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Near Viability of a Set-Valued Map Graph with Respect to a Quasi-Autonomous Nonlinear Inclusion

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Abstract

This paper addresses near viability of a set-valued map graph $\mathcal{G}$ with respect to a quasi-autonomous fully nonlinear differential inclusion of the form $y'(t) \in Ay(t)+F(t, y(t))$. We introduce a new notion of A-quasi-tangency when A is a nonlinear m-dissipative set-valued operator. We give necessary and sufficient conditions for $\mathcal{G}$ to be near viable with respect to the previous differential inclusion. We obtain under weak hypotheses a classical relaxation result stating that each solution of the relaxed differential inclusion can be approximated by a solution of the differential inclusion at any given precision.

Keywords Near viability · Differential inclusion · A-quasi-tangency · Relaxation

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1 Introduction

We consider the quasi-autonomous fully nonlinear differential inclusion

$$y'(t) \in Ay(t) + F(t, y(t)), \tag{1}$$

where X is a real separable Banach space with a norm $\|\cdot\|$, $A : D(A) \subset X \rightsquigarrow X$ is a m-dissipative set-valued operator, and $F : I \times X \rightsquigarrow X$ is a given set-valued map where $I = [a, b) \subset \mathbb{R}$ with $b \leq +\infty$.

In this paper, we are interested in near viability of a graph with respect to Eq. 1. We recall that near viability means the existence of solutions which remain arbitrarily close to a

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given set starting from initial states in that set whereas (exact) viability means the existence of solutions which remain in a given set starting from that set.

Near viability has been characterized in terms of tangency conditions. Unlike the (exact) viability, convexity or compactness of the values of F is not needed. Furthermore, the compactness of the semigroup may be dropped (see, e.g., [3–5]).

In [5], the authors proved that if X is a separable Banach space, $A : D(A) \subset X \rightsquigarrow X$ is m-dissipative, $S \subset \overline{D(A)}$ is a locally closed set, and $F : X \rightsquigarrow X$ is Lipschitz, then the usual tangency condition introduced in [6] is sufficient for S to be near viable with respect to the autonomous fully nonlinear differential inclusion $y'(t) \in Ay(t) + F(y(t))$.

This paper treats the quasi-autonomous fully nonlinear inclusions of the form (1). Two main aspects are concerned. First, we consider a time-dependent $F : I \times X \rightsquigarrow X$ which is only locally Lipschitz with respect to the second variable. Second, the set S is the graph of a set-valued map. We introduce the concept of A-quasi-tangency of a set-valued map when A is nonlinear and m-dissipative. Necessary and sufficient conditions for near viability are established. Then, we apply the obtained results to derive a relaxation theorem under weak conditions which generalizes a well-known classical theorem [12, Theorem 3.3].

The paper is organized as follows. Section 2 is dedicated to background material. Section 3 introduces and characterizes the concept of A-quasi-tangency concept. Section 4 presents the main result of this paper; we give necessary and sufficient conditions for a graph to be near viable with respect to Eq. 1. Finally, we present a relaxation theorem referring to Eq. 1 in Section 5.

2 Notations and Preliminaries

In this section, we present some notations used in the remaining and recall basic concepts and results concerning nonlinear differential inclusions in abstract Banach spaces (see [1, 7, 10] for more details).

We reuse the notations of the problem (1). In addition, we will define later on two set-valued maps, namely, E and K. We denote by $B(x, \rho)$ the closed ball with center x and radius ρ and $\mathbb{B} = B(0, 1)$. The distance between two subsets C and D of X is defined by

$$\mathrm{dist}(C; D) = \inf\{\|c - d\|, c \in C,\ d \in D\}.$$

In particular, for $x \in X$, $\mathrm{dist}(x; C)$ stands for $\mathrm{dist}(\{x\}; C)$. Furthermore, if x and C depend on t, we define the following notation:

$$\mathrm{dist}_I(x; C) = \sup_{t \in I} \mathrm{dist}(x(t); C(t)).$$

In the remaining, we may need to assume without loss of generality (use a translation if necessary) that $0 \in D(A) \cap A0$ and write $0 \in A0$ for shortness.

2.1 Integral Solutions

A Cauchy problem associated to Eq. 1 has the following form

$$y'(t) \in Ay(t) + f(t),\ \ y(t_0) = x \in \overline{D(A)}, \tag{2}$$

where $f \in L^1(t_0, T; X)$, $[t_0, T] \subset [a, b)$.

We recall that the *bracket* $[\cdot,\cdot]_+$ is a "directional derivative" on X defined by (see, e.g., [10] Section 1.2)

$$[x,u]_+ = \lim_{h\to 0^+} \frac{\|x+hu\|-\|x\|}{h}.$$

Definition 1 [2] We say that a continuous function $y:[t_0,T]\to\overline{D(A)}$ is an integral solution of Eq. 2 on $[t_0,T]\subset I$ if $y(t_0)=x$ and

$$\|y(t)-u\| \le \|y(s)-u\| + \int_s^t [y(\theta)-u, f(\theta)+v]_+ d\theta,$$

for every $t_0\le s\le t\le T$, $u\in D(A)$ and $v\in Au$. The function f is called *pseudoderivative* of y and we will write f_y.

Theorem 1 [10] *For each $x\in\overline{D(A)}$ and $f\in L^1(t_0,T;X)$, the Cauchy problem* (1) *has a unique solution on* $[t_0,T]$. *Moreover, if y_1 and y_2 are two solutions of* Eq. 2 *with $y_1(t_0)=x_1$ and $y_2(t_0)=x_2$ then, for every $t\in[t_0,T]$ one has*

$$\|y_1(t)-y_2(t)\| \le \|x_1-x_2\| + \int_{t_0}^t \|f_{y_1}(s)-f_{y_2}(s)\|ds. \tag{3}$$

Corollary 1 *Assume that $0\in A0$. If $y:[t_0,T]\subset I\to\overline{D(A)}$ is solution of* Eq. 2 *with $y(t_0)=x$, then, for every $t\in[t_0,T]$ one has*

$$\|y(t)\| \le \|x\| + \int_{t_0}^t \|f_y(s)\|ds. \tag{4}$$

Proof Apply Theorem 1 with $y_1=y$ and $y_2\equiv 0$. □

Definition 2 A function $y:[t_0,T]\to\overline{D(A)}$ is said to be an *(integral) solution* of Eq. 1 on $[t_0,T]\subset I$ if there exists a pseudoderivative f_y satisfying $f_y(t)\in F(t,y(t))$ for a.e. $t\in[t_0,T]$.

2.2 Some Results About Set-Valued Maps

Consider $E:I\rightsquigarrow X$ measurable, integrably bounded[1] with nonempty and closed values. We denote by $\mathcal{S}_J E$ the set of all integrable selections of E defined on a sub–interval $J\subset I$. Thanks to the Kuratowski-Ryll Nardzewski theorem, we deduce $S_J E\ne\emptyset$.

Lemma 1 [9] *Let $h:[t_0,T]\to X$ and $k:[t_0,T]\to\mathbb{R}_+$ be two measurable functions. Assume that*

$$E(t)\cap(h(t)+k(t)\mathbb{B})\ne\emptyset,$$

for a.e. $t\in[t_0,T]$. Then, there exists a measurable function $g:[t_0,T]\to X$ such that $g(t)\in E(t)\cap(h(t)+k(t)\mathbb{B})$ for a.e. $t\in[t_0,T]$.

[1] E is integrably bounded if $\exists l\in L^1(I;\mathbb{R}_+): E(t)\subset l(t)\mathbb{B}$, for a.e. $t\in I$. In this case, each measurable selection of E is integrable (consequence of Lebesgue theorem).

Lemma 2 *Suppose that X' is uniformly convex. Then, $\mathcal{S}_{[t_0,T]}\overline{co}E$*[2] *is a weakly compact convex subset of $L^1(t_0, T; X)$ and*

$$\mathcal{S}_{[t_0,T]}\overline{co}E = \overline{\mathcal{S}_{[t_0,T]}E}^w$$

where $\overline{\mathcal{S}_{[t_0,T]}E}^w$ denotes the closure of the set $\mathcal{S}_{[t_0,T]}E$ w.r.t. the weak topology.

Proof Suppose that X' is uniformly convex. Then, X' is reflexive (Milman-Pettis' Theorem) and consequently X is reflexive. Therefore, for every $t \in [t_0, T]$, the set $\overline{co}E(t)$ is weakly compact as it is bounded, closed, and convex. The assertion of the lemma holds as a consequence of [12, Lemma 3.1]. □

Definition 3 The operator $Q : \overline{D(A)} \times L^1(t_0, T; X) \to C([t_0, T]; X)$, defined by $Q(x, f) = y$, where y is the unique integral solution of the problem (2), i.e., $f = f_y$ and $y(t_0) = x$, is called an *integral solution operator* attached to the problem (2).

Lemma 3 *Suppose that X' is uniformly convex and A generates a compact semigroup. Then, for every $x \in \overline{D(A)}$, $Q(x, \cdot)$ is continuous on $\mathcal{S}_{[t_0,T]}E$ w.r.t. the weak topology of $L^1(t_0, T; X)$.*

Proof Applying Lemma 3.4 of [12] to $\overline{co}E$, we deduce that $Q(x, \cdot)$ is continuous on $\mathcal{S}_{[t_0,T]}\overline{co}E$ w.r.t. the weak topology of $L^1(t_0, T; X)$. A fortiori, it is also continuous on the smaller set $\mathcal{S}_{[t_0,T]}E$. □

In the following, we give a theorem [5] estimating the distance between a given solution of (2) and the set of solutions of (1), providing a kind of set-valued Gronwall lemma. Such a result has been initially proposed by Fillipov [8] for the particular case $A = 0$, $X = \mathbb{R}^n$, by Frankowska [9] when $A : D(A) \subset X \to X$ generates a C_0–semigroup, and by Zhu [13] when X is not necessarily separable. Capraru and Lazu [5] extended the result of [9] to the nonlinear case.

Theorem 2 [5] *Let $y : [t_0, T] \to \overline{D(A)}$ be a solution of* Eq. 2. *Assume that $F : [t_0, T] \times X \rightsquigarrow X$ takes nonempty closed values and satisfies the following:*

- *(H1)* *For each $x \in \overline{D(A)}$, the set-valued map $F(\cdot, x)$ is measurable.*
- *(H2)* *The function $t \mapsto \gamma(t) := \operatorname{dist}(f_y(t); F(t, y(t)))$ belongs to $L^1(t_0, T; \mathbb{R}_+)$.*
- *(H3)* *There exist $\rho > 0$ and $k \in L^1(t_0, T; \mathbb{R}_+)$ such that for a.e $t \in [t_0, T]$ the set-valued map $F(t, \cdot)$ is $k(t)$–Lipschitz on $y(t) + \rho\mathbb{B}$.*

Let $\delta \geq 0$. We consider two functions defined on $[t_0, T]$ by

$$\forall t \in [t_0, T] : m(t) = \exp\left(\int_{t_0}^{t} k(s)ds\right) \quad \text{and} \quad \eta(t) = m(t)\left(\delta + \int_{t_0}^{t} \gamma(s)ds\right).$$

If $\eta(T) < \rho$, then for all $z \in \overline{D(A)}$ with $\|x - z\| \leq \delta$ and all $\varepsilon > 0$, there exists a solution $u : [t_0, T] \to \overline{D(A)}$ of (1) with $u(t_0) = z$ such that

$$\forall t \in [t_0, T] : \|u(t) - y(t)\| \leq \eta(t) + (t - t_0)\, m(t)\, \varepsilon.$$

[2] $\overline{co}E(t)$ stands for the closure of the convex–hull of $E(t)$.

3 Tangency Conditions

Let $K : I \rightsquigarrow \overline{D(A)}$ be a set-valued map and $\mathcal{G}$ the associated graph. We introduce a new notion of A-quasi-tangent set-valued map which generalizes the definitions given in [3] (for $A = 0$), [4] (when A is linear), and [7, 11] (when E is independent of t).

Let $(t_0, x) \in \mathcal{G}$ and let $h > 0$ be such that $[t_0, t_0 + h] \subset I$. We associate to the Cauchy problem (2) and the set-valued map E, the following set:

$$S^E_{\mathscr{C}}(t_0, h) = \{y(t_0 + h);\ \ f_y \in \mathcal{S}_{[t_0,t_0+h]}E\}.$$

where f_y is the pseudoderivative of y as in Definition 1.

Definition 4 We say that E is A-quasi-tangent to $\mathcal{G}$ at $(t_0, x) \in \mathcal{G}$ if

$$\liminf_{h\to 0^+} \frac{1}{h} \text{dist}\left(S^E_{\mathscr{C}}(t_0, h); K(t_0 + h)\right) = 0.$$

Next, we give two illustrative examples of the tangency notion in the finite and infinite dimensional cases.

Example 1 Let $X = \mathbb{R}$, $D(A) = [1, +\infty)$, and $Ax = \mathbb{R}_+$ if $x = 1$ and $Ax = \{0\}$ if $x \neq 1$. A is m-dissipative.

We define $K : [1, +\infty) \rightsquigarrow [1, +\infty)$ by $K(t) = [t, \infty)$. The graph $\mathcal{G}$ of K is:

$$\mathcal{G} = \{(t, y) \in [1, +\infty] \times \mathbb{R} : y \geq t\}.$$

Let $E : [1, +\infty) \rightsquigarrow \mathbb{R}$ be defined by $E(t) = [-t, 0]$. We can show that E is A-quasi-tangent to K at $(t_0, x) \in \mathcal{G}$ if $t_0 \neq x$. Indeed, let us define $f_y : [t_0, t_0 + h] \to \mathbb{R}$ by $f_y(t) = -t$ where $h > 0$, and $y(t) = \frac{-t^2+t_0^2}{2} + x$, $\forall t \in [t_0, t_0 + h]$ (f_y is a pseudoderivative of y and $f_y \in \mathcal{S}_{[t_0,t_0+h]}E$). Then, we get

$$\text{dist}\left(S^E_{\mathscr{C}}(t_0, h); K(t_0 + h)\right) \leq \text{dist}\left(y(t_0 + h); K(t_0 + h)\right) = 0$$

$h \leq h_0 = \sqrt{t_0^2 + 2x + 1} - t_0 - 1$.

Now we let $t_0 = x > 1$. A straightforward calculus shows that

$$S^E_{\mathscr{C}}(t_0, h) \subset I_{t_0,h} = \left[\frac{2t_0 - 2t_0h - h^2}{2}, t_0\right].$$

Therefore,

$$\text{dist}\left(S^E_{\mathscr{C}}(t_0, h); K(t_0 + h)\right) \geq \text{dist}\left(I_{t_0,h}; K(t_0 + h)\right) = h.$$

Thus,

$$\liminf_{h\to 0^+} \frac{1}{h} \text{dist}\left(S^E_{\mathscr{C}}(t_0, h); K(t_0 + h)\right) \geq 1,$$

which means that E is not A-quasi-tangent to $\mathcal{G}$ at (t_0, x) when $t_0 = x > 1$.

Example 2 Let X be a separable Banach space and let $A : D(A) \subset X \rightsquigarrow X$ be m-dissipative with $0 \in A0$. We define $K : I \rightsquigarrow \overline{D(A)}$ by $K(t) = \{x \in \overline{D(A)};\ \|x\| \leq t\}$, where I is a bounded interval of $\mathbb{R}$. Let $g : I \times X \to X$. Assume that there exist $x_0 \in \overline{D(A)} - \{0\}$ such that $g(\cdot, x_0) \in L^1(I; X)$ and a Lebesgue point t_0 for $g(\cdot, x_0)$ satisfying $\|x_0\| \leq t_0$. We define $E : I \rightsquigarrow X$ such that $E(t) = g(t, x_0) - g(t_0, x_0) + \mathbb{B}$. To show that E is A-quasi-tangent to the graph $\mathcal{G}$ of K at (t_0, x_0), for an arbitrary $h > 0$ satisfying $[t_0, t_0 + h] \subset I$, we consider two elements $z(t_0 + h) \in K(t_0 + h)$ and $y(t_0 + h) \in S^E_{\mathscr{C}}(t_0 + h)$.

Let $f_z : [t_0, t_0+h] \to X$ be a constant function defined by $f_z(t) = \frac{x_0}{\|x_0\|}$. Using Corollary 1, we can write:

$$\|z(t_0+h)\| \leqslant \|x_0\| + \int_{t_0}^{t_0+h} \|f_z(s)\| ds \ \leq t_0 + h.$$

Accordingly, $z(t_0+h) \in K(t_0+h)$.

Now, we let $f_y : [t_0, t_0+h] \to X$ be defined by $f_y(t) = g(t, x_0) - g(t_0, x_0) + \dfrac{x_0}{\|x_0\|}$. From Eq. 3, one gets

$$\begin{aligned}
\liminf_{h\to 0^+} \frac{1}{h} \text{dist}\left(S_{\mathscr{C}}^{E}(t_0, h); K(t_0+h)\right) &\leq \liminf_{h\to 0^+} \frac{1}{h} \|y(t_0+h) - z(t_0+h)\| \\
&\leq \liminf_{h\to 0^+} \frac{1}{h} \int_{t_0}^{t_0+h} \|g(s, x_0) - g(t_0, x_0)\| ds \\
&= 0.
\end{aligned}$$

Theorem 3 *Suppose that X' is uniformly convex and A generates a compact semigroup. Then, E is A-quasi-tangent to $\mathcal{G}$ at a given $(t_0, x) \in \mathcal{G}$ if and only if $\overline{co}E$ satisfies the same property.*

Proof Let $(t_0, x) \in \mathcal{G}$ and $h > 0$ with $[t_0, t_0+h] \subset I$.

On the one hand, since $E \subset \overline{co}E$, we have $S_{\mathscr{C}}^{E}(t_0, h) \subset S_{\mathscr{C}}^{\overline{co}E}(t_0, h)$. Then,

$$\text{dist}\left(S_{\mathscr{C}}^{\overline{co}E}(t_0, h); K(t_0+h)\right) \leq \text{dist}\left(S_{\mathscr{C}}^{E}(t_0, h); K(t_0+h)\right).$$

Passing to limit, we conclude that if E is A-quasi-tangent to $\mathcal{G}$ at $(t_0, x) \in \mathcal{G}$ then $\overline{co}E$ satisfies the same property.

On the other hand, we first prove that $S_{\mathscr{C}}^{\overline{co}E}(t_0, h) \subset \overline{S_{\mathscr{C}}^{E}(t_0, h)}$. Let $y(t_0+h) \in S_{\mathscr{C}}^{\overline{co}E}(t_0, h)$ $(f_y \in \mathcal{S}_{[t_0,T]}\overline{co}E)$. It follows from Lemma 2 that there exists a sequence $(f_n)_n$ in $S_{[t_0,t_0+h]}E$ which converges weakly to f_y. Let $y_n = Q(x, f_n)$ for every $n \in \mathbb{N}$. From Lemma 3, we deduce that $(y_n(t_0+h))_n$ converges to $y(t_0+h)$ in X. Therefore,

$$\begin{aligned}
\text{dist}\left(\overline{S_{\mathscr{C}}^{E}(t_0, h)}; K(t_0+h)\right) &\leq \text{dist}\left(S_{\mathscr{C}}^{\overline{co}E}(t_0, h); K(t_0+h)\right) \\
\text{dist}\left(S_{\mathscr{C}}^{E}(t_0, h); K(t_0+h)\right) &\leq \text{dist}\left(S_{\mathscr{C}}^{\overline{co}E}(t_0, h); K(t_0+h)\right)
\end{aligned}$$

Again, passing to limit, we conclude that if $\overline{co}E$ is A-quasi-tangent to $\mathcal{G}$ at $(t_0, x) \in \mathcal{G}$ then E satisfies the same property, which completes the proof of the theorem. □

4 Criteria for Near Viability

We introduce the definition of near viability of a moving set graph.

Definition 5 The graph $\mathcal{G}$ associated to K is said to be *near viable* with respect to Eq. 1 if, for any $(t_0, x) \in \mathcal{G}$, there exists $T > t_0$ such that $[t_0, T] \subset I$, and for any $\epsilon > 0$, there exists a solution of (1), $y : [t_0, T] \subset I \to \overline{D(A)}$, satisfying $y(t_0) = x$ and $\text{dist}_{[t_0,T]}(y; K) \leq \epsilon$.

If the previous definition condition holds for any $T > t_0$, the graph is said to be *globally near viable*.

For the sake of readability, we first collect some technical hypotheses that we will use in the sequel.

(H3′) For each $(t_0, x) \in \mathcal{G}$, there exist $\rho' > 0$ and $k \in L^1(I; \mathbb{R}_+)$ such that $F(t, \cdot)$ is $k(t)$-Lipschitz on $x + \rho'\mathbb{B}$, i.e.,

$$F(t, y) \subset F(t, z) + k(t)\|y - z\|\mathbb{B}, \quad \forall y, z \in B(x, \rho'), \text{ for a.e.} t \in I. \tag{5}$$

(H4) For each $x \in X$, the set-valued map $F(\cdot, x)$ is integrably bounded with nonempty closed values, i.e., there exists $l_x \in L^1(t_0, T; \mathbb{R}_+)$ such that $F(t, x) \subset l_x(t)\mathbb{B}$ for a.e. $t \in [t_0, T]$;

(H5) For each $x \in X$, the set-valued map $F(\cdot, x)$ has nonempty closed values and satisfies the sub-linear growth condition: There exists $c \in L^1(I; \mathbb{R}_+)$ such that for a.e. $t \in I$ and every $y \in X$:

$$F(t, y) \subset c(t)(1 + \|y\|)\mathbb{B}.$$

(H6) K takes nonempty closed values and is uniformly continuous, i.e., for each $\epsilon > 0$ there exists $\alpha > 0$ such that $K(t) \subset K(s) + \varepsilon\mathbb{B}$, whenever $|t - s| \leq \alpha$.

Proposition 1 *Assume that F satisfies the sub-linear growth condition **(H5)**. Then, for each $(t_0, x) \in \mathcal{G}$ and $T > t_0$ with $[t_0, T] \subset I$, there exists $\mu : [t_0, T] \to \mathbb{R}_+$ with* $\lim_{t \to t_0} \mu(t) = 0$ *such that each solution of (1), $y : [t_0, T] \to \overline{D(A)}$, satisfies:*

$$\|y(t) - x\| \leq \mu(t), \ \forall t \in [t_0, T].$$

Proof Let $(t_0, x) \in \mathcal{G}$, $T > t_0$ with $[t_0, T] \subset I$, and $y : [t_0, T] \to \overline{D(A)}$ a solution of Eq. 1. Without loss of generality, we assume that $0 \in A0$. From Corollary 1 and Gronwall's inequality, one obtains:

$$\begin{aligned}
\forall t \in [t_0, T]: \ \|y(t)\| &\leq \|x\| + \textstyle\int_{t_0}^{t} \|f_y(s)\| ds, \\
&\leq \|x\| + \textstyle\int_{t_0}^{t} c(s)(1 + \|y(s)\|) ds, \quad \text{(from } \textbf{(H5)}\text{)} \\
&\leq \underbrace{\|x\| + \int_{t_0}^{T} c(s) ds}_{\tilde{m}} + \textstyle\int_{t_0}^{t} c(s)\|y(s)\| ds, \\
&\leq \tilde{m} \exp\left(\textstyle\int_{t_0}^{T} c(s) ds\right) = M.
\end{aligned}$$

Let z be a solution of Eq. 2 with $f_z \equiv 0$. Using Theorem 1, one gets

$$\forall t \in [t_0, T]: \quad \big| \|y(t) - x\| - \|z(t) - x\| \big| \leq \|y(t) - z(t)\| \leq \int_{t_0}^{t} \|f_y(s)\| ds.$$

Thus, we get:

$$\begin{aligned}
\forall t \in [t_0, T]: \quad \|y(t) - x\| &\leq \textstyle\int_{t_0}^{t} \|f_y(s)\| ds + \|z(t) - x\|, \\
&\leq \textstyle\int_{t_0}^{t} c(s)(1 + \|y(s)\|) ds + \|z(t) - x\|, \\
&\leq \textstyle\int_{t_0}^{t} c(s)(1 + M) ds + \|z(t) - x\| = \mu(t).
\end{aligned}$$

□

Theorem 4 (Necessary conditions for near viability) *Suppose that F satisfies **(H1)**, **(H3′)**, and **(H5)**. If $\mathcal{G}$ is near viable with respect to (1) then $F(\cdot, x)$ is A-quasi-tangent to $\mathcal{G}$ at (t_0, x) for a.e. $t_0 \in I$ and every $x \in K(t_0)$.*

Proof Let $(t_0, x) \in \mathcal{G}$. As (**H3**$'$) holds, let $T > t_0$ with $[t_0, T] \subset I$ and $h < T - t_0$ verifying $\mu(t) \leq \rho', \forall t \in [t_0, t_0 + h]$.

Suppose that $\mathcal{G}$ is near viable with respect to (1). For $\epsilon = h^2$, there exists a solution $y : [t_0, T] \to \overline{D(A)}$ of (1) satisfying $\text{dist}_{[t_0,T]}(y; K) \leq \epsilon$. We identify y with its restriction to $[t_0, t_0+h]$. The pseudoderivative of y satisfies $f_y(t) \in F(t, y(t))$, for a.e. $t \in [t_0, t_0+h]$.

From Proposition 1, one gets

$$\forall t \in [t_0, t_0 + h] : \|y(t) - x\| \leq \mu(t) \leq \rho',$$

where μ is a function chosen as in Proposition 1. From Eq. 5, we can write:

$$F(t, y(t)) \subset F(t, x) + k(t)\, \|y(t) - x\|\, \mathbb{B}, \;\; for\ a.e.\ t \in [t_0, t_0 + h].$$

Accordingly,

$$f_y(t) \in F(t, x) \cap \left(f_y(t) + k(t)\|y(t) - x\|\mathbb{B}\right) \neq \emptyset, \;\; for\ a.e.\ t \in [t_0, t_0 + h].$$

By virtue of Lemma 1, we deduce that there exist two measurable functions, $g : [t_0, T] \to X$ and $b : [t_0, T] \to \mathbb{B}$ satisfying $g(t) \in F(t, x)$ and

$$g(t) = f_y(t) + k(t)\|y(t) - x\|b(t), \;\; for\ a.e.\ t \in [t_0, t_0 + h]. \tag{6}$$

Now, we let $z : [t_0, t_0 + h] \to \overline{D(A)}$ to be the solution of Eq. 2 with a pseudoderivative $f_z = g$. Applying (3) to the two solutions z and y, we get

$$\begin{aligned}
\forall t \in [t_0, t_0 + h] : \;\; \|z(t) - y(t)\| &\leq 0 + \int_{t_0}^{t} \|f_z(s) - f_y(s)\| ds, \\
&\leq \int_{t_0}^{t} \|f_y(s) + k(s)\|y(s) - x\|b(s) - f_y(s)\| ds, \\
&\leq \int_{t_0}^{t} k(s)\|y(s) - x\| ds.
\end{aligned}$$

As $z(t_0 + h) \in S_{\mathscr{C}}^{F(\cdot,x)}(t_0, h)$, we get:

$$\begin{aligned}
\text{dist}\left(S_{\mathscr{C}}^{F(\cdot,x)}(t_0, h); K(t_0 + h)\right) &\leq \text{dist}(z(t_0 + h), K(t_0 + h)), \\
&\leq \|z(t_0 + h) - y(t_0 + h)\| + \text{dist}(y(t_0 + h); K(t_0 + h)), \\
&\leq \int_{t_0}^{t_0+h} k(s)\|y(s) - x\| ds + h^2, \\
&\leq \int_{t_0}^{t_0+h} k(s)\mu(s) ds + h^2.
\end{aligned}$$

Without loss of generality, we suppose that t_0 is a Lebesgue point for the function $k(\cdot)\mu(\cdot)$. Passing to limit, we get

$$\liminf_{h\to 0^+} \frac{1}{h} \text{dist}\left(S_{\mathscr{C}}^{F(\cdot,x)}(t_0, h); K(t_0 + h)\right) = 0. \tag{7}$$

Thus, $F(\cdot, x)$ is A-quasi-tangent to $\mathcal{G}$ for a.e. $t_0 \in I$ and every $x \in K(t_0)$. □

Remark 1 It is worth noting that the conclusion of the previous theorem 4 holds only almost everywhere. But, if we suppose that every point of I is a Lebesgue point for k, then the conclusion will be true for all $t \in I$, i.e., for any point $(t_0, x) \in \mathcal{G}$.

Next, we provide sufficient conditions for near viability with respect to Eq. 1. But, we first give a technical lemma which constitutes the main step through the proof of the theorem. Before, we recall the definition of X-closed graph $\mathcal{G}$.

Definition 6 The graph $\mathcal{G}$ is said to be X-closed if for each sequence $(t_n, x_n)_n$ in $\mathcal{G}$ with $\lim_n(t_n, x_n) = (t, x)$ and $t \in I$, it follows that $(t, x) \in \mathcal{G}$.

Lemma 4 *Suppose that $\mathcal{G}$ is X-closed graph and F satisfies **(H4)**. If $F(\cdot, x)$ is A-quasi-tangent at every point of $\mathcal{G}$, then for every $(t_0, x) \in \mathcal{G}$ there exists $T > t_0$ with $[t_0, T] \subset I$ such that, for every $\varepsilon \in (0, 1)$, there exist a nondecreasing $\sigma : [t_0, T] \to [t_0, T]$, a solution of (2) $y : [t_0, T] \to \overline{D(A)}$, and a continuous $v : [t_0, T] \to X$ verifying:*

(i) $t - \varepsilon \le \sigma(t) \le t$, *for all* $t \in [t_0, T]$, *and* $\sigma(T) = T$*;*
(ii) $v(\sigma(t)) \in K(\sigma(t))$, *for all* $t \in [t_0, T]$*;*
(iii) $f_y(t) \in F(t, v(\sigma(t)))$ *for a.e.* $t \in [t_0, T]$, *and* $\|f_y(t)\| \le l_x(t)$ *for a.e.* $t \in [t_0, T]$ *where* l_x *is a* L^1 *function associated with* ***(H4)****;*
(iv) $v(t_0) = x$ *and* $\|v(t) - z(t)\| \le (t - \sigma(s))\varepsilon$, *for all* $t_0 \le s \le t \le T$, *where* z *is the solution of (2) with* $z(\sigma(s)) = v(\sigma(s))$*;*
(v) $\|v(t) - v(\sigma(t))\| \le \varepsilon$, *for all* $t \in [t_0, T]$.

For reader's convenience and for the sake of completeness, we give the proof of the lemma in the Appendix.

Remark 2 Notice that if we use **(H5)** rather than **(H4)** in Lemma 4, the time T will be independent of the initial state (t_0, x).

Theorem 5 (Sufficient conditions for near viability) *Suppose that the hypotheses **(H1)**,**(H3′)**, **(H4)**, and **(H6)** are satisfied. If $F(\cdot, x)$ is A-quasi-tangent at every point of $\mathcal{G}$, then $\mathcal{G}$ is near viable with respect to (1).*

Proof Suppose that $F(\cdot, x)$ is A-quasi-tangent at every point of $\mathcal{G}$. From **(H6)**, we deduce that $\mathcal{G}$ is X-closed. Thus, we can apply Lemma 4.

Let $(t_0, x) \in \mathcal{G}$ and let $\bar{T} > t_0$ be as in Lemma 4. According to **(H3′)**, there exist $\rho' > 0$ and $k \in L^1(I; \mathbb{R}_+)$ satisfying (5). We put $\rho = \frac{\rho'}{2}$.

Let us consider $\epsilon > 0$. From **(H6)**, there exists α such that $K(t) \subset K(s) + \varepsilon\mathbb{B}$, whenever $|t - s| \le \alpha$.

Let us consider $\epsilon_1 = \min\{\epsilon, \alpha, \frac{\rho}{\bar{T} - t_0 + 1}, 1/2\} \in (0, 1)$. From Lemma 4, there exist y, a solution of (2) defined on $[t_0, \bar{T}]$, and two functions σ and v such that (i)∼(v) hold true.

Step 0: Choice of T

As y is continuous at t_0, there exists $T_1 \in (t_0, \bar{T}]$ such that $\|y(t) - y(t_0)\| < \rho$, for all $t \in [t_0, T_1]$. On the other hand, as in Theorem 2 (with $\delta = 0$), we consider two functions defined by

$$\forall t \in [t_0, T_1] : m(t) = \exp\left(\int_{t_0}^{t} k(s)ds\right) \text{ and } \eta(t) = m(t)\int_{t_0}^{t} \gamma(s)ds.$$

We will show in Step 2 that γ is integrable. The function η is continuous. Thus, there exists $T_2 > t_0$ such that $\eta(t) < \rho$ when $t < T_2$. We choose $T = \min(T_1, T_2)$.

Step 1: (H3) holds

It is easy to show that $B(y(t),\rho)\subset B(x,\rho')$,for all $t\in[t_0,T]$. From **(H3′)**, we get:

$$F(t,z_1)\subset F(t,z_2)+k(t)\|z_1-z_2\|\mathbb{B},\quad \forall z_1,z_2\in B(y(t),\rho),\ \text{for a.e.} t\in[t_0,T].$$

Step 2: (H2) holds

From (v) and (iv) (with $s=t_0$, i.e., $z=y$), one gets for all $t\in[t_0,T]$:

$$\|v(\sigma(t))-y(t)\|\le\|v(\sigma(t))-v(t)\|+\|v(t)-y(t)\|\le(T-t_0+1)\epsilon_1.$$

So, we get for all $t\in[t_0,T]$:

$$\|v(\sigma(t))-y(t)\|\le(T-t_0+1)\epsilon\ \text{ and } \|v(\sigma(t))-y(t)\|\le\rho.$$

From **(H3)** and the above inequality, we get:

$$\begin{aligned}\text{For a.e}t\in[t_0,T]:\ F(t,v(\sigma(t)))&\subset F(t,y(t))+k(t)\,\|v(\sigma(t))-y(t)\|\,\mathbb{B},\\ &\subset F(t,y(t))+k(t)\,(T-t_0+1)\epsilon\,\mathbb{B}.\end{aligned}$$

On the other hand, we notice that the hypotheses **(H1)** and **(H3)** hold. Consequently, the function γ introduced in hypothesis **(H2)** is measurable (see [9]). In addition, from (iii), we infer that for a.e. $t\in[t_0,T]$,

$$\gamma(t)=\text{dist}(f_y(t),F(t,y(t))\le(T-t_0+1)\,\epsilon\,k(t).$$

Consequently, $\gamma\in L^1(t_0,T;\mathbb{R}_+)$ (in fact, $\gamma\in L^1(t_0,T_1;\mathbb{R}_+)$). In addition, we have

$$\begin{aligned}\eta(T)&=m(T)\int_{t_0}^T\gamma(s)ds,\\ &\le m(T)\int_{t_0}^T(T-t_0+1)\epsilon k(s)\,ds,\\ &\le m(T)(T-t_0+1)\epsilon\int_{t_0}^T k(s)\,ds.\end{aligned}$$

Step 3. From Theorem 2, we deduce that there exists a solution $u:[t_0,T]\to\overline{D(A)}$ of (1) with $u(t_0)=x$, which satisfies

$$\begin{aligned}\forall t\in[t_0,T]:\|u(t)-y(t)\|&\le\eta(t)+(t-t_0)m(t)\varepsilon,\\ &\le\eta(T)+(T-t_0)m(T)\epsilon.\end{aligned}$$

To complete the proof, we check that $\text{dist}_{[t_0,T]}(u,K)\le\epsilon$. According to Lemma 4-(i), we have $t-\sigma(t)<\epsilon_1<\alpha$, $\forall t\in[t_0,T]$. Thus, from **(H6)**, we get

$$\forall t\in[t_0,T]:\text{dist}\,(K(\sigma(t));K(t))<\epsilon.$$

On the other hand, from Lemma 4-(ii), we have

$$\begin{aligned}\forall t\in[t_0,T]:\ \text{dist}\,(u(t);K(\sigma(t)))&\le\|u(t)-y(t)\|+\|y(t)-v(\sigma(t))\|,\\ &\le\eta(T)+(T-t_0)m(T)\epsilon+(T-t_0+1)\epsilon,\\ &\le C\epsilon,\end{aligned}$$

where

$$C=m(T)(T-t_0+1)\int_{t_0}^T k(s)\,ds+(T-t_0)m(T)+(T-t_0+1).$$

Thus, we deduce that $\text{dist}_{[t_0,T]}\,(u;K)\le C\,\epsilon$. □

Theorem 6 (Sufficient conditions for global near viability) *Suppose that the hypotheses **(H1)**, **(H4)**, and **(H6)** are satisfied. Suppose also that there exists* $k\in L^1(I;\mathbb{R}_+)$ *such that*

$F(t,\cdot)$ is $k(t)$-Lipschitz on X. If $F(\cdot,x)$ is A-quasi-tangent at every point of $\mathcal{G}$, then $\mathcal{G}$ is globally near viable with respect to Eq. 1.

Proof The proof follows, except minor modifications, the very same arguments as those of the proof of Theorem 5. □

5 Application

The next result tackles a fundamental question in the qualitative theory of differential equations which is particularly relevant in control theory; the relation between the solutions of Eq. 1 and the solutions of the relaxed (convexified) differential inclusion

$$y'(t) \in Ay(t) + \overline{co}F(t, y(t)). \tag{8}$$

Obviously, solutions of the first problem are also solutions of the second. Here, we will show that, under some weak natural hypotheses for which the relaxation property holds, each solution of the relaxed differential inclusion can be approximated by a solution of the original differential inclusion at any given precision.

Theorem 7 *Suppose that X' is uniformly convex, A generates a compact semigroup and F satisfies **(H1)** and **(H5)**. If $y:[t_0,T]\to\overline{D(A)}$ is a solution of (21), with $y(t_0)=x\in\overline{D(A)}$, and if F satisfies **(H3′)** with $\mathcal{G}$ is the graph of the function y and where every point of I is a Lebesgue point for the functions k. Then, for every $\epsilon>0$, there exists a solution $y_\epsilon:[t_0,T]\to\overline{D(A)}$ of* Eq. 1 *satisfying $y_\epsilon(t_0)=x$, and*

$$\forall t \in [t_0, T]:\ \|y(t) - y_\epsilon(t)\| \le \varepsilon.$$

Proof Let $y:[t_0,T]\to\overline{D(A)}$ be a solution of (21) with $y(t_0)=x\in\overline{D(A)}$. Let $K:[t_0,T)\rightsquigarrow\overline{D(A)}$ be defined by $K(t)=\{y(t)\}$. Let $\epsilon>0$ be arbitrary chosen.

First, we notice that the graph $\mathcal{G}$ of K is near viable with respect to (21). As F satisfies **(H1)** and **(H5)**, $\overline{co}F$ satisfies also the same hypotheses (see [1] for **(H1)**; **(H5)** is easy). Then, from Theorem 4 and Remark 1 applied to $\overline{co}F$, we deduce that $\overline{co}F(\cdot,\xi)$ is A-quasi-tangent to $\mathcal{G}$ at (τ,ξ), for every (τ,ξ) in $\mathcal{G}$. Now from Theorem 3, we infer that $F(\cdot,\xi)$ is A-quasi-tangent to $\mathcal{G}$ at (τ,ξ) for every (τ,ξ) in $\mathcal{G}$.

Notice that assumption **(H6)** holds by the continuity of $y(\cdot)$ and **(H4)** stems from **(H5)**. We can now apply Theorem 5 to conclude that there exists $\overline{T}\in(t_0,T)$ such that for every $\epsilon>0$, there exists a solution $y_\epsilon:[t_0,\overline{T}]\to\overline{D(A)}$ such that

$$\forall t \in [t_0, \overline{T}]:\ \mathrm{dist}(y_\epsilon(t); K(t)) = \|y_\epsilon(t) - y(t)\| \le \epsilon.$$

As in the the proof of Lemma 4, we continue the above construction in a similar way by replacing t_0 by $\overline{T}$. Finally, using a simple application of Zorn's Lemma, we define y_ϵ on $[t_0,T]$. □

Remark 3 A.L. Tolstonogov established a result of this type [12, Theorem 3.3] but for Lipschitz set-valued map F. This theorem imposes weaker condition: for a given solution of the relaxed problem to be approximated, it is enough that F is locally Lipschitz on the graph of that solution **(H3′)**.

6 Conclusion

In this paper, we presented necessary and sufficient conditions for near viability of a graph with respect to a quasi-autonomous fully nonlinear differential inclusion. To do that, we introduced a notion of quasi-tangency of a set-valued map at a given point of the graph. This new notion completely characterized the near viability of a moving graph. We extended some previous notions and results proposed in the autonomous and time-independent graph case. As application, we gave some weak conditions for which solutions of relaxed differential inclusions can be approximated by solutions of the of initial differential inclusion. In the future, we may explore applications of our main result for some control problems.

Appendix

A Proof of Lemma 4

Proof of Lemma 4 Assume that $F(\cdot, x)$ is A-quasi-tangent at every point of $\mathcal{G}$. Let $(t_0, x) \in \mathcal{G}$ and let $T > t_0$ be as in the above definition.

The proof relies on two steps. First, we show that the conclusion of Lemma 4 is true for some σ, y, f_y, and v, defined on $[t_0, t_0 + h]$, where h depends on ϵ. Then, using the Brezis–Browder ordering principle (see, e.g., [7, Theorem 2.1.1]), we prove the existence of σ, y, f_y and v defined on $[t_0, T]$, where T is independent of ϵ.

Step 1.

Let $\epsilon \in (0, 1)$ be arbitrary. Assume that $F(\cdot, x)$ is A-quasi-tangent to $\mathcal{G}$ at (t_0, x). According to the definition, there exist $h > 0$ and a solution of Eq. 2, $y : [t_0, t_0 + h] \to \overline{D(A)}$, such that $f_y(t) \in F(t, x)$ for a.e. $t \in [t_0, t_0 + h]$, and

$$\frac{1}{h}\text{dist}\,(y(t_0 + h); K(t_0 + h)) < \epsilon.$$

Thus, there exists $m \in K(t_0 + h)$ such that:

$$\|m - y(t_0 + h)\| < \epsilon.h\,.$$

We put $p = \frac{1}{h}(m - y(t_0 + h)) \in X$. We have $\|p\| < \epsilon$ and $y(t_0 + h) + hp \in K(t_0 + h)$.

We now define two functions on $[t_0, t_0 + h]$:

$$\sigma(t) = t_0 \text{ for} t < t_0 + h,\ \sigma(t_0 + h) = t_0 + h,\ \text{ and } v(t) = y(t) + (t - t_0)p,\ \forall t \in [t_0, t_0 + h].$$

It is easy to check that σ, v, y and f_y (the pseudoderivative of y) satisfy the conclusion of Lemma 4 on $[t_0, t_0 + h] \subset [t_0, T]$ (we may diminish h if necessary).

Step 2.

Let $\mathcal{S}$ be the set of all triplets (σ, f_y, v) of functions defined on $[t_0, t_0 + h] \subset [t_0, T]$, for some $h > 0$, satisfying the conclusion of Lemma 4. From Step 1, the $\mathcal{S}$ is nonempty.

On $\mathcal{S}$, we introduce a preorder relation $\preceq$ as follows:

$$(\sigma_1, f_{y_1}, v_1) \preceq (\sigma_2, f_{y_2}, v_2) \iff \begin{array}{l}\sigma_2(f_{y_2}, v_2 \text{ respectively) is an extension of}\\ \sigma_1(f_{y_1}, v_1 \text{ respectively) to a lager interval.}\end{array}$$

We define a function $\mathcal{N} : \mathcal{S} \to \mathbb{R}$ by $\mathcal{N}((\sigma, f_y, v)) = h$, where $[t_0, t_0 + h]$ is the domain of definition of the three functions σ, f_y, and v. It is clear that $\mathcal{N}$ is increasing.

Now, we show that each increasing sequence of $\mathcal{S}$ is bounded from above. Let $\big((\sigma_m, f_{y_m}, v_m)\big)_{m\in\mathbb{N}}$ be an increasing sequence of $\mathcal{S}$, where $[t_0, t_0 + h_m]$ is the associated domain of definition. As $(h_m)_{m\in\mathbb{N}}$ is increasing and bounded from above by T, we have $h^\star = \lim_m h_m$. We will define in the following an upper bound $(\sigma, f, v) \in \mathcal{S}$, where the three functions are defined on $[t_0, t_0 + h^\star]$.

If there exists $k \in \mathbb{N}$ such that $h_k = h^*$, then it would be easy to show that (σ_k, f_{y_k}, v_k) is an upper bound of the sequence. So, we assume that $h_m < h^*$ for each $m \in \mathbb{N}$.

Function σ

We define the function $\sigma : [t_0, t_0 + h^\star] \to [t_0, t_0 + h^\star]$ by $\sigma = \sigma_m(t)$ if $t \in [t_0, t_0 + h_m]$ and $\sigma(t_0 + h^\star) = t_0 + h^\star$.

Function f

We let $f : [t_0, t_0 + h^\star] \to X$ to be defined by $f(t) = f_{y_m}(t)$, if $t \in [t_0, t_0 + h_m]$ and $f(t_0 + h^\star) = \eta$ where η is arbitrary but fixed. We have $f \in L^1(t_0, t_0 + h^\star; X)$ because $f(t) \le l_x(t)$ a.e. $t \in [t_0, t_0 + h^\star)$.

Function v

We define $v : [t_0, t_0 + h^\star]$ by $v(t) = v_m(t)$, if $t \in [t_0, t_0 + h_m]$ and $v(t_0 + h^\star) = \lim_{t\to t_0+h^\star} v(t)$. We can prove that the limit exists. Indeed, let $m \in \mathbb{N}$. Denote by z_m the solution of Eq. 2 defined on $[t_0 + h_m, t_0 + h^\star]$ with $z_m(t_0 + h_m) = v_m(\sigma_m(t_0 + h_m)) = v_m(t_0 + h_m)$ and $f_{z_m} = f\,|_{[t_0+h_m, t_0+h^\star]}$. Therefore, for every $t,\ t' \in [t_0 + h_m, t_0 + h^\star)$ one gets:

$$\begin{aligned}
\|v(t) - v(t')\| &\le \|v(t) - z_m(t)\| + \|z_m(t) - z_m(t')\| + \|z_m(t') - v(t')\| \\
&\le (t - t_0 - h_m)\epsilon + \|z_m(t) - z_m(t')\| + (t' - t_0 - h_m))\epsilon \quad \text{(from (iv))} \\
&\le 2(h^\star - (h_m + t_0))\epsilon + \|z_m(t) - z_m(t')\|
\end{aligned}$$

Taking into account that $(h_m)_m$ converges to $h^\star$ and that z_m is continuous on $[t_0 + h_m, t_0 + h^\star]$ we deduce that v satisfies the Cauchy condition for the existence of the limit at $t_0 + h^\star$.

Now, we show that (σ, f, v) verifies the conclusions (i)-(v) of the Lemma 4, to conclude that $(\sigma, f, v) \in \mathcal{S}$.

- Obviously, (i), (iii), and (v) are satisfied.
- $v(\sigma(t)) \in K(\sigma(t))$, for all $t \in [t_0, h^*)$ is straightforward. As $\mathcal{G}$ is X-closed, we get $v(\sigma(t_0 + h^*)) \in K(\sigma(t_0 + h^*))$ by considering the sequence $t_0 + h_m \to t_0 + h^*$. Thus, (ii) is satisfied.
- Thanks to the continuity of v and z, (iv) can be easily proved.

According to the Brezis–Browder ordering principle, there exists at least one $\mathcal{N}$-maximal element of $\mathcal{S}$ that we will denote by (σ^*, f^*, v^*), where the functions are defined on an interval $[t_0, t_0 + h^*]$. Now. we will show that $t_0 + h^* = T$.

By contradiction, assume that $t_0 + h^\star < T$. Since $(t_0 + h^\star, v(t_0 + h^\star)) \in \mathcal{G}$, then it follows that $F(\cdot, v(t_0 + h^\star))$ is A-quasi-tangent to $\mathcal{G}$ at $(t_0 + h^\star, v(t_0 + h^\star))$. Reasoning as the first step, we deduce that there exist $\overline{\sigma}$, $f_{\overline{y}}$, and $\overline{v}$ defined on $[t_0 + h^\star, \overline{T}] \subset [t_0, T]$,

$t_0 + h^\star < \overline{T}$, such that the conclusion of Lemma 4 holds. Now we define σ, f_y, and v as follows:

Function σ

We define the function $\sigma : [t_0, \overline{T}] \to [t_0, \overline{T}]$ by $\sigma(t) = \sigma^\star(t)$, if $t \in [t_0, t_0 + h^\star]$ and $\sigma(t) = \overline{\sigma}(t)$ if $t \in [t_0 + h^\star, \overline{T}]$.

Function f_y

We define $f_y : [t_0, \overline{T}] \to X$ by $f_y(t) = f_{y^\star}(t)$, if $t \in [t_0, t_0 + h^\star]$ and $f_y(t) = f_{\overline{y}}(t)$ if $t \in [t_0 + h^\star, \overline{T}]$. We point out here that the function y is a solution of (2) on $[t_0, \overline{T}]$ (concatenation principle).

Function v

We define $v : [t_0, \overline{T}] \to X$ by $v(t) = v^\star(t)$, if $t \in [t_0, t_0 + h^\star]$ and $v(t) = \overline{v}(t)$ if $t \in [t_0 + h^\star, \overline{T}]$.

It easy to show that that triplet (σ, f_y, v) satisfies (i)∼(v) of Lemma 4. This contradicts the fact that (σ, f_y, v) is a $\mathcal{N}$-maximal element. □

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