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Dedication

* They could neither read nor write, their love allowed me to write my doctoral thesis in mathematics... to my parents

* To my wife and my son

* To all my brothers and sisters

Notations

$\mathcal{L}(X)$	The algebra of all bounded linear operators from X into itself
$\mathcal{N}(A)$	The null space of A
$R(A)$	The range space of A
$G(A)$	The graph of A
$\mathcal{K}(X)$	The ideal of all compact operators on X
$\mathcal{C}(X)$	The set of all closed densly defined
$\mathcal{W}(X,Y)$	The family of weakly compact from X to Y
$\mathcal{S}(X,Y)$	The set of strictly singular operators from X to Y
$C\mathcal{S}(X,Y)$	The set of strictly cosingular operators from X to Y
$\mathcal{I}(X)$	An arbitrary non zero two sided ideal of $\mathcal{L}(X)$
$\sigma(A)$	The spectrum of A,
$\rho(A)$	The resolvent set of A
$F(X)$	The set of Fredholm operators
$\alpha(A)$	The nullity of A is defined as the dimension of $N(A)$
$\beta(A)$	The deficiency of A is defined as the codimension of $R(A)$
I	Operator of identity
$\mathcal{F}(X,Y)$	The set of fredholm perturbation
$\Phi_{+}(X,Y)$	The set of upper semi-Fredholm operators
$\Phi_{-}(X,Y)$	The set of lower semi-Fredholm operators

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Introduction

The theory of the essential spectra of linear operators in Banach space is a section of the spectral analysis widely used in the mathematical and physical sense when resolving a number of applications that can be formulated in terms of linear operators. Since exact computations of the spectrum are almost always impossible, it is relevant to know the spectrum or any of its parts in approximate way. Several authors have studied this problem and other related topics using various types of convergence in $\mathcal{L}(X)$. There are several notions of convergence of a sequence of operators which yield spectral results: the norm convergence ([21], [26], [27], [31], [29], [30], [32], [67], [76], [77], [79]), collectively compact convergence ([12]), compact convergence and regular convergence ([10]), stable and strongly stable convergence ([25]), resolvent operator convergence ([59]), spectral convergence ([5]), convergence of $(T_n)_n$ to T in the sense that the spectral radius $r(T_n - T)$ of $T_n - T$ tends to zero and $\|(T_n - T)T_n\|$ tends to zero ([7]), asymptotically compact convergence ([11]) and also discrete versions of some of these ([84], [40], [85]), to name a few. The principle aim of this thesis is to study the convergence of some classes of the essential spectrum of a sequence of linear operators which converges in some sense. We used the convergence in the generalized sense (see Definition 4.3.4), the convergence compactly (see Definition 4.4.2), and the $v-$convergence. The concept of generalized convergence was introduced automatically from the notion of the "aperture" or "gap" between two closed linear manifolds. The gap between two closed subspaces was introduced in Hilbert space by M.G. Krein and M. A. Krasnoselski in [55]. This notion was later extended to arbitrary Banach spaces in a paper by M. G. Krein, M. A. Krasnoselski, and D. P. Milman in [56]. In recent years, several authors have investigated the notions of generalized convergence of a sequence of closed linear operators in Banach spaces. We can cite as example [54, 68]. In [68], M. Pellicer stated that the notion of generalized convergence can be viewed as the convergence between the graphs of operators and hence it is useful in comparing the corresponding resolvent operators. Essentially, in [68] M. Pellicer used this notion between closed linear operators both to introduce the notion of a gap between linear operators and to study the nonlinear strongly damped wave equation with a dynamic boundary condition. The concept of convergence to zero compactly is appearance dates back to Goldberg's work [38] in 1974, when he studied properties of the nullity, deficiency and index of operators of the form $T + K_n$, where T is semi Fredholm operator and K_n converges to zero compactly, where he obtained theorems analogous to those given in [39] and [54], where the requirement was that K_n be "small enough" in norm, inspired by the concept of convergence to zero compatly we introduce the notion of convergence compactly and study the essential approximate point spectrum and the essential defect spectrum of a sequence of linear operators convergence in this sens. As for the concept of $v-$Convergence is introduced in 1994 by M. Ahues, A. Largillier and B. V. Limaye in [6]. In order to study spectral continuity properties, this type of convergence is useful ([32], [26], [78], [8]).

In ordre to present our results we organise this thesis in the following way: In first chapter we recal the basic properities of the spectrun of bounded and unbounded linear operator in a Banach space, introducing the basic notaions necessary to study linear adjoint operators and we present some classical quantities associated with an operators, this quantities such as the closed and closable of an operators are defined in the first section and are the basic bricks in the construction of one of the most important branches of spectral theory, the theory of Fredholm operators and essential spectrum

The second chapter is devoted to study the semi-regular operator, more precisely, we gives a survey of results concerning the composition of two semi-regular operators. The concept of semi-regular, and essentialy semi regular among the various concepts of regularity, originated from the classical treatment of perturbation theory of Frdholm operators owing to T. kato in (1958), and its development has greatly benefited from the work of many authors in the last years, such as M. Mbekhta and A. Ouahab [62], V. Muller [65], V. Rakocevìc [71], M. berkani, and A. Ouahab [18]. An operator T is said to be semi-regular if $R(A)$ is closed and $\mathcal{N}(A^n) \subseteq R(A)$, for all $n \geq 0$, where $R(A)$ and $N(A)$ denote, respectively, the range and the nul space of A. This concept of classe of operator leads naturally to the semi regular spectrum $\sigma_{es}(A)$, defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not semi-regular and its essential version is the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not essentially semi-regular. The semi regular spectrum was first introduced by C. Apostol [13] for operators on Hilbert spaces and successively studied by several authors mentionned above in the mor general context of operators acting on Banach spaces.

In the third chapter, for the study of perturbation theory, two useful concepts which have extensive applications are those of relative boundedness and relative compactness. These can be defined both with respect to norms and to quadratic forms. The principle application is to selfadjoint second ordre differential operator, for this reason we consider the Sturm-Liouville expression.

The forth chapter provides the investigation of the relationship between the essential approximate point spectrum, noted σ_{eap}, (respectively, the essential defect spectrum, noted $\sigma_{e\delta}$) of a sequence of closed linear operators $(T_n)_{n\in\mathbb{N}}$ on Banach space X, and the essential approximate point spectrum (respectively, the essential defect spectrum) of a linear operator T on X, where $(T_n)_{n\in\mathbb{N}}$ converges to T, in the case of convergence in generalized sense as well as in the case of the convergence compactly.

The last chaptre is devoted to the study of spectral continuity of the essential spectrum using the $v-$Convergence and application on block operator matrix, more precisley we generalise the results of A. Ammar in [9] on 2×2 block operator matrix to the case of 3×3 block operator matrix.

Chaptre 1

Basic Properties

1.1 Generalities about Closed operators

Let X and Y be two Banach spaces. A mapping A which assigns to each element x of a set $D(A) \subset X$ a unique element $y \subset Y$ is called an operator (or transformation), the set $D(A)$ on which A acts is called the domain of A.

Definition 1.1.1 *The operator A is called linear, if $D(A)$ is a subspace of X and if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all scalars α, β and all elements x, y in $D(A)$.*

We note by a pair $(A, D(A))$ the operator A with domain $D(A)$

Definition 1.1.2 *The operator A is called bounded on a Banach space X, if there is a constant M such that $\| Ax \| \leq M \| x \|$, $for\ all$ $x \in X$. The norm of such an operator is dfined by*

$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}$$

Example 1.1.1 *The identity operator $I : X \to X$ defined by $Ix = x$ is obviously a bounded linear operator of norm 1. The diferential operator of first order $A = i\partial_x$ on $\mathcal{L}^2([0;1])$, it readily seen to be unbounded since one can find a sequence of functions $\varphi_n \in \mathcal{L}^2([0;1])$, given by $\varphi_n = e^{inx}$ for $n \in \mathbb{N}$, satisfying $\| \varphi_n \|_{\mathcal{L}^2([0,1])} = 1$ and $\| A\varphi_n \|_{\mathcal{L}^2([0,1])} = n \to \infty$ as $n \to \infty$.*

1.1.1 Unbounded linear operator

Definition 1.1.3 *We define an unbounded linear operator on a Banach space X to be pair consisting of dense linear subspace D of X together with a linear map $A : D \subset X \longrightarrow X$.*

For example, a general continuous functions does not have a continuous derivative, so differential operators are defined on subspace of continuous functions space.

Remark 1.1.1 *If the domain of A is not dense, then we may obtain a densely defined operator by setting A equal to zero on the orthogonal complement of it's domain*

Definition 1.1.4 *We call the graph of A the subspace of $X \times X$, noted $G(A)$, and defined by*

$$G(A) = \{(x, Ax) \in X \times X, \text{ for all } x \in D(A)\}$$

Definition 1.1.5 *An operator $\tilde{A}$ is an extension of A, or A is a restriction of $\tilde{A}$, if*

$$D(A) \subset D\left(\tilde{A}\right), \text{ and } \tilde{A}x = Ax \text{ for all } x \in D(A).$$

We note this relationship by $A \subset \tilde{A}$

Example 1.1.2 *Let $A_k u = u''$, $k = 1, 2, 3, 4$ be diferential operators in $L^2([0,1])$ with domains*

$$\begin{aligned} D(A_1) = &\ \{u \in C^2([0,1]), u(0) = u(1) = 0\} \\ D(A_2) = &\ C^2([0,1]) \\ D(A_3) = &\ \{u \in H^2([0,1]),\ u(0) = u(1) = 0\} \\ D(A_4) = &\ L^2([0,1]). \end{aligned}$$

We have $D(A_k) \subset L^2([0,1])$, $k = 1, 2, 3, 4$ then A_k, $k = 1, 2, 3, 4$ is an unbounded operator on $L^2([0,1])$, and we have $D(A_1) \subset D(A_2) \subset D(A_4)$, then we note $A_1 \subset A_2 \subset A_4$ (on sense of extension), and also $A_3 \subset A_4$

1.1.2 Closed and closable linear operator

Definition 1.1.6 *An unbounded linear operator A on Banach space X is said to be closed in X if and only if it's graph $G(A)$ is closed in X^2, whose topology is determined by the norm $\|(x,y)\| = \left(\|x\|_X^2 + \|y\|_X^2\right)^{\frac{1}{2}}$.*

The closedness of an operator may be caracterized by this property

Proposition 1.1.1 *An unbounded linear operator A on Banach space X is closed if for every sequence (x_n) in $D(A)$ such that $x_n \to x$ and $Ax_n \to y$, we have*

$$x \in D(A) \text{ and } Ax = y.$$

Proposition 1.1.2 *Clearly an operator A is closed if, and only if, $D(A)$ is complete with respect to the norm $\|x\|_A = \left(\|x\|_X^2 + \|Ax\|_X^2\right)^{\frac{1}{2}}$, $x \in D(A)$, If we shall denote $D(A)$ equipped with $\|.\|_A$ by $X(A)$. Thus A is closed if, and only if $X(A)$ is a Banach space.*

We must be make a difference between continuous operator and closed operator, where in continuous operator case the convergence of the sequence (x_n) implies that (Ax_n) converges and we have

$$\lim_n (Ax_n) = A\left(\lim_n x_n\right) \tag{1.1.1}$$

For a closed operator A, the convergence of x_n does not imply the convergence of (Ax_n), but if both (x_n) and (Ax_n) converge, then the equality (1.1.1) holds.

Note that the continuity of A does not, necessarily, imply that A is closed. Conversly, A is closed does not necessarily imply that A is continuous (for example see [39])

As a particular case we have the remark

Remark 1.1.2 *Let A be an unbounded linear operator in Banach space X with domain $D(A)$, if A is closed, it's bounded on $D(A)$ if and only if $D(A)$ is a closed subspace of X.*

Example 1.1.3 *Let the defferential operator $Af = f''$ when $X = L^2([0,1])$,*

We choose $D(A) = AC([0,1])$ the set of functions which are absolutely continuous on $[0,1]$, and with this as domain, A is in fact a closed operator, however $AC([0,1])$ is not a closed subspace of $L^2([0,1])$, and A is not bounded (take the sequence $f_n(t) = t^n$, $n = 1, 2, ...$, then $\dfrac{\|Tf_n\|}{\|f_n\|} \underset{n}{\to} \infty$)

Definition 1.1.7 *An operator $A : X \to X$, with domain $D(A) \subset X$ is said to be closable if $\overline{G(A)}$ is a graph of an operator, In this case, the operator whose graph is $\overline{G(A)}$ is called the closure of A, and denoted by $\overline{A}$, then $G(\overline{A}) = \overline{G(A)}$*

The closure of A is the minimal closed extension of A, in that for any closed extension T of A ($A \subset T$), we have $A \subset \overline{A} \subset T$.

This definition may be caracterised by this property

Proposition 1.1.3 *An operator A is closable if for every sequence (x_n) of elements in $D(A)$ such that $x_n \to 0$ and $Ax_n \to y$, for some $y \in X$, we have $y = 0$.*

Remark 1.1.3 *i) If A is one to one and closed then A^{-1} is closed, then the closed graph theorem implies that A^{-1} is bounded.*

ii) if A is closed, $\mathcal{N}(A)$, the null space of A is closed

iii) If T is bounded on $D(T)$, it is closable, where $D(\overline{T})$ is the closure of $D(T)$ in X and $\overline{T}$ is bounded in $D(\overline{T})$ with $\|\overline{T}\| = \|T\|$.

Example 1.1.4 *In Example 1.1.2, we have A_1 and A_2 are not closed, because we can may choose sequence of functions $u_n \in C^2([0,1])$ such that $u_n \to u$ and $u_n'' \to\ v$ in $L^2([0,1])$, where v is not continuous, hence u is not C^2*

A_3 and A_4 are closed, and we have $\overline{A_1} = A_3$, $\overline{A_2} = A_4$, then A_1, A_2 are closable.

1.1.3 Adjoint and selfadjoint operator

Definition 1.1.8 *suppose that A is a densely defined unbounded linear operator on a Hilbert space H with a domain $D(A)$, The adjoint of A is the operator, noted A^*, of H in H with domain*

$$D(A^*) = \{y \in H, \text{ there is } z \in H \text{ with } \langle Ax, y\rangle = \langle x, z\rangle, \text{ for all } x \in D(A)\}.$$

Let $y \in D(A^*)$, then we define $A^*y = z$, where z is the unique element such that

$$\langle Ax, y\rangle = \langle x, z\rangle \text{ for all } x \in D(A).$$

$D(A^*)$ is not dense necessarely

Definition 1.1.9 *An unbounded linear operator A is selfadjoint if $A = A^*$, meaning that $D(A^*) = D(A)$ and $A^*x = Ax$ for all $x \in D(A)$.*

An unbounded linear operator A is said to be symetric if A^* is an extension of A, meaning that $D(A^*) \supset D(A)$ and $A^*x = Ax$ for all $x \in D(A)$.

Example 1.1.5 *In Example 1.1.2 we have*

$$A_1u = u'',\ D(A_1) = \{u \in C^2([0,1]), u(0) = u(1) = 0\}.$$

$D(A^*) = \{v \in L^2([0,1]),$ there is $w \in L^2([0,1])$ with $\langle A_1u, v\rangle = \langle u, w\rangle,$ for all $u \in D(A_1)\}$

$$\begin{aligned}\langle A_1u, v\rangle = &\ \langle u'', v\rangle \\ &= \int_0^1 u''(t)v(t)dt \\ &= [u'v]_0^1 - [uv']_0^1 + \int_0^1 u(t)v''(t)dt \quad \text{(with integration by parts)} \\ &= [u'v]_0^1 + \int_0^1 u(t)w(t)dt \qquad \text{such that } w(t) = v''(t) \text{ for } t \in [0,1]\end{aligned}$$

This need that $v \in L^2([0,1])$, with $v(0) = v(1) = 0$, then

$$D(A_1^*) = \{v \in L^2([0,1]),\ v(0) = v(1) = 0\} = D(A^3).$$

It means that $A_1^* = A_3$, then $A_1$is symetric but not selfadjoint. We will see that $A_3^* = A_3$, then A_3 is selfadjoint.

For the operator A_2, we have

$$D(A_2^*) = \{v \in L^2([0,1]),\ \text{there is } w \in L^2([0,1]),\ \text{with } \langle A_2u, v\rangle = \langle u, w\rangle,\ \text{for all } u \in D(A_2)\}.$$

Then we have,

$$\begin{aligned}\langle A_2u, v\rangle = &\ \langle u'', v\rangle \\ &= \int_0^1 u''(t)v(t)dt \\ &= [u'v]_0^1 - [uv']_0^1 + \int_0^1 u(t)v''(t)dt \quad \text{(with integration by parts)} \\ &= [u'v - uv']_0^1 + \int_0^1 u(t)w(t)dt \qquad \text{such that } w(t) = v''(t) \text{ for } t \in [0,1]\end{aligned}$$

It needs that $v(0) = v(1) = 0$, and $v'(0) = v'(1) = 0$, then $A_2^* = A_5$, where $A_5u = u''$, and

$$D(A_5) = \{u \in L^2([0,1]),\ u(0) = u(1) = u'(0) = u'(1) = 0\}.$$

We can prove that also, $A_4^* = A_5$.

Proposition 1.1.4 *If $A : D(A) \subset H \longrightarrow H$ is a densely defined linear operator on a Hilbert space H with bounded inverse $A^{-1} : H \longrightarrow H$, then $(A^*)^{-1} = (A^{-1})^*$.*

Proof. Since A^{-1} is bounded, it has a bounded adjoint, if $x \in D(A^*)$ and $y \in H$, then $\left\langle (A^{-1})^* A^*x, y \right\rangle = \langle A^*x, A^{-1}y\rangle = \langle x, AA^{-1}y\rangle = \langle x, y\rangle$

Therefore $(A^{-1})^* A^*x = x$ for all $x \in D(A^*)$, morever if $x \in H$ and $y \in D(A)$, then $\left\langle A^* (A^{-1})^* x, y \right\rangle = \left\langle (A^{-1})^* x, Ay \right\rangle = \langle x, A^{-1}Ay\rangle = \langle x, y\rangle$

Since $D(A)$ dense in H, it follows that $(A^{-1})^* x \in D(A^*)$, and $A^* (A^{-1})^* x = x$. ■

Theorem 1.1.1 *Let be T an unbounded linear operator on a Hilbert space H of domain $D(T)$*

i) If $D(T)$ dense in H and T is closable, $(\overline{T})^* = T^*$.

ii) If T is bounded on $D(T)$, it is closable, where $D(\overline{T})$ is the closure of $D(T)$ in H and $\overline{T}$ is bounded in $D(\overline{T})$ with $\|\overline{T}\| = \|T\|$.

Proposition 1.1.5 *Let be X and Y are two Banach spaces, and $A : D(A) \subset X \to Y$ a densely defined linear operator, then the adjoint A^* is a closed operator, i.e $G(A^*)$ is closed in $Y' \times X'$.*

Proof. ([19]) ■

Thus a self adjoint operator is closed.

1.2 Compact Operators

Definition 1.2.1 ***(Compact linear operators)*** *A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if for every bounded subset G of E, the image $A(G)$ is relatively compact in F. In other words, the closure $\overline{A(G)}$ is compact*

Theorem 1.2.1 ***(Compactness criterion)*** *A linear operator A defined from a normed space E into a normed space F is called a linear compact operator or completely continuous linear operator if and only if for every bounded sequence φ_n in E, the sequence $A\varphi_n$ in F has a convergent subsequence.*

Proof. Let φ_n be a bounded sequence in E since the operator A is compact, then the set $\{A\varphi_n\}$ is relatively compact in F where this property shows that $(A\varphi_n)$ contains a convergent subsequence. Conversely, let us consider any bounded subset G in E and let ψ_n be any sequence in $A(G)$, then there exists a bounded sequence φ_n in G; such that $\psi_n = A\varphi_n$, by assumption, $A\varphi_n = \psi_n$ contains a convergent subsequence ψ_{nk} in F, thus $A(G)$ is relatively compact, because for any bounded sequence ψ_n in $A(G)$ there exits a convergent subsequence ψ_{nk} in F, in other words, for all bounded set $G \subset E$ the set $A(G)$ is relatively compact in F, hence A is compact. ■

Theorem 1.2.2 *The linear combination $A = \alpha A_1 + \beta A_2$ of compact operators A_1 and A_2 is a compact operator, for every scalars α and β*

Proof. Let φ_n be a bounded sequence in E and let $A\varphi_n$ be a sequence in F, then $A\varphi_n(x) = \alpha A_1\varphi_n(x) + \beta A_2\varphi_n(x)$ with $\varphi_n \in E$, $n \in \mathbb{N}$. The operators A_1 and A_2 are compact, one can extract from $A_1\varphi_n$ and $A_2\varphi_n$ two convergent subsequences which give by their sum a convergent subsequence of $A\varphi_n$, hence A is compact. ■

Theorem 1.2.3 *The product AB of two bounded operators A and B is compact if either of operators A or B is compact.*

Proof. Let φ_n be a bounded sequence in E then if we consider B as a bounded operator the sequence $B\varphi_n(x)$ is bounded, and from the compactness of the operator A gives a convergent subsequence $A(B\varphi_{nk}(x))$ of $A(B\varphi_n(x))$, then the operator AB is compact. On the other hand, if we consider B as a compact, one can extract from $B\varphi_n(x)$ a convergent subsequence $B\varphi_n(x)$ and from the boundedness of the operator A gives the convergence of the sequence $A(B\varphi_{nk}(x))$, hence the operator AB is compact ■

Theorem 1.2.4 *A sequence A_n of compact operators defined from a normed space E into a Banach space F converges uniformly to an operator A, say $\lim\limits_{n\to\infty} \| A_n - A \| = 0$, then the limit operator A is compact.*

Proof. Let φ_n be a bounded sequence in E, the operator A_1 is compact, then one can extract from the sequence $A_1\varphi_n$ a convergent subsequence, say φ_n^1 a subsequence from φ_n such that $A_1\varphi_n^1$ converges. In the same way, we can extract from the sequence $A_2\varphi_n^1$ a convergent subsequence, say φ_n^2 a subsequence from φ_n^1 such that $A_2\varphi_n^2$ converges. Noting that, we obtain from the bounded sequence φ_n a subsequence φ_n^2 such that $A_1\varphi_n^2$ and $A_2\varphi_n^2$ both converge, Continuing in this way, we see that, for the compact operators $A_1, A_2, ..., A_p$, there exists a nested subsequences $\varphi_n^p \subset ....\varphi_n^2 \subset \varphi_n^1 \subset \varphi_n$such that the sequences $A_k\varphi_n^p$ converge for all $k = 1, 2, ..., p$: In order to show the compactness of the operator limit A we must use the completeness of the space F and showing that the sequence $A\varphi_n^p$ is Cauchy sequence. Noting that the sequence φ_n is bounded, say $\parallel \varphi_n \parallel \leq M$ for all n hence $\parallel \varphi_n^p \parallel \leq M$ for each n and p hoose $n = p$ so that $\parallel A_n - A \parallel < \frac{\varepsilon}{3M}$

Since the sequence $A_n\varphi_n^p$ is Cauchy, because it converges there is N such that, for all $p > N$ and $q > N$;

we get

$$\parallel A_n\varphi_n^p - A_n\varphi_n^q \parallel < \frac{\varepsilon}{3}$$

Hence we obtain

$$\begin{aligned}\parallel A\varphi_n^p - A\varphi_n^q \parallel &= \parallel A\varphi_n^p - A\varphi_n^p + A\varphi_n^p - A\varphi_n^q + A\varphi_n^q - A\varphi_n^q \parallel \\ &\leq \parallel A_n - A \parallel \parallel \varphi_n^p \parallel + \parallel A_n\varphi_n^p - A_n\varphi_n^q \parallel \parallel A_n\varphi_n^q - A\varphi_n^q \parallel \\ &\leq \frac{\varepsilon}{3M}M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} = \varepsilon\end{aligned}$$

Remembering that, due to the completeness of the space F the Cauchy sequence $A\varphi_n^p$ converges as a subsequence of $A\varphi_n$ where φ_n^p is a subsequence of an arbitrary bounded sequence φ_n hence the compactness of the operator A. ■

1.3 Spectrum

Definition 1.3.1 *Let A be a closable linear operator in a Banach space X. The resolvent set and the spectrum of A are, respectively, defined as*

$$\rho(A) = \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is injective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\right\}, \sigma(A) = \mathbb{C}\backslash\rho(A)$$

and the point spectrum, continuous spectrum, and the residual spectrum are defined as

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is not injective}\},$$

$$\sigma_c(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} = X,\ R(\lambda - A) \neq X\right\},$$

$$\sigma_r(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective, } \overline{R(\lambda - A)} \neq X\right\}.$$

Remark 1.3.1 *Note that, if $\rho(A) \neq \emptyset$ then A is closed. In fact, if $\lambda \in \rho(A)$ then $(\lambda - A)^{-1}$is closed, which is also valid for $\lambda - A$ then, according to the closed graph theorem we deduce that*

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ sucht hat } \lambda - A \text{ is bijective }\}$$

and hence

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Proposition 1.3.1 *Let $(A, D(A))$ be a closed, densely defined, and linear operator with a nonempty resolvent set $\rho(A)$, for each $\lambda_0 \in \rho(A)$ we have*

$$\sigma\left((\lambda_0 - A)^{-1}\right) = (\lambda_0 - \sigma(A))^{-1}$$

1.4 Essential Fredholm spectra

1.4.1 Fredholm Operators

The concept of Fredholm operators is one of the attempts to understand the classical Frdholm operators theory of integral equations. Special types of these operators were cosidered by many authors and treated in the works of F.V. Atkinson [14], I.C. Gohberg ([34], [35], [36]) and B. Yood [89]. These paper considered bounded operators. Generalization to unbouded operators were given by M. G. Krein and M. A. Krasnoselkii [56], B. Sz. Nagy [66], and R. Kress [58]. More complete treatments were given by P. Aiena [1], I. C. Gohberg and G. Krein [37], and T. Kato [54]. A general historical account of the theory of Fredholm operators was done in [37]. Further important contribution were due to M. Schechter [74] who gave a simple and unified treatment of this theory which covered all the basic points while avoiding some of the involved concepts already used by previous authors. These studied of Fredholm theory perturbation results are of a great important in the description of the essential spectrum. This set is not a widely known as other parts of the spectrum, in particular, eigenvalues. Nevertheless, in several application, information about it is interesting for various reasons. In fact, if the whole spectrum lies in a half-plane, then the stability of a physical system is guaranteed. Moreover, near the essential spectrum, numerical calculations of eigenvalues become difficult. Hence, they have to be treated analytically. If the essential spectrum of a closed linear operator is empty, and if its reolvent set is nonempty then the spectrum consists only of isolated eigenvalues with a finite algebraic multiplicity which accumulate, at most, at ∞.

It's well known that, if A is selfadjoint operator on a Hilbert space, then the essential spectrum of A is the set of limite points of the spectrum of A (with eigenvalues counted according to their multiplicities), i.e., all points of the spectrum except some isolated eigenvalues of finite multiplicity (see for example, [86], [87]) Irrespective of wether A is bounded or not on a Banach space X, ther are many ways to define the essential spectrum, most of them are enlargement of the continuous spectrum. Hence, several definition of the essential spectrum may be found in the literature (see for example, [41], [75]) or in the comment in $[73, Chap.11, P.283]$ which coincide for self-adjoint operators on Hilbert spaces.

Definition 1.4.1 *A closed, densely defined linear operator A on a Banach space X is said to be a Fredholm operator if it satisfies*

(i) $\alpha(A) = \dim(\mathcal{N}(A)) < \infty$,

(ii) $R(A)$ *is closed*

(iii) $\beta(A) = co\dim(R(A)) < \infty$

Remark 1.4.1 *The number $i(A) = \alpha(A) - \beta(A)$ is called the index of A densely defined, closed operators on X satisfying (i) and (ii) are said to be upper semi-Fredholm operators, and those satisfying (ii) and (iii) are called lower semi-Fredholm operators.*

- *Let $\Phi(X)$ denote the set of Fredholm operators, $\Phi_+(X)$ denote the set of upper semi-Fredholm operators, and $\Phi_-(X)$ denote the set of lower semi-Fredholm operators.*

Then we have

$\Phi_{+}(X) := \{A \in \mathcal{C}(X)$ such that $\alpha(A) < \infty$ and $R(A)$ is closed in $X\}$,

$\Phi_{-}(X) := \{A \in \mathcal{C}(X)$ such that $\beta(A) < \infty$ and $R(A)$ is closed in $X\}$,

$\Phi(X) := \Phi_{+}(X) \cap \Phi_{-}(X)$,

and $\Phi_{\substack{-\\+}}(X) := \Phi_{+}(X) \cup \Phi_{-}(X)$ the set of semi-Fredholm operators from X into itself

Proposition 1.4.1 *If* $A \in L(X)$ *we consider* $r(A) = \lim\limits_{n\to+\infty} \alpha(A^{n})$ *and* $r'(A) = \lim\limits_{n\to+\infty} \beta(A^{n})$ *It is well known that if* $A = I - K$ *with* $K \in \mathcal{K}(X)$ *then both* $r(A)$ *and* r' *are finite, we next denote by* $\Phi^{*}(X)$ *the set* $\Phi^{*}(X) := \{A \in \Phi(X)$ *such that* $r(A) < \infty, r'(A) < \infty\}$

1.4.2 Schechter essential spectrum

Let A be a self-adjoint operator on a Hilbert space. A spectral singularity is said to be in the essential spectrum if it is a limit point of the spectrum of A (where eigenvalues are counted according to their multi- plicities and hence infinite-dimensional eigenvalues are included.), i.e, all points of the spectrum except isolated eigenvalues of finite multiplicity

- When dealing with non-self-adjoint closed, densely defined linear operators on Banach spaces, there are many ways to define the essential spectrum, densely defined linear operators

Definition 1.4.2 *Let* X *and* Y *be two Banach spaces and let* $A \in \mathcal{C}(X, Y)$. *We define the Schechter essential spectrum of the operator* A *by*

$$\sigma_{ess}(A) = \underset{k \in \mathcal{K}(X)}{\cap} \sigma(A + K)$$

The following proposition gives a characterization of the Schechter essential spectrum by means of Fredholm operators

Proposition 1.4.2 *Let* X *and* Y *be two Banach spaces and let* $A \in \mathcal{C}(X, Y)$, *then*

$$\lambda \notin \sigma_{ess} \text{ if and only if } \lambda \in \Phi_{A}^{0}$$

where $\Phi_{A}^{0} = \{\lambda \in \mathbb{C}$, *such that* $\lambda - A \in \Phi(X)$ *and* $i(\lambda - A) = 0\}$

Remark 1.4.2 *Let $\mathbb{C}$ be the set of complex numbers. If $A \in \mathcal{C}(X)$ we define the sets*

$$\Phi_A := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi(X)\}$$

$$\Phi_{+A} := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_+(X)\}$$

$$\Phi_{-A} := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \in \Phi_-(X)\}$$

As is well known, Φ_A, Φ_{+A} and Φ_{-A} are open subsets of the complex plane

Proposition 1.4.3 *Let $A \in \mathcal{C}(X)$ and define the sets*

$$\sigma_{e1}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+A}$$

$$\sigma_{e2}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X)\} := \mathbb{C}\backslash\Phi_{-A}$$

$$\sigma_{e3}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X) \cup \Phi_+(X)\} := \mathbb{C}\backslash\{\Phi_{-A} \cup \Phi_{+A}\}$$

$$\sigma_{e4}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi(X)\} := \mathbb{C}\backslash\Phi$$

$$\sigma_{e5}(A) := \mathbb{C}\backslash\rho_5(A)$$

$$\sigma_{e6}(A) := \mathbb{C}\backslash\rho_6(A)$$

σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra , σ_{e3} is the Kato essential spectrum, σ_{e4} is the Wolf essential spectrum , σ_{e5} is the Schechter essential spectrum , and σ_{e6} denotes the Browder essential spectrum.

- Note that, in general, we have $\sigma_{e3}(A) \subset \sigma_{e4}(A) \subset \sigma_{e5}(A) \subset \sigma_{e6}(A)$, but if A is a self-adjoint operator on a Hilbert space, then

$$\sigma_{e1}(A) = \sigma_{e2}(A) = \sigma_{e3}(A) = \sigma_{e4}(A) = \sigma_{e5}(A) = \sigma_{e6}(A).$$

1.5 Perturbations classes of operators

In this section we concentrate our attention on some perturbation classes of operators which occur in Fredholm theory, in this theory we find two fundamentally different classes of operators, such as the class of Fredholm operators, the classes of upper and lower semi-Fredholm operators, and ideals of operators, such as the classes of finite-dimensional, compact operators.

1.5.1 Weakly compact

Definition 1.5.1 *Let X and Y be Banach spaces, and let S and A be linear operators from X into Y. S is called relatively weakly compact with respect to A. or (A-weakly compact) if $D(A) \subset D(S)$ and for every bounded sequence $x_n \in D(A)$ such that $(Ax_n)_n$ is bounded, the sequence $(Sx_n)_n$ contains a weakly convergent subsequence*

Definition 1.5.2 *An operator $A \in L(X)$ is said to be weakly compact, if $A(B)$ is relatively weakly compact in Y, for every bounded subset $B \subset X$*

Definition 1.5.3 *Let $A \in L(X)$, A is called a power-compact operator, if there exists $m \in \mathbb{N}^*$ satisfying $A^m \in \mathcal{K}(X)$*

Remark 1.5.1 *The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$ if $X = Y$ the family of weakly compact operators on X, $\mathcal{W}(X,X) = \mathcal{W}(X)$ is a closed two-sided ideal of $L(X)$ containing $\mathcal{K}(X)$.*

1.5.2 Strictly singular and strictly cosingular operator

Definition 1.5.4 *Let X and Y be two Banach spaces. An operator $A \in L(X)$ is called strictly singular if, for every infinite-dimensional subspace M the restriction of A to M is not a homeomorphism, Let $\mathrm{S}(X,Y)$ denote the set of strictly singular operators from X into Y*

Remark 1.5.2 *The concept of strictly singular operators as a generalization of the notion of compact operators.*

Definition 1.5.5 *A normed linear space X is called subprojective if for every closed infinite-dimensional subspace W of X there exists a closed infinite-dimensional subspace W_1 of W and a projection from W into W_1*

Proposition 1.5.1 [87]*Let X, Y and Z be Banach spaces and let A be a closed, densely defined linear operator from X to Y. Suppose $B \in L(Y,Z)$ with $\alpha(B) < \infty$ and $BA \in \Phi(X,Z)$ then $A \in \Phi(X,Y)$*

Proposition 1.5.2 [87]*Let X, Y and Z be Banach spaces and let A be an operator in $\Phi(X, Y)$ Suppose B is a closed, densely defined linear operator from Y to Z such that $BA \in \Phi(X, Z)$ Then $B \in \Phi(Y, Z)$*

- *Let X_p denote the space $L_p(\Omega, d\mu)$ $(1 \leq p \leq \infty)$, where (Ω, Σ, μ) stands for a positive measure space*

Proposition 1.5.3 [87] *Let $A \in \mathcal{C}(X_p)$ and let $S \in L(X)$ then we have:*
(i) *If $A \in \Phi_+(X_p)$ then $A + S \in \Phi_+(X_p)$ for each $1 \leq p \leq \infty$*
(ii) *if $p \in [1, 2]$ and $A \in \Phi_-(X_p)$ then $A + S \in \Phi_-(X_p)$*
(iii) *if $p \in [1, 2]$ and $A \in \Phi_+(X_p) \cup \Phi_-(X_p)$ then $A + S \in \Phi_+(X_p) \cup \Phi_-(X_p)$.*
(iv) *if $A \in \Phi(X_p)$ then $A + S \in \Phi(X_p)$ and $i(A + S) = i(A)$ for each $1 \leq p \leq \infty$.*

1.5.3 Fredholm perturbations

Definition 1.5.6 *Let X and Y be two Banach spaces and let $F \in L(X)$. F is called a Fredholm perturbation if $U + F \in \Phi^b(X, Y)$ whenever $U \in \Phi^b(X, Y)$*

- The set of Fredholm perturbations is denoted by $\mathcal{F}(X, Y)$ this class of operators is shown that $\mathcal{F}^b(X, Y)$ is closed subset of $\mathrm{L}(X, Y)$ and if $X = Y$ then $\mathcal{F}^b(X) := \mathcal{F}^b(X, X)$ is closed two-sided ideal of $\mathrm{L}(X)$

Proposition 1.5.4 *Let X, Y and Z be three Banach spaces. If at least one of the sets $\Phi^b(X, Y)$ and $\Phi^b(Y, Z)$ is not empty, then*
(i) *$F \in \mathcal{F}^b(X, Y), A \in L(Y, Z)$ imply $AF \in \mathcal{F}^b(X, Z)$*
(ii) *$F \in \mathcal{F}^b(Y, Z), A \in L(X)$ imply $FA \in \mathcal{F}^b(X, Z)$*

Definition 1.5.7 *Let X be a Banach space and $A \in L(X)$, R is said to be a Riesz operator if $\Phi_A = \mathbb{C} \setminus \{0\}$*

Remark 1.5.3 *(a) The family of Riesz operators is not an ideal of $L(X)$*

(b) It's proved that $\mathcal{F}^b(X)$ is the largest ideal of $L(X)$ contained in the family of Riesz operators.

Remark 1.5.4 *Let X and Y be two Banach spaces, If in (Definition 1.5.6) we replace $\Phi^b(X,Y)$ by $\Phi(X,X)$ we obtain the set $\mathcal{F}(X,Y)$, let X be an arbitrary Banach space.*

$$\mathcal{F}(X) = \mathcal{F}^b(X)$$

Lemma 1.5.1 *An immediate consequence of this result is that $\mathcal{F}(X)$ is a closed two-sided ideal of $L(X)$*

Definition 1.5.8 *Let X and Y be two Banach spaces and let $F \in L(X)$, F is called a upper (respectively lower) Fredholm perturbation if $U + F \in \Phi^b_+(X,Y)$ (respectively $\Phi^b_-(X,Y)$ whenever $U \in \Phi^b_+(X,Y)$ (respectively $\Phi^b_-(X,Y)$).*

Remark 1.5.5 *The sets of upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}^b_+(X,Y)$and $\mathcal{F}^b_-(X,Y)$ respectively. it is shown that $\mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$are closed subset of $L(X)$ and if $X = Y$,then $\mathcal{F}^b_+ := \mathcal{F}^b_+(X,Y)$ is a closed two-sided ideal of $L(X)$ in general, we have the following inclusions*

$$\begin{aligned} \mathcal{K}(X) &\subset \mathcal{S}(X) \subset \mathcal{F}^b_+ \subset \mathcal{J}(X) \\ \mathcal{K}(X) &\subset C\mathcal{S}(X) \subset \mathcal{F}^b_- \subset \mathcal{J}(X) \end{aligned}$$

where $\mathcal{F}^b_-(X) := \mathcal{F}^b_-(X,X)$ and $\mathcal{J}(X)$ denotes the set

$$\mathcal{J}(X) = \{F \in \mathcal{L}(X) \text{ such that } I - F \in \Phi(X) \text{ and } i(\lambda - F) = 0\}$$

Lemma 1.5.2 *Let $A \in \mathcal{C}(X)$ and let $\mathcal{I}(X)$ be any nonzero ideal of $L(X)$ satisfying $\mathcal{F}_0(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X)$, where $\mathcal{F}_0(X)$ stands for the ideal of finite rank operators, if $A \in \Phi(X)$ and if $J \in \mathcal{I}(X)$ then $A + J \in \Phi(X)$ and $i(A + J) = i(A)$*

Chaptre 2

On Semi Regular Operator

Let A and B be two bounded linear opertaors in a Banach space X, A and B are semi regular operators does not imply, in general, that the product AB is semi regular. We do, however, give conditions under which the above implication is valid.

Introduction

Amoung the various concepts of regularity originating from Fredholm theory, the concept of semi regularity, which will be introduced in this chapter, seems to be the appropriate to investigate some imporatnt aspects of local spectral theory. The concept of semi regularity originates in the classical Kato's treatment (1958) of perturbation theory of Fredholm operator. to introduce our results we need to fix some notations.

If X, and Y are Banach spaces, by $\mathcal{L}(X,Y)$ we denote the Banach space of all bounded linear opertaors from X into Y. Although many of the results in these notes are valid for real Banach spaces.

Recall that if $T \in \mathcal{L}(X,Y)$, the norm of T is defined by

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$$

If $X = Y$ we write $\mathcal{L}(X) = \mathcal{L}(X,X)$, by $X^* = \mathcal{L}(X,\mathbb{C})$ we denote the dual of X, if $T \in \mathcal{L}(X,Y)$, by $T^* \in \mathcal{L}(Y^*,X^*)$ we denote the dual operator of T defined by

$$(T^* f)(x) = f(Tx) \quad \text{for all } x \in X,\ f \in Y^*$$

The identity operator on X will be denoted by I_X, or simply I if no confusion can arise.

Given a bounded opertaor $T \in \mathcal{L}(X,Y)$, the kernel of T is the set

$$\mathcal{N}(T) = \{x \in X,\ Tx = 0\}$$

While the range of T is denoted by $T(X)$ or $R(T)$.

2.1 The closed range operator

Theorem 2.1.1 $[88, Theorem P.205]$ ***(The closed range theorem)*** *Let X and Y be banach spaces and T a closed linear operator defined in X into Y, then the following propositions are all equivalent*

i) $R(T)$ is closed in Y

ii) $R(T^*)$ is closed in X^*

iii) $R(T) = (\mathcal{N}(T^*))^{\perp} = \{y \in Y;\ \langle y, y^*\rangle = 0,$ for all $y^* \in \mathcal{N}(T^*)\}$

iv) $R(T^*) = (\mathcal{N}(T))^{\perp} = \{x^* \in X^*;\ \langle x, x^*\rangle = 0,$ for all $x \in \mathcal{N}(T)\}$

Theorem 2.1.2 [54]*Let X and Y be Banach spaces and $T \in \mathcal{L}(X,Y)$, T is closed range operator if and only if there exists $k > 0$ such that $\|Tx\| > c.d(x, \mathcal{N}(T))$ for all $x \in X$.*

The property of $T(X)$ being closed may be characterized by means of a suitable number associated with T

Definition 2.1.1 *If $T \in \mathcal{L}(X,Y)$, X, Y Banach spaces, the reduced minimal modulus of T is given by*

$$\gamma(T) = \inf_{x \notin \ker T} \frac{\|Tx\|}{dist\,(x, \mathcal{N}(T))}.$$

$dist\,(x, \mathcal{N}(T))$ denotes the distance of x to $\mathcal{N}(T)$.

formally we set $\gamma(0) = \infty$. It easily seen that if T is bijective then $\gamma(T) = \dfrac{1}{\|T^{-1}\|}$. In fact, if T is bijective then $dist\,(x, \mathcal{N}(T)) = dist\,(x, \{0\}) = \|x\|$, thus if $Tx = y$,

$$\gamma(T) = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \left(\sup_{x \neq 0} \frac{\|x\|}{\|Tx\|}\right)^{-1} = \left(\sup_{y \neq 0} \frac{\|T^{-1}y\|}{\|y\|}\right)^{-1} = \frac{1}{\|T^{-1}\|}$$

Theorem 2.1.3 *Let $T \in \mathcal{L}(X,Y)$, X and Y Banach spaces, then we have*

i)$\gamma(T) > 0$ if and only if $T(X)$ is closed.

ii)$\gamma(T) = \gamma(T^*)$

Proof. (i) The statement is clear if $T = 0$. Suppose that $T \neq 0$. Let $\overline{X} = X/\mathcal{N}(T)$ and denote by $\overline{T} : \overline{X} \to Y$ the continous injection corresponding to T, defined by $\overline{T}\overline{x} = Tx$ for every $x \in \overline{x}$.

It's easy to see that $\overline{T}\left(\overline{X}\right) = T(X)$. But it is known that $\overline{T}\left(\overline{X}\right)$ is closed if and only if $\overline{T}$ admits a continuous inverse, it mean that there exists a constant $\delta > 0$ such that $\left\|\overline{T}\overline{x}\right\| \geq \delta \|\overline{x}\|$, for every $x \in X$. From the equality
$\gamma(T) = \inf_{x \notin \ker T} \dfrac{\|Tx\|}{dist\,(x, \mathcal{N}(T))} = \inf_{\overline{x} \neq \overline{0}} \frac{\|\overline{T}\overline{x}\|}{\|\overline{x}\|}$, we then conclude that $\overline{T}\left(\overline{X}\right) = T(X)$ is closed if and only if $\gamma(T) > 0$.

(ii) The assertion is obvious if $\gamma(T) = 0$. Suppose that $\gamma(T) > 0$, then $T(X)$ is closed, if $\overline{T_0} : \overline{X} \to T(X)$ is defined by $\overline{T_0}\overline{x} = Tx$ for every $x \in \overline{x}$, then $\gamma(T) = \gamma\left(\overline{T_0}\right)$ and $T = J\overline{T_0}Q$, where $J : T(X) \to Y$ is the natural embedding, and $Q : X \to \overline{X}$ is the canonical projection defined by $Qx = \overline{x}$. Clearly, that $\overline{T_0}$ is bijective and from $T = J\overline{T_0}Q$ it thenfollows that $T^* = Q^*(\overline{T_0})^*J^*$. From this it easily follows that

$$\gamma(T) = \frac{1}{\left\|(\overline{T_0})^{-1}\right\|} = \frac{1}{\left\|(\overline{T_0}^*)^{-1}\right\|} = \gamma(T^*). \quad \blacksquare$$

A very important class of operators is the class of injective operators having closed range

Definition 2.1.2 *An operator $T \in \mathcal{L}(X)$ is said to be bounded below if T is injective and has closed range.*

Theorem 2.1.4 *$T \in \mathcal{L}(X, Y)$ is bounded below if and only if there exists $k > 0$ such that*

$$\|Tx\| \geq k\|x\| \text{ for all } x \in X \tag{2.1.1}$$

Proof. Indeed, if $\|Tx\| \geq k\|x\|$ for some $k > 0$ and all $x \in X$ then T is injective, Moreover, if (x_n) is a sequence in X for which (Tx_n) converges to $y \in X$ then (x_n) is a Cauchy sequence and hence convergent to some $x \in X$, since $T$is continuous then $Tx = y$ and therfore $T(X)$ is closed.

Conversly, if T is injective and $T(X)$ is closed then, from the open mapping theorem, it easily follows that there exists a $k > 0$ for which the inquality (2.1.1) holds. $\blacksquare$

The equality $j(T) = \inf_{\|x\|=1} \|Tx\| = \inf_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ is called the injective modulus of T and obviously from (1.1) we have

$$T \text{ is bounded below} \Longleftrightarrow j(T) > 0 \text{ and in this case } j(T) = \gamma(T) \tag{2.1.2}$$

The next result shows that the properties to be bounded below or to be surjective are dual each other

Theorem 2.1.5 *Let $T \in \mathcal{L}(X)$, X a Banach space, then*

i) T is surjective (respectively, bounded below) if and only if T^* is bounded below (respectively, surjective)

Proof. Suppose that T is surjective, trivially T has closed range and therefore also T^* has closed range, from the equality $\mathcal{N}(T^*) = T(X)^\perp = X^\perp = \{0\}$ we conclude that T^* is injective.

Conversely, suppose that T^* is bounded below, then T^* has closed range and hence, by the closed range theorem, the operator T has also closed range. From the equality $T(X) = (\mathcal{N}(T^*))^\perp = \{0\}^\perp = X$, we then conclude that T is surjective. ■

2.2 Semi regular operator

To introduce the classe of semi-regular operator, we need first to establish some connections between the kernels and the ranges of the iterates T^n of an operator T on a vector space. The semi regular operator is a bounded operator with a closed range achieved a conditions concerning on hyper range and hyper kernel. The kernels and the ranges of the power T^n of a linear operator T on a vector space X forme the following two sequences of subspaces: $\mathcal{N}(T^0) = \{0\} \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \subseteq ...$, and $T^0(X) = X \supseteq T(X) \supseteq T^2(X) \supseteq ....$ Generally, all these inclusions are strict

Definition 2.2.1 *Given a vector space X and a linear operator T on X, the hyper kernel of T is the subspace $\mathcal{N}^\infty(T) = \underset{n\in\mathbb{N}}{\cup} \mathcal{N}(T^n)$*

The hyper range of T is the subspace $T^\infty(X) = \underset{n\in\mathbb{N}}{\cap} T^n(X)$

Lemma 2.2.1 *for every linear operator on a vector space X, we have*

$$T^m(\mathcal{N}(T^{m+n})) = T^m(X) \cap \mathcal{N}(T^n) \text{ for all } m, n \in \mathbb{N}.$$

Proof. If $x \in \mathcal{N}(T^{m+n})$, then $T^m x \in T^m(X)$ and $T^n(T^m x) = 0$. Hence

$$T^m(\mathcal{N}(T^{m+n})) \subseteq T^m(X) \cap \mathcal{N}(T^n)$$

Conversly, if $y \in T^m(X) \cap \mathcal{N}(T^n)$ then $y = T^m(x)$ and $x \in \mathcal{N}(T^{m+n})$, so the opposite inclusion is verified. ■

The next result exhibits some useful connections between the kernels and the ranges of the iterates T^n of an operator T on a vector space X.

Theorem 2.2.1 *For a linear operator T on a vector space X the following statements are equivalents:*

i) $\mathcal{N}(T) \subseteq T^m(X)$ *for each* $m \in \mathbb{N}$

ii) $\mathcal{N}(T^n) \subseteq T(X)$ *for each* $n \in \mathbb{N}$.

iii) $\mathcal{N}(T^n) \subset T^m(X)$ *for each* $n \in \mathbb{N}$, *and each* $m \in \mathbb{N}$.

iv) $\mathcal{N}(T^n) = T^m(\mathcal{N}(T^{m+n}))$ *for each* $n \in \mathbb{N}$, *and each* $m \in \mathbb{N}$.

Proof. The implication $iv) \Rightarrow (iii) \Longrightarrow (ii)$ are trivial, it's abvious from the lemma precedent and the sequence of ranges $T^n(X)$.

$(ii) \Longrightarrow (i)$ If we apply the inclusion (ii) to the operator T^m we obtain $\mathcal{N}(T^{mn}) \subseteq T^m(X)$, and hence $\mathcal{N}(T) \subseteq T^m(X)$ since $\mathcal{N}(T) \subseteq \mathcal{N}(T^{mn})$.

$(i) \Longrightarrow (iv)$ if we apply the inclusion (i) to the operator T^n, we obtain $\mathcal{N}(T^n) \subseteq T^{mn}(X) \subseteq T^m(X)$, by the lemma 2.2.1, we have

$$T^m(\mathcal{N}(T^{m+n})) = T^m(X) \cap \mathcal{N}(T^n) = \mathcal{N}(T^n).$$

■

We can reformulate the Theorem 2.2.1 by the following corollary

Corollary 2.2.1 *Let T be a linear operator on a vector space X, then the statement of Theorem 2.2.1 are equivalent to each of the following inclusion:*

Theorem 2.2.2 *i)* $\mathcal{N}(T) \subseteq T^\infty(X)$

ii) $\mathcal{N}^\infty(T) \subseteq T(X)$

iii) $\mathcal{N}^\infty(T) \subset T^\infty(X)$.

Definition 2.2.2 *Given a Banach space X, a bounded operator $T \in \mathcal{L}(X)$ is said to be semi regular if $T(X)$ is closed and T verifies one of the equivalent conditions of Theorem2.2.1*

Trivial examples of semi regular operator are surjective operators as well as injective operators with closed range.

It is not difficult to show that T is semi regular if and only if the dual operator T^* is semi regular

Theorem 2.2.3 *Let X a Banach space, suppose that T, $S \in \mathcal{L}(X)$ commute and TS semi regular, then T and S are semi reguar*

Proof. Clearly, we need only to show that one two operators, say T, is semi-regular. From the semi regularity of TS we obtain

$$\mathcal{N}(T) \subseteq \mathcal{N}(TS) \subseteq \bigcap_{n=1}^{\infty} (T^n S^n)(X) \subseteq \bigcap_{n=1}^{\infty} T^n(X) \tag{2.2.1}$$

At this point we need only to show that $T(X)$ is closed.

Let $(y_n) = (Tx_n)$ be a sequence of $T(X)$ which converges to some y_0, then $Sy_n = STx_n = TSx_n \in (TS)(X)$ and (Sy_n) converges to Sy_0, By assumption $(TS)(X)$ is closed, thus $Sy_0 \in (TS)(X) = (ST)(X)$, Hence there exists an element $x_0 \in X$ such that $Sy_0 = STx_0$.

Consequently $y_0 - Tx_0 \in \mathcal{N}(S) \subseteq \mathcal{N}(TS)$ from the inclusion (2.2.1) we obtain $y_0 - Tx_0 \in \bigcap_{n=1}^{\infty} T^n(X) \subseteq T(X)$, From this it follows that $y_0 \in T(X)$, then $T(X)$ is closed, Hence T is semi regular. ∎

2.3 On composition of two semi regular operators

Note carefuly that an operator T has a closed range does not imply that T^2 is a closed range operator see ([16]), but we can introduce that if T is semi regular then T^n is semi regular

So we have the theorem

Theorem 2.3.1 *Suppose that $T \in \mathcal{L}(X)$, X a Banach space, T is semi regular, then $\gamma(T^n) \geq \gamma(T)^n$, for all $n \in \mathbb{N}$.*

Proof. By induction, the case $n = 1$ is trivial.

Suppose that $\gamma(T^n) \geq \gamma(T)^n$, for every element $x \in X$, and $u \in \mathcal{N}(T^{n+1})$, we have $dist\,(x, \mathcal{N}(T^{n+1})) = dist\,(x - u,\ \mathcal{N}(T^{n+1})) \leq dist(x - u, \mathcal{N}(T))$

By assumption T is semi regular, so by Theorem 2.2.1 $\mathcal{N}(T) = T^n(\mathcal{N}(T^{n+1}))$ and therefore

$$\begin{aligned} dist(T^n x, \mathcal{N}(T)) &= dist(T^n x, T^n(\mathcal{N}(T^{n+1})) \\ &= \inf_{u \in \ker T^{n+1}} \|T^n(x-u)\| \\ &\geq \gamma(T^n). \inf_{u \in \ker T^{n+1}} dist(x-u, \mathcal{N}(T^n)) \\ &\geq \gamma(T^n) dist\,(x, \mathcal{N}(T^{n+1}))\,. \end{aligned}$$

From this estimate, it follows that

$\|T^{n+1}x\| \geq \gamma(T) dist(T^n x, \mathcal{N}(T)) \geq \gamma(T)\gamma(T^n) dist\left(x, \mathcal{N}(T^{n+1})\right)$.

Consequently, we obtain that $\gamma(T^{n+1}) \geq \gamma(T)\gamma(T^n) \geq \gamma(T)\gamma(T)^n$ (from our inductive assumption)

then $\gamma(T^{n+1}) \geq \gamma(T)^{n+1}$

Corollary 2.3.1 *If $T \in \mathcal{L}(X)$, X a Banach space, is semi regular, then T^n is semi regular for all $n \in \mathbb{N}$.*

■

Proof. If T is semi regular then by theorem precedent $S = T^n$ has closed range, furthermore, $S^\infty(X) = T^\infty(X)$, and $\mathcal{N}(S) = \mathcal{N}(T^n) \subseteq T^\infty(X) = S^\infty(X)$, from Corollary 1, we conclude that $S = T^n$ is semi regular. ■

Proposition 2.3.1 *The product of two semi-regular operators, also cummuting semi regular operators, need not be semi regular.*

Proof. We have the example:

Let H be a Hilbert space with an orthonormal basis $(e_{i,j})$, where i, j are integers for which $ij \leq 0$, let $T \in \mathcal{L}(H)$ and $S \in \mathcal{L}(H)$ are defined by the assignment

$$T_{e_{i.j}} = \begin{cases} 0 & \text{if } i = 0,\ j > 0 \\ e_{i+1.j} & \text{otherwise} \end{cases}$$

and

$$S_{e_{i.j}} = \begin{cases} 0 & \text{if } j = 0,\ i > 0 \\ e_{i.j+1} & \text{otherwise} \end{cases} .$$

Then

$$TSe_{i.j} = STe_{i,j} \begin{cases} 0 & \text{if } i = 0, j \geq 0 \text{ or } j = 0,\ i \geq 0 \\ e_{i+1.j+1} & \text{otherwise} \end{cases}$$

Hence $TS = ST$ and, we have

$\mathcal{N}(T) = \cup\{e_{0.j}, j > 0\} \subset R^\infty(T)$ and, $\mathcal{N}(T) = \cup\{e_{i.0}, i > 0\} \subset R^\infty(S)$

Moreover, since T, and S have closed range it follows that T, S are semi regular

On the other hand $e_{0,0} \in \mathcal{N}(TS)$ and $e_{0,0} \notin TS(H)$, thus TS is not semi regular ■

Lemma 2.3.1 *Let X, and Y two Banach spaces, and $T \in \mathcal{L}(X,Y)$ a closed range operator, if there existe $k > 0$ such that $\|x\| \leq k \|Ax\|$ for all $x \in X$, then the range of all closed subspace U of X by A is a closed subspace of Y.*

Proof. Let A be a closed operator from X in Y, suppose that there is $k > 0$ such that $\|x\| \leq k \|Ax\|$ for all $x \in X$,

Let U be a closed subspace of X, $A(U)$ denote the range of U in Y, and let (y_n) be a sequence in $A(U)$ converges to $y \in Y$, then there is a sequence $x_n \in U$, such that $y_n = Ax_n$, and $Ax_n \longrightarrow y \in Y$, when $n \longrightarrow \infty$, the sequence (Ax_n) is of Cauchy in Y then $\lim_{n,m \longrightarrow \infty} \|Ax_n - Ax_m\| = 0$, and we have $\|x_n - x_m\| \leq k \|Ax_n - Ax_m\|$, this implies that $\lim_{n,m \longrightarrow \infty} \|x_n - x_m\| = 0$. Hence, the sequence x_n converges (it's of Cauchy in a Banach space)

So $x_n \longrightarrow x \in X$ when $n \longrightarrow \infty$, from the operator A is closed in X, then $x \in U$

Since $x \in U$ and $Ax = y$, then the limit $y \in A(U)$, which implies that $A(U)$ is closed ■

Remark 2.3.1 *From the theorem 2.2.1 and the precedent lemma we conclude that the product of a bounded below operator A in Banach space X and a closed range operator B in X, is a closed range operator, because the range of the closed subspace $B(X)$ by A, which holds the inquality $\|x\| \leq k \|Ax\|$, $k > 0$, for all $x \in X$, $AB(X) = A(B(X))$ is a closed subspace in X.*

Corollary 2.3.2 *Let A, and B be two semi regular operators in Banach space X, such that A is injective, if $B^n(X) \subset (AB)^n(X)$ for all $n \in \mathbb{N}$, then the product operator AB is semi regular.*

Proof. The boundedness is evident, from the inquality $\|AB\| \leq \|A\| \|B\|$.

The closedness of the range of the operator AB is verified by the precedent lemma, we have $B(X)$ is a closed subspace of X because B is semi regular, moreover, A is injective and has a closed range, then there exists $k > 0$ such that $\|Ax\| \geq k \|x\|$ for all $x \in X$, hence $A(B(X)) = AB(X)$ is closed.

For the last condition, A is injective then, $\mathcal{N}(AB) = \mathcal{N}(B) \subset B^n(X)$, for all $n \in \mathbb{N}$

Consequently, if $B^n(X) \subset (AB)^n(X)$, for all $n \in \mathbb{N}$, then, $\mathcal{N}(AB) \subset (AB)^n(X)$, for all $n \in \mathbb{N}$.

Then AB is semi rgular operator ■

Chaptre 3

Relatively Boundedness and compactness for differential operator

For the study of perturbation theory two useful concepts which have extensive applications are those of relative boundedness and relative compactness. These can be defined both with respect to norms and to quadratic forms. The principle application is to selfadjoint second ordre differential operator, for this reason we consider the Sturm-Liouville expression.

3.1 Relatively Boundedness and Relatively Compactness

Definition 3.1.1 *Let X, Y and Z be Banach spaces, and let A and S be two linear operators from X into Y and from X into Z, respectively.*

(i) S is called relatively bounded with respect to A, or $A-$bounded, if $D(A) \subset D(S)$ and there exist two constants $a_s \geq 0$ and $b_s \geq 0$ such that

$$\| Sx \| \leq a_s \| x \| + b_s \| Ax \| \, , \; x \in D(A) \tag{3.1.1}$$

The infimum δ of all b_s that holds for some $a_s \geq 0$ is called relative bound of S with respect to A (or $A-$bound of S).

(ii) S is called relatively compact with respect to A (or $A-$compact), if $D(A) \subset D(S)$ and for every bounded sequence $(X_n)_n \in D(A)$ such that, $(AX_n)_n \subset Y$ is bounded, the sequence $(SX_n)_n \subset Z$ contains a convergent subsequence.

Remark 3.1.1 *The inequality (3.1.1) is equivalent to*

$$\| SX \|^2 \leq a^2 \| x \|^2 + b^2 \| Ax \|^2 \text{ for all } x \in D(A)$$

where $a = \sqrt{a_s^2 + a_s b_s}$ *and* $b = \sqrt{b_s^2 + a_s b_s}$

Lemma 3.1.1 *If S is $A-$bounded with an $A-$bound $\delta < 1$, then S is $(A+S)-$bounded with an $A-$bound* $\leq \dfrac{\delta}{1-\delta}$.

Proof. First of all, it should be mentioned that $A+S$ is will defined as

$$D(A+S) = D(S) \cap D(A) = D(A) \subset D(S)$$

The fact that S is $A-$bounded, there exist $a_s \geq 0$ and $\delta \leq b < 1$ such that, for all $x \in D(A)$ we have

$$\| Sx \| \leq a_s \| x \| + b_s \| Ax \|$$

$$= a_s \| x \| + b_s \| Ax + Sx - Sx \|$$

$$= a_s \| x \| + b_s \| Ax + Sx \| + b_s \| Sx \|$$

Since $b_s < 1$ it follows that

$$\| Sx \| \leq \frac{a_s}{1-b_s} \| x \| + \frac{b_s}{1-b_s} \| (A+S)x \|, x \in D$$

Let $A \in C(X,Y)$, the graph norm of A is defined by $\| x \|_A = \| x \| + \| Ax \|, x \in D(A)$ from the closedness on A it follows that $D(A)$ endowed with the norm $\| \cdot \|_A$ is a Banach space, Let us denote by X_A, the space $D(A)$ equipped with the norm $\| \cdot \|_A$ clearly, the operator A satisfies $\| Ax \| \leq \| x \|_A$ and consequently $A \in \mathrm{L}(X)$

Let $J : X \to Y$ be a linear operator on X if $D(A) \subset D(J)$ then J will be called A-defined. If J is $A-$defined, we will denote by $\hat{J}$ its restriction to $D(A)$ Moreover, if $\hat{J} \in \mathrm{L}(X)$ we say that J is $A-$bounded. We can easily check that, if J is closed (or closable), then J is $A-$bounded. ■

Proposition 3.1.1 *If A is closed and B is closable, then $D(A) \subset D(B)$ implies that B is $A-$bounded.*

Proof. *If A is a closed operator, then $D(A)$ equipped with the graph norm is a Banach space, if we suppose that $D(A) \subset D(B)$ and $(B, D(B))$ is closable, then $D(A) \subset D\left(\bar{B}\right)$ because the graph norm on $D(A)$ is stronger than the norm induced from X, the operator $\bar{B}$, considered as an operator from $D(A)$ into X is every where defined and closed. On the other hand, $\overline{B}_{|D(A)} = B$ hence $B : D(A) \to X$ is bounded by the closed graph and thus B is $A-$bounded.* ■

Definition 3.1.2 *We say that an operator J is $A-$closed, if $x_n \to x, Ax_n \to y$, $Jx_n \to z$ for $(x_n)_n \subseteq D(A)$ implies that $x \in D(J)$ and $Jx = z$. An operator J will be called $A-$closable, if $x_n \to 0, Ax_n \to 0$, $Jx_n \to z$ implies $z = 0$*

Remark 3.1.2 *(i) If J is bounded, then J is $A-$bounded.*

(ii) If J is closed, then J is $A-$closed.

(iii) If J is closable, then J is $A-$closable.

3.2 The differential operator

3.2.1 The adjoint of a differential operator

In this section, we consider a differential operators as an unbouded linear operators on $L^2([a,b])$, where $(-\infty < a < b < \infty)$, with domain D and explain how to define their adjoints.

We discuss the domain of the adjoint operator, and the selfadjointness depending the boundary conditions (BC), which lead to which called the boundary values problems (BVPs).

Definition 3.2.1 *A linear ordinary differential operator of order $n \in \mathbb{N}^*$, is a linear map A that acts on $n-$times continuously differentiable function u by $Au = \sum\limits_{j=0}^{n} a_j u^j$, where u^j denotes the j^{th} derivative of u, and the coefficients a_j are real or complex valued function.*

For $n \in \mathbb{N}^*$ and $a_1, a_2, ..., a_j, ..., a_p \in C^\infty(\mathbb{R}, M_n(\mathbb{C}))$, for all $x \in \mathbb{R}, a_n \in GL_n(\mathbb{C})$, the differential operator $A = \sum_{j=0}^{p} a_j(x)\partial_x^j$, may be considered as an unbounded operator on $L^2(\mathbb{R}^n)$, with domain $D(A) = H^p(\mathbb{R}^n)$ (which is dense in $L^2(\mathbb{R}^n)$, The expression Adefine a linear continuous application of $H^p(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ and we have

$$\|Au\|_{L^2} \leq \sqrt{p+1}\max_j \|a_j\|_{L^\infty} \|u\|_{H^p}$$

Where $\|u\|_{H^p}^2 = \|u\|_{L^2}^2 + \sum_{j=1}^{p} \|\partial_x^j u\|_{L^2}^2$.

Lemma 3.2.1 *Let $(u_m)\, m \in \mathbb{N}$, is a sequence in $H^p(\mathbb{R}^n)$ converge to u on $L^2(\mathbb{R}^n)$, if Au_m converge to f on $L^2(\mathbb{R}^n)$, then $u \in H^p(\mathbb{R}^n)$, and $Au = f$.*

For the proof see [22]

The precedent lemma prove that the differential operator A is closed on $L^2(\mathbb{R}^n)$, So its graph $G(A)$ is closed in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

In the differential operator $A = \sum_{j=0}^{p} a_j(x)\partial_x^j$, the use of integration by parts on $\langle Au, v\rangle$ for

u, $v \in H^p(\mathbb{R}^n)$ yields that $A^*v = \sum_{j=0}^{p} (-1)^j \partial_x^j \left(\overline{a_j(x)}v(x)\right)$, for $v \in D(A^*)$

This expression is called the formal adjoint of A ("formal" because we have not spicified its domain).

3.2.2 Second order differential operator

We consider second order ordinary differential operator A on $L^2([\alpha, \beta])$, $(-\infty < \alpha < \beta < \infty)$, of the form

$$Au = au'' + bu' + cu \tag{3.2.1}$$

Where a, b, and c are sufficiently smooth function on $[\alpha, \beta]$, we assume that $a(x) > 0$ for all $x \in [\alpha, \beta]$, so that A is second order at every point.

For more determination, we consider the boundary conditions $Bu = 0$ is homogenuous system of linear equations that involves the values of u and u' at the endpoint $x = \alpha$, β and higher derivatives of u may be expressed in terms of u and u', by use of the differential equation. Some common types of boundary conditions are:

$u(\alpha) = u(\beta) = 0$ *Dirichlet conditions*

$u'(\alpha) = u'(\beta) = 0$ *Neumann conditions*

$u(\alpha) = u(\beta),\ u'(\alpha) = u'(\beta)$ *peridic conditions*

$\delta u(\alpha) + \mu u'(\alpha) = 0,\ \lambda u(\beta) + \theta u'(\beta) = 0$ *mixed conditions*

And also, we can impose two conditions at one of the endpoints

$u(\alpha) = 0,\ u'(\alpha) = 0$ *initial conditions*

$u(\beta) = 0,\ u'(\beta) = 0$ *final conditions*

Now, we formulate the adjoint boundary values problem.

Suppose that A is given by (3.2.1), where $a,\ b,\ c \in C^2([\alpha,\beta])$ then, the integration by parts yields that

$$\langle v, Au\rangle = \int_\alpha^\beta \overline{v}((x))(au'' + bu' + cu)(x)\,dx$$
$$= \int_\alpha^\beta [\overline{v}(au'' + bu') + c\overline{v}u](x)\,dx$$
$$= [a\overline{v}u' + b\overline{v}u]_\alpha^\beta - \int_\alpha^\beta [(a\overline{v})'u' - (b\overline{v})'u + c\overline{v}u](x)\,dx$$
$$= [a\overline{v}u' + b\overline{v}u - (a\overline{v})'u]_\alpha^\beta + \int_\alpha^\beta [(a\overline{v})''u - (b\overline{v})'u + c\overline{v}u](x)\,dx$$
$$= [a(\overline{v}u' - \overline{v}'u) + (b - a')\overline{v}u]_\alpha^\beta + \int_\alpha^\beta \left[\overline{((\overline{a}v)'' - (\overline{b}v)' + \overline{c}v)}u\right](x)\,dx$$

Suppose that

$$A^*v = (\overline{a}v)'' - (\overline{b}v)' + \overline{c}v \tag{3.2.2}$$

This expression is the formal adjoint of A.

So we have

$$\langle v, Au\rangle - \langle A^*v, u\rangle = [a(\overline{v}u' - \overline{v}'u) + (b - a')\overline{v}u]_\alpha^\beta \tag{3.2.3}$$

The equality (3.2.3) is called "The Green's formula"

If we note the boundary terms in" Green's formula" $S(\alpha,\beta) = [a(\overline{v}u' - \overline{v}'u) + (b - a')\overline{v}u]_\alpha^\beta$, given boundary conditions $Bu = 0$, for A, we define adjoint boundary conditions $(B^*v = 0)$ for A^* by the requirement that the boundary terms $S(\alpha,\beta)$ vanish. Thus for $v \in C^2([\alpha,\beta])$ we say that $(B^*v = 0)$ if and only if $S(\alpha,\beta) = 0$ for all $u \in C^2([\alpha,\beta])$, such that $Bu = 0$.

We say that the (BVP) is selfadjoint ($B = B^*$) if $S(\alpha, \beta) = 0$ for all $u \in C^2([\alpha, \beta])$, such that $Bu = 0$ if and only if $Bv = 0$. Then, we say that A is selfadjoint if $Au = A^*u$, and $B = B^*$.

Example 3.2.1 *Suppose that $A = D^2$, then the Green's formula written by*

$$\langle v, Au\rangle - \langle v'', u\rangle = [\overline{v}u' - \overline{v}'u]_\alpha^\beta$$

If we have $u(\alpha) = u(\beta) = 0$ (Dirichlet conditions), then $S(\alpha, \beta) = \overline{v}u'(\beta) - \overline{v}u'(\alpha) = 0$ for all $u'(\beta)$ and $u'(\alpha)$ if and only if $v(\alpha) = v(\beta) = 0$. Then $B = B^*$ and $Au = A^*u = u''$. Then the operator $A = D^2$ with the dirichlet conditions is selfedjoint.

If we have the (BC) $u(\alpha) = 0$, $u'(\alpha) = 0$ $(initial\ conditions)$ then

$$S(\alpha, \beta) = [\overline{v}u' - \overline{v}'u]_\alpha^\beta = \overline{v}(\beta)\, u'(\beta) - \overline{v}'(\beta)\, u(\beta).$$

$S(\alpha, \beta) = 0$ for all $u'(\beta)$ and $u(\beta)$ if and only if $v(\beta) = 0$, and $v'(\beta) = 0$ $(final\ conditions)$. Final conditions are adjoint of initial conditions ($B \neq B^*$). Thus the operator $A = D^2$, with initial conditions is not selfadjoint.

3.2.3 Sturm-Liouville expression

In this section, let us find all the formally selfadjoint second order ordinary differential operators ($Au = au'' + bu' + cu$) for a, b, and c are complex values functions and $u \in C^2([\alpha, \beta])$.

Green's formula gives that

$$A^*u = (\overline{a}u)'' - (\overline{b}u)' + \overline{c}u = \overline{a}u'' + (2\overline{a}' - \overline{b})u' + \left(\overline{a}'' - \overline{b}' + \overline{c}\right) u$$

For the equality ($Au = A^*u$), which implies

$$au'' + bu' + cu = \overline{a}u'' + (2\overline{a}' - \overline{b})u' + \left(\overline{a}'' - \overline{b}' + \overline{c}\right) u.$$

We must therefore have

$$a = \overline{a}, b = 2\overline{a}' - \overline{b}, c = \overline{a}'' - \overline{b}' + \overline{c}.$$

These relations are satisfied if and only if a is real, $\operatorname{Re} b = a'$, and $\operatorname{Im} c = \frac{\operatorname{Im} b'}{2}$. For operators with real values coefficients, which we denote by p and q where

$$(a = -p,\ b = -p',\ \text{and } c = q)$$

we infer that

$$Au = -pu'' - p'u' + qu.$$

The resulting formally selfadjoint operator, called a Sturm-Liouville operator, is given by

$$Au = -\left(pu'\right)' + qu, or A = -D(pD) + q$$

We need only to impose selfadjoint boundary conditions on functions in the domain of a Sturm-Liouville operator, to obtain a selfa-djoint operator.

3.3 Relatively form boundeness and compactness of Sturm-Liouville operator

In quantum mechanics, the position of a single particle living in space is described by a complex-valued function $\psi(x,t)$, $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ called the "wave function". The physical meaning of the wave function is that the probability to find it in a region Ω at time t is $E(\chi_\Omega) = \int |\psi(x,t)|\, dx$. In contradistinction to classical mechanics, the particle is no longer localized at a certain point. In particular, the mean-square deviation (or variance) $\Delta(x)^2 = E_\psi(x^2) - E_\psi(x)^2$ always nonzero.

In general, the configuration space (or phase space) of a quantum system is a (complex) Hilbert space H. An observable a corresponds to a linear operator A in this Hilbert, and its expectation, is given by the real number

$$E_{\psi}(A) = \langle \psi, A\psi \rangle = \langle A\psi, \psi \rangle \tag{3.3.1}$$

where $\langle\ ,\rangle$ denotes the scalar product of H. Similarly, the mean-square deviation is given by

$$\Delta(A)^2 = E_{\psi}(A^2) - E_{\psi}(A)^2 = \|(A - E_{\psi}(A))\,\psi\|^2 .$$

Note that $\Delta(A)$ vanishes if and only if ψ is an eiguenfunction corresponding to the eigenvalue $E_{\psi}(A)$, that is $A = E_{\psi}(A)$.

3.3.1 Schrödinger equation

Given an initial state $\psi(0)$ of the system, there should be a unique $\psi(t)$ representing the state of the system at time t. We will write $\psi(t) = U(t)\psi(0)$, $U(t)$ should be a linear operator. Moreover, since $\psi(t)$ is a state (i.e $\|\psi\| = 1$), we have $\|U(t)\psi\| = \|\psi\|$, since we have assumed uniqueness of solutions to the initial value problem, we must have $U(0) = I_H$, and $U(t+s) = U(t)U(s)$. In addition, it is natural to assume that the family of **unitary** operators group $U(t)$ is continuous; that is, $\lim_{t_0} U(t)\psi = u(t_0)\psi$, for all $\psi \in H$. Each such group has an in nitesimal generator defined by

$$H\psi = \lim_{0} \frac{i}{t}(U(t)\psi - \psi),\ D(H) = \left\{ \psi \in H,\ \lim_{0} \frac{1}{t}(U(t)\psi - \psi \text{ exists} \right\}$$

This operator is called "Hamiltonian", and corresponds to the energy of the system, then $\psi(t)$ is a solution of the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H\psi(t)$$

If initially the wave function is given by $\psi(x;0) = \psi_0(x)$, the Schrödinger equation says that its evolution is given by

$$\psi(x,t) = \psi_0 \exp(-itH)$$

But how to compute $\exp(-itH)$?, For exemple if $H = \Delta$, this is where Fourier transform comes in handy. Diferentiation becomes multiplication on the Fourier transform side, and so $\exp(-itH)\psi_0(x) = \int\limits_{\mathbb{R}^n} \exp\left(i\xi x - i|\xi|^2 t\right)\hat{\psi}_0(\xi)\, d\xi$, where

$$\hat{\psi}_0(\xi) = \int\limits_{\mathbb{R}^n} \exp(-i\xi x)\,\psi_0(x)\,dx$$

Also, Stationary phase methods lead to very precise estimates on free Schrödinger evolution. In particular, for large time, $\exp(-itH)\psi_0(x) \backsim \frac{1}{t^{\frac{n}{2}}}\hat{\psi}_0\left(\frac{x}{t}\right) + \circ(t^{\frac{-n}{2}})$.

An other direction is extention the property of diagonalizable matrix on finite dimentional setting to infinite dimentional setting, which requires that the operator is self-adjoint. This lead to the spectrum theory for self-adjoint operator. For exemple the equation

$$i\frac{dv}{dt} = Av, v(0) = \sum c_j v_j \tag{3.3.2}$$

has solution $v(t) = \exp(-iAt) = \sum \exp(-i\lambda_j t) c_j v_j$

this time λ_j =frequency of the solution in direction v_j. Looking ahead in PDEs, we suppose that $A = -\Delta$ in a domain $\Omega \subset \mathbb{R}^n$, assume boudary conditions that make the operator self-adjoint. Then the eigenvalues λ_j and eigenfunctions $v_j(x)$ of the lablacian satisfy $-\Delta v_j = \lambda_j v_j$ in Ω, and the spectrum increase to infinity: $\lambda_1 < \lambda_2 < \lambda_3 < ... \to \infty$ and the eigenfunctions form an ONB for $L^2(\Omega)$

Substituting $A = -\Delta$ into (3.3.2) transform them into famous PDE for the function $v(x,t)$ called Schrödinger equation. Separation of variables gives

$$v = \exp(i\Delta t)v(.,0) = \sum \exp(-i\lambda_j t) c_j v_j$$

Here λ_j is the frequency or energy level, and v_j is the quantum state.

Recall that the symmetric second order differential expressions take the form

$$S(u) = \omega^{-1}(-D(pDu) + qu)$$

on an interval $I = [c, \infty[$, $c > 0$, where D is the first derivative, the coefficient p, q are real, $p(x) > 0$ on I and $\frac{1}{p}$, q are locally lebesgue integrable on I, and $\omega > 0$ is the weight function of the Hilbert space setting $L^2(\omega, I)$.

3.4 Relatively form boundedness and compactness

A much more interesting problem is when we add electric potential, some sort of effective interaction with other particles. The operator describing this system is now: $H_V = H + V$, how to analyse this operator?. One possibility is to consider V as a perturbation to the operator H. For the study of perturabtion theory, two useful concepts which have extensive application are those of relative boundedness and compactness. Many important spectral properties of operators are preserved under this property.

Note that A being $L-$ *bounded* (respectively, $L-$ *compact*) is equivalent to A being a bounded (respectively, compact) operator acting on the graph G_L of L with graph norm $\|x\| = \|x\| + \|Lx\|$. Then, as a consequence of the closed graph theorem, we have the following theorem.

Theorem 3.4.1 ([64]) *If A and L are closed operators and $\mathcal{D}(L) \subset \mathcal{D}(A)$, then A is $L-$ bounded*

Many important spectral properties of operators are preserved under relatively bounded or relatively compact perturbation. For example the essential spectrum of L is invariant under a relatively compact perturbation as is self-adjointness if L is self-adjoint, A is symmetric and $\beta_0 < 1$ (Kato-rellich Theorem). The property of a purely discrete spectrum is also preserved for a symmetric perturbation A of a self-adjoint opeartor L when $\beta_0 < 1$ [20]. Also, deficiency indices are unchanged for perturabation where $\beta_0 < 1$ [17].

These concepts can be defined also with respect to quadratic forms. When we first consider a self-adjoint operator $A \geq 0$, and a densely defined symmetric operator L which satisfies $L \geq \varepsilon I$, for some $\varepsilon > 0$, ε is called the lower bound of L, which means $\langle Lx, x \rangle \geq \varepsilon \langle x, x \rangle$ for all $x \in \mathcal{D}(L)$. Form relative boundedness and compactness of A with repect to L can be defined as follows.

Definition 3.4.1 *If $\mathcal{D}(L) \subset \mathcal{D}(A)$, and there exist nonegative constant α and β such that*

$$\langle Ax, x \rangle \leq \alpha \langle x, x \rangle + \beta \langle Lx, x \rangle \text{ , } \textit{for all } x \in \mathcal{D}(L), \tag{3.4.1}$$

then the form $\langle A.,.\rangle$ is bounded with respect to the form $\langle L.,.\rangle$

If L is self-adjoint, then closed, applying Theorem 3.4.1, this definition is equivalent to say that the self-djoint operator A is $L-$ *bounded.* For this reason it is costumary to reformulate (3.4.1) in terms of the closures of the forms.

In a finit dimensional space, the notion of sesquilinear form and that of a linear operator are equivalent, any symmertic form corresponding to a symmetric operator. This is valid on infinite dimensional space as long as one is concerned with bounded forms. In the case of unbounded forms and unbounded operators, however, there is no such abvious relationship. Nevertheless there exists a closed theory on a relationship between semi-bounded symmetric forms and semi-bounded self-adjoint operators, this theory is due to Friederichs [33], on which a self-adjoint extention of the symmetric operator L, noted L_f, is constructed and has the same lower bounds ε as L, called the Friederichs extension, if L is already self-adjoint then $L = L_f$.

For the sturm-liouville operator, a characterization of L_f is given in [53], and the closure of the form $\langle L.,.\rangle$ is defined by $\left\langle L_f^{\frac{1}{2}}.,L_f^{\frac{1}{2}}.\right\rangle$[53], where $L_f^{\frac{1}{2}}$ is the unique positive square roote of L_f . Which lead to reformulate (3.4.1) by the inquality

$$\left\|A^{\frac{1}{2}}x\right\|^2 \leq \alpha \left\|x\right\|^2 + \beta \left\|L_f^{\frac{1}{2}}x\right\|^2, \ for\ allx \in \mathcal{D}(L)$$

From which montioned above, we can rephrase the definition 3.4.1 on the form version, rectricted to particular operators as following

Definition 3.4.2 *If $A \geq 0$, and $L \geq \varepsilon I$, for some $\varepsilon > 0$, are self-adjoint operators, then A is said to be relatively form bounded with respect to L if $D(L_f^{\frac{1}{2}}) \subset D(A^{\frac{1}{2}})$.*

Chaptre 4

On Convergence of Essential Approximate Point Spectrum And Essential Defect Spectrum

This chapter is devoted to the investigation of the relationship between the essential approximate point spectrum (respectively, the essential defect spectrum) of a sequence of closed linear operators $(T_n)_{n\in\mathbb{N}}$ on Banach space X, and the essential approximate point spectrum (respectively, the essential defect spectrum) of a linear operator T on X, where $(T_n)_{n\in\mathbb{N}}$ converges to T, in the case of convergence in generalized sense as well as in the case of the convergence compactly.

4.1 Introduction

The central objective of this work is to make an extensive study about the essential approximate point spectrum, noted $\sigma_{eap}(.)$, and the essential defect spectrum, noted $\sigma_{e\delta}(.)$, of a sequence of linear operators in Banach spaces. According to the convergence sense of the sequence of operators, our work may be devised in two parts, the first is concerned on the case of a sequence of closed linear operators T_n converges in the generalized sense to a closed linear operator T (see Definition 4.3.4), and the second is concerned on the case of a sequence of bounded linear operators T_n converges compactly to a bounded linear operator T (see Definition 4.4.2). The case of a sequence of bounded linear operator T_n converges in the operator norm to a bounded operator T was studied by C.Schmoeger in $[79, Proposition 5]$.

In the first part, we study the essential approximate point spectrum and the essential defect spectrum of a sequence of closed linear operators T_n converges in the generalized sense to a closed linear operator T. Using some of the basic properties of the notion of convergence in the generalized sense which is extended to the case of a sequence of closable linear operator in $[8, Theorem$ 2.3], we show that the essential approximate point spectrum of $(T_n + B - \lambda_0)$ is included in the essential approximate point spectrum of $(T + B - \lambda_0)$ for sufficiently large ($n \in \mathbb{N}$) where T_n converges to T in the generalized sense, B, and $(T + B - \lambda_0)^{-1}$ are bounded in X, the same statement is proved for the essential defect spectrum (see Theorem 4.3.3), a particular case is presented in (Corollary 4.3.1) which is a generalization to the result obtained by C. Schmoeger in $[79, Proposition 5]$. In the second part, in contrast to the results due to S. Goldberg on perturbation of semi fredholm operators by operators converging to zero compactly (see [38]), and the characterization of the essential approximate point spectrum and the essential defect spectrum by means of semi fredholm operators (see Proposition 4.2.1), we examine the relationship between the essential approximate point spectrum of a sequence of linear operators T_n and the essential approximate point spectrum of an operator T such that T_n converges to T compactly, and the same statement is obtained for the essential defect spectrum, (see Theorem 5.2.2).

In order to introduce our results, we organize this part as follows. The next section is a preliminaries, Where we review some of the concepts and properties that concern us in our study. In section 3 we present our main results to investigate the essential approximate point spectrum and the essential defect spectrum of a sequence of linear operators T_n on Banach space X, where T_n converges in the generalized sense, and the same when T_n converges compactly.

4.2 Preliminaries and auxiliary results

In this section we gather some notations and results of each of convergence in generalized sense and the convergence compactly, that we need to prove our results later. In order to present our results, we need to fix some notation and assumptions. Let X be Banach space. By an operator T on X we mean a linear operator with domain $\mathcal{D}(T) \subset X$, and a range $\mathrm{R}(T) \subset X$. $\mathcal{N}(T)$ denote the null space of T, the graph of T is the set defined by $G(T) := \{(x, Tx) \in X \times X, \text{for all } x \in \mathcal{D}(T)\}$. T is said to be closed if it's graph $G(T)$ is closed in the product space $X \times X$. T is said to be compact if, for every M a bounded subset of $\mathcal{D}(T)$, $T(M) \subset \mathrm{R}(T)$ is relatively compact, so that $\overline{T(M)}$ is compact. We denote by $\mathcal{C}(X)$ the set of all closed, densely defined linear operators on X, and let $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$) denote the Banach algebra of all bounded linear operators (respectively, the ideal of all compact operators) on X. The nullity, $\alpha(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathrm{R}(T)$ in X. For $T \in \mathcal{C}(X)$, we let $\sigma(T)$, $\rho(T)$ respectively the spectrum, and the resolvent set of T. The reduced minimum modulus $\gamma(T)$ of T is defined by $\gamma(T) := \inf \Big\{ \|Tx\| : dist(x, \mathcal{N}(T)) = 1,\ x \in D(T) \Big\}$, we set $\gamma(T) = \infty$ if $T = 0$. A useful classes of linear operators which have extensive application in spectrum theory are those of:

The set of upper semi-Fredholm operators on X is defined by

$$\Phi_+(X) := \Big\{ T \in \mathcal{C}(X) :\ \alpha(T) < \infty \text{ and } \ R(T) \text{ is closed in } X \Big\}.$$

The set of lower semi-Fredholm operators on X is defined by

$$\Phi_-(X) := \Big\{ T \in \mathcal{C}(X) :\ \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X \Big\}.$$

The set of semi-Fredholm operators on X is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X).$$

The set of Fredholm operators on X is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

For $T \in \Phi_\pm(X)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T.

Let $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$ denote the set $\Phi(X)\cap\mathcal{L}(X)$, $\Phi_+(X)\cap\mathcal{L}(X)$ and $\Phi_-(X)\cap\mathcal{L}(X)$, respectively.

Moreover, the set of Frefholm perturbations on X is defined by

$$\mathcal{F}(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi(X); \text{whenever } T \in \Phi(X)\Big\},$$

the set of lower semi-Fredholm perturbations on X is defined by

$$\mathcal{F}_-(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi_-(X); \text{whenever } T \in \Phi_-(X)\Big\},$$

and the set of upper semi-Fredholm perturbations on X is given by

$$\mathcal{F}_+(X) := \Big\{F \in \mathcal{L}(X) : T + F \in \Phi_+(X); \text{whenever } T \in \Phi_+(X)\Big\}.$$

Let $\mathcal{F}^b(X)$, $\mathcal{F}^b_+(X)$ and $\mathcal{F}^b_-(X)$ denote the set $\mathcal{F}(X)\cap\mathcal{L}(X)$, $\mathcal{F}_+(X)\cap\mathcal{L}(X)$ and $\mathcal{F}_-(X)\cap\mathcal{L}(X)$, respectively.

Let we now recall definitions and notions of concepts which we are interested throughout this study.

Definition 4.2.1 *Let X be Banach space. For each $T \in \mathcal{C}(X)$. We define*

(i) The Weyl essential spectrum of the operator T by

$$\sigma_w(T) := \bigcap_{K \in \ \mathcal{K}(X)} \sigma(A + K),$$

(ii) The essential approximate point spectrum of the operator T by

$$\sigma_{eap}(T) := \bigcap_{K \in \ \mathcal{K}(X)} \sigma_{ap}(T + K)$$

where, $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \inf_{\|x\|=1, x\in\mathcal{D}(T)} \|\lambda - Tx\| = 0)\}$, (*iii*) The essential defect spectrum of the operator T by

$$\sigma_{e\delta}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma_\delta(T+K)$$

where, $\sigma_\delta(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{is not surjective}\}$.

It's clear that, for $T \in \mathcal{C}(X)$, it holds $\sigma_w(T) := \sigma_{eap}(T) \cup \sigma_{e\delta}(T)$. The essential approximate point spectrum was introduced by V. Rakočević in [69], and the essential defect spectrum was introduced by C. Schmoeger in [79].

A characterization of the essential approximate point spectrum, and the essential defect spectrum by means of upper and lower semi-Fredholm operators is given by the following proposition

Proposition 4.2.1 [44, Proposition 3.1] *Let $T \in \mathcal{C}(X)$, then*

(*i*) $\lambda \notin \sigma_{eap}(T)$ *if, and only if,* $\lambda - T \in \Phi_+(X)$ *and* $i(\lambda - T) \leq 0$.

(*ii*) $\lambda \notin \sigma_{e\delta}(T)$ *if, and only if,* $\lambda - T \in \Phi_-(X)$ *and* $i(\lambda - T) \geq 0$.

4.3 The convergence in the generalized sense

4.3.1 The gap topology

While the distance between two bounded linear operators can be defined as the norm of their difference, the distance between two unbounded linear operators has to be measured in a different ways. One possibility is to use the gap between their graphs, which leads to the notion of convergence in the generalized sense, which essentially represents the convergence between their graphs. This concept of convergence can be found in the literature (see [54])

Definition 4.3.1 *The gap between two linear subspaces M and N of a normed space X is defined by*

$$\widehat{\delta}(M,N) := \max\left\{\delta(M,N), \delta(N,M)\right\}$$

where

$$\delta(M,N) := \begin{cases} \sup\limits_{\|x\|=1} dist(x,N), & \text{if } M \neq \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

From the Definition 4.3.1, it follows that $\delta(M,N) = 0$ if, and only if, $\overline{M} \subset \overline{N}$. The set of all closed linear subspaces of X equipped with the distance $\widehat{\delta}(\cdot,\cdot)$ forms a metric space, and a sequence of closed linear subspaces M_n converges to M if $\widehat{\delta}(M_n, M) \to 0$. The gap between two closed subspaces was introduced in Hilbert space by M.G. Krein and M. A. Krasnoselski in [55]. This notion was later extended to arbitrary Banach spaces in a paper by M. G. Krein, M. A. Krasnoselski, and D. P. Milman in [56].
If T, S $\in \mathcal{C}(X)$, their graphs $G(T), G(S)$ are closed linear subspaces in the product space X × X. Thus the distance between T and S can be measured by the "gap" between the closed linear subspaces $G(T), G(S)$.

Definition 4.3.2 *Let X be a Banach space, and let T, S be two closed linear operators on X. Let us define*

$$\delta(T,S) = \delta\big(G(T), G(S)\big) \text{ and } \widehat{\delta}(T,S) = \widehat{\delta}\big(G(T), G(S)\big).$$

$\widehat{\delta}\big(T,S\big)$ *is called the gap between S and T.*

Remark 4.3.1 *(i)* $\delta\big(G(T), G(S)\big) = \sup\limits_{\substack{x \in \mathcal{D}(T) \\ \|x\|^2 + \|Tx\|^2 = 1}} \left[\inf\limits_{y \in \mathcal{D}(S)} \Big(\|x-y\|^2 + \|Tx - Sy\|^2 \Big)^{\frac{1}{2}} \right].$

(ii) The function $\widehat{\delta}(\cdot,\cdot)$ defines a metric on $\mathcal{C}(X,Y)$ which is called the gap metric, and the topology induced by this metric is called the Gap topology or Kato topology.
(iii) For more properties of $\delta(\cdot,\cdot)$ and $\widehat{\delta}(\cdot,\cdot)$ we refer to [54, Pages 197, 200, 201].

Now, let us study some basic properties of the convergence in the generalized sense.

Theorem 4.3.1 *[54, Chapter IV Section 2] Let T and S be two closed densely defined linear operators. Then, we have:*

(i) If S and T are one-to-one, then $\delta(S,T) = \delta(S^{-1}, T^{-1})$ and $\widehat{\delta}(S,T) = \widehat{\delta}(S^{-1}, T^{-1})$.

(ii) Let $A \in \mathcal{L}(X)$. Then $\widehat{\delta}(A+S, A+T) \leq 2(1 + \|A\|^2)\widehat{\delta}(S,T)$.

(iii) Let T be Fredholm operator (respectively semi-Fredholm operator). If $\widehat{\delta}(T,S) < \gamma(T)(1 + [\gamma(T)]^2)^{\frac{-1}{2}}$, then S is Fredholm operator (respectively semi-Fredholm operator), $\alpha(S) \leq \alpha(T)$ and $\beta(S) \leq \beta(T)$. Furthermore, there exists $b > 0$ such that $\widehat{\delta}(T,S) < b$, which implies $i(S) = i(T)$.

(iv) Let $T \in \mathcal{L}(X)$. If $S \in \mathcal{C}(X)$ and $\widehat{\delta}(T,S) \leq \left[1 + \|T\|^2\right]^{-\frac{1}{2}}$, then S is bounded operator (so that $\mathcal{D}(S)$ is closed).

A complete discussion of all the above definitions and properties may be found in T. Kato [54]. For the case of closable linear operators, the authors A. Ammar, and A. Jeribi have introduced in [8] the following definitions and results

Definition 4.3.3 *Let S and T be two closable operators. We define the gap between T and S by $\delta(T,S) = \delta(\overline{T}, \overline{S})$ and $\widehat{\delta}(T,S) = \widehat{\delta}(\overline{T}, \overline{S})$.*

4.3.2 The convergence in the generalized sense

Definition 4.3.4 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators on X and let T be a closable linear operator on X. $(T_n)_{n\in\mathbb{N}}$ is said to be convergent in the generalized sense to T, written $T_n \xrightarrow{g} T$, if $\widehat{\delta}(T_n, T)$ converges to 0 when $n \to \infty$.*

In recent years, several authors have investigated the notions of generalized convergence of a sequence of closed linear operators in Banach spaces. We can cite as example [54, 68]. In [68], M. Pellicer stated that the notion of generalized convergence can be viewed as the convergence between the graphs of operators and hence it is useful in comparing the corresponding resolvent operators. Essentially, in [68] M. Pellicer used this notion between closed linear operators both to introduce the notion of a gap between linear operators and to study the nonlinear strongly damped wave equation with a dynamic boundary condition. It should be remarked that the notion of generalized convergence introduced above for closed and closable operators can be thought as a generalization of convergence in norm for linear operators that may be unbounded. Moreover, an important passageway between these two notions is developed in the following theorem

Theorem 4.3.2 [8, Theorem 2.3] *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators on X and let T be a closable linear operator on X.*

(i) $T_n \xrightarrow{g} T$, if, and only if, $T_n + S \xrightarrow{g} T + S$, for all $S \in \mathcal{L}(X)$.

(ii) Let $T \in \mathcal{L}(X)$. $T_n \xrightarrow{g} T$ if, and only if, $T_n \in \mathcal{L}(X)$ for sufficiently larger n and T_n converges to T.

(iii) Let $T_n \xrightarrow{g} T$. Then, T^{-1} exists and $T^{-1} \in \mathcal{L}(Y)$, if, and only if, T_n^{-1} exists and $T_n^{-1} \in \mathcal{L}(X)$ for sufficiently large n and T_n^{-1} converges to T^{-1}.

Proof. (i) Let $S \in \mathcal{L}(X,Y)$, then

$$\begin{aligned}\widehat{\delta}(T_n+S, T+S) &= \widehat{\delta}(\overline{T_n+S}, \overline{T+S}) \\ &= \widehat{\delta}(\overline{T_n}+S, \overline{T}+S).\end{aligned}$$

By using Theorem 4.3.1 (iii), we have $\widehat{\delta}(\overline{T_n}+S, \overline{T}+S) \leq 2\left(1+\|S\|^2\right)\widehat{\delta}(\overline{T_n}, \overline{T})$. Hence,

$$\widehat{\delta}(T_n+S, T+S) \leq 2(1+\|S\|^2)\widehat{\delta}(T_n, T). \tag{4.3.1}$$

In other terms, $\widehat{\delta}(\overline{T_n}, \overline{T}) = \widehat{\delta}\left(\overline{T_n+S} - S, \overline{T+S} - S\right)$, hence,

$$\widehat{\delta}(T_n, T) \leq 2(1+\|S\|^2)\widehat{\delta}(T_n+S, T+S). \tag{4.3.2}$$

If T_n converges in the generalized sense to T then $\widehat{\delta}(T_n, T) \to 0$. So, by using (4.3.1) we have, $\widehat{\delta}(T_n+S, T+S) \to 0$. Then, $T_n + S$ converges in the generalized sense to $T+S$. Conversely, if $T_n + S$ converges in the generalized sense to $T + S$ and according to (4.3.2) we have, $\widehat{\delta}(T_n+S, T+S) \to 0$ as $n \to \infty$. Hence, T_n converges in the generalized sense to T.

(ii) Let us assume that T_n converges in the generalized sense to T and that $T \in \mathcal{L}(X,Y)$. Let $n_0 \in \mathbb{N}$ such that $\widehat{\delta}(T, T_n) < \frac{1}{\sqrt{1+\|T\|^2}}$ holds for all $n \geq n_0$. Suppose that $x \in \mathcal{D}(\overline{T_n})$ where $n \geq n_0$ is fixed. First, we show that

$$\|(T-\overline{T_n})x\| \leq \delta(\overline{T_n}, T)\left(1+\|T^2\|\right)^{\frac{1}{2}}. \tag{4.3.3}$$

Indeed,

$$\|(T-\overline{T_n})x\| \leq \|Ty-\overline{T_n}x\| + \|T\|\|x-y\|, \ \forall y \in X.$$

By using the Cauchy-Schwarz inequality, we deduce that

$$\|(T-\overline{T_n})x\| \leq (\|x-y\|^2 + \|Ty-\overline{T_n}x\|^2)^{\frac{1}{2}} \left(1+\|T\|^2\right)^{\frac{1}{2}}.$$

So, we have

$$\left\|(T-\overline{T_n})x\right\| \leq \left[\inf_{y\in Y}\left(\|x-y\|^2 + \ |Ty-\overline{T_n}x\|^2\right)^{\frac{1}{2}}\right]\left(1+\|T\|^2\right)^{\frac{1}{2}}. \tag{4.3.4}$$

Hence, the inequality (4.3.3) follows immediately by using both (4.3.4) and Remark 4.3.1 (i). Now, let $x \in \mathcal{D}(\overline{T_n})$, such that $\|x\|^2 + \|\overline{T_n}x\|^2 = 1$. So, it is easy to prove that

$$1 \leq \left(1+\|T\|^2\right)^{\frac{1}{2}}\left\|x\right\| + \left\|(T-\overline{T_n})x\right\|. \tag{4.3.5}$$

Let $x \in \mathcal{D}(\overline{T}_n)$ such that $\|x\| \leq 1$. Then we have

$$\left\|(T-\overline{T_n})x\right\| \leq \left(\frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(\overline{T_n},T)}\delta(\overline{T_n},T)\right)\left\|x\right\|. \tag{4.3.6}$$

Since this inequality is homogeneous in x, then (4.3.6) is also true for any $x \in \mathcal{D}(\overline{T_n})$. The fact that $\delta(T_n,T) = \delta(\overline{T_n},\overline{T}) = \delta(\overline{T_n},T)$ and $\mathcal{D}(T_n) \subset \mathcal{D}(\overline{T_n})$, allows us to conclude that

$$\left\|(T-T_n)x\right\| \leq \left(\frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(T_n,T)}\delta(T_n,T)\right)\left\|x\right\|, \forall x \in \mathcal{D}(T_n). \tag{4.3.7}$$

By virtue of (4.3.7), the operator T_n is bounded on $\mathcal{D}(T_n)$. This implies that $\mathcal{D}(T_n)$ is closed. Hence, $\overline{\mathcal{D}(T_n)} = \mathcal{D}(T_n) = X$, $T_n \in \mathcal{L}(X,Y)$, and

$$\left\|T_n-T\right\| \leq \left(\frac{1+\|T\|^2}{1-\sqrt{1+\|T\|^2}\delta(T_n,T)}\right)\delta(T_n,T), \ \forall n \geq n_0. \tag{4.3.8}$$

The relation (4.3.8) enables us to conclude that T_n converges to T. Conversely, we suppose that T_n converges to T. So, T is bounded. Now, we can write $\widehat{\delta}(T_n,T) = \widehat{\delta}\Big((T_n-T)+T, 0+T\Big)$. This implies that

$$\widehat{\delta}(T_n,T) \leq 2\Big(1+\|T\|^2\Big)\widehat{\delta}\big(T_n-T,0\big). \tag{4.3.9}$$

Theorem 4.2.1 leads to the following inequality $\widehat{\delta}(T_n, T) \leq 2\Big(1 + \|T\|^2\Big) \frac{\|T_n - T\|}{\sqrt{1 + \|T_n - T\|^2}}$. As a result, T_n converges in the generalized sense to T.

(iii) Now, let us argue by contradiction. We assume that there exists $x \in \mathcal{D}(T_n)$ such that $\|x\| = 1$ and $T_n x = 0$, for all $n \in \mathbb{N}$. In other words, T_n converges in the generalized sense to T then there exists $N \in \mathbb{N}$ such that $\widehat{\delta}(T_n, T) < \frac{1}{\sqrt{1+\|T^{-1}\|^2}}$, $\forall n \geq N$. Then there exists $\delta > 0$ such that

$$\widehat{\delta}(T_n, T) < \delta < \frac{1}{\sqrt{1 + \|T^{-1}\|^2}}.$$

Since $(x, 0) \in G(T_n)$, then there exists $y \in \mathcal{D}(T)$ such that

$$\|x - y\|^2 + \|\overline{T_n} x - Ty\|^2 < \delta^2.$$

Hence, we have

$$1 = \|x\|^2 \leq \Big(\|x - y\| + \|y\|\Big)^2 \leq \Big(\|x - y\| + \|T^{-1}\|\|Ty\|\Big)^2, \tag{4.3.10}$$

and, by using both the Schwarz inequality and (4.3.10) we infer that

$$1 \leq \delta^2 \Big(1 + \|T^{-1}\|^2\Big) < 1$$

is a contradiction. So, T_n^{-1} exists for sufficiently larger n and by virtue of (4.3.8), we deduce that $T_n^{-1} \in \mathcal{L}(Y, X)$, and ■

$$\|T^{-1} - T_n^{-1}\| \leq \left(\frac{1 + \|T^{-1}\|^2}{1 - \sqrt{1 + \|T^{-1}\|^2}\delta(T_n^{-1}, T^{-1})}\right) \delta(T_n^{-1}, T^{-1}). \tag{4.3.11}$$

By using Theorem 4.2.1 (ii), and also the estimation (4.3.11) we infer that

$$\|T^{-1} - T_n^{-1}\| \leq \left(\frac{1 + \|T^{-1}\|^2}{1 - \sqrt{1 + \|T^{-1}\|^2}\delta(T_n, T)}\right) \delta(T_n, T). \tag{4.3.12}$$

According to (4.3.12), we also emphasize that T_n^{-1} converges to T^{-1}. Conversely, for the reasoning of the proof of (ii), it is sufficient to just replace T and T_n by T^{-1} and T_n^{-1} respectively.

4.3.3 Convergence of Essential Approximate Point Spectrum And Essential Defect Spectrum via the generalized convergence

The first main result is embodied in the following theorem, when we discuss and study the essential approximate point spectrum, and the essential defect spectrum of a sequence of closed linear operators perturbed by a bounded operator, and converges in the generalized sense to a closed linear operator in a Banach space

Theorem 4.3.3 *Let X be Banach space, Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators converges in the generalized sense in $\mathcal{C}(X)$ to a closed linear operator T, and let B a bounded linear operator mapping on X such that $\rho(T+B)\neq\emptyset$. For $\lambda_0\in\rho(T+B)$, we have*

(i) If $\mathcal{U}\subset\mathbb{C}$ is open and $0\in\mathcal{U}$, then there exists $n_0\in\mathbb{N}$ such that, for every $n\geq n_0$, we have

$$\sigma_{eap}(T_n+B-\lambda_0)\subseteq\sigma_{eap}(T+B-\lambda_0)+\mathcal{U}, \tag{4.3.13}$$

and

$$\sigma_{e\delta}(T_n+B-\lambda_0)\subseteq\sigma_{e\delta}(T+B-\lambda_0)+\mathcal{U}, \tag{4.3.14}$$

(ii) If $\mathcal{U}\subset\mathbb{C}$ is open and $0\in\mathcal{U}$, then there exists $\varepsilon>0$, and $n_0\in\mathbb{N}$, such that, for all $S\in B(X)$, and $\|S\|<\varepsilon$, we have

$$\sigma_{eap}(T_n+B+S-\lambda_0)\subseteq\sigma_{eap}(T+B-\lambda_0)+\mathcal{U},\ \textit{for all } n\geq n_0,$$

and

$$\sigma_{e\delta}(T_n+B+S-\lambda_0)\subseteq\sigma_{e\delta}(T+B-\lambda_0)+\mathcal{U},\ \textit{for all } n\geq n_0.$$

Proof. For (i), before proof, we make some preliminary observations. Since $T_n\overset{g}{\longrightarrow}T$, then by Theorem 4.3.2 (i), $(T_n+B-\lambda_0)\overset{g}{\longrightarrow}(T+B-\lambda_0)$, furthermore, we have $(T+B-\lambda_0)^{-1}\in\mathcal{L}(X)$, which implies according to Theorem 4.3.2 (iii), that $\lambda_0\in\rho(T_n+B)$ for a sufficiently large n and $(T_n+B-\lambda_0)^{-1}$ converges to $(T+B-\lambda_0)^{-1}$. We recall that the essential approximate point spectrum of a bounded linear operator is compact, but this property is not valid for the case of unbounded operators, for this reason, using the compactness of $\sigma_{eap}(T+B-\lambda_0)^{-1}$ because $(T+B-\lambda_0)^{-1}$ is bounded, as a first step, we will prove the existence of $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, we have

$$\sigma_{eap}(T_n+B-\lambda_0)^{-1} \subseteq \sigma_{eap}(T+B-\lambda_0)^{-1} + \mathcal{U}. \tag{4.3.15}$$

The proof by contradiction. Suppose that (4.3.15) does not hold. Then, by studying a subsequence (if necessary) we may assume that, for each n there exists $\lambda_n \in \sigma_{eap}(T_n+B-\lambda_0)^{-1}$ such that $\lambda_n \notin \sigma_{eap}(T+B-\lambda_0)^{-1} + \mathcal{U}$. Since (λ_n) is bounded, we may assume that $\lim\limits_{n\to+\infty} \lambda_n = \lambda$, which implies that $\lambda \notin \sigma_{eap}(T+B-\lambda_0)^{-1} + \mathcal{U}$. Since $0 \in \mathcal{U}$ then we have $\lambda \notin \sigma_{eap}(T+B-\lambda_0)^{-1}$. Therefore $\lambda-(T+B-\lambda_0)^{-1} \in \Phi_+^b(X)$ and $i\big(\lambda-(T+B-\lambda_0)^{-1}\big) \leq 0$. As $\big(\lambda_n-(T_n+B-\lambda_0)^{-1}\big)$ converges to $\big(\lambda-(T+B-\lambda_0)^{-1}\big)$, we deduce that

$$\widehat{\delta}\big(\lambda_n-(T_n+B-\lambda_0)^{-1}, \lambda-(T+B-\lambda_0)^{-1}\big) \to 0 \text{ when } n \to \infty.$$

Let $\delta = \gamma\big(\lambda-(T+B-\lambda_0)^{-1}\big) > 0$, then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ we have $\widehat{\delta}\big(\lambda_n-(T_n+B-\lambda_0)^{-1}, \lambda-(T+B-\lambda_0)^{-1}\big) \leq \frac{\delta}{\sqrt{1+\delta^2}}$. By using Theorem 4.3.1 (iv) we infer that $\lambda_n-(T_n+B-\lambda_0)^{-1} \in \Phi_+(X)$. Furthermore, there exists $b > 0$ such that

$$\widehat{\delta}\big(\lambda_n-(T_n+B-\lambda_0)^{-1}, \lambda-(T+B-\lambda_0)^{-1}\big) < b,$$

which implies $i\big(\lambda_n-(T_n+B-\lambda_0)^{-1}\big) = i\big(\lambda-(T+B-\lambda_0)^{-1}\big) \leq 0$. Then we obtain $\lambda_n \notin \sigma_{eap}\big((T_n+B-\lambda_0)^{-1}\big)$, which is a contradiction. Hence (4.3.15) holds. Now, take $\gamma_n \in \sigma_{eap}(T_n+B-\lambda_0)$ such that $\gamma_n \notin \sigma_{eap}(T+B-\lambda_0) + \mathcal{U}$. From σ_{eap} is upper semi continuous at $(T+B-\lambda_0)^{-1}$, (see[69, 70]), there exists $k > 0$ such that $k^{-1} \leq |\gamma_n-\lambda_0|^{-1}$, so (γ_n) is bounded. Therefore it can be assumed that $\gamma_n \to \gamma$. Then $\gamma \notin \sigma_{eap}(T+B-\lambda_0)+\mathcal{U}$ and hence $\gamma \notin \sigma_{eap}(T+B-\lambda_0)$. This implies that $(\gamma)^{-1} \notin \sigma_{eap}((T+B-\lambda_0)^{-1})$.

We set $\lambda_n = (\gamma_n)^{-1}$, $\lambda = (\gamma)^{-1}$. Then by using (4.3.15), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\lambda_n \notin \sigma_{eap}((T_n+B-\lambda_0)^{-1})$, which implies $\gamma_n \notin \sigma_{eap}((T_n+B-\lambda_0))$, is a contradiction.

(ii) Since S is bounded, it's clear by using Theorem 4.3.2 (i), that the operators sequence $A_n = (T_n+B+S-\lambda_0)$ converges in the generalized sense to the operator $A = (T+B+S-\lambda_0)$, then we need, for applying (i), to prove that $\rho(T+B) \subset \rho(T+B+S)$. Let $\lambda_0 \in \rho(T+B)$, for $S \in \mathcal{L}(X)$ such that $\|S\| < \dfrac{1}{\|(T+B-\lambda_0)^{-1}\|} = \varepsilon_1$, we have

$\|S(T+B-\lambda_0)^{-1}\| < 1$, which gives that $\big(I+S(T+B-\lambda_0)^{-1}\big)^{-1}$ exists and bounded, when the existence is given by the convergence of the Neumann serie $\sum_{k=0}^{\infty}\big(-S(T+B-\lambda_0)^{-1}\big)^k$, and the boundedness is immediately from the inequality

$$\|\big(I+S(T+B-\lambda_0)^{-1}\big)^{-1}\| < \frac{1}{1-\|S\|\|(T+B-\lambda_0)^{-1}\|},$$

which implies that the operator

$$\big((T+B-\lambda_0)+S\big)^{-1} = (T+B-\lambda_0)^{-1}\big(I+S(T+B-\lambda_0)^{-1}\big)^{-1}$$

exists and bounded, then $0 \in \rho(T+B+S-\lambda_0)$. Now applying (i) to A_n and A, we deduce that there exists $n_0 \in \mathbb{N}$, such that $\sigma_{eap}(T_n+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B+S-\lambda_0) + \mathcal{U}$, for all $n \geq n_0$, and $\mathcal{U} \subset \mathbb{C}$ is an open containing 0, we will prove that $\sigma_{eap}(T+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B$ by contradiction. Let $\lambda \notin \sigma_{eap}(T+B-\lambda_0)$, then $\big(\lambda-(T+B-\lambda_0)\big) \in \Phi_+(X)$ and $i\big(\lambda-(T+B-\lambda_0)\big) \leq 0$. From [54, $Chapter IV.Theoreme 5.22, p236$], we deduce that there exists $\varepsilon_2 > 0$ such that for $\|S\| < \varepsilon_2$, one has $\big(\lambda-(T+B+S-\lambda_0)\big) \in \phi_+(X)$ and $i\big(\lambda-(T+B+S-\lambda_0)\big) = i\big(\lambda-(T+B-\lambda_0)\big) \leq 0$. This implies that $\lambda \notin \sigma_{eap}(T+B+S-\lambda_0)$. Then by transitivity

$$\sigma_{eap}(T_n+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B-\lambda_0)$$

From what has been mentioned and if we take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then for all $\|S\| < \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that $\sigma_{eap}(T_n+B+S-\lambda_0) \subseteq \sigma_{eap}(T+B-\lambda_0) + \mathcal{U}$, for all $n \geq n_0$. ■

As a straightforward consequence of Theorem 4.3.3 we can easily obtain the following result.

Corollary 4.3.1 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators and T a closed linear operator such that $(T_n) \xrightarrow{g} T$, we suppose that $0 \in \rho(T)$ then,*
(i) If $\mathcal{U} \subset \mathbb{C}$ is open and $0 \in \mathcal{U}$, then there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we have

$$\sigma_{eap}(T_n) \subseteq \sigma_{eap}(T) + \mathcal{U}. \tag{4.3.16}$$

and

$$\sigma_{e\delta}(T_n) \subseteq \sigma_{e\delta}(T) + \mathcal{U}. \tag{4.3.17}$$

4.4 The convergence compactly

In this part, motivated by the analysis started by S. Goldberg in 1973 [38, *Theorem* 4], we examine the convergence of the essential approximate point spectrum and essential defect spectrum of a sequence of linear operators converges compactly

4.4.1 Convergence to zero compactly

Definition 4.4.1 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators mapping on X, $(T_n)_{n\in\mathbb{N}}$ is said to be convergent to zero compactly, written $T_n \xrightarrow{c} 0$, if for all $x \in X$, $T_n x \to 0$ and $(T_n x_n)_n$ is relatively compact for every bounded sequence $(x_n)_n \subset X$.*

Theorem 4.4.1 [*38, Theorem* 4] *Let K_n be a sequence of bounded linear operators such that $K_n \xrightarrow{c} 0$, and let T be a closed linear operator. If T is a semi-Fredholm operator, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $(T + K_n)$ is semi-Fredholm,

(ii) $\alpha(T + K_n) < \alpha(T)$,

(iii) $\beta(T + K_n) < \beta(T)$, and

(iv) $i(T + K_n) = i(T)$.

4.4.2 Convergence compactly

Inspired by the notion of convergence to zero compactly, we examine the following definition.

Definition 4.4.2 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators mapping on X and let $T \in \mathcal{L}(X)$, $(T_n)_{n\in\mathbb{N}}$ is said to be converge to T compactly, written $T_n \xrightarrow{c} T$ if, and only if, $T_n - T$ converges to zero compactly.*

Proposition 4.4.1 [*8, Proposition* 3.1] *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators which converges compactly to a bounded operator T. Then,*

(i) if $T_n \in \mathcal{F}^b(X)$, then $T \in \mathcal{F}^b(X)$,

(ii) if $T_n \in \mathcal{F}^b_+(X)$, then $T \in \mathcal{F}^b_+(X)$, and

(iii) if $T_n \in \mathcal{F}^b_-(X)$, then $T \in \mathcal{F}^b_-(X)$.

Proof. Let $A \in \Phi^b(X)$. Using the fact that $T_n - T$ converges to zero compactly, and by virtue of Theorem 4.4.1, there exists $n_0 \in \mathbb{N}$ such that $A-(T_n-T) \in \Phi^b(X)$ for all $n \geq n_0$. Since $T_n \in \mathcal{F}^b(X)$, we have $A-(T_n-T)+T_n \in \Phi^b(X)$ for all $n_0 \in \mathbb{N}$. This shows that $T \in \mathcal{F}^b(X)$.

The proofs of (ii) and (iii) may be achieved by following the same reasoning as for (i). ■

M. Schechter in [72], proved the following related theorem: Let T_n converges to T. If $T_n \in \mathcal{F}^b(X)$ (respectively $T_n \in \mathcal{F}^b_-(X)$) then $T \in \mathcal{F}^b(X)$ (respectively $T \in \mathcal{F}^b_-(X)$). The above result is a very special case of Proposition 4.4.1. Indeed, if T_n converges to T this implies that $T_n - T$ converges to zero compactly.

4.4.3 Convergence of Essential Approximate Point Spectrum And Essential Defect Spectrum via the convergence compactly

In the next theorem we discuss the essential approximate point spectrum and the essential defect spectrum of a sequence of linear operators converges compactly

Theorem 4.4.2 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X)$ and let T be a bounded linear operator on X.*

(i) If T_n converges to T compactly, $\mathcal{U} \subseteq \mathbb{C}$ is open and $0 \in \mathcal{U}$, then there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$

$$\sigma_{eap}(T_n) \subseteq \sigma_{eap}(T) + \mathcal{U}, \tag{4.4.1}$$

and

$$\sigma_{e\delta}(T_n) \subseteq \sigma_{e\delta}(T) + \mathcal{U}. \tag{4.4.2}$$

(ii) If T_n converges to zero compactly then there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$

$$\sigma_{eap}(T + T_n) \subseteq \sigma_{eap}(T), \tag{4.4.3}$$

and

$$\sigma_{e\delta}(T + T_n) \subseteq \sigma_{e\delta}(T). \tag{4.4.4}$$

Proof. (i) The proof by contradiction. Assume that the inclusion is fails. Then by passing to a subsequence (if necessary) it may be assumed that, for each n, there exists

$\lambda_n \in \sigma_{eap}(T_n)$ such that $\lambda_n \notin \sigma_{eap}(T_n) + \mathcal{U}$, since λ_n is bounded, we suppose (if necessary pass to a subsequence) that $\lim\limits_{n\to+\infty} \lambda_n = \lambda$, which implies that $\lambda \notin \sigma_{eap}(T) + \mathcal{U}$. Using the fact that $0 \in \mathcal{U}$, we have $\lambda \notin \sigma_{eap}(T)$. Therefore $(\lambda - T) \in \Phi_+(X)$, and $i(\lambda - T) \leq 0$. As $(\lambda_n - T_n) - (\lambda - T) \xrightarrow{c} 0$, which implies by Theorem **??** (i) and (iv) that $(\lambda_n - T_n) \in \Phi_+(X)$, and $i(\lambda_n - T_n) = i(\lambda - T) \leq 0$, then $\lambda_n \notin \sigma_{eap}(T_n)$, which is a contradiction, hence the inclusion (4.4.1) holds. The statement for the essential defect spectrum can be proved similarly.

(ii) We have T is bounded, and $T_n \xrightarrow{c} 0$, then for $\lambda \notin \sigma_{eap}(T)$, $\lambda - T \in \Phi^b_+(X)$, and $i(\lambda - T) \leq 0$. Since $T_n \xrightarrow{c} 0$, then if we apply Theorem **??** (i) and (iv) we obtain that there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $(\lambda - T) - T_n = \big(\lambda - (T + T_n)\big) \in \Phi_+(X)$, and $i(\lambda - T) = i\big(\lambda - (T + T_n)\big) \leq 0$, which implies that $\lambda \notin \sigma_{eap}(T_n)$, then the inclusion (4.4.3) is valid. For the inclusion (4.4.4) the proof is similarly. ∎

From the above result we deduce the following consequence concerned on a sequence of closed linear operators

Corollary 4.4.1 *Let T be a closed linear operator and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators on X, such that $\rho(T_n) \cap \rho(T) \neq \phi$, let $\eta \in \rho(T_n) \cap \rho(T)$, if $(T_n - \eta)^{-1} - (T - \eta)^{-1}$ converges to zero compactly then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$*

$$\sigma_{eap}(T_n - \eta) \subseteq \sigma_{eap}(T - \eta),$$

and

$$\sigma_{e\delta}(T_n - \eta) \subseteq \sigma_{e\delta}(T - \eta).$$

Proof. If we put $K_n = (T_n - \eta)^{-1} - (T - \eta)^{-1}$, we have $K_n \xrightarrow{c} 0$, and $(T_n - \eta)^{-1} = (T - \eta)^{-1} + K_n$, by using the inclusions (4.4.3) and (4.4.4) respectively, we infer that $\sigma_{eap}\big((T_n - \eta)^{-1}\big) = \sigma_{eap}\big((T - \eta)^{-1} + K_n\big) \subseteq \sigma_{eap}\big((T - \eta)^{-1}\big)$, and $\sigma_{e\delta}\big((T_n - \eta)^{-1}\big) = \sigma_{e\delta}\big((T - \eta)^{-1} + K_n\big) \subseteq \sigma_{e\delta}\big((T - \eta)^{-1}\big)$. Then $\sigma_{eap}(T_n - \eta) \subseteq \sigma_{eap}(T - \eta)$, and $\sigma_{e\delta}(T_n - \eta) \subseteq \sigma_{e\delta}(T - \eta)$. ∎

Chaptre 5

The spectrum continuity using the $\upsilon-$convergence

This chapter is devoted to the investigation of some properties of the essential spectra of a sequence of bounded linear operators by referring back to ν-convergence in a Banach space. In addition to that, we shed light on the notion of ν-continuity. In fact, if U is a bounded linear operator then the Wolf essential spectrum and the Weyl essential spectrum of U are ν-upper semi-continuous at U. Similarly we study the ν-continuity of some 3×3 block operator matrices (see Theorem 5.4.2), more precisely we generalized the result of A. Ammar in 3×3 block operator matrices to 3×3 block operator matrices .

5.1 Introduction

Let X be Banach space. By an operator T on X we mean a linear operator with domain $\mathcal{D}(T) \subset X$, and a range $\mathrm{R}(T) \subset X$. $\mathcal{N}(T)$ denote the null space of T, the graph of T is

the set defined by $G(T) := \{(x, Tx) \in X \times X, \text{for all } x \in \mathcal{D}(T)\}$. T is said to be closed if its graph $G(T)$ is closed in the product space $X \times X$. T is said to be compact if, for every M a bounded subset of $\mathcal{D}(T)$, $T(M) \subset \mathrm{R}(T)$ is relatively compact, so that $\overline{T(M)}$ is compact. We denote by $\mathcal{C}(X)$ the set of all closed, densely defined linear operators on X, and let $\mathcal{L}(X)$ (respectively, $\mathcal{K}(X)$) denote the Banach algebra of all bounded linear operators (respectively, the ideal of all compact operators) on X. The nullity, $\alpha(T)$, of T is defined as the dimension of $\mathcal{N}(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathrm{R}(T)$ in X. For $T \in \mathcal{C}(X)$, we let $\sigma(T)$, $\rho(T)$ respectively the spectrum, and the resolvent set of T. A useful classes of linear operators which have extensive application in spectrum theory are those of:
The set of upper semi-Fredholm operators on X is defined by

$$\Phi_+(X) := \Big\{T \in \mathcal{C}(X) : \ \alpha(T) < \infty \text{ and } \ R(T) \text{ is closed in } X\Big\}.$$

The set of lower semi-Fredholm operators on X is defined by

$$\Phi_-(X) := \Big\{T \in \mathcal{C}(X) : \ \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X\Big\}.$$

The set of semi-Fredholm operators on X is defined by

$$\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X).$$

The set of Fredholm operators on X is defined by

$$\Phi(X) := \Phi_+(X) \cap \Phi_-(X).$$

For $T \in \Phi_\pm(X)$, the number $i(T) = \alpha(T) - \beta(T)$ is called the index of T.

Let $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$ denote the set $\Phi(X) \cap \mathcal{L}(X)$, $\Phi_+(X) \cap \mathcal{L}(X)$ and $\Phi_-(X) \cap \mathcal{L}(X)$, respectively.
For each $T \in \mathcal{C}(X)$, we define the Weyl essential spectrum of the operator T by

$$\sigma_w(T) := \bigcap_{K \in \ \mathcal{K}(X)} \sigma(A + K).$$

and the Wolf essential spectrum of the operator T by

$$\sigma_f(U) := \mathbb{C} \backslash \left\{\lambda \in \mathbb{C} : \ \lambda - U \in \Phi(X)\right\}.$$

A characterization of the Weyl essential spectrum by means of Fredholm operators is given by the following proposition

Proposition 5.1.1 [44, Proposition 3.1] *Let $T \in \mathcal{C}(X)$, then*

$\lambda \notin \sigma_w(T)$ *if, and only if,* $\lambda - T \in \Phi(X)$ *and* $i(\lambda - T) = 0$.

It's clear that $\sigma_w(U) = \sigma_f(U) \cup \{\lambda \in \mathbb{C} : \; i(\lambda - U) \neq 0\}$.

5.2 The $\upsilon-$convergence

The spectrum of a linear operator is one of the most useful objects in functional analysis. However, since exact computations of the spectrum are almost always impossible, it is relevant to know the spectrum or any of its parts in approximate way. Several authors have studied this problem and other related topics using various types of convergence in $\mathcal{L}(X)$. There are several notions of convergence of a sequence of operators which yield spectral results: the norm convergence ([21], [26], [27], [31], [29], [30], [32], [67], [76], [77], [79]), collectively compact convergence ([12]), compact convergence and regular convergence ([10]), stable and strongly stable convergence ([25]), resolvent operator convergence ([59]), spectral convergence ([5]), convergence of $(T_n)_n$ to T in the sense that the spectral radius $r(T_n - T)$ of $T_n - T$ tends to zero and $\|(T_n - T)T_n\|$ tends to zero ([7]), asymptotically compact convergence ([11]) and also discrete versions of some of these ([84], [40], [85]), to name a few. Moreover, in 1994 M. Ahues, and A. Largillier introduced the ν-convergence in ([6]), this mode of convergence encompasses a wide variety of approximation methods, also this convergence allows to approximate non-compact operators through finite rank operators. The ν-convergence is defined as
$T_n \xrightarrow{\nu} T \Leftrightarrow \{(\|T_n\|)\}$ is bounded, $\|(T_n - T)T\| \to 0$, and $\|(T_n - T)T_n\| \to 0$. The condition on the norms of the operators T_n, $(T_n - T)T$, and $(T_n - T)T_n$ have evolved from ([6]) and ([82]). The concept ν-convergence emerged as a new mode of convergence to approximate the noncompact opertaors resorting to finite rank operators. Note that the ν-convergence is a pseudo-convergence in sense that we can find T_n ν-convergent to T and T_n ν-convergent

to V but $T \neq V$(see [8, *Remark* 2.1]). However the spectrum of T and V are the same (see [6, *Exercise* 2.12]). In order to study spectral continuity properties, this type of convergence is useful ([32], [26], [78], [8]).

Lemma 5.2.1 [*6, Lemma* 2.2]

(i) If $U_n \to U$ *then* $U_n \xrightarrow{\nu} U$*. Conversely, if* $0 \in \rho(U)$ *and* $U_n \xrightarrow{\nu} U$*, then* $U_n \to U$*.*

(ii)Let $U_n \xrightarrow{\nu} U$ *and* $V_n \to V$*. Then* $U_n + V_n \xrightarrow{\nu} U + V$ *if, and only if,* $(U_n - U)V \to 0$*.*

Theorem 5.2.1 [*9, Theorem* 2.2] *Let* U*,* $V \in \mathcal{L}(X)$ *be two operators such that* $R(U) \cap R(V)$ *is closed and let* (U_n) *be a sequence of bounded linear operators commuting with both* U *and* V*. If* $U_n \xrightarrow{\nu} U$ *and* $U_n \xrightarrow{\nu} V$ *then*

$$\sigma_w(U) = \sigma_w(V), \tag{5.2.1}$$

and

$$\sigma_f(U) = \sigma_f(V). \tag{5.2.2}$$

Theorem 5.2.2 [*9, Theorem* 3.1]*Let* $U \in \mathcal{L}(X)$ *and* (U_n) *be a sequence in* $\mathcal{L}(X)$ *such that* $0 \in \Phi^b_U$*. If* $U_n \xrightarrow{\nu} U$ *and* $\mathcal{O}$ *is an open set of* $\mathbb{C}$ *with* $0 \in \mathcal{O}$*, then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$*, we have*

$$\sigma_w(U_n) \subseteq \sigma_w(U) + \mathcal{O}, \tag{5.2.3}$$

and

$$\sigma_f(U_n) \subseteq \sigma_f(U) + \mathcal{O}, \tag{5.2.4}$$

Proof. For (5.2.3), assume that the assertion fails. Then by passing to a subsequence, it may be assumed that, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \sigma_w(U_n)$ such that $\lambda_n \notin \sigma_w(U) + \mathcal{O}$. Since (λ_n) is bounded, we may assume that $\lim\limits_{n \to +\infty} \lambda_n = \lambda$. This implies that $\lambda \notin \sigma_w(U) + \mathcal{O}$. Using the fact that $0 \in \mathcal{O}$, we have $\lambda \notin \sigma_w(U)$ and therefore, $\lambda - U \in \Phi^b(X)$ and $i(\lambda - U) = 0$. Let

$$V_n = (\lambda_n - \lambda)U + (U - U_n)U. \tag{5.2.5}$$

It follows from Eq. (5.2.5), that the operator $(\lambda_n - U_n)U$ can be expressed in the form

$$(\lambda_n - U_n)U = (\lambda - U)U + V_n.$$

Using the fact that $V_n \to 0$ and $(\lambda - U)U \in \Phi^b(X)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $(\lambda_n - U_n)U \in \Phi^b(X)$ and $i\,[(\lambda_n - U_n)U] = i[(\lambda - U)U]$. Moreover $\beta(U) < \infty$, then by [78, *Theorem* 1.1(*b*)], we conclude that $(\lambda_n - U_n) \in \Phi^b(X)$ with $i(\lambda_n - U_n) = i(\lambda - U)$. We get $(\lambda_n - U_n) \in \Phi^b(X)$ and $i(\lambda_n - U_n) = i(\lambda - U) = 0$. So $\lambda_n \notin \sigma_w(U_n)$, which is a contradiction. This enables us to conclude that, $\sigma_w(U_n) \subseteq \sigma_w(U) + \mathcal{O}$, $\forall n \geq n_0$.
The proof of the inclusion (5.2.4) follows the same reasoning. ∎

Theorem 5.2.3 *Let $U \in \mathcal{L}(X)$ such that $0 \in \Phi^b_U$, then σ_i is ν-upper semi continuous at U for $i = f,\ w$.*

Furthermore, σ_w is ν-continuous at U in each one of the following cases:

(a) σ_f is ν-continuous at U.

(b) If there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\sigma_f(U) \subset \sigma_w(U_n)$.

Proof. Let $\lambda \in \limsup(\sigma_i(U_n))$ for $i = f,\ w$. Then we may suppose that there exists a sequence (λ_n) such that $\lambda_n \in \sigma_i(U_n)$ for all $n \in \mathbb{N}$ and $\lambda_n \to \lambda$. From Theorem 5.2.2, for each $p \in \mathbb{N}\backslash\{0\}$, there exists $n_p \in \mathbb{N}$ such that $\sigma_i(U_n) \subseteq \sigma_i(U) + \mathcal{B}(0, \frac{1}{p})$ for $i = f,\ w$ and for all $n \geq n_p$. This implies that for each $p \geq 1$, there exist $\beta_p \in \sigma_i(U)$ and $\gamma_p \in \mathcal{B}(0, \frac{1}{p})$ such that $\lambda_{np} = \beta_p + \gamma_p$. Therefore, $\lim \beta_p = \lim \lambda_{np} - \lim \gamma_p = \lambda$. Since $\sigma_i(U)$ is a closed set, it follows that $\lambda \in \sigma_i(U)$ for $i = f,\ w$. We conclude that

$$\limsup(\sigma_i(U_n)) \subseteq \sigma_i(U) \text{ for } \ i = f,\ w.$$

(a) Now, we prove that $\sigma_w(U) \subseteq \liminf(\sigma_w(U_n))$. Suppose that $\lambda \notin \liminf(\sigma_w(U_n))$ then there is a neighbourhood $\mathcal{V}$ of λ with does not intersect infinitely many $\sigma_w(U_n)$. Since, $\sigma_f(U_n) \subset \sigma_w(U_n)$, then $\mathcal{V}$ does not intersect infinitely with many $\sigma_f(U_n)$. Hence $\lambda \notin \liminf(\sigma_f(U_n))$. Now using σ_f is ν-continuous at U. Then $\liminf(\sigma_f(U_n)) = \sigma_f\,(U)$. This implies that $\lambda \notin \sigma_f(U)$. This shows that

$$\lambda - U \in \Phi^b(X). \tag{5.2.6}$$

Now, we prove that $i(\lambda - U) = 0$. Since $\lambda \notin \liminf(\sigma_w(U_n))$ there exists an increasing sequence of natural numbers $n_1 < n_2 < \cdots$ such that $\lambda \notin \sigma_w(U_{n_j})$ for all $j \in \mathbb{N}$. Then $\lambda - U_{n_j} \in \Phi^b(X)$ and $i(\lambda - U_{n_j}) = 0$. From $[78, Theorem 3.4]$ there exists $j_0 \in \mathbb{N}$ such that $i(\lambda - U_{n_j}) = i(\lambda - U)$ for all $j \geq j_0$. Therefore

$$i(\lambda - U) = 0. \tag{5.2.7}$$

It follows from Eqs (5.2.6), (5.2.7) and Proposition 5.1.1, that $\lambda \notin \sigma_w(U)$. So we get that

$$\sigma_w(U) \subset \liminf(\sigma_w(U_n)).$$

(b) Assume that the assertion does not hold. Then there exists $\lambda \in \sigma_w(U)$ such that $\lambda \notin \liminf(\sigma_w(U_n))$. We discuss two cases.

1^{st}*case*. If $\lambda \in \sigma_w(U)\backslash\sigma_f(U)$ then by using a similar reasoning of (a) we have $i(\lambda - U) = 0$ which is a contradiction.

2^{nd}*case*. If $\lambda \in \sigma_f(U)$, since $\lambda \notin \liminf(\sigma_w(U_n))$. Arguing as above, there exists an increasing sequence of natural numbers $n_1 < n_2 < \ldots$, and $\varepsilon > 0$ such that for all $j \in \mathbb{N}$ we have $\mathcal{B}(\lambda, \varepsilon) \cap \sigma_w(U_{n_j}) = \emptyset$. Then $\mathcal{B}(\lambda, \varepsilon) \cap \sigma_f(U) = \emptyset$ which is a contradiction. ■

5.3 The spectrum continuity

5.3.1 Limit inferior and limit superior

Let $\mathcal{C}$ be the collection of all non-empty compact subsets of $\mathbb{C}$. For the study of convergence of a sequence in $\mathcal{C}$ with respect to the Hausdorff metric, two useful concepts which have extensive applications are those of limit inferior and limit superior.

Definition 5.3.1 *For E_n be a sequence in $\mathcal{C}$, the limit inferior and limit superior of E_n, denoted respectively by* $\liminf E_n$, *and* $\limsup E_n$ *can be defined as follow:*

$$\liminf E_n := \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \textit{ such that } \mathcal{B}(\lambda, \varepsilon) \cap E_n \neq \emptyset\ \forall n \geq n_0\}, \textit{ and}$$

$$\limsup E_n := \{\lambda \in \mathbb{C} : \forall \varepsilon > 0, \exists\, \mathbb{I} \subseteq \mathbb{N} \textit{ infinite such that } \mathcal{B}(\lambda, \varepsilon) \cap E_n \neq \emptyset\ \forall n \in \mathbb{I}\}.$$

This definition can be characterized by the following

Remark 5.3.1 *(i)* $\lambda \in \limsup E_n$ *if and only if there exists an increasing sequence of natural numbers* $n_1 < n_2 < n_3 < \ldots$ *and points* $\lambda_{n_k} \in E_{n_k}$, *for all* $k \in \mathbb{N}$, *such that* $\lim \lambda_{n_k} = \lambda$.

(ii) $\lambda \in \liminf E_n$ *if and only if there exists a sequence* λ_n *such that* $\lambda_n \in E_n$ *for all* $n \in \mathbb{N}$, *and* $\lim \lambda_n = \lambda$.

Inspired by the notion of ν-convergence, we examine the following definitions

Definition 5.3.2 *A mapping* τ *on* $\mathcal{L}(X)$ *whose values are compact subsets of* $\mathbb{C}$ *is said to be* ν*-upper semi continuous at* U *when*

$$U_n \xrightarrow{\nu} U \implies \limsup \tau(U_n) \subset \tau(U),$$

and to be ν*-lower semi continuous at* U *when*

$$U_n \xrightarrow{\nu} U \implies \tau(U) \subset \liminf \tau(U_n).$$

If τ *is both* ν*-upper and* ν*-lower semi continuous, then it is said to be* ν*-continuous.*

5.4 Application to 3×3 block operator matrices

Many problems in mathematical physics are described by the system of partial or ordinary differential equations or linearizations thereof. In application, the time evolution of a physical system is governed by block operator matrices. Hence the spectral theory of these matrices plays a crucial role. During the last years F.V. Atkinson, H. langer, R. Mennicken, and A. A. Shkalikov ([15], [81]) studied the Wolf essential spectrum of operator defined by a block operator matrix:

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which acts on the product space $X \times Y$ of Banach spaces. An account of the research a wide panorama of methods to investigate the spectrum of block operator matrices were presented by C. Tretter in ([83]).

Systems of linear evolution equations as well as linear initial value problems with more than one set of initial data lead, in a natural way, to an abstract Cauchy problem involving an operator matrix defined in a product of n Banach spaces. In ([48]), A. Jeribi, N. Moalla, and I. Walha treated a 3×3 block operator matrix

$$\mathcal{M} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & L \end{pmatrix}$$

in a product of Banach spaces, where the entries of the matrix are generally unbounded operators. moreover, A. Ammar studied the spectrum continuity of the Weyl and the Wolf essential spectrum of 2×2 block operator matrices using the $\nu-$convergence ([8]), it is the result that we want to extend in the next section to 3×3 block operator matrices. Particularly we focus on the relationship between the ν-continuity of 3×3 block operator matrices and the ν-continuity of its components.

Theorem 5.4.1 *Let X be a Banach space. In the product space $X \otimes X \otimes X$, we consider a sequence of operators formally defined by a matrix:*
$\mathcal{M}_n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & L_n \end{pmatrix}$, $\mathcal{M} = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & L \end{pmatrix}$ *and* $\mathcal{M}_C = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & L \end{pmatrix}$ *when* A, $A_n, B, B_n, C, C_n, D_n, E, E_n, F, F_n, G_n, L$ *and* $L_n \in \mathcal{L}(X)$.

(a) If $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$, $L_n \xrightarrow{\nu} L$, $B_n \to 0$, $C_n \to 0$, $D_n \to 0$, $F_n \to 0$, $G_n \to 0$ *and* $H_n \to 0$, *then* $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}$. *Moreover, if* $0 \in \Phi_A^b \cap \Phi_E^b \cap \Phi_L^b$ *and* $\mathcal{O}$ *is an open set of* $\mathbb{C}$ *with* $0 \in \mathcal{O}$, *then there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$

$$\sigma_w(\mathcal{M}_n) \subseteq [\sigma_w(A) \cup \sigma_w(E) \cup \sigma_w(L)] + \mathcal{O}, \tag{5.4.1}$$

and

$$\sigma_f(\mathcal{M}_n) \subseteq [\sigma_f(A) \cup \sigma_f(E) \cup \sigma_f(L)] + \mathcal{O}. \tag{5.4.2}$$

(a)If $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$, $L_n \xrightarrow{\nu} L$, $B_n \to B$, $C_n \to C$, $D_n \to 0$, $F_n \to F$, $G_n \to 0$ *and ,* $H_n \to 0$, $A_nB \to AB$, $A_nC \to AC$, $E_nF \to EF$. *if* $0 \in \Phi_A^b \cap \Phi_E^b \cap \Phi_L^b$ *and* $\mathcal{O}$ *is an open set*

of $\mathbb{C}$ *with* $0 \in \mathcal{O}$*, then there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ $n \geq n_0$

$$\sigma_w(\mathcal{M}_n) \subseteq [\sigma_w(A) \cup \sigma_w(E) \cup \sigma_w(L)] + \mathcal{O}, \tag{5.4.3}$$

and

$$\sigma_f(\mathcal{M}_n) \subseteq [\sigma_f(A) \cup \sigma_f(E) \cup \sigma_f(L)] + \mathcal{O}. \tag{5.4.4}$$

Proof. (a) Let $\mathcal{U}_n = \begin{pmatrix} A_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & L_n \end{pmatrix}$, $\mathcal{V}_n = \begin{pmatrix} 0 & 0 & G_n \\ 0 & 0 & F_n \\ 0 & 0 & 0 \end{pmatrix}$, and $\mathcal{W}_n = \begin{pmatrix} 0 & 0 & 0 \\ D_n & 0 & 0 \\ G_n & H_n & 0 \end{pmatrix}$.

First we prove that $\mathcal{U}_n \xrightarrow{\nu} \mathcal{M}$. Indeed $\|\mathcal{U}_n\| = \max\Big\{\|A_n\|, \|E_n\|, \|L_n\|\Big\}$ then $(\|U_n\|)$ is bounded,

$$\|(\mathcal{U}_n - \mathcal{M})\mathcal{M}\| = \max\Big\{\|(A_n - A)A\|, \|(E_n - E)E\|, \|(L_n - L)L\|\Big\} \to 0,$$
$$\|(\mathcal{U}_n - \mathcal{M})\mathcal{U}_n\| = \max\Big\{\|(A_n - A)A_n\|, \|(E_n - E)E_n\|, \|(L_n - L)L_n\Big\} \to 0, \text{ and}$$
$$\|(\mathcal{U}_n - \mathcal{M})\mathcal{U}_n\| = \max\Big\{\|(A_n - A)A_n\|, \|(E_n - E)E_n\|, \|(L_n - L)L_n\Big\} \to 0.$$

Now, by representing $\mathcal{M}_n$ as $\mathcal{M}_n = \mathcal{U}_n + \mathcal{V}_n + \mathcal{W}_n$. It follows immediately from the fact that $\mathcal{U}_n \xrightarrow{\nu} \mathcal{M}$, $\mathcal{V}_n \to 0$, $\mathcal{W}_n \to 0$ and Lemma 5.2.1 that

$$\mathcal{U}_n + \mathcal{V}_n + \mathcal{W}_n \xrightarrow{\nu} 0 + 0 + \mathcal{M}.$$

Then $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}$. Now, if we suppose that $0 \in \Phi^b_A \cap \Phi^b_E \cap \Phi^b_L$ then $0 \in \Phi^b_{\mathcal{M}}$. According to **??**, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\sigma_i(\mathcal{M}_n) \subseteq [\sigma_i(A) \cup \sigma_i(E) \cup \sigma_i(L)] + \mathcal{O}$, for $i = f,\ w$.

(b) By using (a) we have

$$\begin{pmatrix} A_n & B_n - B & C_n - C \\ D_n & E_n & F_n \\ G_n & H_n & L_n \end{pmatrix} = \left[\begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n - F \\ G_n & H_n & L_n \end{pmatrix} + \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix}\right] \xrightarrow{\nu} \mathcal{M}.$$

Since

$$\left[\begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & L_n \end{pmatrix} + \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_nB & A_nC + (B_n - B)F \\ 0 & D_nB & D_nC + E_nF \\ 0 & G_nB & G_nC + G_nF \end{pmatrix}$$

and

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & L \end{pmatrix} \times \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & AB & AC \\ 0 & 0 & EF \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\left[\begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & L_n \end{pmatrix} + \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & L \end{pmatrix} \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$$

Now by using Lemma 5.2.1 , and Theorem 5.2.1 we have

$$\left[\begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & L_n \end{pmatrix} + \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\nu} \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & L \end{pmatrix} \begin{pmatrix} 0 & B & C \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}.$$

So, $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$. Now, if we suppose that $0 \in \Phi^b_A \cap \Phi^b_E \cap \Phi^b_L$ then $0 \in \Phi^b_{\mathcal{M}}$. Applying Theorem 5.2.1 , there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\sigma_i(\mathcal{M}_n) \subseteq [\sigma_i(A) \cup \sigma_i(E) \cup \sigma_i(L)] + \mathcal{O}$, for $i = f,\ w$. ■

Remark 5.4.1 *If $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$ and $A_nC \to AC$, $A_nB \to AB$, $B_nF \to BF$, $E_nF \to EF$ and $B_n \to B$, $C_n \to C$, $F_n \to F$, $D_n \to 0$, $G_n \to 0$, $H_n \to 0$, then $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$, and $L_n \xrightarrow{\nu} L$. In fact, let $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$,*

$$\begin{pmatrix} 0 & -B_n & -C_n \\ -D_n & 0 & -F_n \\ -G_n & -H_n & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\Big(\mathcal{M}_n - \mathcal{M}_C\Big) \begin{pmatrix} 0 & -B & -C \\ 0 & 0 & -F \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(A_n - A)B & -(A_n - A)C - (B_n - B)F \\ 0 & 0 & -(E_n - E)F \\ 0 & 0 & 0 \end{pmatrix} \to 0.$$

Corollary 5.4.1 *Let X be a Banach space. In the product space $X \otimes X \otimes X$, we consider a sequence of operators formally defined by a matrix $\mathcal{M}_n = \begin{pmatrix} A_n & B_n & C_n \\ 0 & E_n & F_n \\ 0 & 0 & L_n \end{pmatrix}$. Let A_n and*

A be two closed operators acting on the Banach space X and having the domain $\mathcal{D}(A_n)$ and $\mathcal{D}(A)$ respectively. E_n and E acting on the Banach space X and having the domain $\mathcal{D}(B_n)$, and $\mathcal{D}(E)$ respectively. L_n and L acting on the Banach space X and having the domain $\mathcal{D}(L_n)$, and $\mathcal{D}(L)$ respectively and let B_n, C_n, $F_n \in \mathcal{L}(X)$ and $\mathcal{M} = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & L \end{pmatrix}$ and let $\lambda \in \rho(A) \cap \rho(E) \cap \rho(L) \cap \rho(A_n) \cap \rho(E_n) \cap \rho(L_n)$. If $(\lambda - A_n)^{-1} \xrightarrow{\nu} (\lambda - A)^{-1}$, $(\lambda - E_n)^{-1} \xrightarrow{\nu} (\lambda - E)^{-1}$, $(\lambda - L_n)^{-1} \xrightarrow{\nu} (\lambda - L)^{-1}$, $B_n \to 0$, $C_n \to 0$,and $F_n \to 0$, then for each open set $\mathcal{O} \subseteq \mathbb{C}$ with $0 \in \mathcal{O}$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\sigma_w(\mathcal{M}_n) \subseteq [\sigma_w(A) \cup \sigma_w(E) \cup \sigma_w(L)] + \mathcal{O}, \tag{5.4.5}$$

and

$$\sigma_f(\mathcal{M}_n) \subseteq [\sigma_f(A) \cup \sigma_f(E) \cup \sigma_f(L)] + \mathcal{O}. \tag{5.4.6}$$

Proof. Obviously, we can verify that,

$$(\lambda - \mathcal{M}_n)^{-1} = \begin{pmatrix} (\lambda - A_n)^{-1} & B_n(\lambda - A_n)^{-1}(\lambda - E_n)^{-1} & (B_nF_n(\lambda - E_n)^{-1} + C_n)(\lambda - A_n)^{-1}(\lambda - L_n)^{-1} \\ 0 & (\lambda - E_n)^{-1} & F_n(\lambda - E_n)^{-1}(\lambda - L_n)^{-1} \\ 0 & 0 & (\lambda - L_n)^{-1} \end{pmatrix}$$

Since $B_n(\lambda - A_n)^{-1}(\lambda - E_n)^{-1} \to 0$, $(B_nF_n(\lambda - E_n)^{-1} + C_n)(\lambda - A_n)^{-1}(\lambda - L_n)^{-1} \to 0$, and $F_n(\lambda - E_n)^{-1}(\lambda - L_n)^{-1} \to 0$, then by using Theorem 5.4.1, we get
$(\lambda - \mathcal{M}_n)^{-1} \xrightarrow{\nu} (\lambda - \mathcal{M})^{-1}$. Now, using [30, *Theorem* 2.1] we deduce that

$$\sigma_i(\mathcal{M}_n) \subseteq [\sigma_i(A) \cup \sigma_i(E) \cup \sigma_i(L)] + \mathcal{O}, \text{ for } i = f,\ w.$$

■

Theorem 5.4.2 *Let X be a Banach space. In the product space $X \otimes X \otimes X$, we consider a sequence of operators formally defined by a matrix*

$$\mathcal{M}_n = \begin{pmatrix} A_n & B_n & C_n \\ 0 & E_n & F_n \\ 0 & 0 & L_n \end{pmatrix} \text{ and } \mathcal{M}_C = \begin{pmatrix} A & B & C \\ 0 & E & F \\ 0 & 0 & L \end{pmatrix}$$

such that A, $A_n \in \mathcal{L}(X)$, B, $B_n \in \mathcal{L}(X)$, C, $C_n \in \mathcal{L}(X)$, E, $E_n \in \mathcal{L}(X)$, F, $F_n \in \mathcal{L}(X)$, L, $L_n \in \mathcal{L}(X)$, $B_n \to B$, $C_n \to C$ and $F_n \to F$. If $0 \in \Phi^b_A \cap \Phi^b_E \cap \Phi^b_L$ and $\sigma_i(A) \cap \sigma_i(E) \cap \sigma_i(L) = \emptyset$, for $i = f,\ w$. Then σ_i is ν-continuous at A, E and L if, and only if, σ_i is ν-continuous at $\mathcal{M}_C$, for $i = f,\ w$.

Proof. Since $\sigma_i(A) \cap \sigma_i(E) \cap \sigma_i(L) = \emptyset$, for $i = f,\ w$, then from the ν-upper semi continuity of σ_i at A, E and L, for $i = f,\ w$ (see 5.2.3), and by using a similar reasoning of [30, *Theorem* 2.1], for every sequence $(A_n) \in \mathcal{L}(X)$, every sequence $(E_n) \in \mathcal{L}(X)$ and every sequence $(L_n) \in \mathcal{L}(X)$ such that $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$ and $L_n \xrightarrow{\nu} L$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\sigma_i(\mathcal{M}_n) = \sigma_i(A_n) \cup \sigma_i(E_n) \cup \sigma_i(L_n), \quad \text{for } i = f,\ w. \tag{5.4.7}$$

"$\Longrightarrow$" Assume that σ_i is ν-continuous at A, E and L and we show that σ_i is ν-continuous at $\mathcal{M}_C$ for $i = f,\ w$. Let $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$. First by using Theorem 5.2.3, σ_i is ν-upper semi continuous at $\mathcal{M}_C$, for $i = f,\ w$. It is sufficient to show that σ_i is ν-lower semi continuous at $\mathcal{M}_C$ for $i = f,\ w$. We suppose $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$, then by using [8, *Remarque* 3.4] we have $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$ and $L_n \xrightarrow{\nu} L$. Since σ_i is ν-lower semi continuous at A, E and L we have $\sigma_i(A) \subset \liminf \sigma_i(A_n)$, $\sigma_i(E) \subset \liminf \sigma_i(E_n)$ and $\sigma_i(L) \subset \liminf \sigma_i(L_n)$ for $i = f, w$. Then

$$\Big[\sigma_i(A) \cup \sigma_i(E) \cup \sigma_i(L)\Big] \subset \Big[\big(\liminf \sigma_i(A_n)\big) \cup \big(\liminf \sigma_i(E_n)\big) \cup \big(\liminf \sigma_i(L_n)\big)\Big], \text{ for } i = f,\ w.$$

Hence $(\sigma_i(A) \cup \sigma_i(E) \cup \sigma_i(L)) \subset \liminf(\sigma_i(A_n) \cup \sigma_i(E_n) \cup \sigma_i(L_n))$ for $i = f,\ w$. This implies that, $\sigma_i(\mathcal{M}_C) \subset \liminf(\sigma_i(\mathcal{M}_n))$ for $i = f,\ w$. Hence, σ_i is ν-lower semi continuous at $\mathcal{M}_C$ for $i = f,\ w$.

"$\Longleftarrow$" Assume that σ_i is ν-continuous at $\mathcal{M}_C$. We show that σ_i is ν-continuous at A, E and L for $i = f,\ w$. Let $A_n \xrightarrow{\nu} A$, $E_n \xrightarrow{\nu} E$ and $L_n \xrightarrow{\nu} L$. By using 5.2.3, σ_i is ν-upper semi continuous at A, E and L for $i = f,\ w$. then continuous at A, E and L for $i = f,\ w$. So,

$$\begin{aligned} &\limsup \sigma_i(A_n) \subset \sigma_i(A) \text{ for } i = f,\ w, \\ &\limsup \sigma_i(E_n) \subset \sigma_i(E) \text{ for } i = f,\ w, \text{ and} \\ &\limsup \sigma_i(L_n) \subset \sigma_i(L) \text{ for } i = f,\ w. \end{aligned} \tag{5.4.8}$$

Now, we prove that σ_i is ν-lower semi continuous at A for $i = f,\ w$. Let $\lambda \in \sigma_i(A)$ for $i = f,\ w$. Since $\sigma_i(A) \subset \sigma_i(\mathcal{M}_C)$, then$\lambda \in \sigma_i(\mathcal{M}_C)$ for $i = f,\ w$. By using Theorem 5.4.1 we have $\mathcal{M}_n \xrightarrow{\nu} \mathcal{M}_C$. It follows from ν-lower semi continuity of σ_i at $\mathcal{M}_C$

$$\lambda \in \liminf\left(\sigma_i(\mathcal{M}_n)\right) \text{ for } i = f,\ w.$$

Then there exists a sequence (λ_n) such that $\lambda_n \in \sigma_i(\mathcal{M}_n)$ and $\lambda_n \to \lambda$ for $i = f,\ w$. On the other hand, by Eq. (5.4.7), there exists $N \in \mathbb{N}$ such that $\lambda_n \in \sigma_i(\mathcal{M}_n) = \sigma_i(A_n) \cup \sigma_i(E_n) \cup \sigma_i(L_n)$, for all $n \geq N$ and $i = f,\ w$. We discuss two cases.

<u>1^{st}*case*</u>. If $\lambda_n \in \sigma_i(A_n)$, for all $n \geq N$ and $i = f,\ w$ thus $\lambda \in \liminf \sigma_i(A_n)$ and hence σ_i is ν-continuous at A for $i = f,\ w$.

<u>2^{nd}*case*</u>. If there exists a subsequence λ_{n_j} of λ_n such that $\lambda_{n_j} \in (\sigma_i(E_{n_j}) \cap \sigma_i(L_{n_j}))$, then we have $\lambda \in \limsup \sigma_i(E_n) \cap \sigma_i(L_n)$ for $i = f,\ w$. Using Eq. (5.4.8) we obtain $\lambda \in \sigma_i(E) \cap \sigma_i(L)$ for $i = f,\ w$. This shows that $\lambda \in \sigma_i(E) \cap \sigma_i(L) \cap \sigma_i(A)$ for $i = f,\ w$. This is a contradiction.

■

Conclusion and perspectives

The main thrust of this thesis is in the spirit of the operator theory and spectral theory, its aim is to give a study of some parts of the essential spectrum for a sequence of linear operators (The essential approximate point spectrum, the essential defect spectrum, the Weyl essetial spectrum and the Wolf essential spectrum) using different types of convergences (Generalized convergence sense, the convergence compactly, and the $v-$convergece). The results obtained are used to study the spectrum continuity.

By the preceding chapters, a question do, however, remain open to further investigation. We point out that the operations of addition and composition in $\mathcal{L}(X)$ are not compatible with $v-$convergence. What conditions do we need to put in place to get compatibility?. Do we Can introduce a new type of convergence that preserves the characteristics of the $v-$convergence and achieves compatibility?

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