 3.7 and (i).

□

Remark 3.4. *The converse of the above proposition does not necessarily hold. Indeed, let us consider the trellis T given in Figure 1.4 and $S = \{d, c, b, a\}$. Therefore, $\downharpoonleft S = \{d, c, b, a\} = S$ which is not is an ideal on T.*

3.1.3. Principal ideals and filters in a trellis

In this subsection, we provide some interesting characterizations of principal ideals (resp. filters) on a trellis in terms of left-transitive element (resp. right-transitive element). First, we need to recall the following definition of down-set (resp. an up-set) on a trellis.

Definition 3.5. *Let T be a trellis and S be a non-empty subset of T. The ideal generated by S, denoted by $\langle S$, is the intersection of all ideals of T containing S. The filter generated by S, denoted by $S\rangle$, is the intersection of all filters of T containing S. If S is a singleton $\{x\}$, then we write $\langle x$ the ideal generated by x and $x\rangle$ the filter generated by x.*

Definition 3.6. *Let T be a trellis. I and F be two subsets on T. Then it hold that*

(i) The set I is a principal ideal, if there exists an element $x \in T$ such that $I = \langle x$.

(ii) The set F is a principal filter, if there exists an element $x \in T$ such that $I = x\rangle$.

Remark 3.5. *For the case of lattice $\langle x = \{x \in T \mid x \trianglelefteq a\}$. For the general case of a trellis it is not true.*

Lemma 3.2. [67] *Let T be a trellis, $a, b \in T$, $a \trianglelefteq b$ implies $\langle a \subseteq \langle b$.*

Proof. From Definition 3.1. We obtain the assertion. □

Proposition 3.12. *Let T be a trellis and x be an element on T. Then it holds that*

(i) The set $\downarrow x$ is an ideal if and only if x is left-transitive element;

(ii) The set $\uparrow x$ *is a filter if and only if* x *is right-transitive element.*

Proof. It is easy to check that if $\downarrow x$ is an ideal (resp. $\uparrow x$ is a filter), then x is left-transitive (resp. x is right-transitive). Conversely,

(i) Suppose that x is a left-transitive element on T. We show that $\downarrow x$ is an ideal.

  (a) Let $y \in \downarrow x$ and $z \trianglelefteq y$. Since x is left-transitive it follows that $z \trianglelefteq x$. Hence, $z \in \downarrow x$.

  (b) Let $y, z \in \downarrow x$, then it holds that $y \trianglelefteq x$ and $z \trianglelefteq x$. This implies that $y \sqcup z \trianglelefteq x$. Hence, $y \sqcup z \in \downarrow x$. Thus, $\downarrow x$ is an ideal.

(ii) Follows from Proposition 3.7 and (i).

□

Remark 3.6. *For an element* x *of a trellis* T*.*

(i) The set $\langle x = \downarrow x$ *if and only if* x *is left-transitive.*

(ii) The set $x\rangle = \uparrow x$ *if and only if* x *is right-transitive.*

Theorem 3.2. *Let* T *be a trellis,* x *an element of* T*. The following equivalences hold*

(i) The set $\rfloor x = \langle x$ *if and only if* $\rfloor x$ *is a* $\sqcup$*-semi-trellis of* T*.*

(ii) The set $\lceil x = x\rangle$ *if and only if* $\lceil x$ *in is an* $\sqcap$*-semi-trellis of* T*.*

Proof. (i) We only show that $\rfloor x$ is a principal ideal, as $\lceil x$ can be proved analogously.

  (a) Let $y \in \rfloor x$ and $z \in T$ such that $z \trianglelefteq y$. $y \in \rfloor x$ this implies that $y \lesssim x$. Since $\lesssim$ is transitive, it follows that $z \lesssim x$. Hence, $z \in \rfloor x$.

  (b) Let $y, z \in \rfloor x$. Then it holds that, $y \lesssim x$ and $z \lesssim x$. Since $\rfloor x$ is a $\sqcup$-semi-trellis then $\{a \sqcup b\}$ exists in $\rfloor x$ for any $a, b \in \rfloor x$, it follows that $y \sqcup z \in \rfloor x$. Hence, $\rfloor x$ is an ideal on T.

  Now, we show that $\rfloor x$ is the smallest ideal containing x. Suppose that there exists another ideal J containing x such that $J \subseteq \rfloor x$. then there exist $z \in T$ such that $z \in \rfloor x$ and $z \notin J$. $z \in \rfloor x$ implies that $z \lesssim x$ then there exist a finite sequence of elements $(a_1, ..., a_n)$ from T such that $z \trianglelefteq a_1 \trianglelefteq ... \trianglelefteq a_n \trianglelefteq x$. Since J is an ideal containing x, then it holds that $z \in J$. Hence, $\downharpoonleft x \subseteq J$ and $\downharpoonleft x = J$, contradiction with the supposition. Thus, $\downharpoonleft x$ is the smallest ideal containing x.

(ii) Follows from Proposition 3.7 and (i).

□

Corollary 3.2. *Let T be a trellis, x an element on T. The following implications hold*

(i) If x is a left-transitive, then $\downharpoonleft x = \downarrow x$.

(ii) If x is a right-transitive, then $\upharpoonleft x = \uparrow x$.

Example 3.3. *Let T be the trellis depicted below in the following Figure.*

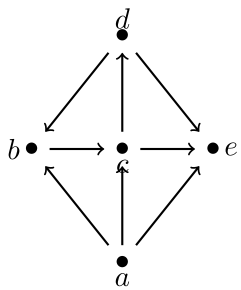

Figure 3.2: Illustration of an ideal on a trellis T.

Note that d is not a left-transitive element (i.e., $b \trianglelefteq c$ and $c \trianglelefteq d$ but $b \ntrianglelefteq d$) and the set I defined by $I = \downharpoonleft d = \{d, c, b, a\}$ is an ideal on T because of $\{x \sqcup y\} \in \downharpoonleft d$, for any $x, y \in \downharpoonleft d$. Furthermore, since e is a left-transitive element we conclude that $\downharpoonleft e = \downarrow e$.

Lemma 3.3. [13] *Let T be an acyclic trellis satisfying the ascending p-chain condition. Then every ideal I of T is a principal ideal written as $I = \downarrow a$ and a is a left-transitive element of T.*

Proof. Let I is an ideal of T. Since T is acyclic and satisfying the ascending p-chain condition, every non-empty set of T has a maximal element. Hence, I has a maximal element say a. As I is a subtrellis of T, a is the greatest element of I and clearly $I = \langle a$. Let $x \trianglelefteq y \trianglelefteq a$. As I is an ideal containing a, both y and x belong to I. Since a is the greatest element of I, then it follows that $x \trianglelefteq a$. Therefore a must be left-transitive. Thus, $\langle a = \downarrow a$. □

3.1.4. Fuzzy trellises on a trellis

In this subsection, inspired from the concept of fuzzy lattice introduced in [1]. We define the concept of fuzzy trellis on a crisp trellis as an L-set stable by the supremum and the infimum of the binary operations $\sqcap$ *and* $\sqcup$. *We generally use the sign* $\wedge$ *for min operation and* $\vee$ *for max operation on a bounded lattice* L.

Definition 3.7. *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis and* A *be an* L*-set on* T *(i.e.,* $A : T \to L$*). Then* A *is called an* L*-trellis if for all* $x, y \in T$*, the following conditions are satisfied*

(i) $A(x \sqcup y) \geq A(x) \wedge A(y)$;

(ii) $A(x \sqcap y) \geq A(x) \wedge A(y)$.

Example 3.4. *In Figure 1.4, the Hasse diagram of a trellis* T *with* $T = \{a, b, c, d, e\}$ *is depicted. The* L*-set* A *on* T *given by:* $A = \{< a, 0.5 >, < b, 0.4 >, < c, 0.4 >, < d, 0.7 >, < e, 0.5 >\}$ *is an* L*-trellis.*

3.1.5. Fuzzy ideals and filters in a trellis

Inspired from the concept of fuzzy ideal (resp. filter) on a lattice introduced by Davvaz [32] and , we define the concept of fuzzy ideal (resp. fuzzy filter) as an L*-sets on a trellis.*

Definition 3.8. *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis and* I *be an* L*-set on* T*. Then* I *is called a* L*-fuzzy ideal on* T *if for all* $x, y \in T$*, the following conditions are satisfied:*

(i) $I(x \sqcup y) \geq I(x) \wedge I(y)$;

(ii) $I(x \sqcap y) \geq I(x) \vee I(y)$.

Example 3.5. *Let* T *be the trellis given by the Hasse diagram in Figure 1.4. The* L*-set* I *on* T *defined by:* $I = \{< a, 0.5 >, < b, 0.3 >, < c, 0.2 >, < d, 0.1 >, < e, 0.1 >\}$ *is an* L*-fuzzy ideal.*

Definition 3.9. *Let* $(T, \trianglelefteq, \sqcap, \sqcup)$ *be a trellis and* F *be an* L*-set on* T*. Then* F *is called a fuzzy filter on* T *if for all* $x, y \in T$*, the following conditions are satisfied*

(i) $F(x \sqcup y) \geq F(x) \vee F(y)$;

(ii) $F(x \sqcap y) \geq F(x) \wedge F(y)$.

Example 3.6. *Let T be the trellis given by the Hasse diagram in Figure 1.4. The L-set F on T defined by: $F = \{< a, 0.1 >, < b, 0.2 >, < c, 0.2 >, < d, 0.1 >, < e, 0.5 >\}$ is an L-fuzzy filter.*

Remark 3.7. *Notice that every L-fuzzy ideal on T is an L-trellis, but the converse is not true in general. Indeed, Let T be the trellis given by the Hasse diagram in Figure 1.4, and A be an L-set on T defined by: $A = \{< a, 0.5 >, < b, 0.4 >, < c, 0.4 >, < d, 0.7 >, < e, 0.5 >\}$. Trivially, A is an L-trellis. But, since $A(a) = A(b \sqcap d) = 0.5 \ngeq \max\{0.4; 0.7\}$ then A isn't an L-fuzzy ideal on X. Also, since $A(e) = A(b \sqcup d) = 0.5 \ngeq \max\{0.4; 0.7\}$ then it holds that A isn't an L-fuzzy filter on T.*

The following proposition is immediate.

Proposition 3.13. *Let T be a trellis, T^d be its pseudo order-dual trellis and A be an L-set on T. Then A is a fuzzy ideal on T if and only if A is a fuzzy filter on T^d, and conversely.*

3.1.6. Fuzzy down-sets and up-sets of a trellis

In this subsection, we introduce the notion of L-fuzzy down-set (resp. L-fuzzy up-set) on a trellis analogously to the crisp down-set (resp. up-set), and study their interesting properties.

Definition 3.10. *Let T be a trellis and S be an L-set on T.*

(i) The set S is called L-fuzzy down-set if $S(x) \geq S(y)$ for all $x \trianglelefteq y$, such that $x, y \in S$.

(ii) Dually, S is called L-fuzzy up-set if $S(x) \leq S(y)$ for all $x \trianglelefteq y$, such that $x, y \in S$.

Definition 3.11. *For a given an L-set S on a trellis T, we denote by:*

(i) $\Downarrow S$ the L-set associated with S defined by:

$$\Downarrow S(x) = \sup_{y \in \uparrow x} S(y).$$

(ii) $\Uparrow S$ the L-set associated with S defined by:

$$\Uparrow S(x) = \sup_{y \in \downarrow x} S(y).$$

(iii) $\downharpoonleft\downharpoonleft S$ *the L-set associated with S defined by:*

$$\downharpoonleft\downharpoonleft S(x) = \sup_{y \in \upharpoonleft x} S(y).$$

(iv) $\upharpoonleft\upharpoonleft S$ *the L-set associated with S defined by:*

$$\upharpoonleft\upharpoonleft S(x) = \sup_{y \in \downharpoonleft x} S(y).$$

Remark 3.8. *For any crisp set S on a given trellis T, it holds that*

(i) $\Downarrow S = \downarrow S$;

(ii) $\Uparrow S = \uparrow S$;

(iii) $\downharpoonleft\downharpoonleft S = \downharpoonleft S$;

(iv) $\upharpoonleft\upharpoonleft S = \upharpoonleft S$.

Now, we show some interesting properties of L-fuzzy down-sets (resp. L-fuzzy up-sets) on a trellis.

Proposition 3.14. *Let T be a trellis, T^d be its pseudo order-dual trellis and S be an L-set on T. The following statements hold.*

(i) *S is an L-fuzzy down-set on T if and only if S is an L-fuzzy up-set on T^d;*

(ii) *S is an L-fuzzy up-set on T if and only if S is an L-fuzzy down-set on T^d;*

(iii) $\Downarrow S$ *on T coincides with* $\Uparrow S$ *on T^d;*

(iv) $\Uparrow S$ *on T coincides with* $\Downarrow S$ *on T^d;*

(v) $\downharpoonleft\downharpoonleft S$ *on T coincides with* $\upharpoonleft\upharpoonleft S$ *on T^d;*

(vi) $\upharpoonleft\upharpoonleft S$ *on T coincides with* $\downharpoonleft\downharpoonleft S$ *on T^d.*

Proof. Straightforward. □

Remark 3.9. *For a given trellis T, one easily verifies that the elimination of the property of transitivity of $\trianglelefteq$ makes that $\Downarrow S$ (resp. $\Uparrow S$) would not necessarily L-fuzzy down-set (resp. L-fuzzy up-set) on T, for any set S on T.*

The following proposition shows that L-fuzzy down-sets (resp. L-fuzzy up-sets) on a trellis are closed under the union and intersection of L-sets.

Proposition 3.15. *Let T be a trellis and R, S be two L-sets on T. It holds that*

(i) If R and S are two L-fuzzy down-sets, then $R \cup S$ and $R \cap S$ are L-fuzzy down-sets.

(ii) If R and S are two L-fuzzy up-sets, then $R \cup S$ and $R \cap S$ are L-fuzzy up-sets.

Proof. (i) We only show that $R \cup S$ is an L-fuzzy down-set, as $R \cap S$ can be proved analogously. Let $x, y \in T$ such that $x \trianglelefteq y$. Since R and S are L-fuzzy down-sets, and $R \cup S(x) = R(x) \vee S(x)$, it follows that $R \cup S(x) \geq R(y) \vee S(y) = R \cup S(y)$. Thus, $R \cup S$ is an L-fuzzy down-set.

(ii) Follows from Proposition 3.14 and (i).

□

Proposition 3.16. *Let T be a trellis and S be an L-set on T. It holds that*

(i) $\downharpoonright\downharpoonleft S$ is the smallest L-fuzzy down-set containing S;

(ii) $\upharpoonleft\upharpoonright S$ is the smallest L-fuzzy up-set containing S.

Proof. (i) Let $x, y \in X$ such that $x \lesssim y$. We show that $\downharpoonright\downharpoonleft S(y) \geq \downharpoonright\downharpoonleft S(x)$.

$$\begin{aligned} \downharpoonright\downharpoonleft S(y) &= \sup_{y \in \upharpoonleft t} S(t) = \sup_{y \lesssim t} S(t) \\ &\leq \sup_{x \lesssim y \lesssim t} S(t) = \sup_{x \lesssim t} S(t) \\ &= \downharpoonright\downharpoonleft S(x). \end{aligned}$$

Hence, $\downharpoonright\downharpoonleft S$ is an L-fuzzy down-set.

Furthermore, let $\downharpoonright\downharpoonleft R$ be another L-fuzzy down-set containing S such that $S \subseteq \downharpoonright\downharpoonleft R \subseteq \downharpoonright\downharpoonleft S$. Since $S \subseteq \downharpoonright\downharpoonleft R$ then it holds that $\downharpoonright\downharpoonleft S(x) = \sup_{t \in \upharpoonleft x} S(t) \leq \sup_{t \in \upharpoonleft x} R(t) \leq \downharpoonright\downharpoonleft R(x)$. Thus, $\downharpoonright\downharpoonleft S \subseteq \downharpoonright\downharpoonleft R$. Therefore, $\downharpoonright\downharpoonleft S = \downharpoonright\downharpoonleft R$. Finally, we conclude that $\downharpoonright\downharpoonleft S$ is the smallest L-fuzzy down-set containing S.

(ii) Follows and from Proposition 3.14 and (i).

□

Corollary 3.3. *let T be a trellis and S be an L-set on T. Then, it holds that*

$$S \subseteq \Downarrow S \subseteq \downharpoonleft\downharpoonleft S \text{ and } S \subseteq \Uparrow S \subseteq \upharpoonleft\upharpoonleft S.$$

The following Corollary given a characterization of L-fuzzy down-sets and L-fuzzy up-sets.

Corollary 3.4. *Let T be a trellis and S be an L-set on T. The following equivalences hold:*

(i) S is an L-fuzzy down-set if and only if $S = \downharpoonleft\downharpoonleft S$;

(ii) S is an L-fuzzy up-set if and only if $S = \upharpoonleft\upharpoonleft S$.

The following propositions lists some properties of fuzzy down-sets and fuzzy up-sets.

Proposition 3.17. *Let T be a trellis and S, R be two L-sets on T. The following statements hold*

(i) If $S \subseteq R$, then $\downharpoonleft\downharpoonleft S \subseteq \downharpoonleft\downharpoonleft R$ and $\Downarrow S \subseteq \Downarrow R$.

(ii) $\downharpoonleft\downharpoonleft (S \cup R) = \downharpoonleft\downharpoonleft S \cup \downharpoonleft\downharpoonleft R$ and $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$.

(iii) $\downharpoonleft\downharpoonleft (S \cap R) \subseteq \downharpoonleft\downharpoonleft S \cap \downharpoonleft\downharpoonleft R$ and $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

Proof. We only give the proof for $\downharpoonleft\downharpoonleft$ as for $\Downarrow$ can be proved analogously.

(i) Suppose that $S \subseteq R$, then $\mu_{\downharpoonleft\downharpoonleft S}(x) = \sup\limits_{y \in \upharpoonleft x} \mu_S(y) \leq \sup\limits_{y \in \upharpoonleft x} \mu_R(y) = \mu_{\downharpoonleft\downharpoonleft R}(x)$. Hence, $\downharpoonleft\downharpoonleft S \subseteq \downharpoonleft\downharpoonleft R$.

(ii) On the one hand, we easily verify from (i) that $\downharpoonleft\downharpoonleft S \cup \downharpoonleft\downharpoonleft R \subseteq \downharpoonleft\downharpoonleft (S \cup R)$. On the other hand, from Propositions 3.16 and 3.15 it holds that $\downharpoonleft\downharpoonleft S \cup \downharpoonleft\downharpoonleft R$ is a fuzzy down-set. Now the fact that $S \cup R \subseteq \downharpoonleft\downharpoonleft S \cup \downharpoonleft\downharpoonleft R$ implies that $\downharpoonleft\downharpoonleft (S \cup R) = \downharpoonleft\downharpoonleft S \cup \downharpoonleft\downharpoonleft R$.

(iii) The proof is similar to that of (ii).

□

In the same direction, a dual version of Proposition 3.17 can also be obtained for fuzzy up-sets. Its proof follows from Propositions 3.17 and 3.14.

Proposition 3.18. *Let T be a trellis and S, R be two fuzzy sets on T. The following statements hold*

(i) *If $S \subseteq R$, then $\upharpoonleft\upharpoonleft S \subseteq \upharpoonleft\upharpoonleft R$ and $\Uparrow S \subseteq \Uparrow R$.*

(ii) $\upharpoonleft\upharpoonleft (S \cup R) = \upharpoonleft\upharpoonleft S \cup \upharpoonleft\upharpoonleft R$ *and* $\Uparrow (S \cup R) = \Uparrow S \cup \Uparrow R$.

(iii) $\upharpoonleft\upharpoonleft (S \cap R) \subseteq \upharpoonleft\upharpoonleft S \cap \upharpoonleft\upharpoonleft R$ *and* $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

Remark 3.10. *The previous propositions remain true for the L-fuzzy down-set $\downharpoonleft\downharpoonleft S$ (resp. L-fuzzy up-set $\upharpoonleft\upharpoonleft S$). Note that if S be a fuzzy set on a trellis T, then*

$\Downarrow (\Downarrow S) \neq \Downarrow S($ *resp.* $\Uparrow (\Uparrow S) \neq \Uparrow S)$ *but* $\downharpoonleft\downharpoonleft (\downharpoonleft\downharpoonleft S) = \downharpoonleft\downharpoonleft S($*resp.* $\upharpoonleft\upharpoonleft (\upharpoonleft\upharpoonleft S) = \upharpoonleft\upharpoonleft S)$.

Example 3.7. *Let T be the trellis given by the Hasse diagram in Figure 1.4 and S be the fuzzy set on T given by:*

$S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.2 >, < d, 0.7 >, < e, 0.1 >\}$. *It holds that*

$\Downarrow S(a) = \sup_{y \in \uparrow a} S(y) = 0.8$ *such that* $\uparrow a = \{a, b, c, d, e\}$,

$\Downarrow S(b) = \sup_{y \in \uparrow b} S(y) = 0.6$ *such that* $\uparrow b = \{b, c, e\}$,

$\Downarrow S(c) = \sup_{y \in \uparrow c} S(y) = 0.7$ *such that* $\uparrow c = \{c, d, e\}$,

$\Downarrow S(d) = \sup_{y \in \uparrow d} S(y) = 0.7$ *such that* $\uparrow d = \{d, e\}$.

$\Downarrow S(e) = \sup_{y \in \uparrow e} S(y) = 0.1$ *such that* $\uparrow e = \{e\}$.

Therefore, $\Downarrow S = \{< a, 0.8 >, < b, 0.6 >, < c, 0.7 >, < d, 0.7 >, < e, 0.1 >\}$. *In the same way, we obtain that* $\Downarrow (\Downarrow S) = \{< a, 0.8 >, < b, 0.7 >, < c, 0.7 >, < d, 0.7 >, < e, 0.1 >\}$. *Thus,* $\Downarrow (\Downarrow S) \neq \Downarrow S$. *Otherwise,* $\downharpoonleft\downharpoonleft (\downharpoonleft\downharpoonleft S) = \downharpoonleft\downharpoonleft S$.

The following proposition shows the interaction of the support and the kernel with the notions of L-fuzzy down-set and up-set.

Proposition 3.19. *Let T be a trellis and S be an L-set on T. It holds that*

(i) $Supp(\downharpoonleft\downharpoonleft S) = \downharpoonleft Supp(S)$ *and* $Ker(\downharpoonleft\downharpoonleft S) = \downharpoonleft Ker(S)$;

(ii) $Supp(\upharpoonleft\upharpoonleft S) = \upharpoonleft Supp(S)$ *and* $Ker(\upharpoonleft\upharpoonleft S) = \upharpoonleft Ker(S)$.

Proof. (i) First, we prove that $Supp(\downharpoonleft\downharpoonleft S) = \downharpoonleft Supp(S)$. On the one hand, let $x \in Supp(\downharpoonleft\downharpoonleft S)$. Then it holds that $\downharpoonleft\downharpoonleft S(x) > 0$, so $\sup\limits_{y \in \upharpoonleft x} S(y) > 0$. This implies that there exists $t \in \upharpoonleft x$ such that $S(t) > 0$. Hence, $t \in Supp(S)$. Since $t \in \upharpoonleft x$, it follows that $x \in \downharpoonleft Supp(S)$. Thus, $Supp(\downharpoonleft\downharpoonleft S) \subseteq \downharpoonleft Supp(S)$.

On the other hand, let $x \in \downharpoonleft Supp(S)$. Then it holds that $y \in \upharpoonleft x$ such that $S(y) > 0$, and then $\sup\limits_{y \in \upharpoonleft x} S(y) = \downharpoonleft\downharpoonleft S(x) > 0$. Hence, $x \in Supp(\downharpoonleft\downharpoonleft S)$.

Thus, $\downharpoonleft Supp(S) \subseteq Supp(\downharpoonleft\downharpoonleft S)$. Therefore, $Supp(\downharpoonleft\downharpoonleft S) = \downharpoonleft Supp(S)$.

The proof of $Ker(\downharpoonleft\downharpoonleft S) = \downharpoonleft Ker(S)$ is similar.

(ii) Follows from Proposition 3.14 and (i).

□

In the following result, we show that any L-fuzzy ideal (resp. L-fuzzy filter) on a trellis T is an L-fuzzy down-set (resp. L-fuzzy up-set) on T.

Theorem 3.3. *Let T be a trellis and S be an L-set on T. The following implications hold*

(i) If S is an L-fuzzy ideal, then S is an L-fuzzy down-set.

(ii) If S is an L-fuzzy filter, then S is an L-fuzzy up-set.

Proof. (i) Let $x, y \in T$ such that $x \trianglelefteq y$. Since S is an L-fuzzy ideal, it follows that $S(x) = S(x \sqcap y) \geq S(x) \vee S(y)$. Hence, $S(x) \geq S(y)$. Thus, S is an L-fuzzy down-set.

(ii) Follows from Proposition 3.14 and (i).

□

In view of Theorem 3.3 and Corollary 3.3, we obtain the following corollary.

Corollary 3.5. *Let T be a trellis and S be an L-set on T. The following implications hold*

(i) If S is an L-fuzzy ideal, then $S = \Downarrow S = \downharpoonleft\downharpoonleft S$.

(ii) If S is an L-fuzzy filter, then $S = \Uparrow S = \upharpoonleft\upharpoonleft S$.

Remark 3.11. *The converse of the above theorem does not necessarily hold. Indeed, let us consider the trellis T depicted in Figure 1.4 and S is an L-set on T given by $S = \{< a, 0.7 >, < b, 0.4 >, < c, 0.3 >, < d, 0.2 >, < e, 0.1 >\}$. Then, it follows that*

$\Downarrow S(a) = \sup_{y \in \uparrow a} S(y) = 0.7$ *such that* $\uparrow a = \{a, b, c, d, e\}$,

$\Downarrow S(b) = \sup_{y \in \uparrow b} S(y) = 0.4$ *such that* $\uparrow b = \{b, c, e\}$,

$\Downarrow S(c) = \sup_{y \in \uparrow c} S(y) = 0.3$ *such that* $\uparrow c = \{c, e, d\}$,

$\Downarrow S(d) = \sup_{y \in \uparrow d} S(y) = 0.2$ *such that* $\uparrow d = \{d, e\}$.

$\Downarrow S(e) = \sup_{y \in \uparrow e} S(y) = 0.1$ *such that* $\uparrow e = \{e\}$.

Therefore, $\Downarrow S = \{< a, 0.7 >, < b, 0.4 >, < c, 0.3 >, < d, 0.2 >, < e, 0.1 >\}$.

Thus, $\Downarrow S = S$ *but* S *is not an* L*-fuzzy ideal on* T.

Remark 3.12. *Let T be a trellis and S be an L-set on T. If S is not an L-fuzzy ideal (resp. L-fuzzy filter), then it holds that $\Downarrow S \neq S$ (resp. $\Uparrow S \neq S$), as can be seen in Example 3.8.*

Example 3.8. *Let T be the trellis depicted in Figure 1.2 and S be the L-set on T given by:*

$S = \{< a, 0.5 >, < b, 0.4 >, < c, 0.4 >, < d, 0.7 >, < e, 0.5 >\}$. *It holds that*

$\Downarrow S(a) = \sup_{y \in \uparrow a} S(y) = 0.7$ *such that* $\uparrow a = \{a, b, c, d, e\}$,

$\Downarrow S(b) = \sup_{y \in \uparrow b} S(y) = 0.5$ *such that* $\uparrow b = \{b, c, e\}$,

$\Downarrow S(c) = \sup_{y \in \uparrow c} S(y) = 0.7$ *such that* $\uparrow c = \{c, d, e\}$,

$\Downarrow S(d) = \sup_{y \in \uparrow d} S(y) = 0.7$ *such that* $\uparrow d = \{d, e\}$.

$\Downarrow S(e) = \sup_{y \in \uparrow e} S(y) = 0.5$ *such that* $\uparrow e = \{e\}$.

Therefore, $\Downarrow S = \{< a, 0.7 >, < b, 0.5 >, < c, 0.7 >, < d, 0.7 >, < e, 0.5 >\}$. *In the same way, we obtain that* $\Uparrow S = \{< a, 0.5 >, < b, 0.5 >, < c, 0.5 >, < d, 0.7 >, < e, 0.7 >\}$. *Thus,* $\Downarrow S \neq S$ *and* $\Uparrow S \neq S$.

3.2. Characterizations of fuzzy ideals and fuzzy filters on a trellis

In this section, following the work of Milles, Zedam and Rak [53, 54], we provide interesting characterizations of fuzzy ideals and fuzzy filters on a (crisp) trellis.

3.2.1. Basic characterization of fuzzy ideals and fuzzy filters on a trellis

In this subsection, we characterize the notion of fuzzy ideals and fuzzy filters as an L-sets on a trellis in terms of the trellis operations. We start with the key results.

Theorem 3.4. *Let T be a trellis and A be an L-set on T. Then for all $x, y \in T$, the following statements hold*

(i) $(A(x \sqcap y) \geq A(x) \vee A(y))$ *if and only if* $(x \trianglelefteq y \Rightarrow A(x) \geq A(y))$;

(ii) $(A(x \sqcup y) \geq A(x) \vee A(y))$ *if and only if* $(x \trianglelefteq y \Rightarrow A(x) \leq A(y))$.

Proof. Let $x, y \in T$.

(i) Suppose that $A(x \sqcap y) \geq A(x) \vee A(y)$. If $x \trianglelefteq y$ then $x \sqcap y = x$. Since $A(x \sqcap y) \geq A(x) \vee A(y)$, it follows that $A(x) = A(x \sqcap y) \geq A(x) \vee A(y)$. Hence, $A(x) \geq A(y)$.

Conversely, suppose that $(x \trianglelefteq y \Rightarrow A(x) \geq A(y))$. Then it follows that $A(x \sqcap y) \geq A(x)$ and $A(x \sqcap y) \geq A(y)$. Hence, $A(x \sqcap y) \geq A(x) \vee A(y)$;

(ii) Suppose that $A(x \sqcup y) \geq A(x) \vee A(y)$. If $x \trianglelefteq y$ then $x \sqcup y = y$. Since $A(x \sqcup y) \geq A(x) \vee A(y)$, it follows that $A(y) = A(x \sqcup y) \geq A(x) \vee A(y)$. Hence, $A(x) \leq A(y)$.

Conversely, suppose that $(x \trianglelefteq y \Rightarrow A(x) \leq A(y))$. Then it follows that $A(x) \leq A(x \sqcup y)$ and $A(y) \leq A(x \sqcup y)$. Hence, $A(x \sqcup y) \geq A(x) \vee A(y)$.

□

As corollaries, we obtain the following interesting properties of L-fuzzy ideals and L-fuzzy filters.

Corollary 3.6. *Let T be a trellis and I be an L-fuzzy ideal on T. Then for any $x, y \in T$ it holds that*

(i) If $x \trianglelefteq y$, then $I(x) \geq I(y)$, (i.e., the mapping $I : T \to L$ is antitone).

Corollary 3.7. *Let T be a trellis and F be an L-fuzzy filter on T. Then for any $x, y \in T$ it holds that*

(i) If $x \trianglelefteq y$, then $F(x) \leq F(y)$, (i.e., the mapping $F : X \to L$ is monotonic).

In the following theorem we provide a basic characterization of L-fuzzy ideals on a trellis.

Theorem 3.5. *Let T be a trellis and I be an L-set on T. Then it holds that I is an L-fuzzy ideal on T if and only if*

$$I(x \sqcup y) = I(x) \wedge I(y), \text{ for any } x, y \in T.$$

Proof. Suppose that I is an L-fuzzy ideal on T, then for any $x, y \in T$ it holds that $I(x \sqcup y) \geq I(x) \wedge I(y)$. Since $x \trianglelefteq x \sqcup y$ and $y \trianglelefteq x \sqcup y$, it follows from Corollary 3.6 that

$$I(x) \geq I(x \sqcup y)$$

and

$$I(y) \geq I(x \sqcup y)\,.$$

Hence, $I(x) \wedge I(y) \geq I(x \sqcup y)$. Thus, $I(x \sqcup y) = I(x) \wedge I(y)$.

Conversely, suppose that $I(x \sqcup y) = I(x) \wedge I(y)$ for all $x, y \in T$. Then it is easy to see that

$$I(x \sqcup y) \geq I(x) \wedge I(y)$$

for all $x, y \in T$. Next, we will show that $I(x \sqcap y) \geq I(x) \vee I(y)$ for all $x, y \in T$. Let $x, y \in T$, since $x \sqcup (x \sqcap y) = x$ and $y \sqcup (x \sqcap y) = y$ then it holds that $I(x \sqcup (x \sqcap y)) = I(x)$ and $I(y \sqcup (x \sqcap y)) = I(y)$. From hypothesis (i) and (ii), it follows that $I(x) \wedge I(x \sqcap y) = I(x)$ and $I(y) \wedge I(x \sqcap y) = I(y)$. Hence, $I(x \sqcap y) \geq I(x)$ and $I(x \sqcap y) \geq I(y)$. Thus, $I(x \sqcap y) \geq I(x) \vee I(y)$, for all $x, y \in T$. Therefore, I is an L-fuzzy ideal on T. □

In the same manner, the following theorem provides a basic characterization of L-fuzzy filters on a trellis.

Theorem 3.6. *Let T be a trellis and F be an L-set on T. Then it holds that F is an L-fuzzy filter on T if and only if*

$$F(x \sqcap y) = F(x) \wedge F(y), \text{ for any } x, y \in T.$$

Proof. The proof is a direct application of Proposition 4.1 and Theorem 3.5. □

Proposition 3.20. *Let $(T, \trianglelefteq)$ be a trellis and $\mathcal{I}(T)$ be the family of all L-fuzzy ideals on T. Let Z be a finite subset of $\mathcal{I}(T)$, then $\bigcap Z \in \mathcal{I}(T)$, where $\bigcap Z(x) = \bigwedge_{Z_i \in Z} Z_i(x)$.*

Proof. Let $x, y \in T$, then

$$\begin{aligned} \bigcap Z(x \sqcup y) &= \bigwedge_{Z_i \in Z} Z_i(x \sqcup y) \\ &= \bigwedge_{Z_i \in Z} Z_i(x) \wedge Z_i(y) \\ &= \bigwedge_{Z_i \in Z} Z_i(x) \wedge \bigwedge_{Z_i \in Z} Z_i(y) \\ &= \bigcap Z(x) \wedge \bigcap Z(y) \end{aligned}$$

Hence, $\bigcap Z$ is an L-fuzzy ideal on T. □

3.2.2. Characterizations of fuzzy ideals and fuzzy filters in terms of their level sets

In this subsection, we provide some interesting characterizations and properties of L-fuzzy ideals and L-fuzzy filters in terms of their level sets.

Proposition 3.21. *Let T be a trellis and A be an L-set on T. The following statements hold*

(i) If A is an L-fuzzy ideal, then its support $Supp(A)$ is an ideal on T;

(ii) If A is an L-fuzzy filter, then its support $Supp(A)$ is a filter on T.

Proof. Let A be an L-set on T.

(i) Suppose that A is an L-fuzzy ideal. We show that $Supp(A)$ is an ideal on T.

(a) Let $x \in Supp(A)$ and $y \trianglelefteq x$, then it holds that $x \sqcup y = x$ and $A(x) > 0$. This implies that $A(x) = A(x \sqcup y) > 0$. From Theorem 3.5, it follows that $A(x \sqcup y) = A(x) \wedge A(y) > 0$. Hence, $A(y) > 0$. Thus, $y \in Supp(A)$.

(b) Let $x, y \in Supp(A)$, then it holds that $A(x) > 0$ and $A(y) > 0$. Since A is an L-fuzzy ideal, then it follows that $A(x \sqcup y) = A(x) \wedge A(y) > 0$. Hence, $A(x \sqcup y) > 0$. Thus, $x \sqcup y \in Supp(A)$.

Therefore, $Supp(A)$ is an ideal on T.

(ii) Follows from Proposition 4.1 and (i).

□

Remark 3.13. *The converse of the above proposition is not necessarily hold. Indeed, let us consider T the trellis depicted in Figure 1.4 and A is an L-set given by $A = \{< a, 0.5 >, < b, 0.4 >, < c, 0.4 >, < d, 0.7 >, < e, 0.5 >\}$. It is easy to verify that $Supp(A) = T$ is an ideal and a filter on T, but A is neither an L-fuzzy ideal nor an L-fuzzy filter on T.*

The following theorem provides a characterization of L-fuzzy ideals (resp. L-fuzzy filters) in terms of their level sets.

Theorem 3.7. *Let T be a trellis and A be an L-set on T. The following statements hold*

(i) A is an L-fuzzy ideal if and only if its level sets are ideals on T.

(ii) A is an L-fuzzy filter if and only if its level sets are filters on T.

Proof. Let A be an L-set on T and A_α its level set, where $\alpha \in]0, 1]$.

(i) Suppose that A is an L-fuzzy ideal on T and we show that A_α is an ideal on T, for any $\alpha \in]0, 1]$;

(a) Let $\alpha \in]0, 1]$, $x \in A_\alpha$ and $y \in T$ such that $y \trianglelefteq x$. Since $x \in A_\alpha$, then it holds that $A(x) \geq \alpha$. Since $y \trianglelefteq x$, it follows from Corollary 3.6 that $A(y) \geq A(x)$. This implies that $A(y) \geq \alpha$. Hence, $y \in A_\alpha$, for any $\alpha \in]0, 1]$.

(b) Let $\alpha \in]0, 1]$ and $x, y \in A_\alpha$. Then it holds that $(A(x) \geq \alpha$ and $A(y) \geq \alpha)$. From Theorem 3.5 it follows that $A(x \sqcup y) =$

$A(x) \wedge A(y) \geq \alpha$. Hence, $x \sqcup y \in A_\alpha$, for any $\alpha \in]0,1]$. Thus, A_α is an ideal on T, for any $\alpha \in]0,1]$.

Conversely, suppose that all level sets of A are ideals on T. Let us show that A is an L-fuzzy ideal on T. Let $x, y \in T$, $\alpha = A(x) \wedge A(y)$. Then it follows that $(A(x) \geq \alpha$ and $A(y) \geq \alpha)$. The case $\alpha = 0$ is obvious. So, let $\alpha \in]0,1]$ and $x, y \in A_\alpha$. Since A_α is an ideal on T, then it holds that $x \sqcup y \in A_\alpha$, for any $\alpha \in]0,1]$. This implies that $A(x \sqcup y) \geq \alpha$. Hence, $A(x \sqcup y) \geq A(x) \wedge A(y)$.

On the other hand, let $\alpha = A(x \sqcup y)$. The case $\alpha = 0$ is also obvious. So, let $\alpha \in]0,1]$ and $x \sqcup y \in A_\alpha$. Since A_α is an ideal on T, $x \trianglelefteq x \sqcup y$ and $y \trianglelefteq x \sqcup y$, it follows that $x, y \in A_\alpha$, for any $\alpha \in]0,1]$. This implies that $(A(x) \geq \alpha)$ and $(A(y) \geq \alpha)$. Hence, $A(x) \wedge A(y) \geq A(x \sqcup y)$. Thus, $A(x) \wedge A(y) = A(x \sqcup y)$. Therefore, A is an L-fuzzy ideal on T.

(ii) Follows from Proposition 4.1 and (i).

□

3.3. Prime fuzzy ideals and filters on a trellis

In this section, we introduce and characterize the prime L-fuzzy ideals (resp. prime L-fuzzy filters) on a trellis.

Definition 3.12. *An L-fuzzy ideal I on a trellis T is called a prime L-fuzzy ideal if, for any $x, y \in T$,*

$$I(x \sqcap y) \leq I(x) \vee I(y).$$

Definition 3.13. *An L-fuzzy filter F on a trellis T is called a prime L-fuzzy filter if, for any $x, y \in T$,*

$$F(x \sqcup y) \leq F(x) \vee F(y).$$

The following proposition shows that the support of a prime L-fuzzy ideal (resp. prime L-fuzzy filter) on a trellis is a prime ideal (resp. prime filter) on that trellis.

Proposition 3.22. *Let T be a trellis and A be an L-set on T. Then it holds that*

(i) If A is a prime L-fuzzy ideal, then its $Supp(A)$ is a prime ideal on T.

(ii) If A is a prime L-fuzzy filter, then its $Supp(A)$ is a prime filter on T.

Proof. (i) Suppose that A is a prime L-fuzzy ideal on a trellis T. From Proposition 3.21, it holds that $Supp(A)$ is an ideal on T. Next we prove that $Supp(A)$ is prime. Let $x, y \in T$ such that $x \sqcap y \in Supp(A)$. It then holds that $A(x \sqcap y) > 0$. The fact that A is prime L-fuzzy ideal on T implies that

$$0 < A(x \sqcap y) \leq A(x) \vee A(y)$$

this implies that either $A(x) > 0$ or $A(y) > 0$. Hence, either $x \in Supp(A)$ or $y \in Supp(A)$. Therefore, $Supp(A)$ is a prime ideal on T.

(ii) Follows from Proposition 4.1 and (i).

□

In the same direction, we get the following theorem which provides a characterization of prime L-fuzzy ideals (resp. prime L-fuzzy filters) in terms of their level sets.

Theorem 3.8. *Let T be a trellis and A be an L-set on T. Then it holds that:*

(i) A is a prime L-fuzzy ideal if and only if their level sets are prime ideals;

(ii) A is a prime L-fuzzy filter if and only if their level sets are prime filters.

Proof. (i) From Theorem 3.7, we have that A is an L-fuzzy ideal of a trellis $(T, \trianglelefteq)$ if and only if for some $\alpha \in]0, 1]$, A_α is an ideal on $(T, \trianglelefteq)$. For the primality, suppose that A is a prime L-fuzzy ideal. We first, show that A_α is a prime ideal. Let $x \sqcap y \in A_\alpha$ for some $\alpha \in]0, 1]$, then $A(x \sqcap y) \geq \alpha$, since $A(x) \vee A(y) \geq A(x \sqcap y)$ this implies that $A(x) \vee A(y) \geq \alpha$, thus $A(x) \geq \alpha$ or $A(y) \geq \alpha$. Hence, $x \in A_\alpha$ or $y \in A_\alpha$.

Conversely, suppose that $A(x \sqcap y) = \alpha$ this implies that $x \sqcap y \in A_\alpha$. Since A_α is a prime ideal, then it follows that $x \in A_\alpha$ or $y \in A_\alpha$, for each $\alpha \in]0,1]$. Then it holds that, $A(x) \geq \alpha$ or $A(y) \geq \alpha$. Thus, $A(x) \vee A(y) \geq A(x \sqcap y) = \alpha$. Hence, A is a prime L-fuzzy ideal.

(ii) Follows from Proposition 4.1 and (i).

□

3.4. Principal fuzzy ideals and filters on a trellis

In this section, we extend the notion of principal ideal (resp. filter) on a trellis to the setting of fuzzy sets. More specifically, we introduce the notions of L-fuzzy ideals and L-fuzzy filters on a trellis structure, and provide a basic characterizations of these notions based on the weakly associative meet and join operations of this trellis. We pay particular attention to the kind of principal L-fuzzy ideals (resp. L-fuzzy filters) on a given trellis, which is more complicated than the same notions on a lattice. Similarly to the crisp case, we characterize these notions in terms of down and up-sets generated by an L-fuzzy singletons. First, we introduce the notion of L-fuzzy singleton.

Definition 3.14. *An L-fuzzy singleton on a trellis T is an L-set on T given by $\tilde{x} = \{\langle t, \tilde{x}(t)\rangle \mid t \in T\}$, where*

$$\tilde{x}(t) = \begin{cases} 1 \ , \ if \ x = t \\ \alpha(t) \ , \quad otherwise \end{cases}$$

such that $\alpha : T \to L - \{1\}$ is a given mapping.

For a given L-fuzzy singleton $\tilde{x}$ on a trellis T, the associated sets $\Downarrow \tilde{x}$ and $\Uparrow \tilde{x}$ are defined by:

$$\Downarrow \tilde{x} = \{\langle t, \Downarrow \tilde{x}(t)\rangle \mid t \in T\} \text{ and } \Uparrow \tilde{x} = \{\langle t, \Uparrow \tilde{x}(t)\rangle \mid t \in T\},$$

where

$$\Downarrow \tilde{x}(t) = \sup_{y \in \uparrow t} \tilde{x}(y) \text{ and } \Uparrow \tilde{x}(t) = \sup_{y \in \downarrow t} \tilde{x}(y).$$

One easily verifies that

$$\Downarrow \tilde{x}(t) = \begin{cases} 1 \ , \ if \ t \trianglelefteq x \\ \sup\limits_{y \in \uparrow t} \alpha(y) \ , \quad otherwise \end{cases}$$

and

$$\Uparrow \tilde{x}(t) = \begin{cases} 1 \ , \ if \ x \trianglelefteq t \\ \sup\limits_{y \in \downarrow t} \alpha(y) \ , \quad otherwise \end{cases} .$$

In the same line, the L-sets $\downharpoonleft\downharpoonleft \tilde{x}$ and $\upharpoonright\upharpoonright \tilde{x}$ associated with a given L-fuzzy singleton $\tilde{x}$ on a trellis T are defined as:

$$\downharpoonleft\downharpoonleft \tilde{x} = \{\langle t, \downharpoonleft\downharpoonleft \tilde{x}(t)\rangle \mid t \in T\} \text{ and } \upharpoonright\upharpoonright \tilde{x} = \{\langle t, \upharpoonright\upharpoonright \tilde{x}(t)\rangle \mid t \in T\} ,$$

where

$$\downharpoonleft\downharpoonleft \tilde{x}(t) = \sup_{y \in \upharpoonright t} \tilde{x}(y) \text{ and } \upharpoonright\upharpoonright \tilde{x}(t) = \sup_{y \in \downharpoonleft t} \tilde{x}(y) .$$

we have:

$$\downharpoonleft\downharpoonleft \tilde{x}(t) = \begin{cases} 1 \ , \ if \ t \lesssim x \\ \sup\limits_{y \in \upharpoonright t} \alpha(y) \ , \quad otherwise \end{cases}$$

and

$$\upharpoonright\upharpoonright \tilde{x}(t) = \begin{cases} 1 \ , \ if \ x \lesssim t \\ \sup\limits_{y \in \downharpoonleft t} \alpha(y) \ , \quad otherwise \end{cases} .$$

Note that for a given crisp singleton x on a given trellis T, it holds that

(i) $\Downarrow x = \downarrow x = \{y \in T \ : \ y \trianglelefteq x\}$;

(ii) $\Uparrow x = \uparrow x = \{y \in T \ : \ x \trianglelefteq y\}$;

(i) $\downharpoonleft\downharpoonleft x = \downharpoonleft x = \{y \in T \ : \ y \lesssim x\}$;

(ii) $\upharpoonright\upharpoonright x = \upharpoonright x = \{y \in T \ : \ x \lesssim y\}$.

The following proposition provides a necessary and sufficient condition that $\Downarrow \tilde{x}$ (resp. $\Uparrow \tilde{x}$) would be coincides with $\downharpoonleft\downharpoonleft \tilde{x}$ (resp. $\upharpoonright\upharpoonright \tilde{x}$).

Proposition 3.23. *Let T be a trellis and $x \in T$. The following equivalences hold:*

(i) $\Downarrow \tilde{x} = \downharpoonleft\downharpoonright \tilde{x}$ *if and only if* x *is a left-transitive element;*

(ii) $\Uparrow \tilde{x} = \upharpoonleft\upharpoonright \tilde{x}$ *if and only if* x *is a right-transitive element.*

Proof. (i) Let T be a trellis and $x \in T$. Then

$$\downharpoonleft\downharpoonright \tilde{x}(t) = \begin{cases} 1\ , & if\ t \lesssim x \\ \sup\limits_{y \in \upharpoonleft t} \alpha(y)\ , & otherwise \end{cases}$$

Since x is a left-transitive element, it follows from Theorem 1.4 that $t \lesssim x$, implies $t \trianglelefteq x$. Hence,

$$\begin{aligned} \downharpoonleft\downharpoonright \tilde{x}(t) &= \begin{cases} 1\ , & if\ t \trianglelefteq x \\ \sup\limits_{y \in \uparrow t} \alpha(y)\ , & otherwise \end{cases} \\ &= \mu_{\Downarrow \tilde{x}}(t). \end{aligned}$$

(ii) Follows from Proposition 3.14 and (i).

∎

The following corollary provides a necessary and sufficient condition that $\Downarrow \tilde{x}$ (resp. $\Uparrow \tilde{x}$) would be an L-fuzzy down-set (resp. an L-fuzzy up-set).

Corollary 3.8. *Let* $(X, \trianglelefteq)$ *be a psoset and* $x \in X$. *The following equivalences hold:*

(i) $\Downarrow \tilde{x}$ *is an* L*-fuzzy down-set if and only if* x *is a left-transitive element;*

(ii) $\Uparrow \tilde{x}$ *is an* L*-fuzzy up-set if and only if* x *is a right-transitive element.*

Definition 3.15. *Let* T *be a trellis and* $\tilde{x}$ *be an* L*-fuzzy singleton on* T*, Then*

(i) The principal L*-fuzzy ideal generated by an* L*-fuzzy singleton* $\tilde{x}$ *is the smallest* L*-fuzzy ideal contains* $\tilde{x}$*, denoted by* $\langle\langle \tilde{x}$.

(ii) The principal L*-fuzzy filter generated by an* L*-fuzzy singleton* $\tilde{x}$ *is the smallest* L*-fuzzy filter contains* $\tilde{x}$*, denoted by* $\tilde{x}\rangle\rangle$.

Next, we characterize principal L-fuzzy ideals (resp. L-fuzzy filters) on a trellis in terms of the down-sets (resp. up-sets) generated by L-fuzzy singletons on that trellis. We start with the following key results.

Proposition 3.24. *Let T be a trellis and $\tilde{x}$ be an L-fuzzy singleton on T. The following equivalences hold:*

(i) $\Downarrow \tilde{x}$ is a principal L-fuzzy ideal on T if and only if x is a left-transitive element;

(ii) $\Uparrow \tilde{x}$ is a principal L-fuzzy filter on T if and only if x is a right-transitive element.

Proof. (i) Let $a, b \in T$. First, suppose that x a left-transitive element, we show that $\Downarrow \tilde{x}$ is an L-fuzzy ideal on T. Since x is a left-transitive and from Proposition 3.23, it hold that $\Downarrow \tilde{x} = \downharpoonleft\downharpoonright \tilde{x}$. Hence,

$$\begin{aligned} \Downarrow \tilde{x}(a \sqcup b) = \downharpoonleft\downharpoonright \tilde{x}(a \sqcup b) &= \sup_{t \in \upharpoonleft(a \sqcup b)} \tilde{x}(t) \\ &= \sup_{a \sqcup b \lesssim x} \tilde{x}(t) \\ &= \sup_{a \sqcup b \trianglelefteq x} \tilde{x}(t). \end{aligned}$$

Also, for the same reason it holds that $a \sqcup b \trianglelefteq x$ if and only if $a \trianglelefteq x$ and $b \trianglelefteq x$. Hence,

$$\begin{aligned} \Downarrow \tilde{x}(a \sqcup b) = \downharpoonleft\downharpoonright \tilde{x}(a \sqcup b) &= \sup_{a \lesssim t} \tilde{x}(t) \wedge \sup_{b \lesssim t} \tilde{x}(t) \\ &= \sup_{a \trianglelefteq x} \tilde{x}(t) \wedge \sup_{b \trianglelefteq x} \tilde{x}(t) \\ &= \Downarrow \tilde{x}(a) \wedge \Downarrow \tilde{x}(b). \end{aligned}$$

Thus, $\Downarrow \tilde{x}$ is an L-fuzzy ideal on T.
Second, suppose that $\Downarrow \tilde{x}$ is an L-fuzzy ideal on T. From Corollaries 3.5, 3.8, it follows that $\Downarrow \tilde{x}$ is a principal L-fuzzy down-set and x is a left-transitive element.

(ii) Follows dually by using Proposition 3.14 and (i).

□

Theorem 3.9. *Let T be a trellis, x an element on T. The following equivalences hold:*

(i) $\downharpoonleft\downharpoonright \tilde{x} = \langle\langle \tilde{x}$ if and only if $\downharpoonleft\downharpoonright \tilde{x}$ is a $\sqcup$-semi L-fuzzy trellis of T.

(ii) $\upharpoonleft\upharpoonright \tilde{x} = \tilde{x}\rangle\rangle$ if and only if $\upharpoonleft\upharpoonright \tilde{x}$ is an $\sqcap$-semi L-fuzzy trellis of T.

Proof. (i) Let $a, b \in T$. First, we show that $\downharpoonleft\downharpoonleft \tilde{x}$ is an L-fuzzy ideal on T.

$$\begin{aligned} \downharpoonleft\downharpoonleft \tilde{x}(a \sqcup b) &= \sup_{t \in \upharpoonleft(a \sqcup b)} \tilde{x}(t) \\ &= \sup_{a \sqcup b \lesssim x} \tilde{x}(t). \end{aligned}$$

Since $\downharpoonleft\downharpoonleft \tilde{x}$ is a $\sqcup$-semi L-fuzzy trellis, then it holds that $a \sqcup b \lesssim t$ if and only if $a \lesssim t$ and $b \lesssim t$. Hence,

$$\begin{aligned} \downharpoonleft\downharpoonleft \tilde{x}(a \sqcup b) &= \sup_{t \in \upharpoonleft a} \tilde{x}(t) \wedge \sup_{t \in \upharpoonleft b} \tilde{x}(t) \\ &= \downharpoonleft\downharpoonleft \tilde{x}(a) \wedge \downharpoonleft\downharpoonleft \tilde{x}(b). \end{aligned}$$

Thus, $\downharpoonleft\downharpoonleft \tilde{x}$ is an L-fuzzy ideal containing x.
Now, we show that $\downharpoonleft\downharpoonleft \tilde{x}$ is the smallest L-fuzzy ideal containing $\tilde{x}$. Let J be another L-fuzzy ideal contain $\tilde{x}$ i.e., $\tilde{x}(t) \leq J(t)$, for any $t \in T$.

$$\begin{aligned} \downharpoonleft\downharpoonleft \tilde{x}(t) &= \sup_{y \in \upharpoonleft t} \tilde{x}(y) \leq \sup_{y \in \upharpoonleft t} J(y) \\ &= \downharpoonleft\downharpoonleft J(t) = J(t). \end{aligned}$$

Thus, $\downharpoonleft\downharpoonleft \tilde{x} \subseteq J$.

(ii) Follows dually by using Proposition 3.14 and (i).

□

In the following proposition, we show that the kernel of a principal L-fuzzy ideal (resp. L-fuzzy filter) is a crisp principal ideal (resp. filter).

Proposition 3.25. *Let T be a trellis and x be an element of T. Then it holds that*

(i) $Ker(\downharpoonleft\downharpoonleft \tilde{x}) = \downharpoonleft x$;

(ii) $Ker(\upharpoonleft\upharpoonleft \tilde{x}) = \upharpoonleft x$.

Proof. (i) From Proposition 3.19, it holds that $Ker(\downharpoonleft\downharpoonleft \tilde{x}) = \downharpoonleft Ker(\tilde{x})$. This means that

$$Ker(\downharpoonleft\downharpoonleft \tilde{x}) = \downharpoonleft \{t \in T \mid \tilde{x}(t) = 1\}.$$

By the definition of L-fuzzy singleton, we know that $\tilde{x}(t) = 1$ if and only if $t = x$. Hence, $Ker(\downharpoonleft\downharpoonleft \tilde{x}) = \downharpoonleft \{x\} = \downharpoonleft x$.

(ii) Follows from Proposition 3.14 and (i).

□

Proposition 3.26. *Let T be a trellis and $\downharpoonleft\downharpoonleft \tilde{x}$ and $\downharpoonleft\downharpoonleft \tilde{y}$ be two principal L-fuzzy ideals on T. Then*

$$x \lesssim y \text{ if and only if } \downharpoonleft\downharpoonleft \tilde{x} \subseteq \downharpoonleft\downharpoonleft \tilde{y}.$$

Proof. First, suppose that $x \lesssim y$ and we show that $\downharpoonleft\downharpoonleft \tilde{x} \subseteq \downharpoonleft\downharpoonleft \tilde{y}$. Since, $x \lesssim y$ then it holds that

$$\begin{aligned}
\downharpoonleft\downharpoonleft \tilde{x}(z) &= \sup_{z \lesssim t} \tilde{x}(t) \\
&= \sup_{z \lesssim x \lesssim y} \tilde{x}(t) \\
&\leq \sup_{z \lesssim y} \tilde{x}(t) \\
&\leq \downharpoonleft\downharpoonleft \tilde{y}(t).
\end{aligned}$$

Hence, $\downharpoonleft\downharpoonleft \tilde{x} \subseteq \downharpoonleft\downharpoonleft \tilde{y}$.

Conversely, suppose that $\downharpoonleft\downharpoonleft \tilde{x} \subseteq \downharpoonleft\downharpoonleft \tilde{y}$, then

$$1 = \downharpoonleft\downharpoonleft \tilde{x}(x) \leq \downharpoonleft\downharpoonleft \tilde{y}(x) = \sup_{x \lesssim t} \tilde{y}(t)$$

This implies that $\tilde{y}(t) = 1$ and $t = y$. Hence, $x \lesssim y$. □

4 Principal intuitionistic fuzzy ideals and filters on lattices

In this chapter, we generalize the notion of principal ideal (resp. filter) on a lattice to the setting of intuitionistic fuzzy sets and investigate their various characterizations and properties. More specifically, we show that any principal intuitionistic fuzzy ideal (resp. filter) coincides with an intuitionistic fuzzy down-set (resp. up-set) generated by an intuitionistic fuzzy singleton. Afterwards, for a given intuitionistic fuzzy set, we introduce two intuitionistic fuzzy sets: its intuitionistic fuzzy down-set and up-set, and we investigate their properties.

4.1. Intuitionistic fuzzy sets

This section contains the basic definitions and properties of intuitionistic fuzzy sets, as well some operations on intuitionistic fuzzy sets. In 1983, Atanassov [4] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 4.1. [4] *Let X be a non-empty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfies the condition:*

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \ \textit{for each } x \in X.$$

For each $x \in X$ the number $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the hesitation degree or the intuitionistic index of x to A.

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

Certainly, fuzzy sets are intuitionistic fuzzy sets by setting $\nu_A(x) = 1 - \mu_A(x)$.

Example 4.1. [6] *Let X be the set of all countries with elective governments. Assume that we know for every country $x \in X$ the percentage of the electorate* *that have voted for the corresponding government. Denote it by* $M(x)$ *and let* $\mu(x) = \frac{M(x)}{100}$ *(degree of membership, validity, etc.). Let* $\nu(x) = 1 - \mu(x)$*. This number corresponds to the part of electorate who have not voted for the government. Using only the fuzzy set theory, we cannot consider this value in more detail. However, if we define* $\nu(x)$ *(degree of non-membership, non-validity, etc.) as the number of votes given to parties or persons outside the government, then we can show the part of electorate who have not voted at all or who have given bad voting-paper and the corresponding number will be* $\pi(x) = 1 - \mu(x) - \nu(x)$ *(degree of indeterminacy, uncertainty, etc.). Thus, we can construct the set* $\{\langle x, \mu(x), \nu(x)\rangle \mid x \in X\}$ *and obviously,*

$$0 \leqslant \mu(x) + \nu(x) \leqslant 1.$$

4.1.1. Operations on intuitionistic fuzzy sets

For two intuitionistic fuzzy sets A and B on a set X, several operations are defined (see, e.g., Atanassov [5, 6, 7], Biswas [17] and Gy [78]). Here we will present only those which are related to the present work.

(i) $A \subseteq B$ if $\mu_A(x) \leqslant \mu_B(x)$ and $\nu_A(x) \geqslant \nu_B(x)$ for each $x \in X$;

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ for each $x \in X$;

(iii) $A \cap B = \{\langle x, \mu_A(x) \barwedge \mu_B(x), \nu_A(x) \veebar \nu_B(x)\rangle \mid x \in X\}$;

(iv) $A \cup B = \{\langle x, \mu_A(x) \veebar \mu_B(x), \nu_A(x) \barwedge \nu_B(x)\rangle \mid x \in X\}$;

(v) $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in X\}$;

(vi) $A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in X\}$;

(vii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(viii) $[A] = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$;

(ix) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \mid x \in X\}$.

In the sequel, we need the following definition of level sets (which is also often called (α, β)-cuts) of an intuitionistic fuzzy set.

Definition 4.2. [82] *Let* A *be an intuitionistic fuzzy set on a set* X*. The* (α, β)*-cut of* A *is a crisp subset*

$$A_{\alpha,\beta} = \{x \in X \mid \mu_A(x) \geqslant \alpha \text{ and } \nu_A(x) \leqslant \beta\},$$

where $\alpha, \beta \in]0,1]$ *with* $\alpha + \beta \leqslant 1$.

Definition 4.3. [6] *Let* A *be an intuitionistic fuzzy set on a set* X*. The support of* A *is the crisp subset on* X *given by*

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0 \text{ or } (\mu_A(x) = 0 \text{ and } \nu_A(x) < 1)\}.$$

The notion of fuzzy relations was first introduced by Zadeh [84] as a natural extension to a fuzzy set and plays an important role in the theory of such sets and their applications.

Definition 4.4. [84] *Let* X *and* Y *be two nonempty sets. A binary fuzzy relation from* X *to* Y*, is a fuzzy subset of* $X \times Y$ *characterized by a membership function* μ_R *which associates with each pair* (x, y) *its grade of membership* $\mu_R(x, y)$ *in the interval* $[0, 1]$.

4.2. Intuitionistic fuzzy relations

In this section, we recall the basic definitions and properties of intuitionistic fuzzy relations introduced by Burillo and Bustince [23, 24]. This is a natural generalization of a fuzzy relation.

Definition 4.5. [23, 24] *Let* X *and* Y *be two non-empty sets. An intuitionistic fuzzy binary relation (an intuitionistic fuzzy relation, for short) from* X *to* Y *is an intuitionistic fuzzy subset of* $X \times Y$*, i.e., is an expression* R *given by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where

$$\mu_R : X \times Y \to [0, 1], \text{ and } \nu_A : X \times Y \to [0, 1]$$

satisfy the condition

$$0 \leqslant \mu_R(x, y) + \nu_R(x, y) \leqslant 1,$$

for each $(x, y) \in X \times Y$*. The value* $\mu_R(x, y)$ *is called the degree of membership of* (x, y) *in* R *and* $\nu_R(x, y)$ *is called the degree of non-membership of* (x, y) *in* R.

Let R be an intuitionistic fuzzy relation on X. The following properties are crucial in (see e.g., [23, 24, 75, 85]):

(i) *Reflexivity*: $\mu_R(x,x) = 1$ for each $x \in X$. In this case, we note that $\nu_R(x,x) = 0$ for each $x \in X$.

(ii) *Antisymmetry*: for all $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \\ \pi_R(x,y) = \pi_R(y,x) \end{cases},$$

where $\pi_R(x,y) = 1 - \mu_R(x,y) - \nu_R(x,y)$.

(iii) *Perfect antisymmetry*: for each $x, y \in X$ with $x \neq y$ and

$$\begin{cases} \mu_R(x,y) > 0 \\ \quad \text{or} \\ \mu_R(x,y) = 0 \ \text{ and } \nu_R(x,y) < 1 \,, \end{cases}$$

then

$$\begin{cases} \mu_R(y,x) = 0 \\ \quad \text{and} \\ \nu_R(y,x) = 1 \,. \end{cases}$$

(iv) *Transitivity*:

$$R \supseteq R \circ^{\alpha,\beta}_{\lambda,\rho} R.$$

In the above definition, the composition $R \circ^{\alpha,\beta}_{\lambda,\rho} R$ used in the transitivity means that

$$R \circ^{\alpha,\beta}_{\lambda,\rho} R = \{\langle (x,z),\ \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\},$$

$$\lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\}\rangle \mid x, z \in X\}\,,$$

where α, β, λ and ρ are t-norms or t-conorms taken under the intuitionistic fuzzy condition

$$0 \leq \alpha_{y\in X}\{\beta[\mu_R(x,y), \mu_R(y,z)]\} + \lambda_{y\in X}\{\rho[\nu_R(x,y), \nu_R(y,z)]\} \leq 1\,,$$

for each $x, z \in X$.

The properties of this composition and the choice of α, β, λ and ρ, for which this composition fulfills a maximal number of properties, are investigated in [23]-[25], [33].

If no other conditions are imposed, in the following we will take $\alpha = \sup$, $\beta = \min$, $\lambda = \inf$ and $\rho = \max$.

Note that Bustince and Burillo in [26], mentioned that the definition of intuitionistic fuzzy antisymmetry does not recover the fuzzy antisymmetry for the case in which the considered relation R is fuzzy. However, the definition of intuitionistic fuzzy perfect antisymmetry recovers the definition of fuzzy antisymmetry given by Zadeh [84] when the considered relation is fuzzy. This note justifies the following definition of intuitionistic fuzzy order.

Definition 4.6. [23, 24] *Let X be a non-empty crisp set and R be an intuitionistic fuzzy relation on X. R is called an intuitionistic fuzzy order or a partial intuitionistic fuzzy order if it is reflexive, transitive and perfect antisymmetric.*

A non-empty set X with an intuitionistic fuzzy order R, defined on it, is called an intuitionistic fuzzy ordered set and is denoted by (X, μ_R, ν_R). It easily follows that each partially ordered set $(X, \leq)$ and each fuzzy ordered set (X, R) can be viewed as intuitionistic fuzzy ordered sets.

Example 4.2. *Let $X = \{a, b, c, d, e\}$. Then the intuitionistic fuzzy relation R defined on X by*

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\},$$

where μ_R and ν_R are given by the following tables:

$\mu_R(.,.)$	a	b	c	d	e
a	1	0	0	0.55	0.40
b	0	1	0	0.35	0.45
c	0	0	1	0	0.70
d	0	0	0	1	0
e	0	0	0	0	1

$\nu_R(.,.)$	a	b	c	d	e
a	*0*	*1*	*0.40*	*0.45*	*0.25*
b	*0.30*	*0*	*0.20*	*0.35*	*0.10*
c	*1*	*1*	*0*	*0.85*	*0.15*
d	*1*	*1*	*1*	*0*	*1*
e	*1*	*1*	*1*	*0.90*	*0*

is an intuitionistic fuzzy order on X.

On the basis of the above definition of perfect antisymmetry, we define a complete (or total) intuitionistic fuzzy order as follows:

Definition 4.7. [85] *An intuitionistic fuzzy order* R *on a universe* X *is called complete (or total) if for all* $x, y \in X$ *it holds that*

$$[\mu_R(x,y) > 0 \text{ or } (\mu_R(x,y) = 0 \text{ and } \nu_R(x,y) < 1)]$$

or

$$[\mu_R(y,x) > 0 \text{ or } (\mu_R(y,x) = 0 \text{ and } \nu_R(y,x) < 1)].$$

Definition 4.8. [85] *An intuitionistic fuzzy ordered set* (X, μ_R, ν_R) *in which* R *is linear is called a linearly intuitionistic fuzzy ordered set or an intuitionistic fuzzy chain.*

4.2.1. Intuitionistic fuzzy ideals and filters

The notion of intuitionistic fuzzy ideal (resp. filter) in a lattice was first introduced by Thomas and Nair [73].

Definition 4.9. [73] *Let* $(L, \leq, \wedge, \vee)$ *be a lattice and* $I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in L\}$ *be an IFS on* L*. Then* I *is called an intuitionistic fuzzy ideal on* L *(IF-ideal, for short) if for all* $x, y \in L$*, the following conditions hold:*

(i) $\mu_I(x \vee y) \geqslant \mu_I(x) \barwedge \mu_I(y)$;

(ii) $\mu_I(x \wedge y) \geqslant \mu_I(x) \veebar \mu_I(y)$;

(iii) $\nu_I(x \vee y) \leqslant \nu_I(x) \veebar \nu_I(y)$;

(iv) $\nu_I(x \wedge y) \leqslant \nu_I(x) \barwedge \nu_I(y)$.

Definition 4.10. [73] *Let $(L, \leq, \wedge, \vee)$ be a lattice and $F = \{\langle x, \mu_F(x), \nu_F(x)\rangle \mid x \in L\}$ be an IFS on L. Then F is called an intuitionistic fuzzy filter on L (IF-filter, for short) if for all $x, y \in L$, the following conditions hold:*

(i) $\mu_F(x \vee y) \geqslant \mu_F(x) \veebar \mu_F(y)$;

(ii) $\mu_F(x \wedge y) \geqslant \mu_F(x) \barwedge \mu_F(y)$;

(iii) $\nu_F(x \vee y) \leqslant \nu_F(x) \barwedge \nu_F(y)$;

(iv) $\nu_F(x \wedge y) \leqslant \nu_F(x) \veebar \nu_F(y)$.

For further details on intuitionistic fuzzy ideals and filters, we refer to [53, 72, 73].

Example 4.3. *Let L be the lattice depicted below in Figure 1. The intuitionistic fuzzy set I on L defined by: $I = \{< 0, 0.5, 0.1 >, < a, 0.4, 0.3 >, < b, 0.1, 0.2 >, < c, 0.1, 0.7 >, < 1, 0.1, 0.7 >\}$ is an IF-ideal.*

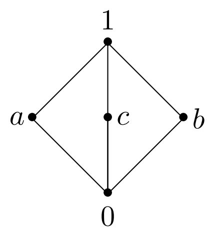

Figure 4.1: Hasse diagram of the lattice $(L, \wedge, \vee)$ with $L = \{0, a, b, c, 1\}$.

Example 4.4. *Let L be the lattice given by the Hasse diagram in Figure 1. The intuitionistic fuzzy set F on L defined by: $F = \{< 0, 0.1, 0.5 >, < a, 0.1, 0.6 >, < b, 0.1, 0.5 >, < c, 0.3, 0.4 >, < 1, 0.4, 0.3 >\}$ is an IF-filter.*

The following proposition is straightforward.

Proposition 4.1. *Let L be a lattice, L^d be its order-dual lattice and $S \in IFS(L)$. The following equivalences hold:*

(i) The IF-set S is an IF-ideal on L if and only if S is an IF-filter on L^d;

(ii) The IF-set S is an IF-filter on L if and only if S is an IF-ideal on L^d.

4.3. Intuitionistic fuzzy down-sets and up-sets

Analogously to a crisp down-set and up-set on a lattice L, we introduce the notions of an intuitionistic fuzzy down-set and an intuitionistic fuzzy up-set. Also, the $\Downarrow S$ and $\Uparrow S$, for any intuitionistic fuzzy set S on L.

4.3.1. Definitions

Let L be a lattice and $S \in IFS(L)$.

(i) S is called an intuitionistic fuzzy down-set (IF-down-set, for short) if $\mu_S(x) \geqslant \mu_S(y)$ and $\nu_S(x) \leqslant \nu_S(y)$ for all $x \leq y$.

(ii) Dually, S is called an intuitionistic fuzzy up-set (IF-up-set, for short) if $\mu_S(x) \leqslant \mu_S(y)$ and $\nu_S(x) \geqslant \nu_S(y)$ for all $x \leq y$.

For a given intuitionistic fuzzy set S on a lattice L we denote by:

(i) $\Downarrow S$ the intuitionistic fuzzy set associated with S defined as

$$\mu_{\Downarrow S}(x) = \sup_{y \in \uparrow x} \mu_S(y),$$

$$\nu_{\Downarrow S}(x) = \inf_{y \in \uparrow x} \nu_S(y).$$

(ii) $\Uparrow S$ the intuitionistic fuzzy set associated with S defined as

$$\mu_{\Uparrow S}(x) = \sup_{y \in \downarrow x} \mu_S(y),$$

$$\nu_{\Uparrow S}(x) = \inf_{y \in \downarrow x} \nu_S(y).$$

Remark 4.1. *For any crisp set S on a given lattice L, it holds that*

(i) $\Downarrow S = \downarrow S$;

(ii) $\Uparrow S = \uparrow S$.

For a given lattice L and $S \in IFS(L)$, it is clear that

(i) $\mu_{\Downarrow S}$ is an antitone mapping and $\nu_{\Downarrow S}$ is a monotone mapping;

(ii) $\mu_{\Uparrow S}$ is a monotone mapping and $\nu_{\Uparrow S}$ is an antitone mapping.

4.3.2. Properties of intuitionistic fuzzy-down-sets and intuitionistic fuzzy-up-sets

In this subsection, we show some interesting properties of intuitionistic fuzzy down-sets and up-sets on a lattice. We start with the easiest one.

Proposition 4.2. *Let L be a lattice, L^d be its order-dual lattice and $S \in IFS(L)$. The following statements hold:*

(i) S is an IF-down-set on L if and only if S is an IF-up-set on L^d;

(ii) S is an IF-up-set on L if and only if S is an IF-down-set on L^d;

(iii) $\Downarrow S$ on L coincides with $\Uparrow S$ on L^d ;

(iv) $\Uparrow S$ on L coincides with $\Downarrow S$ on L^d.

The following proposition shows that IF-down-sets (resp. IF-up-sets) on a lattice are closed under the union and intersection of intuitionistic fuzzy sets.

Proposition 4.3. *Let L be a lattice and $R, S \in IFS(L)$. It holds that*

(i) If R and S are IF-down-sets, then $R \cup S$ and $R \cap S$ are IF-down-sets;

(ii) If R and S are IF-up-sets, then $R \cup S$ and $R \cap S$ are IF-up-sets.

Proof. (i) We only show that $R \cup S$ is an IF-down-set, as $R \cap S$ can be proved analogously. Let $x, y \in L$ such that $x \leq y$. Since R and S are IF-down-sets, and $\mu_{R\cup S}(x) = \mu_R(x) \veebar \mu_S(x)$, it follows that $\mu_{R\cup S}(x) \geqslant \mu_R(y) \veebar \mu_S(y) = \mu_{R\cup S}(y)$. Similarly, we prove that $\nu_{R\cup S}(x) \leqslant \nu_{R\cup S}(y)$. Thus, $R \cup S$ is an IF-down-set.

(ii) Follows from Proposition 4.2 and (i).

□

Proposition 4.4. *Let L be a lattice and $S \in IFS(L)$. It holds that*

(i) $\Downarrow S$ is the smallest IF-down-set containing S;

(ii) $\Uparrow S$ is the smallest IF-up-set containing S.

Proof. (i) First we show $S \subseteq \Downarrow S$. Let $a \in L$ and $b \in \uparrow a$, then it holds that $\mu_S(a) \leqslant \sup\limits_{b\in\uparrow a} \mu_S(b) = \mu_{\Downarrow S}(a)$. Similarly, $\nu_S(a) \geqslant \inf\limits_{b\in\uparrow a} \nu_S(b) = \nu_{\Downarrow S}(a)$.

Hence, $S \subseteq \Downarrow S$. Now, we show that $\Downarrow S$ is an IF-down-set. Let $x, y \in L$ such that $x \leq y$. We have $\mu_{\Downarrow S}(x) = \sup_{t \in \uparrow x} \mu_S(t) \geqslant \sup_{t \in \uparrow y} \mu_S(t)$. Hence, $\mu_{\Downarrow S}(x) \geqslant \mu_{\Downarrow S}(y)$. Similarly, we obtain that $\nu_{\Downarrow S}(x) \leqslant \mu_{\Downarrow S}(y)$. Thus, $\Downarrow S$ is an IF-down-set. Furthermore, let R be an IF-down-set containing S. This implies that $\mu_S(x) \leqslant \mu_R(x)$ and $\nu_S(x) \geqslant \nu_R(x)$, for any $x \in L$. Hence, $\sup_{t \in \uparrow x} \mu_S(t) \leqslant \sup_{t \in \uparrow x} \mu_R(t)$ and $\inf_{t \in \uparrow x} \nu_S(t) \geqslant \inf_{t \in \uparrow x} \nu_R(t)$. Thus, $\Downarrow S \subseteq \Downarrow R$. Finally, we conclude that $\Downarrow S$ is the smallest IF-down-set containing S.

(ii) Follows from Proposition 4.2 and (i).

□

From Proposition 4.4, we obtain the following corollary. It shows a characterization of IF-down-sets and IF-up-sets.

Corollary 4.1. *Let L be a lattice and $S \in IFS(L)$. The following equivalences hold:*

(i) S is an IF-down-set if and only if $S = \Downarrow S$;

(ii) S is an IF-up-set if and only if $S = \Uparrow S$.

The following propositions list some properties of IF-down and IF-up-sets.

Proposition 4.5. *Let L be a lattice and $R, S \in IFS(L)$. The following statements hold:*

(i) If $S \subseteq R$, then $\Downarrow S \subseteq \Downarrow R$;

(ii) $\Downarrow (\Downarrow S) = \Downarrow S$;

(iii) $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$;

(iv) $\Downarrow (S \cap R) \subseteq \Downarrow S \cap \Downarrow R$.

Proof. (i) Since $R \subseteq \Downarrow R$, it holds that $S \subseteq \Downarrow R$. From Proposition 4.4, it trivially holds that $\Downarrow S \subseteq \Downarrow R$.

(ii) Follows from Proposition 4.4 and Corollary 4.1.

(iii) On the one hand, we easily verify from (i) that $\Downarrow S \cup \Downarrow R \subseteq \Downarrow (S \cup R)$. On the other hand, since $\Downarrow S \cup \Downarrow R$ is an IF-down-set and $S \cup R \subseteq \Downarrow$

$S \cup \Downarrow R$, it follows from Proposition 4.4 that $\Downarrow (S \cup R) \subseteq \Downarrow S \cup \Downarrow R$. Thus, $\Downarrow (S \cup R) = \Downarrow S \cup \Downarrow R$.

(iv) Follows from Proposition 4.4 and (i).

□

In the same direction, a dual version of Proposition 4.5 can also obtained for intuitionistic fuzzy up-sets. Its proof follows from Propositions 4.2 and 4.5.

Proposition 4.6. *Let L be a lattice and $R, S \in IFS(L)$. The following statements hold:*

(i) If $S \subseteq R$, then $\Uparrow S \subseteq \Uparrow R$;

(ii) $\Uparrow (\Uparrow S) = \Uparrow S$;

(iii) $\Uparrow (S \cup R) = \Uparrow S \cup \Uparrow R$;

(iv) $\Uparrow (S \cap R) \subseteq \Uparrow S \cap \Uparrow R$.

The following result follows immediately from Propositions 4.4, 4.5 and 4.6.

Proposition 4.7. *Let L be a lattice. Then the mappings $\Downarrow$ and $\Uparrow$ define topological closures on the set $IFS(L)$ of intuitionistic fuzzy sets on L.*

The following proposition shows the interaction of the *Support* and the *Kernel* with the notions of IF-down-set and IF-up-set.

Proposition 4.8. *Let L be a lattice and $S \in IFS(L)$. It holds that*

(i) $Supp(\Downarrow S) = \downarrow Supp(S)$ and $Ker(\Downarrow S) = \downarrow Ker(S)$;

(ii) $Supp(\Uparrow S) = \uparrow Supp(S)$ and $Ker(\Uparrow S) = \uparrow Ker(S)$.

Proof. (i) First, we prove that $Supp(\Downarrow S) = \downarrow Supp(S)$. On the one hand, let $x \in Supp(\Downarrow S)$. Then it holds that

$$\mu_{\Downarrow S}(x) > 0 \text{ or } (\mu_{\Downarrow S}(x) = 0 \text{ and } \nu_{\Downarrow S}(x) < 1).$$

Two cases to consider:

(a) If $\mu_{\Downarrow S}(x) > 0$, then $\sup_{y \in \uparrow x} \mu_S(y) > 0$. This implies that there exists $t \in \uparrow x$ such that $\mu_S(t) > 0$. Hence, $t \in Supp(S)$. Since $t \in \uparrow x$, it follows that $x \in \downarrow Supp(S)$.

(b) If $\mu_{\Downarrow S}(x) = 0$ and $\nu_{\Downarrow S}(x) < 1$, then $\mu_S(y) = 0$ for any $y \in\uparrow x$ and $\inf_{y\in\uparrow x} \nu_S(y) < 1$. This implies that there exists $t \in\uparrow x$ such that $\mu_S(t) = 0$ and $\nu_S(t) < 1$. Hence, $t \in Supp(S)$. Since $t \in\uparrow x$, it follows that $x \in\downarrow Supp(S)$.

Thus, $Supp(\Downarrow S) \subseteq\downarrow Supp(S)$. On the other hand, let $x \in\downarrow Supp(S)$. Then it holds that

$$\mu_S(x) > 0 \text{ or } (\mu_S(x) = 0 \text{ and } \nu_S(x) < 1).$$

Two cases to consider:

(c) If $\mu_S(x) > 0$, then $\sup_{y\in\uparrow x} \mu_S(y) = \mu_{\Downarrow S}(x) > 0$. Hence, $x \in Supp(\Downarrow S)$.

(d) If $\mu_S(x) = 0$ and $\nu_S(x) < 1$, then $\mu_{\Downarrow S}(x) > 0$ or $(\mu_{\Downarrow S}(x) = 0$ and $\inf_{y\in\uparrow x} \nu_S(y) = \nu_{\Downarrow S}(x) < 1)$. Hence, $x \in Supp(\Downarrow S)$.

Thus, $\downarrow Supp(S) \subseteq Supp(\Downarrow S)$. Therefore, $Supp(\Downarrow S) =\downarrow Supp(S)$.

The proof of $Ker(\Downarrow S) =\downarrow Ker(S)$ is analogous.

(ii) follows from Proposition 4.2 and (i).

□

In the following result, we show that any IF-ideal (resp. IF-filter) on a lattice L is an IF-down-set (resp. IF-up-set) on L.

Theorem 4.1. *Let L be a lattice and $S \in IFS(L)$. The following implications hold:*

(i) If S is an IF-ideal, then S is an IF-down-set;

(ii) If S is an IF-filter, then S is an IF-up-set.

Proof. (i) Let $x, y \in L$ such that $x \leq y$. Since S is an IF-ideal, it follows that $\mu_S(x) = \mu_S(x \wedge y) \geqslant \mu_S(x) \veebar \mu_S(y)$. Hence, $\mu_S(x) \geqslant \mu_S(y)$. Similarly, we obtain that $\nu_S(x) \leqslant \nu_S(y)$. Thus, S is an IF-down-set.

(ii) follows from Proposition 4.1, (i) and Proposition 4.2.

□

Combining Theorem 4.1 and Corollary 4.1 leads to the following corollary.

Corollary 4.2. *Let L be a lattice and $S \in IFS(L)$. The following implications hold:*

(i) If S is an IF-ideal, then $\Downarrow S = S$;

(ii) If S is an IF-filter, then $\Uparrow S = S$.

Remark 4.2. *The converse of the implications in the above Theorem 4.1 and Corollary 4.2 does not necessarily hold. Indeed, consider L the lattice given by the Hasse diagram in Figure 1 and $S \in IFS(L)$ given by $S = \{< 0, 0.7, 0.1 >, < a, 0.4, 0.2 >, < b, 0.3, 0.1 >, < c, 0.2, 0.1 >, < 1, 0.1, 0.3 >\}$. We easily verify that*

x	0	a	b	c	1
$\mu_{\Downarrow S}(x)$	0.7	0.4	0.3	0.2	0.1
$\nu_{\Downarrow S}(x)$	0.1	0.2	0.1	0.1	0.3

Then $\Downarrow S = \{< 0, 0.7, 0.1 >, < a, 0.4, 0.2 >, < b, 0.3, 0.1 >, < c, 0.2, 0.1 >, < 1, 0.1, 0.3 >\}$. Hence, $\Downarrow S = S$, i.e., S is an IF-down-set. But, $\mu_S(1) = \mu_S(a \vee b) \ngeq \min\{0.4, 0.3\}$, which implies that S is not an intuitionistic fuzzy ideal on L .

4.4. Principal intuitionistic fuzzy-ideals and intuitionistic fuzzy-filters on a lattice

In this section, we introduce the notion of principal IF-ideal (resp. IF-filter) on a lattice. Similarly to the crisp case, we characterize these notions in terms of a down set and an up-set generated by intuitionistic fuzzy singletons. First, we need to recall the following definition of crisp principal ideal (resp. filter), and the definition of intuitionistic fuzzy singleton.

Definition 4.11. [31] *Let L be a lattice. Then*

(i) The principal ideal generated by an element $x \in L$, is the smallest ideal contains x, and is given by

$$\downarrow x = \{y \in L \mid y \leq x\};$$

(ii) The principal filter generated by an element $x \in L$, is the smallest filter contains x, and is given by

$$\uparrow x = \{y \in L \mid x \leq y\}.$$

Definition 4.12. [22] *Let L be a lattice. For any $x \in L$, an intuitionistic fuzzy singleton (IF- singleton, for short) $\tilde{x}$ is an intuitionistic fuzzy set on L given by $\tilde{x} = \{\langle t, \mu_{\tilde{x}}(t), \nu_{\tilde{x}}(t)\rangle \mid t \in L\}$, where*

$$\mu_{\tilde{x}}(t) = \begin{cases} 1\ , & if\ x = t \\ f(t)\ , & otherwise, \end{cases}$$

and

$$\nu_{\tilde{x}}(t) = \begin{cases} 0\ , & if\ x = t \\ g(t)\ , & otherwise, \end{cases}$$

such that f (resp. g) is a monotonic (resp. antitone) mapping on $[0,1[$ and $f(t) + g(t) < 1$, for any $t \in L$.

Definition 4.13. [22] *Let L be a lattice, Then*

(i) The principal IF-ideal generated by an IF-singleton $\tilde{x}$ is the smallest IF-ideal contains $\tilde{x}$.

(ii) The principal IF-filter generated by an IF-singleton $\tilde{x}$ is the smallest IF-filter contains $\tilde{x}$.

Next, we characterize the principal IF-ideals (resp. IF-filters) on a lattice in terms of the down-sets (resp. up-sets) generated by IF-singletons on that lattice. First, we need to recall the following characterization theorem of IF-ideals and IF-filters on a lattice.

Theorem 4.2. [53, 54] *Let L be a lattice and $I, F \in IFS(L)$. It holds that*

(1) I is an IF-ideal on L if and only if the following two conditions are satisfied:

(i) $\mu_I(x \vee y) = \mu_I(x) \,\overline{\wedge}\, \mu_I(y)$, for any $x, y \in L$;

(ii) $\nu_I(x \vee y) = \nu_I(x) \,\underline{\vee}\, \nu_I(y)$, for any $x, y \in L$.

(2) F is an IF-filter on L if and only if the following two conditions are satisfied:

(i) $\mu_F(x \wedge y) = \mu_F(x) \,\overline{\wedge}\, \mu_F(y)$, for any $x, y \in L$;

(ii) $\nu_F(x \wedge y) = \nu_F(x) \veebar \nu_F(y)$, *for any* $x, y \in L$.

The following theorem shows that the IF-down-set (resp. the IF-up-set) generated by an intuitionistic fuzzy singleton on a lattice L is an IF-ideal (resp. is an IF-filter) on L.

Theorem 4.3. *Let* L *be a lattice and* x *be an element on* L. *Then it holds that*

(i) $\Downarrow \tilde{x}$ *is an IF-ideal on* L*;*

(ii) $\Uparrow \tilde{x}$ *is an IF-filter on* L.

Proof. (i) From Theorem 4.2, it suffices to show for any $x, y \in L$ that

$$\mu_{\Downarrow\tilde{x}}(x \vee y) = \mu_{\Downarrow\tilde{x}}(x) \barwedge \mu_{\Downarrow\tilde{x}}(y) \text{ and } \nu_{\Downarrow\tilde{x}}(x \vee y) = \nu_{\Downarrow\tilde{x}}(x) \veebar \nu_{\Downarrow\tilde{x}}(y).$$

Let $a, b \in L$. On the one hand, by Proposition 4.4, $\Downarrow \tilde{x}$ is an IF-down-set, which implies that $\mu_{\Downarrow\tilde{x}}(a) \geqslant \mu_{\Downarrow\tilde{x}}(a \vee b)$ and $\mu_{\Downarrow\tilde{x}}(b) \geqslant \mu_{\Downarrow\tilde{x}}(a \vee b)$. Hence, $\mu_{\Downarrow\tilde{x}}(a) \barwedge \mu_{\Downarrow\tilde{x}}(a) \geqslant \mu_{\Downarrow\tilde{x}}(a \vee b)$. On the other hand, since $\mu_{\tilde{x}}$ is a monotonic mapping, it holds that $\mu_{\tilde{x}}(a) \leqslant \mu_{\tilde{x}}(a \vee b)$ and $\mu_{\tilde{x}}(b) \leqslant \mu_{\tilde{x}}(a \vee b)$. This implies that $\sup\limits_{a \leq t} \mu_{\tilde{x}}(t) \leqslant \sup\limits_{a \vee b \leq t} \mu_{\tilde{x}}(t)$ and $\sup\limits_{b \leq t} \mu_{\tilde{x}}(t) \leqslant \sup\limits_{a \vee b \leq t} \mu_{\tilde{x}}(t)$. Hence, $\sup\limits_{a \leq t} \mu_{\tilde{x}}(t) \barwedge \sup\limits_{b \leq t} \mu_{\tilde{x}}(t) \leqslant \sup\limits_{a \vee b \leq t} \mu_{\tilde{x}}(t)$. Thus, $\mu_{\Downarrow\tilde{x}}(a) \barwedge \mu_{\Downarrow\tilde{x}}(b) \leqslant \mu_{\Downarrow\tilde{x}}(a \vee b)$. Therefore, $\mu_{\Downarrow\tilde{x}}(a \vee b) = \mu_{\Downarrow\tilde{x}}(a) \barwedge \mu_{\Downarrow\tilde{x}}(b)$, for all $a, b \in L$. In an analogous way, we easily prove that $\nu_{\Downarrow\tilde{x}}(a \vee b) = \nu_{\Downarrow\tilde{x}}(a) \veebar \nu_{\Downarrow\tilde{x}}(b)$. Finally, we conclude that $\Downarrow \tilde{x}$ is an IF-ideal on L.

(ii) Follows dually by using Proposition 4.2, (i) and Proposition 4.1.

□

In the following result, we show a characterization of a principal IF-ideal (resp. IF-filter) in terms of a down-set (resp. up-set) generated by an IF-singleton.

Theorem 4.4. *Let* L *be a lattice and* I *(resp.* F*) be an IF-ideal (resp. IF-filter) on* L. *Then it holds that*

(i) I *is a principal IF-ideal on* L *if and only if there exists* $x \in L$ *such that* $I = \Downarrow \tilde{x}$*;*

(ii) F *is a principal IF-filter on* L *if and only if there exists* $x \in L$ *such that* $F = \Uparrow \tilde{x}$.

Proof. We only prove (i), as (ii) can be proved analogously by using Proposition 4.2 and Proposition 4.1. Suppose that I is a principal IF-ideal on L. Then there exists an IF-singleton $\tilde{x}$ such that I is the smallest IF-ideal contains $\tilde{x}$. Since $\tilde{x} \subseteq I$, it follows from Proposition 4.5 that $\Downarrow \tilde{x} \subseteq \Downarrow I = I$. On the other hand, Theorem 4.3 guarantees that $\Downarrow \tilde{x}$ is an ideal. Then the fact that I is the smallest ideal contains $\tilde{x}$ implies that $I \subseteq \Downarrow \tilde{x}$. Thus, $I = \Downarrow \tilde{x}$.

Conversely, $I = \Downarrow \tilde{x}$ is an IF-ideal contains $\tilde{x}$. Now, suppose that J is an other IF-ideal contains $\tilde{x}$. From Proposition 4.5, it holds that $\Downarrow \tilde{x} \subseteq \Downarrow J = J$. Hence, $I = \Downarrow \tilde{x}$ is the smallest IF-ideal contains $\tilde{x}$. Thus, I is a principal IF-ideal. □

In the following proposition, we show that the kernel of a principal IF-ideal (resp. IF-filter) is a crisp principal ideal (resp. filter).

Proposition 4.9. *Let L be a lattice and x be an element on L. Then we have*

(i) $Ker(\Downarrow \tilde{x}) = \downarrow x$;

(ii) $Ker(\Uparrow \tilde{x}) = \uparrow x$.

Proof. (i) From Proposition 4.8, it holds that $Ker(\Downarrow \tilde{x}) = \downarrow Ker(\tilde{x})$. This means that

$$Ker(\Downarrow \tilde{x}) = \downarrow \{t \in L \mid \mu_{\tilde{x}}(t) = 1 \text{ or } \nu_{\tilde{x}}(t) = 0\}.$$

By the definition of IF-singleton, we know that $\mu_{\tilde{x}}(t) = 1$ or $\nu_{\tilde{x}}(t) = 0$ if and only if $t = x$. Hence,

$$Ker(\Downarrow \tilde{x}) = \downarrow \{x\} = \downarrow x.$$

(ii) Follows from Proposition 4.2 and (i).

□

General conclusion and future research

In this thesis, we studied some properties of fuzzy pseudo order and the possibility of fuzzification of certain notions and results of the classical pseudo orders. We have introduced the notion of fuzzy pseudo ordered set and fuzzy trellis. In particular, we proved a representation theorem, namely, any fuzzy pseudo ordered set is isomorphic with a contraction set of fuzzy ordered set. Also, by fuzzifications of the classical three ordering axioms [39, 84], we generalized many results known in classical pseudo orders. We followed the steps of Davvaz [32], we define the concept of fuzzy ideal (resp. filter) on a trellis as a fuzzy set closed under the operations the supremum and the infimum of the binary operations $\sqcap$ and $\sqcup$.

We used the concepts of left-transitive (resp. right-transitive) elements on a trellis in order to characterize the notion of principal ideal (resp. filter) on a trellis. Also, we introduced the notion of fuzzy down-set (resp. up-set) on a trellis to investigate their properties, and characterized principal fuzzy ideals and filters on it.

Also, we introduced the notion of an intuitionistic fuzzy down-set (resp. up-set) on a lattice, and investigated their properties. Using the previous notions, We characterized principal intuitionistic fuzzy ideals and filters on a lattice.

Future work will be directed towards the extension of the notions of fuzzy pseudo order and fuzzy trellis. The notions of an L-E-pseudo order and L-E-trellis, is anticipated in multiple directions. We think it makes sense to study the notions of an intuitionistic fuzzy ideal and an intuitionistic fuzzy filter for other types of lattices and intuitionistic fuzzy ordered lattice. Moreover, we intend to extend this work to other kinds of intuitionistic fuzzy ideals and filters. Also, we will investigate other classes of intuitionistic fuzzy ordered sets satisfying the fixed point property, in order to combine the properties of these classes with the properties of its intuitionistic fuzzy monotonic mappings.

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خلاصة

في هذه الأطروحة، درسنا مفهوم المجموعات شبه المرتبة (مجموعات مزودة بعلاقة انعكاسية و ضد تناظرية و لكن ليست متعدية) وناقشنا بعض خصائصها الأساسية. ومنه قدمنا مفهوم الشبكية الضبابية كتعميم طبيعي للشبكية التي قدمتها Skala و التي هي حالة خاصة من المجموعات شبه المرتبة. بالإضافة إلى ذلك، قدمنا مفهوم المثالي الضبابي و(المرشح الضبابي، على التوالي) كمجموعات ضبابية على الشبكية وعرضنا خصائصها المهمة. وأولينا اهتمامًا خاصًا لنوع المثالي الأساسي الضبابي و(المرشح الأساسي، على التوالي) على شبكية معينة، والتي تكون أكثر تعقيدًا من نفس المفاهيم على الشبكة. أخيرًا، نأخذ في الاعتبار نفس الخصائص للمُثُل الضبابية و(المرشحات، على التوالي) على شبكة في الحالة الضبابية الحدسية.

الكلمات المفتاحية:

علاقة شبه ترتيب، مجموعة شبه مرتبة، الشبكية، المثاليات الرئيسية، المرشحات الرئيسية ،المجموعة الضبابية و المجموعة الضبابية الحدسية.

Abstract

In this thesis, we study the notion of fuzzy pseudo ordered sets and some of its interesting properties, and then we introduce the concept of fuzzy trellis as a natural generalization of the trellis introduced by Skala with respect to a fuzzy pseudo order relation. In addition, we introduce the notion of fuzzy ideal (resp. fuzzy filter) as fuzzy sets on a crisp trellis and we present interesting characterizations of these notions. We pay particular attention to the principal fuzzy ideals (resp. filters) on a given trellis, which is more complicated than the same notions on a lattice. At the end, we provide various characterizations of intuitionistic fuzzy ideals (resp. fuzzy filters) on a lattice.

Keywords:

Pseudo order relation, Pseudo ordered set, Trellis, Lattice, Principal ideal, Principal filter, Fuzzy set , Intuitionistic fuzzy set.

Résumé

Dans cette thèse, nous avons étudié la notion d'ensembles pseudo ordonnés flous et ses intéressantes propriétés, puis, nous avons introduit la notion de trellis flou comme une généralisation naturelle de celle de trellis introduite par Skala. De plus, on a considéré la notion d'idéal (resp. filtre) flou comme des ensembles flous sur un trellis et nous avons présenté leurs caractérisations. Nous avons accordé une attention particulière au type d'idéaux (resp. filtres) flous principaux sur un trellis donné, ce qui est plus compliqué que les mêmes notions sur un treillis (lattice). Pour finir, nous considérons les mêmes caractérisations d'idéaux flous (resp. filtres) sur un treillis au cas flou intuitionniste.

Mots clés:

Relation de pseudo ordre, Ensemble pseudo ordonné, Trellis, Treillis, Idéal principal, Filtre principal, Ensemble flou, Ensemble flou intuitionniste.