University of M'sila
Faculty of Mathematics and Informatics

(F, G)-derivations on lattices

Mourad Yettou

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Jury president: Prof. dr. Douadi Mihoubi
Faculty of of Mathematics and Informatics
Department of Mathematics
University of M'sila-Algeria

Supervisor: Prof. dr. Abdelaziz Amroune
Faculty of Mathematics and Informatics
Department of Mathematics
University of M'sila-Algeria

Co-Supervisor: Prof. dr. Lemnaouar Zedam
Faculty of Mathematics and Informatics
Department of Mathematics
University of M'sila-Algeria

Examination committee: Prof. dr. Abdelmadjid Boudaoud
Faculty of Mathematics and Informatics
Department of Mathematics
University of M'sila-Algeria

Dr. Yacine Halim
Department of Mathematics and Informatics
University Center of Mila-Algeria

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Introduction

The notion of derivation introduced on function spaces as a linear function d with an additional property $d(f \cdot g) = d(f) \cdot g + f \cdot d(g)$. Based on this property, several authors generalize this notion of derivation to other mathematical structures. First, Posner in [28] introduced the notion of derivation on a prime ring $(R, *, +)$ as a function d from R into itself ($d : R \to R$) satisfying the following two conditions:

$$d(x * y) = (d(x) * y) + (x * d(y)) \text{ and } d(x + y) = d(x) + d(y), \text{ for any } x, y \in R\,.$$

This notion of derivation on ring structures has plied an important role in various fields of mathematics and it has many applications (see, e.g. [3]).

Later on, Szász [32] extended the notion of derivation to the lattice structures based on the meet ($\wedge$) and the join ($\vee$) operations. A $(\wedge, \vee)$-derivation (derivation, for short) on a given lattice $(L, \leqslant, \wedge, \vee)$ with respect to the meet ($\wedge$) and the join ($\vee$) operations is a function $d : L \longrightarrow L$ satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y), \text{ for any } x, y \in L\,.$$

After that, Ferrari [10] has investigated some properties of this notion of derivation on lattices and provided some interesting examples in particular classes of lattices. Xin et al. [35] have extended the notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on distributive lattices. In the same paper, they have characterized distributive and modular lattices in terms of their isotone derivations. Later, Xin [36] has focused his attention on the structure of set of fixed points of a derivation on a lattice and has shown some relationships between this set of fixed points and the lattice ideals.

In the same direction, Çeven and Öztürk [6] have generalized the notion of derivation to f-derivation on a lattice by using an arbitrary function f of that lattice. For a given function f from a lattice L into itself, an

f-derivation on a lattice L is a function $d : L \longrightarrow L$ satisfying:

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L\,.$$

In this context, they also characterized the distributive and modular lattices based on the notion of isotone f-derivations.

These notions of derivations and f-derivations on a lattice are witnessing increased attention. They have studied, among others, in partially ordered sets [1, 40], in distributive lattices [39], in semilattices [38], in bounded hyperlattices [33], in quantales and residuated lattices [14, 34] and in several kinds of algebras [18, 21, 26]. Furthermore, they were used to define congruences and ideals of a distributive lattice [29].

Inspired by the above extensions and developments of the notion of derivations and their applications, we aim in this thesis to introduce and study further generalizations. The main objective is to generalize the notion of $(\wedge, \vee)$-derivation to (F, G)-derivation on a lattice, and investigate its properties in detail[1]. This generalization is based on two arbitrary binary operations F and G instead of the lattice meet $(\wedge)$ and join $(\vee)$ operations. To that end, a lot of preparatory work is required, in particular several properties and characterizations of binary operations on an arbitrary lattice are investigated[2]. We complete this thesis by studying the most important properties of isotone and principal f-derivations on a lattice. In this contex, we pay particular attention to the lattice structure of isotone f-derivations on a lattice, and to the ideal structures of the sets of their f-fixed points[3].

This thesis is organized as follows:

- In Chapter 1, we present generalities on binary operations, lattices and derivations on lattices that will be needed in this thesis.
- In Chapter 2, we study and characterize some important properties of binary operations on an arbitrary lattice $(L, \leqslant, \wedge, \vee)$ (not necessarily bounded). More specifically, we show necessary and sufficient conditions under which a given binary operation on a lattice coincides with

[1] This part is accepted for publication in the Kragujevac Journal of Mathematics [2].
[2] This part is the subject of an international publication in the journal of Kybernetika [37].
[3] This part is the subject of an international publication in the Discussiones Mathematicae-General Algebra and Applications journal [39].

its meet ($\wedge$) (resp. its join ($\vee$)) operation. Furthermore, we construct two new posets of functions based on a binary operation on a given lattice, and investigate their lattice structures under some conditions. Also, we provide representations of any lattice in terms of these newly constructed lattices.

- In Chapter 3, we introduce the notion of (F, G)-derivation on a lattice as a generalization to $(\wedge, \vee)$-derivations, where F and G are arbitrary binary operations on that lattice instead of the meet ($\wedge$) and join ($\vee$) operations. Also, we investigate their properties in detail and we give the notion of principal (F, G)-derivations on a lattice as a particular class of (F, G)-derivations, and we study their various properties. Specific attention is paid to the lattice structure of the poset of principal (F, G)-derivations on a lattice. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. These representations are draw upon some properties of binary operations on a lattice that we have investigated in [37].

- In Chapter 4, we investigate the most important properties of isotone and principal f-derivations on a lattice. We focus on the lattice structure of isotone f-derivations on a lattice, and to the ideal structure of the sets of their f-fixed points. More specifically, we show some cases that the set of principal f-derivations on a lattice has a lattice structure. Moreover, we investigate the structure of the set of f-fixed points of an isotone f-derivation on a lattice, and we show some cases that this set is an ideal (resp. a principal ideal). As applications, we provide a representation of a lattice and a characterization of a distributive lattice in terms of their principal f-derivations. Also, we show a characterization theorem of principal ideals of a lattice in terms of principal f-derivations on that lattice.

- Finally, we give a conclusion including the important results of this thesis and some future works.

The most parts of this theses have already been published or accepted for publication in peer-reviewed international journals. The results included in Chapter 2, Chapter 3 and Chapter 4 have been described in [37], [2] and in [39] respectively.

1 Generalities on lattices and derivations on lattices

In this chapter, we recall the necessary basic concepts and properties of binary operations, partially ordered sets, lattices and derivations on lattices that will be needed throughout this thesis. Further information on posets and lattices can be found in [8, 19, 22, 27, 30, 31].

1.1. Binary operations on a non-empty set

A *binary operation* on a non-empty set X is any function from the Cartesian product $X \times X$ into X, i.e., $F : X \times X \to X$. A binary operation F on X is called:

(i) *commutative*, if $F(x, y) = F(y, x)$, for any $x, y \in X$;

(ii) *associative*, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$;

An element $e \in X$ is called

(i) a *left-neutral* (resp. *right-neutral*) element of F, if $F(e, x) = x$ (resp. $F(x, e) = x$), for any $x \in X$;

(ii) a *neutral* element of F, if it is left- and right-neutral element, i.e., $F(e, x) = F(x, e) = x$, for any $x \in X$;

(iii) a *left-* (resp. *right-*) *absorbing* element of F, if $F(e, x) = e$ (resp. $F(x, e) = e$), for any $x \in X$;

(iv) an *absorbing* element of F, if it is left- and right-absorbing element, i.e., $F(e, x) = F(x, e) = e$, for any $x \in X$.

Example 1.1. *On the set of real numbers* $\mathbb{R}$*, the addition* $(+)$ *and the multiplication* $(\times)$ *are commutative and associative binary operations. Where,* 0 *is the neural element of* $(+)$ *and* 1 *is the neural element of* $(\times)$*. Also,* 0 *is the absorbing element of* $(\times)$*.*

1.2. Lattices

This section is devoted to recalling some definitions, examples and fundamental properties of lattices.

1.2.1. Partially ordered sets

Definition 1.1. *A binary relation on a set X is a subset of the Cartesian product $X \times X$.*

Definition 1.2. *An order relation (a partial order) $\leqslant$ on X is a binary relation on X that is reflexive (i.e., $x \leqslant x$, for any $x \in X$), antisymmetric (i.e., $x \leqslant y$ and $y \leqslant x$ imply $x = y$, for any $x, y \in X$) and transitive (i.e., $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$, for any $x, y, z \in X$).*

Definition 1.3. *A set P equipped with an order relation $\leqslant$ on P is called a partially ordered set (poset, for short) and denoted by $(P, \leqslant)$.*

Definition 1.4. *Let $(P, \leqslant)$ be a poset and let A be a subset of P. An element $x_0 \in P$ is called a lower bound of A if $x_0 \leqslant x$, for any $x \in A$. x_0 is called the greatest lower bound (or the infimum) of A if x_0 is a lower bound of A and $m \leqslant x_0$, for any lower bound m of A. Upper bound and least upper bound (or supremum) are defined dually.*

Example 1.2. *Let $\mathbb{N}^*$ be the set of positive integers and $|$ be the divisibility relation. One can easily verify that the divisibility relation $|$ is reflexive, antisymmetric and transitive on $\mathbb{N}^*$. Thus, $|$ is an order relation (a partial order) on $\mathbb{N}^*$, i.e., the structure $(\mathbb{N}^*, |)$ is a poset.*

Definition 1.5. *A poset $(P, \leqslant)$ is called bounded, if it has a least and a greatest element, respectively denoted by 0 and 1, i.e., $0 \leqslant x \leqslant 1$, for any $x \in L$.*

Usually, the notation $(P, \leqslant, 0, 1)$ is used to describe a bounded poset.
Example 1.3. *Let $D(30)$ be the set of positive divisors of 30 and $|$ be the divisibility order. The poset $(D(30), |)$ has 1 as the least element and 30 as the greatest element. Indeed, 1 divides all the elements of $D(30)$, and any element of $D(30)$ divides 30. Thus, the structure $(D(30), |, 1, 30)$ is a bounded poset.*

Definition 1.6. *Let $(P, \leqslant)$ be a poset. An element $y \in P$ covers an element $x \in P$ if $x < y$ (i.e., $x \leqslant y$ and $x \neq y$) and there is no element $z \in P$ such* *that $x < z < y$. The set of pairs (x, y) such that y covers x is called the covering relation of $(P, \leqslant)$.*

Definition 1.7. *The Hasse diagram of a finite poset $(P, \leqslant)$ is a picture of the digraph whose vertexes are the elements of P and which has line segments between some their vertexes. Where, if an element $y \in P$ covers an element $x \in P$, we get the vertex y is higher up than the vertex x and they are connected with a line segment.*

Example 1.4. *The Figure 1.1 presents the Hasse diagram of the bounded poset $(D(30), |, 1, 30)$ given in Example 1.3.*

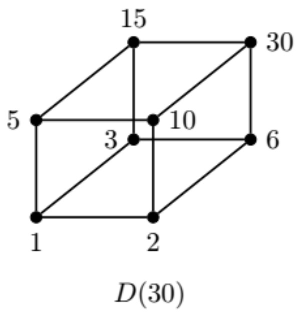

Figure 1.1: The Hasse diagram of the pounded poset $(D(30), |, 1, 30)$.

Theorem 1.1. *Any finite poset has a Hasse diagram.*

Let $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ be two posets. A mapping φ from P_1 into P_2 is called an *order isomorphism* if it is surjective and satisfies

$$x \leqslant_1 y \text{ if and only if } \varphi(x) \leqslant_2 \varphi(y), \text{ for any } x, y \in P_1.$$

Example 1.5. *Let $(D(30), |, 1, 30)$ be the poset of the positive divisors of* 30 *ordered by the divisibility order and let $f : D(30) \to D(30)$ be a mapping given by the following table:*

x	1	2	3	5	6	10	15	30
fx	1	2	5	3	10	6	15	30

One can easily verify that f is an order isomorphism.

1.2.2. Lattices: definitions and examples

Definition 1.8. *A poset* $(P, \leqslant)$ *is called a meet-semilattice if any two elements* x *and* y *of* P *have a greatest lower bound, denoted by* $x \wedge y$ *and called the meet (infimum) of* x *and* y*.*

Example 1.6. *Let* $(P_1 = \{\{1,2\},\{1,2,3\},\{1,2,4\}\}, \subseteq)$ *and* $(P_2 = \{\{1\},\{2\},\{1,2\},\{1,2,3\},\{1,2,4\}\}, \subseteq)$ *be two posets ordered by the inclusion relation and given by the Hasse diagrams in Figure 1.6. One can easily verify that* $(P_1, \subseteq)$ *is a meet-semilattice but* $(P_2, \subseteq)$ *is not.*

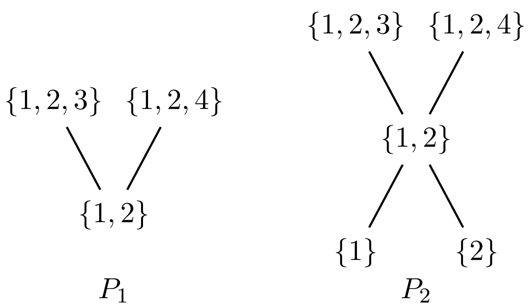

Figure 1.2: The Hasse diagrams of posets $(P_1, \subseteq)$ and $(P_2, \subseteq)$

Definition 1.9. *A poset* $(P, \leqslant)$ *is called a join-semilattice if any two elements* x *and* y *of* P *have a least upper bound, denoted by* $x \vee y$ *and called the join (supremum) of* x *and* y*.*

Example 1.7. *Let* $(P_1 = \{[1,2],[2,3],[0,4]\}, \subseteq)$ *and* $(P_2 = \{[1,2],[2,3],[0,4],[2,5]\}, \subseteq)$ *be two posets ordered by the inclusion relation and given by the Hasse diagrams in Figure 1.7. It is not difficult to check that* $(P_1, \subseteq)$ *is a join-semilattice but* $(P_2, \subseteq)$ *is not.*

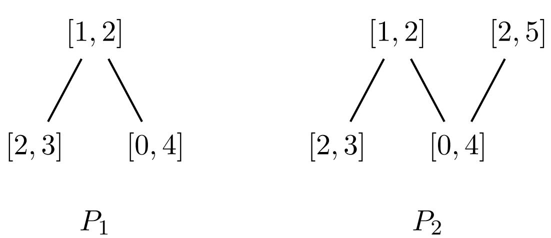

Figure 1.3: The Hasse diagrams of posets $(P_1, \subseteq)$ and $(P_2, \subseteq)$

Definition 1.10. *A poset $(L, \leqslant)$ is called a lattice if it is both a meet- and a join-semilattice.*

We recall that a lattice can also be defined as an algebraic structure. A set L equipped with two binary operations $(\wedge)$ and $(\vee)$ that are commutative, associative and idempotent (i.e., $x \wedge x = x$ and $x \vee x = x$, for any $x \in L$), and they satisfy the absorption laws (i.e., $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for any $x, y \in L$). In this case, the order relation $\leqslant$ associated to L is related with the meet and the join operations as follows: $x \wedge y = x$ if and only if $x \leqslant y$ if and only if $x \vee y = y$, for any $x, y \in L$.

Usually, the structure $(L, \leqslant, \wedge, \vee)$ is used to describe a lattice.

Example 1.8. *(i) The set of real numbers $\mathbb{R}$ ordered by the usual order $\leqslant$ is a lattice, where min and max are its meet and joint operations;*

(ii) The set of positive integers $\mathbb{N}^*$ equipped with the divisibility order $|$ has a structure of a lattice, where gcd (i.e., the greatest common divisor) and lcm (i.e., the least common multiple) are its meet and joint operations respectively;

(iii) The bounded poset given in Example 1.4 is a lattice;

(iv) The set of integers $\mathbb{Z}$ equipped with the divisibility order $|$ has not a structure of a lattice. Indeed, the divisibility relation $|$ is not a partial order on $\mathbb{Z}$.

Definition 1.11. *A lattice $(L, \leqslant, \wedge, \vee)$ is called bounded if it has a least and a greatest element denoted by 0 and 1 respectively.*

Usually, the structure $(L, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice.

Definition 1.12. *A lattice $(L, \leqslant, \wedge, \vee)$ is called distributive, if one of the following two equivalent conditions holds:*

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, *for any* $x, y, z \in L$;

(a^δ) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *for any* $x, y, z \in L$.

Example 1.9. *(i) The lattice $(D(30), |, gcd, lcm, 1, 30)$ given by the Hasse diagram in Figure 1.1 is bounded and distributive;*

(ii) The lattice of real numbers $(\mathbb{R}, \leqslant, min, max)$ is distributive, but is not bounded.

Definition 1.13. *A poset $(L, \leqslant)$ is called a complete lattice if every subset A of L has both a greatest lower bound denoted by $\bigwedge A$ and called the infimum of A, and a least upper bound denoted by $\bigvee A$ and called the supremum of A.*

Example 1.10. *(i) The poset of real numbers $(\mathbb{R}, \leqslant)$ is a complete lattice;*

(ii) Let E be a non-empty set and $\mathcal{P}(L)$ be its power set. The poset $(\mathcal{P}(L), \subseteq)$ is a complete lattice.

Theorem 1.2. *[8] Any finite lattice is complete.*

Remark 1.1. *[30] Note that any complete lattice is bounded.*

1.2.3. Ideals and homomorphisms of lattices

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let I be a proper non-empty subset of L ($I \subsetneq L$ and $I \neq \emptyset$). The set I is called:

(i) a *meet-subsemilattice* (resp. a *join-subsemilattice*) of L if $x \wedge y \in I$ (resp. $x \vee y \in I$), for any $x, y \in I$;

(ii) a *sublattice* of L if it is both a *meet*- and a *join*-subsemilattice of L;

(iii) a *down-set* of L if $x \in I$ and $y \in L$ such that $y \leqslant x$ implies $y \in I$;

(iv) an *ideal* of L if it is both a down-set and a *join*-subsemilattice of L;

(v) a *prime ideal* of L if it is an ideal of L and $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for any $x, y \in L$.

Example 1.11. *Let $(D(30), |, gcd, lcm)$ be the lattice given by the Hasse diagram in Figure 1.1 and let I_1, I_2 be two subsets of $D(30)$ such that $I_1 = \{1, 2, 3, 6\}$ and $I_2 = \{1, 3\}$. One can easily verify that I_1 is a prime ideal of $D(30)$ and I_2 is an ideal of L, but it is not prime. Indeed, $gcd(6, 15) = 3 \in I_2$, but $6 \notin I_2$ and $15 \notin I_2$.*

For each $x \in L$, the set $\downarrow x = \{y \in L \mid y \leqslant x\}$ is an ideal of L called the *principal ideal* generated by x. It is the smallest ideal of L contains x.

Let L_1 and L_2 be two lattices. A mapping $\varphi : L_1 \to L_2$ is called:

(i) a $\wedge$-*homomorphism* (resp. $\vee$-*homomorphism*) if it satisfies $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ (resp. $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$), for any $x, y \in L$;

(ii) a *lattice homomorphism* if it is both a $\wedge$- and a $\vee$-homomorphism;

(iii) a $\wedge$-*monomorphism* (resp. $\vee$-*monomorphism*) if it is an injective $\wedge$-homomorphism (resp. $\vee$-homomorphism);

(iv) a $\wedge$-*epimorphism* (resp. $\vee$-*epimorphism*) if it is a surjective $\wedge$-homomorphism (resp. $\vee$-homomorphism);

(v) a *lattice isomorphism* if it is a bijective lattice homomorphism.

If $L = M$, a lattice isomorphism $\varphi : L \to L$ is called a *lattice automorphism*.

The following proposition shows that an order isomorphism between to lattices is equivalent to a lattice isomorphism.

Proposition 1.1. *[8] Let L, M be two lattices and let φ be a mapping from L into M. The following statements are equivalent:*

(i) φ is an order isomorphism;

(ii) φ is a lattice isomorphism.

1.3. Algebraic derivations on lattices

This section contains basic definitions and properties of algebraic derivations and f-derivations on lattices.

1.3.1. Definitions and examples

The notion of algebraic derivation on a lattice was first introduced by Szász in [32].

Definition 1.14. *[32] Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on L if it satisfies the following two conditions:*

(D_1) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$*, for any* $x, y \in L$*;*

(D_2) $d(x \vee y) = d(x) \vee d(y)$*, for any* $x, y \in L$*.*

Later on, Xin et al. [35] have reduced the above conditions by considering only the condition (D_1).

Definition 1.15. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \to L$ is called a derivation on L if it satisfies the following condition:*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \text{ for any } x, y \in L\,.$$

In the rest of this thesis, we shortly write dx instead of $d(x)$.

Example 1.12. *(i) Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be the identity function of L (i.e., $d(x) = x$, for any $x \in L$). d is a derivation on L;*

(ii) Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with the least element $0 \in L$. The null function of L (i.e., $dx = 0$, for any $x \in L$) is a derivation on L;

(iii) Let $(L = \{0, a, b, 1\}, \leqslant, \wedge, \vee)$ be a lattice given by the Hasse diagram in Figure 1.4. Let d_1 and d_2 be two functions on L defined in the following table:

x	0	a	b	1
$d_1(x)$	0	a	a	0
$d_2(x)$	0	b	a	0

One can easily verify that d_1 is a derivation on L, but d_2 is not a derivation on L. Indeed, $d_2(a \wedge a) = b \neq (d_2(a) \wedge a) \vee (a \wedge d_2(a)) = a$.

Figure 1.4: The Hasse diagram of the lattice $(L = \{0, a, b, 1\}, \leqslant, \wedge, \vee)$.

Definition 1.16. *[32] A $\wedge$-translation on a lattice $(L, \leqslant, \wedge, \vee)$ is a function $f : L \to L$ satisfies $f(x \wedge y) = x \wedge f(y)$, for any $x, y \in L$.*

Example 1.13. *Any $\wedge$-translation on a lattice L is a derivation on L.*

Definition 1.17. *[10] The radical of a positive integer n is denoted $r(n)$. It is the product of the primes of its factorization, i.e., $r(n) = P_1 \cdot P_2 \cdot \ldots \cdot P_t$, where $n = P_1^{\alpha_1} \cdot P_2^{\alpha_2} \cdot \ldots \cdot P_t^{\alpha_t}$.*

Example 1.14. *[10] Let $(\mathbb{N}^*, |, gcd, lcm)$ be the lattice of positive integers ordered by the divisibility order. Let f be the radical function of $\mathbb{N}^*$ (i.e., $f : \mathbb{N}^* \to \mathbb{N}^*$ such that $f(n) = r(n)$, for any $n \in \mathbb{N}^*$), then f is a derivation on $\mathbb{N}^*$.*

Definition 1.18. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be a derivation on L. d is called isotone if it satisfies the following condition:*

$$x \leqslant y \text{ implies } dx \leqslant dy, \text{ for any } x, y \in L\,.$$

1.3.2. Derivations on lattices: fundamental properties

The following proposition shows fundamental properties of derivations on a lattice.

Proposition 1.2. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be a derivation on L. Then the following holds:*

(i) $dx \leqslant x$, *for any* $x \in L$;

(ii) $d(dx) = dx$, *for any* $x \in L$;

(iii) If L *has a least element* $0 \in L$, *then* $d0 = 0$;

(iv) d *is isotone if and only if* $d(x \wedge y) = dx \wedge dy$, *for any* $x, y \in L$;

(v) If L *is distributive and* d *is isotone, then* $d(x \vee y) = dx \vee dy$, *for any* $x, y \in L$.

Proposition 1.3. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $\alpha \in L$ and let $d_\alpha : L \to L$ be a function such that $d_\alpha(x) = \alpha \wedge x$, for any $x \in L$. Then d_α is an isotone derivation on L.*

Definition 1.19. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $\alpha \in L$. The derivation d_α given in Proposition 1.3 is called a principal derivation on L.*

Example 1.15. *(i) Any $\wedge$-translation on a given lattice $(L, \leqslant, \wedge, \vee)$ is an isotone derivation on L;*

(ii) Let $(\mathbb{R} \leqslant, min, max)$ be the lattice of real numbers ordered by the usual order. Let $d_7 : \mathbb{R} \to \mathbb{R}$ be a principal derivation on $\mathbb{R}$ defined as:

$$d_7(x) = min(7, x), \text{ for any } x \in \mathbb{R}.$$

Proposition 1.3 guarantees that d_7 is an isotone derivation on $\mathbb{R}$.

Proposition 1.4. *[35] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be a derivation on L. If $Fix_d(L) = \{x \in L \mid dx = x\}$ the set of fixed points of d is non-empty, then it is a down-set. Moreover, if d is isotone, then $Fix_d(L)$ is an ideal of L.*

Proposition 1.5. *[35] Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with a greatest element 1 and d be a derivation on L. Then $d1 = 1$ if and only if d is the identity function on L.*

1.3.3. f-derivations on lattices

In the same direction, Çeven and Öztürk have generalized the notion of derivation to f-derivation on a lattice.

Definition 1.20. *[6] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let $f : L \to L$ be a function. A function $d : L \to L$ is called an f-derivation on L if it satisfies the following condition:*

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

Throughout this thesis, we shortly write fx instead of $f(x)$.

Example 1.16. *(i) Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let f be the identity function on L (i.e., $f(x) = x$, for any $x \in L$). Any derivation d on L is an f-derivation on L;*

(ii) Any $\wedge$-homomorphism f on a given lattice $(L, \leqslant, \wedge, \vee)$ is an f-derivation on L;

(iii) Let $(L = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$ be a lattice given by the Hasse diagram in Figure 1.5. Let f and d be two functions on L defined in the following table:

x	0	a	b	c	1
fx	0	a	a	1	1
dx	0	a	a	c	c

It is not difficult to check that d is an f-derivation on L.

Figure 1.5: The Hasse diagram of the lattice $(L = \{0, a, b, c, 1\}, \leqslant, \wedge, \vee)$.

Definition 1.21. *[6] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and let* d *be an* f*-derivation on* L*.* d *is called isotone if it satisfies the following condition:*

$$x \leqslant y \text{ implies } dx \leqslant dy, \text{ for any } x, y \in L.$$

Example 1.17. *The* f*-derivation* d *given in Example 1.16 (iii) is an isotone* f*-derivation on* L*.*

The following proposition gives some properties of f-derivations on a lattice.

Proposition 1.6. *[6] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and let* d *be an* f*-derivation on* L*. Then the following holds.*

(i) $dx \leqslant fx$*, for any* $x \in L$*;*

(ii) If $(L, \leqslant, \wedge, \vee)$ *is distributive,* f *is a* $\vee$*-homomorphism and* d *is isotone, then* $d(x \vee y) = dx \vee dy$*.*

Proposition 1.7. *[6] Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* f *be a* $\wedge$*-homomorphism on* L*. Let* $\alpha \in L$ *and* $d_{(\alpha,f)} : L \to L$ *be a function defined as*

$$d_{(\alpha,f)}(x) = \alpha \wedge fx, \text{ for any } x \in L.$$

Then $d_{(\alpha,f)}$ *is an isotone* f*-derivation on* L*.*

The following sets are the key notions of Chapter 4.

Notation 1.1. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and let* $f : L \to L$ *be a function. We denote by:*

(i) $\mathfrak{I}_f(L)$ *the set of isotone* f*-derivations on* L*;*

(ii) $\mathcal{P}_f(L) := \{d_{(\alpha,f)} \mid \alpha \in L\}$*.*

The following theorem gives a lattice structure to the set of isotone f-derivations on a distributive lattice.

Theorem 1.3. *[38] Let* $(L, \leqslant, \wedge, \vee)$ *be a distributive lattice and let* d_1, d_2 *be two isotone* f*-derivations on* L*. Define* $(d_1 \sqcap d_2)(x) = d_1x \wedge d_2x$ *and* $(d_1 \sqcup d_2)(x) = d_1x \vee d_2x$*, for any* $x \in L$*. Then the structure* $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ *is a distributive lattice, where the order relation* $\preceq$ *is defined as:*

$$d_1 \preceq d_2 \text{ if and only if } d_1 \sqcup d_2 = d_2, \text{ for any } d_1, d_2 \in \mathfrak{I}_f(L).$$

For more detail on derivations and f-derivations on lattices, we refer to [6, 35, 36, 38].

2 A binary operation-based representation of a lattice

In this chapter, we study and characterize some properties of a given binary operation on a lattice. These properties are the basic results to generalize the notion of derivation to (F, G)-derivation on lattices. More specifically, we show necessary and sufficient conditions under which a binary operation on a lattice coincides with its meet (resp. join) operation. Importantly, we construct two new posets based on a given binary operation on a lattice and investigate some cases that these posets have a lattice structure. We finish this chapter by providing some representations of a given lattice based on these newly constructed lattices.

2.1. Binary operations on a lattice

Binary operations are among the oldest fundamental concept in algebraic structures. Since their introduction, they have become the key notion in the concepts of semigroups, monoids, groups, rings, and in more algebraic structures studied in abstract algebra [9, 19, 23]. Binary operations have become essential tools in lattice theory and its applications [11, 12]. Several notions and properties, and the notion of the lattice itself can be interpreted in terms of binary operations [8, 30]. Further, it is not surprising that binary operations with specific properties appear in various theoretical and application fields. For instance, aggregation functions on bounded lattices and their wide use in various fields of applied sciences, including, computer and information sciences, economics, and social sciences (see, e.g. [13, 15, 16, 17, 20, 24, 25]). Also, they play an important role (as a generalization of the basic connectives between fuzzy sets) in theories of fuzzy sets and logic [4].

2.1.1. Properties of binary operations on a lattice

In this subsection, we present the properties of conjunctive, disjunctive, idempotent and increasing binary operations on a lattice. These properties are adopted from those of aggregation functions on bounded lattices (see, e.g. [13, 17, 20, 25]).

Definition 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. F is called:*

(i) idempotent, if $F(x, x) = x$, for any $x \in L$;

(ii) conjunctive (resp. disjunctive), if $F(x, y) \leqslant x \wedge y$ (resp. $x \vee y \leqslant F(x, y)$), for any $x, y \in L$;

(iii) left-increasing (resp. right-increasing), if $x \leqslant y$, implies $F(x, z) \leqslant F(y, z)$ (resp. $F(z, x) \leqslant F(z, y)$), for any $x, y, z \in L$;

(iv) increasing, if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ imply $F(x_1, y_1) \leqslant F(x_2, y_2)$, for any $x_1, y_1, x_2, y_2 \in L$.

We note that F is increasing if and only if it is left- and right-increasing. Also, if F is commutative, then F is left-increasing if and only if F is right-increasing.

Example 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. The meet $(\wedge)$ and the join $(\vee)$ operations of L are idempotent and increasing. Moreover, $\wedge$ is conjunctive and $\vee$ is disjunctive.*

Inspired by the notion of transpose of a binary aggregation on the real interval $[0, 1]$ (see, e.g. [5]), the following definition extends this notion to any binary operation on a lattice.

Definition 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. The transpose of F is the binary operation F^t defined as*

$$F^t(x, y) = F(y, x), \text{ for any } x, y \in L.$$

The following proposition expresses relationships between the properties of a binary operation and those of its transpose. The proof is straightforward.

Proposition 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. The following equivalences hold:*

(i) F is commutative if and only if $F^t = F$;

(ii) F is conjunctive if and only if F^t is conjunctive;

(iii) F is disjunctive if and only if F^t is disjunctive;

(iv) F is left-increasing if and only if F^t is right-increasing;

(v) *F is increasing if and only if F^t is increasing;*

(vi) *an element $e \in L$ is a left-neutral element of F if and only if e is a right-neutral element of F^t;*

(vii) *an element $e \in L$ is the neutral element of F if and only if e is the neutral element of F^t.*

2.1.2. Characterizations of some properties of binary operations on a lattice

In this subsection, we study and characterize some properties of a binary operation on a lattice. More precisely, we give necessary and sufficient conditions under which a binary operation on a lattice coincides with its meet (resp. join) operation. These characterizations give us alternative conditions to those known for aggregation functions on bounded lattices (see, e.g. [5, 7, 25]).

Proposition 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then it holds that*

(i) *if F is increasing, idempotent and conjunctive, then $F(x, y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha) = F(x \wedge \alpha, y) = F(x, y \wedge \alpha)$ for any $x, y, \alpha \in L$;*

(ii) *if F is increasing, idempotent and disjunctive, then $F(x, y) \vee \alpha = F(x \vee \alpha, y \vee \alpha) = F(x \vee \alpha, y) = F(x, y \vee \alpha)$ for any $x, y, \alpha \in L$.*

Proof. Let $x, y, \alpha \in L$.

(i) We only prove that $F(x, y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha)$, as the other equalities can be proved similarly. Since F is increasing, it follows that $F(x \wedge \alpha, y \wedge \alpha) \leqslant F(x, y)$. The fact that F is conjunctive implies that $F(x \wedge \alpha, y \wedge \alpha) \leqslant \alpha$. Hence, $F(x \wedge \alpha, y \wedge \alpha)$ is a lower bound of $\{F(x, y), \alpha\}$. On the other hand, let $m \in L$ a lower bound of $\{F(x, y), \alpha\}$, i.e., $m \leqslant F(x, y)$ and $m \leqslant \alpha$. Since F is conjunctive, we have $m \leqslant x$ and $m \leqslant y$. Hence, $m \leqslant x \wedge \alpha$ and $m \leqslant y \wedge \alpha$. Since F is idempotent and increasing, we obtain that $m = F(m, m) \leqslant F(x \wedge \alpha, y \wedge \alpha)$. Thus, we conclude that $F(x \wedge \alpha, y \wedge \alpha)$ is the greatest lower bound of $\{F(x, y), \alpha\}$, i.e., $F(x, y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha)$.

(ii) The proof is analogous to that of (i).

□

The following lemma generalizes a remarkable result of Cooman and Kerre (Proposition 3.7 in [7]) to any lattice.

Lemma 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. If an idempotent binary operation F on L is increasing and conjunctive (resp. disjunctive), then F is the meet $(\wedge)$ (resp. join $(\vee)$) operation of L.*

Proof. Suppose that F is idempotent, increasing and conjunctive binary operation on L. Using Proposition 2.2 we obtain that

$$F(x,y) \wedge (x \wedge y) = F(x \wedge (x \wedge y), y \wedge (x \wedge y)) = F(x \wedge y, x \wedge y) = x \wedge y\,,$$

for any $x, y \in L$. Hence, $(x \wedge y) \leqslant F(x,y)$, for any $x, y \in L$. Now, the fact that F is conjunctive guarantees that $F(x,y) = x \wedge y$, for any $x, y \in L$. Thus, $F = \wedge$, i.e., F is the meet operation of L. In a similar way, we prove that F is the join operation $(\vee)$ of L. □

The following proposition shows an equivalent condition to both properties of increasing and conjunctive binary operations on a lattice.

Proposition 2.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is idempotent, then the following conditions are equivalent:*

(i) *F is increasing and conjunctive;*

(ii) *$F(x,y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha) = F(x \wedge \alpha, y) = F(x, y \wedge \alpha)$, for any $x, y, \alpha \in L$.*

Proof. $(i) \Rightarrow (ii)$: The proof is directly from Proposition 2.2.
$(ii) \Rightarrow (i)$: Firstly, we aim to prove that F is conjunctive. Let $x, y \in L$, from (ii) it follows that $F(x,y) \wedge x = F(x \wedge x, y) = F(x,y)$ and $F(x,y) \wedge y = F(x, y \wedge y) = F(x,y)$. Hence, $F(x,y) \leqslant x$ and $F(x,y) \leqslant y$. Thus, $F(x,y) \leqslant x \wedge y$. Consequently, F is conjunctive. Secondly, we aim to show that F is increasing. Let $x, y, t, z \in L$ such that $x \leqslant y$. Since F is idempotent, it follows that $F(x,z) \wedge F(y,z) = F(F(x,z) \wedge y, F(x,z) \wedge z) = F(F(x \wedge y, z), F(x, z \wedge z)) = F(F(x,z), F(x,z)) = F(x,z)$. Hence, $F(x,z) \leqslant F(y,z)$. Thus, F is left-increasing. Moreover, $F(t,x) \wedge F(t,y) = F(F(t,x) \wedge t, F(t,x) \wedge y) = F(F(t \wedge t, x), F(t, x \wedge y)) = F(F(t,x), F(t,x)) = F(t,x)$. This implies that $F(t,x) \leqslant F(t,y)$. Therefore, F is right-increasing. Now, since F is left- and right-increasing, we conclude that F is increasing. □

In the same line, we obtain the following proposition.

Proposition 2.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is idempotent, then the following conditions are equivalent:*

(i) *F is increasing and disjunctive;*

(ii) *$F(x,y) \vee \alpha = F(x \vee \alpha, y \vee \alpha) = F(x \vee \alpha, y) = F(x, y \vee \alpha)$, for any $x, y, \alpha \in L$.*

The following theorem shows necessary and sufficient conditions under which a binary operation on a lattice coincides with its meet (resp. join) operation. We note that this result establishes a simple criterion to check; if a binary operation on a given lattice coincides with its meet (resp. join) operation.

Theorem 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be an idempotent binary operation on L. Then it holds that*

(i) *F is the meet operation $\wedge$ if and only if $F(x,y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha) = F(x \wedge \alpha, y) = F(x, y \wedge \alpha)$, for any $x, y, \alpha \in L$;*

(ii) *F is the join operation $\vee$ if and only if $F(x,y) \vee \alpha = F(x \vee \alpha, y \vee \alpha) = F(x \vee \alpha, y) = F(x, y \vee \alpha)$, for any $x, y, \alpha \in L$.*

Proof. The proof can be easily obtained from Lemma 2.1, Propositions 2.3 and 2.4. □

Remark 2.1. *In view of Lemma 2.1, if F is idempotent, conjunctive (resp. disjunctive) and increasing, then $F = \wedge$ (resp. $F = \vee$). But, if F is only left- or right-increasing, then F is not necessarily the meet (resp. join) operation of L. Indeed, let $(L = \{0, 1, 2\}, \leqslant)$ be the lattice given by the Hasse diagram in Figure 2.1. Let F and G be two binary operations on L defined by the following tables:*

$F(x,y)$	0	1	2
0	0	0	0
1	0	1	0
2	0	1	2

$G(x,y)$	0	1	2
0	0	1	2
1	2	1	2
2	2	2	2

One can easily verify that F is left-increasing, conjunctive and idempotent, but it is not the meet operation ($\wedge$) of L. Analogously, the transpose F^t of F is right-increasing, conjunctive and idempotent, but also it is not the *meet operation* ($\wedge$)*. Moreover,* G *(resp.* G^t*) is left-increasing (resp. right-increasing), disjunctive and idempotent, but it is not the join operation* ($\vee$) *of* L*.*

Figure 2.1: The Hasse diagram of the lattice $(L = \{0, 1, 2\}, \leqslant)$.

2.2. Lattices constructed by a binary operation

In this section, we construct two posets based on a binary operation on a lattice. also, we study some cases that these posets have lattice structures. These lattices will turn out to be useful technical tools in the next section.

2.2.1. Posets constructed by a binary operation

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. For any $\alpha \in L$, we define two specific F-based functions f_α and f^α from L into L as:

$$f_\alpha(x) = F(\alpha, x) \text{ and } f^\alpha(x) = F(x, \alpha), \text{ for any } x \in L\,.$$

We denote by $\mathcal{A}_F^\ell(L)$ the set of f_α functions and by $\mathcal{A}_F^\ell(L)$ the set of f^α functions, i.e.,

$$\mathcal{A}_F^\ell(L) = \{f_\alpha \mid \alpha \in L\} \text{ and } \mathcal{A}_F^r(L) = \{f^\alpha \mid \alpha \in L\}.$$

The following proposition is immediate.

Proposition 2.5. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L*. If* F *is commutative, then* $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$*.*

Remark 2.2. *The converse of the above proposition does not hold in general. Indeed, consider the lattice* $(L = \{0, 1, 2\}, \leqslant)$ *given by the Hasse diagram in Figure 2.1. Let* F *be a binary operation on* L *defined by the following table:*

$F(x,y)$	0	1	2
0	1	1	2
1	2	1	1
2	1	2	1

From the above table, we see that $f_0 = f^1$, $f_1 = f^2$ *and* $f_2 = f^0$. *Indeed,*

x	0	1	2
$f_0(x)$	$F(0,0)=1$	$F(0,1)=1$	$F(0,2)=2$
$f^1(x)$	$F(0,1)=1$	$F(1,1)=1$	$F(2,1)=2$
$f_1(x)$	$F(1,0)=2$	$F(1,1)=1$	$F(1,2)=1$
$f^2(x)$	$F(0,2)=2$	$F(1,2)=1$	$F(2,2)=1$
$f_2(x)$	$F(2,0)=1$	$F(2,1)=2$	$F(2,2)=1$
$f^0(x)$	$F(0,0)=1$	$F(1,0)=2$	$F(2,0)=1$

Then $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$. *But, since* $F(0,1) = 1 \neq F(1,0) = 2$, *it holds that* F *is not commutative.*

Proposition 2.6. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice,* F *be a binary operation on* L *and* $\alpha, \beta \in L$. *If* F *has a neutral element* $e \in L$ *and* $f_\alpha = f^\beta$, *then* $\alpha = \beta$.

Proof. Suppose that F has a neutral element $e \in L$. Let $\alpha, \beta \in L$ such that $f_\alpha = f^\beta$. Then $F(\alpha, x) = F(x, \beta)$, for any $x \in L$. Setting $x = e$, we obtain that $\alpha = \beta$. □

In view of Remark 2.2, the following proposition states a sufficient condition to make the converse implication in Proposition 2.5 holds.

Proposition 2.7. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L. *If* F *has a neutral element* $e \in L$ *and* $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$, *then* F *is commutative.*

Proof. Suppose that F has a neutral element $e \in L$ and $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$. We aim to show that F is commutative, i.e., $F(\alpha, \beta) = F(\beta, \alpha)$, for any $\alpha, \beta \in L$. So let $\alpha, \beta \in L$, since $f_\alpha \in \mathcal{A}_F^\ell(L)$ and $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$, then there exists $\gamma \in L$ such that $f_\alpha = f^\gamma$. Applying Proposition 2.6 gives that $\alpha = \gamma$. Hence, $f_\alpha = f^\alpha$. Thus, $F(\alpha, \beta) = f_\alpha(\beta) = f^\alpha(\beta) = F(\beta, \alpha)$. Consequently, F is commutative. □

The sets $\mathcal{A}_F^\ell(L)$ and $\mathcal{A}_F^r(L)$ equipped with the usual order of functions have poset structures. We denote by $\leqslant^\ell$ to the usual order of functions defined on $\mathcal{A}_F^\ell(L)$ as:

$$f_\alpha \leqslant^\ell f_\beta \text{ if and only if } f_\alpha(x) \leqslant f_\beta(x), \text{ for any } x \in L\,.$$

In the same way, we denote by $\leqslant^r$ to the usual order of functions defined on $\mathcal{A}_F^r(L)$ as:

$$f^\alpha \leqslant^r f^\beta \text{ if and only if } f^\alpha(x) \leqslant f^\beta(x), \text{ for any } x \in L\,.$$

Remark 2.3. *In general, the posets $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r)$ are not necessarily lattices. Indeed, let $(D(6), |, gcd, lcm)$ be the lattice of positive divisors of 6 ordered by the divisibility relation and given by the Hasse diagram in Figure 2.2. Let F be a binary operation on $D(6)$ defined by the following table:*

$F(x,y)$	1	2	3	6
1	3	2	3	2
2	2	1	6	3
3	6	2	6	6
6	2	3	6	6

The Hasse diagrams given in Figure 2.2 show that the posets $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell)$ and $(\mathcal{A}_F^r(D(6)), \leqslant^r)$ are not lattices.

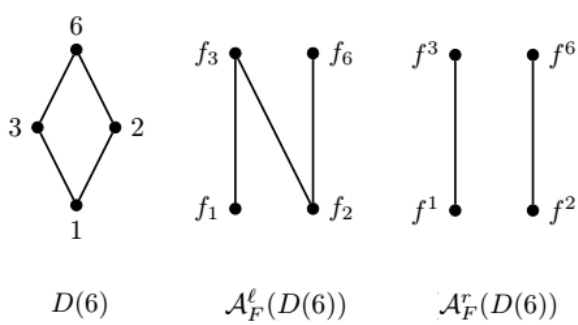

Figure 2.2: The Hasse diagrams of $(D(6), |, gcd, lcm)$, $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell)$ and $(\mathcal{A}_F^r(D(6)), \leqslant^r)$.

2.2.2. First lattice structures

In this subsection, we show some cases that the posets $(\mathcal{A}^{\ell}_F(L), \leqslant^{\ell})$ and $(\mathcal{A}^{r}_F(L), \leqslant^{r})$ have a lattice structure. First, we prove the following key results.

Proposition 2.8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then it holds that*

(i) if F is left-increasing and having a right-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f_\alpha \leqslant^{\ell} f_\beta\,;$$

(ii) if F is right-increasing and having a left-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f^\alpha \leqslant^{r} f^\beta\,.$$

Proof.

(i) The fact that F is left-increasing guarantees the first implication. Conversely, let $f_\alpha, f_\beta \in \mathcal{A}^{\ell}_F(L)$ such that $f_\alpha \leqslant^{\ell} f_\beta$. This implies that $F(\alpha, x) \leqslant F(\beta, x)$, for any $x \in L$. Using the fact that e is a right-neutral element of F and setting $x = e$, it follows that $F(\alpha, e) \leqslant F(\beta, e)$. Thus, $\alpha \leqslant \beta$.

(ii) Follows from Proposition 2.1 and (i).

□

Proposition 2.8 leads to the following corollary.

Corollary 2.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is increasing and having a neutral element $e \in L$, then for any $\alpha, \beta \in L$, the following statements are equivalent:*

(i) $\alpha \leqslant \beta$;

(ii) $f_\alpha \leqslant^{\ell} f_\beta$;

(iii) $f^\alpha \leqslant^{r} f^\beta$.

In the following result, we provide sufficient conditions under which the posets $(\mathcal{A}^{\ell}_F(L), \leqslant^{\ell})$ and $(\mathcal{A}^{r}_F(L), \leqslant^{r})$ have lattice structures.

Theorem 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. Then it holds that*

(i) if F is left-increasing and having a right-neutral element $e \in L$, then $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ is a lattice;

(ii) if F is right-increasing and having a left-neutral element $e \in L$, then $(\mathcal{A}_F^r(L), \leqslant^r)$ is a lattice.

Proof.

(i) Let $f_\alpha, f_\beta \in \mathcal{A}_F^\ell(L)$. Using Proposition 2.8 (i), we obtain that $f_{\alpha\wedge\beta}$ and $f_{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of f_α and f_β. Thus, $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ is a lattice.

(ii) Let $f^\alpha, f^\beta \in \mathcal{A}_F^r(L)$. From Proposition 2.1 and (i), we get that $f^{\alpha\wedge\beta}$ and $f^{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of f^α and f^β. Thus, $(\mathcal{A}_F^r(L), \leqslant^r)$ is also a lattice.

$\square$

Next, we denote by $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ the lattice structure associated to the poset $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$, where $f_\alpha \wedge^\ell f_\beta = f_{\alpha\wedge\beta}$ and $f_\alpha \vee^\ell f_\beta = f_{\alpha\vee\beta}$, for any $f_\alpha, f_\beta \in \mathcal{A}_F^\ell(L)$. Also, $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ denotes the lattice structure associated to the poset $(\mathcal{A}_F^r(L), \leqslant^r)$, where $f^\alpha \wedge^r f^\beta = f^{\alpha\wedge\beta}$ and $f^\alpha \vee^r f^\beta = f^{\alpha\vee\beta}$, for any $f^\alpha, f^\beta \in \mathcal{A}_F^r(L)$.

In the following, we give a counter-example to show that the converse implications of Theorem 2.2 do not necessarily hold.
Example 2.2. *Let $(L = \{0, 1, 2\}, \leqslant, min, max)$ be the lattice given by the Hasse diagram in Figure 2.3 and F be a binary operation on L defined by the following table:*

$F(x,y)$	0	1	2
0	1	0	2
1	0	0	0
2	2	0	2

Figure 2.3 and the above table guarantee that:

(i) $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice, but F is not left-increasing and has not a right-neutral element $e \in L$;

(ii) $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$ *is a lattice, but* F *is not right-increasing and has not a left-neutral element* $e \in L$.

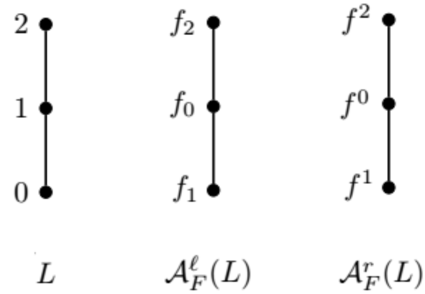

Figure 2.3: The Hasse diagrams of the lattices $(L = \{0, 1, 2\}, \leqslant, min, max)$, $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$.

Remark 2.4. *We mention that, if* $(L, \leqslant, \wedge, \vee, 0, 1)$ *is a bounded lattice, then the lattices* $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell, f_0, f_1)$ *and* $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r, f^0, f^1)$ *are also bounded.*

2.2.3. Second lattice structures

In this subsection, we show other sufficient conditions on the binary operation F under which the posets $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r)$ have lattice structures. First, we start by proving the following key results.

Proposition 2.9. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice. Let* F *be an idempotent and conjunctive or disjunctive binary operation on* L. *Then it holds that*

(i) if F *is left-increasing, then for any* $\alpha, \beta \in L$, *we obtain that*

$$\alpha \leqslant \beta \text{ if and only if } f_\alpha \leqslant^\ell f_\beta\,;$$

(ii) if F *is right-increasing, then for any* $\alpha, \beta \in L$, *we obtain that*

$$\alpha \leqslant \beta \text{ if and only if } f^\alpha \leqslant^r f^\beta\,.$$

Proof.

(i) The proof of the direct implication follows from the fact that F is left-increasing. For the converse implication, let $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$ such that $f_\alpha \leqslant^\ell f_\beta$. Then $F(\alpha, x) \leqslant F(\beta, x)$, for any $x \in L$. Setting

$x = \alpha$ (resp. $x = \beta$). Since F is idempotent and conjunctive (resp. disjunctive), we obtain that $\alpha = F(\alpha, \alpha) \leqslant F(\beta, \alpha) \leqslant \beta$ (resp. $\alpha \leqslant F(\alpha, \beta) \leqslant F(\beta, \beta) = \beta$). Thus, $\alpha \leqslant \beta$.

(ii) Follows from Proposition 2.1 and (i).

□

Similarly to the Theorem 2.2, the following result shows other sufficient conditions under which the posets $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r)$ have lattice structures.

Theorem 2.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be an idempotent and conjunctive or disjunctive binary operation on L. Then it holds that*

(i) if F is left-increasing, then $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice;

(ii) if F is right-increasing, then $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ is a lattice.

Proof.

(i) Let $f_\alpha, f_\beta \in \mathcal{A}_F^\ell(L)$. From Proposition 2.9, one can easily verify that $f_{\alpha\wedge\beta}$ and $f_{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of f_α and f_β. Thus, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice.

(ii) Let $f^\alpha, f^\beta \in \mathcal{A}_F^r(L)$. From Proposition 2.1 and (i), it holds that $f^{\alpha\wedge\beta}$ and $f^{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of f^α and f^β. Thus, $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ is a lattice.

□

Note that the converse implications of Theorem 2.3 do not necessarily hold, as can be seen the following example.

Example 2.3. *Let $(L = \{0, 1, 2, 3\}, \leqslant, min, max)$ be the lattice given by the Hasse diagram in Figure 2.4. Let F and G be two idempotent binary operations on L such that F is conjunctive and G is disjunctive, which are given by the following tables:*

$F(x,y)$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	1	2	0
3	0	1	2	3

$G(x,y)$	0	1	2	3
0	0	1	2	3
1	2	1	3	3
2	2	3	2	3
3	3	3	3	3

Figure 2.4 and the above tables guarantee that $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ *and* $(\mathcal{A}_G^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ *are lattices, but* F *and* G *are not left- or right-increasing. Indeed,* $1 \leqslant 2$, *but*

(i) $F(1,3) = 1 \nleqslant F(2,3) = 0$ *and* $G(1,2) = 3 \nleqslant G(2,2) = 2$;

(ii) $F(1,1) = 1 \nleqslant F(1,2) = 0$ *and* $G(2,1) = 3 \nleqslant G(2,2) = 2$.

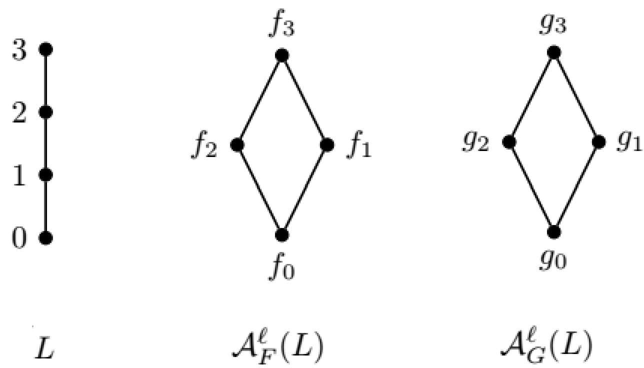

Figure 2.4: The Hasse diagrams of $(L = \{0,1,2,3\}, \leqslant, min, max)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_G^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$.

2.3. Representations of a lattice based on a binary operation

In this section, we provide two representations of a given lattice in terms of an arbitrary binary operation.

2.3.1. A first representation theorem

Based on Theorem 2.2, we provide a first representation of a lattice in terms of a given binary operation.

Theorem 2.4. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L. *Then it holds that*

(i) if F *is left-increasing and having a right-neutral element* $e \in L$, *then the lattices* $(L, \leqslant, \wedge, \vee)$ *and* $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ *are isomorphic;*

(ii) if F *is right-increasing and having a left-neutral element* $e \in L$, *then the lattices* $(L, \leqslant, \wedge, \vee)$ *and* $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ *are isomorphic.*

Proof.

(i) Theorem 2.2 guarantees that $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice. Next, let φ be a mapping from L into $\mathcal{A}_F^\ell(L)$ such that $\varphi(\alpha) = f_\alpha$, for any $\alpha \in L$. Obvious that φ is surjective. Moreover, since F is left-increasing and having a right-neutral element $e \in L$, it follows from Proposition 2.8 that

$$\alpha \leqslant \beta \text{ if and only if } \varphi(\alpha) \leqslant^\ell \varphi(\beta), \text{ for any } \alpha, \beta \in L\,.$$

Hence, φ is an order isomorphism between these two lattices L and $\mathcal{A}_F^\ell(L)$. Using Proposition 1.1 we conclude that φ is a lattice isomorphism. Therefore, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ are isomorphic.

(ii) Follows from Proposition 2.1 and (i).

□

Remark 2.5. *Note that Example 2.2 shows that the converse implications of Theorem 2.4 do not necessarily hold.*

In view of Theorem 2.4, we obtain the following result.

Theorem 2.5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If F is increasing having a neutral element $e \in L$, then the lattices $(L, \leqslant, \wedge, \vee)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic.*

Example 2.4. *Let $(L = \{0, 1, 2\}, \leqslant, min, max)$ be the lattice given by the Hasse diagram in Figure 2.5 and F be a binary operation on L defined by the following table:*

$F(x,y)$	0	1	2
0	0	0	1
1	0	1	2
2	0	2	2

The above table guarantees that F is increasing and 1 its neutral element. Then from Theorem 2.5, we conclude that the lattices $(L = \{0, 1, 2\}, \leqslant, min, max)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic. Figure 2.5 shows this isomorphism.

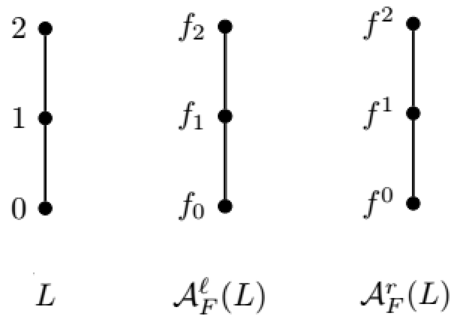

Figure 2.5: The Hasse diagrams of the lattices $(L = \{0, 1, 2\}, \leqslant, min, max)$, $(\mathcal{A}^{\ell}_F(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ and $(\mathcal{A}^{r}_F(L), \leqslant^{r}, \wedge^{r}, \vee^{r})$.

2.3.2. A second representation theorem

Based on Theorem 2.3, we provide a second representation of a lattice in terms of a given binary operation.

Theorem 2.6. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be an idempotent and conjunctive or disjunctive binary operation on L. Then it holds that*

(i) if F is left-increasing, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}^{\ell}_F(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ are isomorphic.

(ii) if F is right-increasing, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}^{r}_F(L), \leqslant^{r}, \wedge^{r}, \vee^{r})$ are isomorphic.

Proof.

(i) Firstly, Theorem 2.3 guarantees that $(\mathcal{A}^{\ell}_F(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ is a lattice. Secondly, let φ be a mapping from L into $\mathcal{A}^{\ell}_F(L)$ such that $\varphi(\alpha) = f_\alpha$, for any $\alpha \in L$. Then the surjectivity of φ is immediate. Moreover, from Proposition 2.9, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } \varphi(\alpha) \leqslant^{\ell} \varphi(\beta), \text{ for any } \alpha, \beta \in L\,.$$

Hence, φ is an order isomorphism between these two lattices. Applying Proposition 1.1 guarantees that φ is a lattice isomorphism. Therefore, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}^{r}_F(L), \leqslant^{r}, \wedge^{r}, \vee^{r})$ are isomorphic.

(ii) Follows from Proposition 2.1 and (i).

□

Theorem 2.6 leads to the following result.

Theorem 2.7. *Let* $(L, \leqslant, \wedge, \vee)$ *be a lattice and* F *be a binary operation on* L*. If* F *is the meet operation* $(\wedge)$ *or the join operation* $(\vee)$ *of* L*, then the lattices* $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ *and* $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ *are identical and isomorphic with* $(L, \leqslant, \wedge, \vee)$*.*

In the following, we illustrate two examples justify the conditions of left- and right-increasing in Theorem 2.6.

Example 2.5. *Let* $(D(6), |, gcd, lcm)$ *be the lattice given by the Hasse diagram in Figure 2.6 and* F *be a binary operation on* $D(6)$ *defined by the following table:*

$F(x,y)$	1	2	3	6
1	1	1	1	1
2	1	2	1	1
3	1	1	3	1
6	1	2	3	6

One can easily verify that F *is idempotent, conjunctive and left-increasing. Then from Theorem 2.6, we obtain that the lattice* $(D(6), |, gcd, lcm)$ *and* $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ *are isomorphic. But in this case, the poset* $(\mathcal{A}_F^r(D(6)), \leqslant^r$ *) has not the structure of a lattice. Then it is not isomorphic with* $(D(6), |, gcd, lcm)$*.*

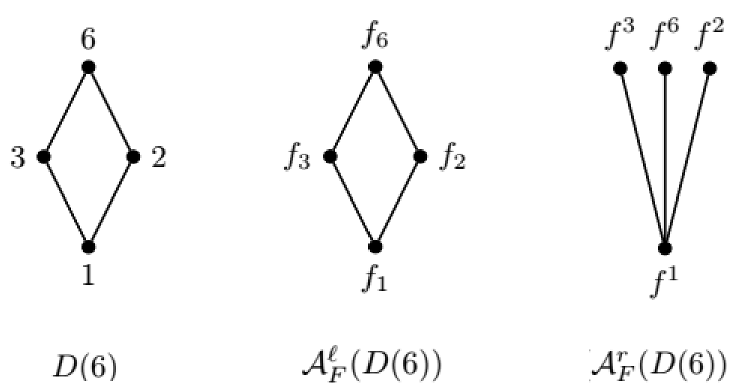

Figure 2.6: The Hasse diagrams of the lattices $(D(6), |, gcd, lcm)$, $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell, \wedge^\ell, \vee^\ell)$, and the poset $(\mathcal{A}_F^r(D(6)), \leqslant^r)$.

Example 2.6. *Let $(D(12), |, gcd, lcm)$ be the lattice given by the Hasse diagram in Figure 2.7 and F be a binary operation on $D(12)$ defined by the following table:*

$F(x,y)$	1	2	3	4	6	12
1	1	4	6	4	12	12
2	2	2	12	4	12	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

It is not difficult to check that F is idempotent, disjunctive and right-increasing. Then Theorem 2.6 guarantees that the lattices $(D(12), |, gcd, lcm)$ and $(\mathcal{A}_F^r(D(12)), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic. But, the poset $(\mathcal{A}_F^\ell(D(12)), \leqslant^\ell)$ is not a lattice. Then it is not isomorphic with $(D(12), |, gcd, lcm)$.

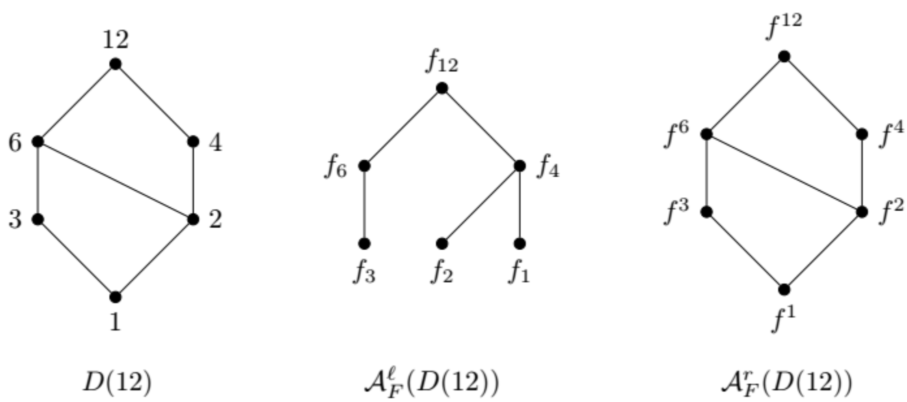

Figure 2.7: The Hasse diagrams of the lattices $(D(12), |, gcd, lcm)$, $(\mathcal{A}_F^r(D(12)), \leqslant^r, \wedge^r, \vee^r)$, and the poset $(\mathcal{A}_F^\ell(D(12)), \leqslant^\ell)$.

2.3.3. Links between the two representation theorems of a lattice

In this subsection, we show the links between the different conditions of the above two representation theorems of a given lattice based on a binary operation. First, we need the following sufficient condition for the uniqueness of a left- (resp. right-) neutral element of a binary operation on a lattice.

Proposition 2.10. ***(Uniqueness)*** *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a conjunctive or a disjunctive binary operation on L. If F has a left- (resp. right-) neutral element $e \in L$, then e is unique.*

Proof. Let $e_1, e_2 \in L$ be two left- (resp. right-) neutral elements of F. Here, we discuss the following two possible cases:

(i) if F is conjunctive, then it holds that $e_2 = F(e_1, e_2) \leqslant e_1$ and $e_1 = F(e_2, e_1) \leqslant e_2$ (resp. $e_2 = F(e_2, e_1) \leqslant e_1$ and $e_1 = F(e_1, e_2) \leqslant e_2$). Hence, $e_1 = e_2$.

(ii) if F is disjunctive, the proof is analogous to that of (i).

□

Proposition 2.11. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L.*

(i) If F is conjunctive and has a left- or a right-neutral element $e \in L$, then $e = 1$;

(ii) If F is disjunctive and has a left- or a right-neutral element $e \in L$, then $e = 0$.

Proof.

(i) Suppose that F has a left- (resp. a right-) neutral element $e \in L$. Since F is conjunctive, we get $1 = F(e, 1) \leqslant e$ (resp. $1 = F(1, e) \leqslant e$). Hence, $e = 1$.

(ii) The proof is a similar to that of (i).

□

In view of Proposition 2.11, we obtain the following corollary.

Corollary 2.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and F be a binary operation on L. If $(L, \leqslant, \wedge, \vee)$ has not a greatest (resp. least) element and F is conjunctive (resp. disjunctive), then F has not a left- and a right-neutral element.*

The following result characterizes the conjunction and the disjunction properties of a binary operation on a bounded lattice in terms of its neutral element.

Proposition 2.12. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and F be a binary operation on L. If F is increasing and having a neutral element $e \in L$, then the following statements hold:*

(i) F is conjunctive if and only if $e = 1$;

(ii) F is disjunctive if and only if $e = 0$.

Proof.

(i) Proposition 2.11 guarantees that if F is conjunctive, then $e = 1$. Conversely, suppose that $e = 1$. Let $x, y \in L$, then $x \leqslant e$ and $y \leqslant e$. Since F is increasing, it follows that $F(x, y) \leqslant F(x, e) = x$ and $F(x, y) \leqslant F(e, y) = y$. Hence, $F(x, y) \leqslant x \wedge y$. Thus, F is conjunctive.

(ii) The proof is a similar to that of (i).

□

The following propositions list some conditions for the existence and the uniqueness of a left- (resp. right-) neutral element of a binary operation on a lattice.

Proposition 2.13. ***(Existence and uniqueness)*** *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with a greatest element $1 \in L$ and F be a binary operation on L. If F is idempotent, conjunctive and left-increasing (resp. right-increasing), then 1 is the unique left-neutral (resp. the unique right-neutral) element of F.*

Proof. Since F is idempotent, conjunctive and left-increasing (resp. right-increasing), we have $x = F(x, x) \leqslant F(1, x) \leqslant x$ (resp. $x = F(x, x) \leqslant F(x, 1) \leqslant x$), for any $x \in L$. This implies that $F(1, x) = x$ (resp. $F(x, 1) = x$), for any $x \in L$. Hence, 1 is a left-neutral (resp. right-neutral) element of F. Moreover, since F is conjunctive, it holds from Proposition 2.10 that 1 is the unique left-neutral (resp. the unique right-neutral) element of F. □

Dually, we get the following proposition.

Proposition 2.14. ***(Existence and uniqueness)*** *Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with a least element $0 \in L$ and F be a binary operation on L. If F is idempotent, disjunctive and left-increasing (resp. right-increasing), then 0 is the unique left-neutral (resp. the unique right-neutral) element of F.*

Remark 2.6. *From Propositions 2.13 and 2.14, and Corollary 2.2, we conclude the following links between the different conditions of the above two representations of a given lattice based on a binary operation.*

(i) If $(L, \leqslant, \wedge, \vee, 1)$ *is a lattice with a greatest element* 1 *and* F *is the meet operation of* L*, then* 1 *is the neutral element of* F*. In this case,* $(L, \leqslant, \wedge, \vee)$ *can be represented by using both Theorems 2.5 and 2.7.*

(ii) If $(L, \leqslant, \wedge, \vee)$ *has not a greatest element and* F *is the meet operation of* L*, then* F *has not a neutral element. In this case,* $(L, \leqslant, \wedge, \vee)$ *can be represented only by using Theorem 2.7.*

(iii) If $(L, \leqslant, \wedge, \vee, 0)$ *is a lattice with a least element* 0 *and* F *is the join operation of* L*, then* 0 *is the neutral element of* F*. In this case,* $(L, \leqslant, \wedge, \vee)$ *can be represented by using both Theorems 2.5 and 2.7.*

(iv) If $(L, \leqslant, \wedge, \vee)$ *has not a least element and* F *is the join operation of* L*, then* F *has not a neutral element. In this case,* $(L, \leqslant, \wedge, \vee)$ *can be represented only by using Theorem 2.7.*

Now, after these studies and investigations of some several properties of binary operations on lattices, we have the principal tools to introduce the notion of (F, G)-derivation on a lattice and to investigate their various properties in detail.

3 (F, G)-derivations on lattices

In this chapter, we introduce the notion of (F, G)-derivation on a lattice as a generalization of the notion of derivation ($(\wedge, \vee)$-derivation). This newly notion is based on two arbitrary binary operations F and G instead of the meet ($\wedge$) and the join ($\vee$) operations. Also, we investigate several properties of (F, G)-derivations on a lattice in detail. In the same context, we define and study the notion of principal (F, G)-derivation as a particular class of (F, G)-derivations. As applications, we provide two representations of a given lattice in terms of its principal (F, G)-derivations.

3.1. The notion of (F, G)-derivation on lattices

In this section, we introduce the notion of (F, G)-derivation on a lattice and investigate their properties. This newly notion is a natural generalization of the notion of derivation on a lattice given by Xin et al. [35] with respect to the meet and the join operations.

3.1.1. Definitions and examples

Inspired by Definition 1.15, we introduce the core definition of this chapter. It based on two arbitrary binary operations on a lattice.

Definition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L. A function $d : L \to L$ is called an (F, G)-derivation on L if it satisfies the following condition:*

$$d(F(x, y)) = G(F(d(x), y), F(x, d(y))), \text{ for any } x, y \in L\,.$$

In the rest of this theses, we shortly write $dF(x, y)$ instead of $d(F(x, y))$.

In the following, we present some illustrative examples of (F, G)-derivations on lattices.

Example 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L such that G is idempotent. The identity function of L is an (F, G)-derivation on L. Indeed, suppose that d is the identity function of L. The fact that G is idempotent implies that $dF(x, y) = F(x, y) =$*

$G(F(x, y), F(x, y)) = G(F(dx, y), F(x, dy))$, for any $x, y \in L$. Hence, d is an (F, G)-derivation on L.

Example 3.2. *Let $(\mathbb{N}^*, \leqslant, min, max)$ be the lattice of positive integers and let $\alpha \in \mathbb{N}^*$. Then it holds that:*

(i) the null function of $\mathbb{N}$ is a $(\cdot, +)$-derivation on $\mathbb{N}$, but it is not a $(+, \cdot)$-derivation on $\mathbb{N}$;

(ii) the translation function d_1 of $\mathbb{N}^*$ defined by $d_1(x) = x + \alpha$, for any $x \in \mathbb{N}^*$, it is both $(+, gcd)$-derivation and $(+, lcm)$-derivation on $\mathbb{N}^*$;

(iii) the homothety function d_2 of $\mathbb{N}^*$ defined by $d_2(x) = \alpha \cdot x$, for any $x \in \mathbb{N}^*$, it is both $(\cdot, gcd)$-derivation and $(\cdot, lcm)$-derivation on $\mathbb{N}^*$.

Remark 3.1. *We note that a derivation on a lattice L with respect to the Definition 1.15 of Xin et al. [35], it is exactly a $(\wedge, \vee)$-derivation on L.*

Definition 3.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice. An (F, G)-derivation d on L is called isotone if it satisfies the following condition:*

$$x \leqslant y \text{ implies } dx \leqslant dy, \text{ for any } x, y \in L\,.$$

Example 3.3. *(i) The translation function given in Example 3.2 is both isotone $(+, gcd)$-derivation and isotone $(+, lcm)$-derivation;*

(ii) The homothety function given in Example 3.2 is both isotone $(\cdot, gcd)$-derivation and isotone $(\cdot, lcm)$-derivation on $\mathbb{N}^*$.

3.1.2. Properties of (F, G)-derivations on a lattice

In this subsection, we investigate some properties of (F, G)-derivations on a lattice.

Proposition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. The following implications hold:*

(i) if G is conjunctive, then $dF(x, y) \leqslant F(dx, y) \wedge F(x, dy)$, for any $x, y \in L$;

(ii) if G is disjunctive, then $F(dx, y) \vee F(x, dy) \leqslant dF(x, y)$, for any $x, y \in L$.

Proof.

(i) The fact that d is an (F, G)-derivation on L and G is conjunctive imply that $dF(x, y) = G(F(dx, y), F(x, dy)) \leqslant F(dx, y) \wedge F(x, dy)$, for any $x, y \in L$. Thus, $dF(x, y) \leqslant F(dx, y) \wedge F(x, dy)$, for any $x, y \in L$.

(ii) The proof is a similar to that of (i)

□

The above proposition leads to the following corollary.

Corollary 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. The following implications hold:*

(i) if F and G are conjunctive, then $dF(x, y) \leqslant dx \wedge dy \wedge x \wedge y$, for any $x, y \in L$;

(ii) if F and G are disjunctive, then $dx \vee dy \vee x \vee y \leqslant dF(x, y)$, for any $x, y \in L$.

Proposition 3.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. The following implications hold:*

(i) if F is conjunctive and G is increasing, then $dF(x, y) \leqslant G(dx, dy) \wedge G(y, x) \wedge G(dx, x) \wedge G(y, dy)$, for any $x, y \in L$;

(ii) if F is disjunctive and G is increasing, then $G(dx, dy) \vee G(y, x) \vee G(dx, x) \vee G(y, dy) \leqslant dF(x, y)$, for any $x, y \in L$.

Proof. (i) Let $x, y \in L$, the conjunctivity of F guarantees that $F(dx, y) \leqslant dx \wedge y$ and $F(x, dy) \leqslant x \wedge dy$. Since G is increasing, we obtain that

$$G(F(dx, y), F(x, dy)) \leqslant G(dx, dy) \wedge G(y, x) \wedge G(dx, x) \wedge G(y, dy)\,.$$

Using the fact that d is an (F, G)-derivation on L, we conclude that

$$dF(x, y) \leqslant G(dx, dy) \wedge G(y, x) \wedge G(dx, x) \wedge G(y, dy).$$

(ii) The proof is a similar to that of (i).

□

Theorem 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. If F and G are increasing and conjunctive, then the following statements hold for any $x, y \in L$:*

(i) $G(F(dx, dF(y, y)), F(dF(x, x), dy)) \leqslant dF(x, y) \wedge F(dx, dy)$;

(ii) $G(F(dF(x, x), y), F(x, dF(y, y))) \leqslant dF(x, y) \wedge F(x, y)$;

(iii) $G(F(dF(x,x), dF(y,y)), F(dF(x,x), dF(y,y))) \leqslant dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy)$.

Proof. To prove (i), suppose that d is an (F, G)-derivation on L and let $x, y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then,

$$\begin{cases} ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)) \\ ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2\times L^2} ((dx,dy),(dx,dy)) . \end{cases}$$

The fact that F is increasing implies that

$$\begin{cases} (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx,y), F(x,dy)) \\ (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx,dy), F(dx,dy)) . \end{cases}$$

Since G is increasing and conjunctive, and d is an (F, G)-derivation on L, then it follows that

$$\begin{cases} G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx,y), F(x,dy)) = dF(x,y) \\ G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx,dy), F(dx,dy)) \leqslant F(dx,dy) . \end{cases}$$

Thus, $G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant dF(x,y) \wedge F(dx,dy)$.

To demonstrate (ii), let $x, y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then,

$$\begin{cases} ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)) \\ ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((x,y),(x,y)) . \end{cases}$$

Since F is increasing, it holds that

$$\begin{cases} (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(dx,y), F(x,dy)) \\ (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(x,y), F(x,y)) . \end{cases}$$

The fact that G is increasing and conjunctive, and d is an (F, G)-derivation on L, it implies that

$$\begin{cases} G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(dx,y), F(x,dy)) = dF(x,y) \\ G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(x,y), F(x,y)) \leqslant F(x,y) . \end{cases}$$

Therefore, $G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant dF(x,y) \wedge F(x,y)$.

For proving (iii), let $x, y \in L$. Corollary 3.1 (i) guarantees that $dF(x,x) \leqslant dx \wedge x$ and $dF(y,y) \leqslant dy \wedge y$. Then,

$$\begin{cases} ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)) \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((x,y),(x,y)) \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((x,dy),(x,dy)) \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(dx,y)) \\ ((dF(x,x), dF(y,y)), (dF(x,x), dF(y,y))) \leqslant_{L^2\times L^2} ((dx,dy),(dx,dy)) . \end{cases}$$

The fact that F is increasing implies that

$$\begin{cases}(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,y),F(x,dy))\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(x,y),F(x,y))\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(x,dy),F(x,dy))\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,y),F(dx,y))\\(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,dy),F(dx,dy))\,.\end{cases}$$

Since G is increasing, it holds that

$$\begin{cases}G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(x,dy))\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,y),F(x,y))\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,dy),F(x,dy))\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(dx,y))\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,dy),F(dx,dy))\end{cases}$$

The conjunctivity of G and the fact that d is an (F,G)-derivation on L assure that

$$\begin{cases}G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y)\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,y)\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(x,dy)\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,y)\\G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant F(dx,dy)\,.\end{cases}$$

Thus, $G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant dF(x,y)\wedge F(x,y)\wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy)\,.$ □

Analogously, we obtain the following result for increasing and disjunctive binary operations.

Theorem 3.2. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and let d be an (F,G)-derivation on L. If F and G are increasing and disjunctive, then the following statements hold for any $x,y\in L$:*

(i) $dF(x,y)\vee F(dx,dy) \leqslant G(F(dx,dF(y,y)),F(dF(x,x),dy))$;

(ii) $dF(x,y)\vee F(x,y) \leqslant G(F(dF(x,x),y),F(x,dF(y,y)))$;

(iii) $dF(x,y)\vee F(x,y)\vee F(x,dy)\vee F(dx,y)\vee F(dx,dy) \leqslant G(F(dF(x,x),dF(y,y)), F(dF(x,x),dF(y,y)))$.

Proposition 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. The following implications hold:*

(i) if F is conjunctive, G is increasing and idempotent, then $dF(x, x) \leqslant x \wedge dx$, for any $x \in L$;

(ii) if F is disjunctive, G is increasing and idempotent, then $x \vee dx \leqslant dF(x, x)$, for any $x \in L$;

Proof.

(i) Let $x \in L$, since F is conjunctive we have $(F(dx, x), F(x, dx)) \leqslant_{L\times L} (x \wedge dx, x \wedge dx)$. Applying the fact that G is increasing we obtain that

$$G(F(dx, x), F(x, dx)) \leqslant G(x \wedge dx, x \wedge dx).$$

Since d is an (F, G)-derivation on L and G is idempotent, we conclude that $dF(x, x) \leqslant x \wedge dx$.

(ii) The proof is a similar to that of (i).

□

The following theorem gives alternative conditions to that of Theorem 3.1.

Theorem 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F, G)-derivation on L. If F is increasing and conjunctive, G is increasing and idempotent, then the following statements hold for any $x, y \in L$:*

(i) $G(F(dx, dF(y, y)), F(dF(x, x), dy)) \leqslant dF(x, y) \wedge F(dx, dy)$;

(ii) $G(F(dF(x, x), y), F(x, dF(y, y))) \leqslant dF(x, y) \wedge F(x, y)$;

(iii) $F(dF(x, x), dF(y, y)) \leqslant dF(x, y) \wedge F(x, y) \wedge F(x, dy) \wedge F(dx, y) \wedge F(dx, dy)$.

Proof. To prove (i), let $x, y \in L$. Using the facts that F is conjunctive, G is increasing and idempotent, we obtain from Proposition 3.3 that

$$\begin{cases} \begin{cases} dF(y, y) \leqslant y \\ dF(x, x) \leqslant x \end{cases} \\ \begin{cases} dF(y, y) \leqslant dy \\ dF(x, x) \leqslant dx \end{cases} \end{cases}$$

Then it holds that

$$\begin{cases} ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2\times L^2} ((dx, y), (x, dy)) \\ ((dx, dF(y,y)), (dF(x,x), dy)) \leqslant_{L^2\times L^2} ((dx, dy), (dx, dy)) \end{cases}$$

Since F is increasing, we have

$$\begin{cases} (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx, y), F(x, dy)) \\ (F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant_{L\times L} (F(dx, dy), F(dx, dy)) \end{cases}$$

Since G is increasing and idempotent, and d is an (F, G)-derivation on L, we obtain that

$$\begin{cases} G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx, y), F(x, dy)) = dF(x, y) \\ G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant G(F(dx, dy), F(dx, dy)) = F(dx, dy)\,. \end{cases}$$

Therefore, $G(F(dx, dF(y,y)), F(dF(x,x), dy)) \leqslant dF(x,y) \wedge F(dx, dy)$.

For proving (ii), let $x, y \in L$. Applying the facts that F is conjunctive, G is increasing and idempotent, we get from Proposition 3.3 that

$$\begin{cases} \begin{cases} dF(x,x) \leqslant dx \\ dF(y,y) \leqslant dy \end{cases} \\ \begin{cases} dF(x,x) \leqslant x \\ dF(y,y) \leqslant y \end{cases} \end{cases}$$

Then

$$\begin{cases} ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((dx, y), (x, dy)) \\ ((dF(x,x), y), (x, dF(y,y))) \leqslant_{L^2\times L^2} ((x, y), (x, y)) \end{cases}$$

Since F is increasing, we have

$$\begin{cases} (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(dx, y), F(x, dy)) \\ (F(dF(x,x), y), F(x, dF(y,y))) \leqslant_{L\times L} (F(x, y), F(x, y)) \end{cases}$$

Since G is increasing and idempotent, and d is an (F, G)-derivation on L, we obtain that

$$\begin{cases} G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(dx, y), F(x, dy)) = dF(x, y) \\ G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant G(F(x, y), F(x, y)) = F(x, y)\,. \end{cases}$$

Thus, $G(F(dF(x,x), y), F(x, dF(y,y))) \leqslant dF(x,y) \wedge F(x,y)$.

To demonstrate (iii), let $x, y \in L$. Using the facts that F is conjunctive, G is increasing and idempotent, we get from Proposition 3.3 that

$$\begin{cases} dF(x,x) \leqslant dx \\ dF(y,y) \leqslant y \\ dF(x,x) \leqslant x \\ dF(y,y) \leqslant dy \end{cases}$$

Then it holds that

$$\begin{cases} ((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(x,dy)) \\ ((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y))) \leqslant_{L^2\times L^2} ((x,y),(x,y)) \\ ((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y))) \leqslant_{L^2\times L^2} ((x,dy),(x,dy)) \\ ((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y))) \leqslant_{L^2\times L^2} ((dx,y),(dx,y)) \\ ((dF(x,x),dF(y,y)),(dF(x,x),dF(y,y))) \leqslant_{L^2\times L^2} ((dx,dy),(dx,dy)) \end{cases}$$

Since F is increasing, we have

$$\begin{cases} (F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,y),F(x,dy)) \\ (F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(x,y),F(x,y)) \\ (F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(x,dy),F(x,dy)) \\ (F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,y),F(dx,y)) \\ (F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant_{L\times L} (F(dx,dy),F(dx,dy)) \end{cases}$$

Since G is increasing, we have

$$\begin{cases} G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(x,dy)) \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,y),F(x,y)) \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(x,dy),F(x,dy)) \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,y),F(dx,y)) \\ G(F(dF(x,x),dF(y,y)),F(dF(x,x),dF(y,y))) \leqslant G(F(dx,dy),F(dx,dy)) \end{cases}$$

This equivalent to

$$\begin{cases} F(dF(x,x),dF(y,y)) \leqslant dF(x,y) \\ F(dF(x,x),dF(y,y)) \leqslant F(x,y) \\ F(dF(x,x),dF(y,y)) \leqslant F(x,dy) \\ F(dF(x,x),dF(y,y)) \leqslant F(dx,y) \\ F(dF(x,x),dF(y,y)) \leqslant F(dx,dy)\,. \end{cases}$$

Thus, $F(dF(x,x),dF(y,y)) \leqslant dF(x,y) \wedge F(x,y) \wedge F(x,dy) \wedge F(dx,y) \wedge F(dx,dy)$. □

The following theorem shows alternative conditions to that of Theorem 3.2. Its proof can be done in similar way.

Theorem 3.4. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and let d be an (F,G)-derivation on L. If F is increasing and disjunctive, G is increasing and idempotent, then the following statements hold for any $x,y\in L$:*

(i) $dF(x,y) \vee F(dx,dy) \leqslant G(F(dx,dF(y,y)),F(dF(x,x),dy))$*;*

(ii) $dF(x,y) \vee F(x,y) \leqslant G(F(dF(x,x),y),F(x,dF(y,y)))$*;*

(iii) $dF(x,y) \vee F(x,y) \wedge F(x,dy) \vee F(dx,y) \vee F(dx,dy) \leqslant F(dF(x,x),dF(y,y))$.

Proposition 3.4. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and let d be an (F,G)-derivation on L. If F is commutative and G is idempotent, then*

$$dF(x,x) = F(dx,x) = F(x,dx), \text{ for any } x \in L\,.$$

Proof. Using the facts that d is an (F,G)-derivation on L, F is commutative and G is idempotent, we obtain

$$dF(x,x) = G(F(dx,x),F(x,dx)) = G(F(dx,x),F(dx,x)) = F(dx,x),$$

for any $x \in L$. □

Proposition 3.4 leads to the following corollary.

Corollary 3.2. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and let d be an (F,G)-derivation on L. The following implications hold:*

(i) if F is commutative and conjunctive, and G is idempotent, then

$$dF(x,x) \leqslant x \wedge dx, \text{ for any } x \in L;$$

(ii) if F is commutative and disjunctive, and G is idempotent, then

$$x \vee dx \leqslant dF(x,x), \text{ for any } x \in L.$$

Proposition 3.5. *Let $(L,\leqslant,\wedge,\vee,0)$ be a lattice with the least element $0 \in L$ and let d be an (F,G)-derivation on L. The following implications hold:*

(i) if F and G are conjunctive, then $d0 = 0$ (i.e., 0 is a fixed point of d);

(ii) if F is conjunctive, G is increasing and idempotent, then $d0 = 0$;

(iii) if F is commutative and conjunctive, and G is idempotent, then $d0 = 0$.

Proof. (i) The conjunctivity of F implies that $F(0,0) = 0$. Then $d0 = dF(0,0)$. From Corollary 3.1 (i), it holds that $dF(0,0) \leqslant 0$. Thus, $d0 = 0$.

(ii) By using Proposition 3.3 (i), the proof is a similar to that of (i) .

(iii) By using Corollary 3.2 (i), the proof is a similar to that of (i) .

□

In the same line, we obtain the following result.

Proposition 3.6. *Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and let d be an (F,G)-derivation on L. The following implications hold:*

(i) if F and G are disjunctive, then $d1 = 1$ (i.e., 1 is a fixed point of d);

(ii) if F is disjunctive, G is increasing and idempotent, then $d1 = 1$;

(iii) if F is commutative and disjunctive, and G is idempotent, then $d1 = 1$.

Proposition 3.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F,G)-derivation on L. If F has a right- (resp. a left-) neutral element $e \in L$, then $dx = G(dx, F(x, de))$ (resp. $dx = G(F(de, x), dx)$), for any $x \in L$.*

Proof. The fact that e is a right- (resp. a left-) neutral element of F and d is an (F,G)-derivation on L imply that $dx = dF(x,e) = G(F(dx,e), F(x,de)) = G(dx, F(x,de))$ (resp. $dx = dF(e,x) = G(F(de,x), F(e,dx)) = G(F(de,x), dx)$), for any $x \in L$. □

Proposition 3.8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F,G)-derivation on L. If F has a right- (resp. a left-) absorbing element $k \in L$, then $dk = G(k, F(x, dk))$ (resp. $dk = G(F(dk, x), k)$), for any $x \in L$.*

Proof. Since k is a right- (resp. a left-) absorbing element of F and d is an (F,G)-derivation on L, then $dk = dF(x,k) = G(F(dx,k), F(x,dk)) = G(k, F(x,dk))$ (resp. $dk = dF(k,x) = G(F(dk,x), F(k,dx)) = G(F(de,x), k)$), for any $x \in L$. □

The above Proposition 3.8 leads to the following corollary.

Corollary 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let d be an (F,G)-derivation on L. If F and G have a right- or a left-absorbing element $k \in L$, then $dk = k$ (i.e., k is a fixed point of d).*

3.2. Principal (F,G)-derivations on lattices

In this section, we introduce the notion of principal (F,G)-derivation as a particular class of (F,G)-derivations on a lattice and investigate their various properties.

3.2.1. Definitions and auxiliary results

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F be a binary operation on L. For any element $\alpha \in L$, there exists an F-function $f_\alpha : L \to L$ defined as:

$$f_\alpha(x) = F(\alpha, x), \text{ for any } x \in L\,.$$

Let $\mathcal{A}_F(L)$ be the set of the f_α functions on L, i.e., $\mathcal{A}_F(L) = \{f_\alpha \mid \alpha \in L\}$. One can easily verify that $\mathcal{A}_F(L)$ equipped with the usual order of functions (i.e., $f_\alpha \preceq f_\beta$ if and only if $f_\alpha(x) \leqslant f_\beta(x)$, for any $x \in L$) is a poset.

The following propositions show some cases that the poset $(\mathcal{A}_F(L), \preceq)$ has a lattice structure.

Proposition 3.9. *[37] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F be a binary operation on L. If F is left-increasing and having a right-neutral element $e \in L$, then the poset $(\mathcal{A}_F(L), \preceq)$ is a lattice. Where, the meet $\frown$ and the join $\smile$ operations of $\mathcal{A}_F(L)$ are defined as $f_\alpha \frown f_\beta = f_{\alpha\wedge\beta}$ and $f_\alpha \smile f_\beta = f_{\alpha\vee\beta}$, for any $f_\alpha, f_\beta \in \mathcal{A}_F(L)$.*

Proposition 3.10. *[37] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F be a binary operation on L. If F is the meet (resp. the join) operation of L, then $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ is a lattice.*

Proposition 3.11. *[37] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F be a binary operation on L. Then it holds that*

(i) if F is left-increasing and having a right-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \textit{ if and only if } f_\alpha \preceq f_\beta\,;$$

(ii) if F is idempotent, conjunctive or disjunctive and left-increasing, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \textit{ if and only if } f_\alpha \preceq f_\beta\,.$$

The following proposition provides some conditions under which the elements of $\mathcal{A}_F(L)$ are (F, G)-derivations on L.

Proposition 3.12. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. Then any elements of $\mathcal{A}_F(L)$ is an (F, G)-derivation on L.*

Proof. Let $f_\alpha \in \mathcal{A}_F(L)$ and $x, y \in L$. We aim to show

$$f_\alpha(F(x,y)) = G(F(f_\alpha(x), y), F(x, f_\alpha(y)))\,.$$

Since F is commutative and associative, and G is idempotent, it follows that

$$\begin{aligned} f_\alpha(F(x,y)) &= F(\alpha, F(x,y)) \\ &= G(F(\alpha, F(x,y)), F(\alpha, F(x,y))) \\ &= G(F(\alpha, F(x,y)), F(F(\alpha, x), y)) \\ &= G(F(\alpha, F(x,y)), F(F(x, \alpha), y)) \\ &= G(F(F(\alpha, x), y), F(x, F(\alpha, y))) \\ &= G(F(f_\alpha(x), y), F(x, f_\alpha(y)))\,. \end{aligned}$$

Thus, f_α is an (F, G)-derivation on L. □

In view of Proposition 3.12, we introduce the notion of principal (F, G)-derivation on a given lattice.

Definition 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. Any f_α function is called principal (F, G)-derivation on L. In this case, $\mathcal{A}_F(L)$ denotes the set of the principal (F, G)-derivations on L.*

Example 3.4. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, $\alpha \in L$, T be a t-norm on L and let S be a t-conorm on L. The function t_α (resp. s_α) defined for any $x \in L$ by $t_\alpha(x) = T(\alpha, x)$ (resp. $s_\alpha(x) = S(\alpha, x)$) is a principal (T, G)-derivation (resp. principal (S, G)-derivation) on L, for any idempotent binary operation G on L.*

3.2.2. Properties of principal (F, G)-derivations on a lattice

In this subsection, we investigate some properties of principal (F, G)-derivations on a given lattice.

Proposition 3.13. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F, G)-derivation on L. Then it holds that*

$$f_\alpha \circ f_\alpha(F(x,y)) = F(f_\alpha(x), f_\alpha(y)),\ \textit{for any } x, y \in L\,.$$

Proof. Let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F,G)-derivation on L and let $x, y \in L$. The facts that F is commutative and associative imply that

$$\begin{aligned} f_\alpha \circ f_\alpha(F(x,y)) &= f_\alpha(f_\alpha(F(x,y))) \\ &= f_\alpha F(\alpha, F(x,y)) \\ &= F(\alpha, F(\alpha, F(x,y))) \\ &= F(\alpha, F(F(\alpha,x),y)) \\ &= F(F(\alpha, F(\alpha,x)),y) \\ &= F(F(F(\alpha,x),\alpha),y) \\ &= F(F(\alpha,x),F(\alpha,y)) \\ &= F(f_\alpha(x), f_\alpha(y))\,. \end{aligned}$$

□

Proposition 3.14. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F,G)-derivation on L. If F is increasing, then f_α is isotone, i.e.,*

$$\textit{if } x \leqslant y, \textit{ then } f_\alpha(x) \leqslant f_\alpha(y), \textit{ for any } x, y \in L.$$

Proof. Let $x, y \in L$ such that $x \leqslant y$. Since F is increasing, it holds that $F(\alpha, x) \leqslant F(\alpha, y)$, i.e., $f_\alpha(x) \leqslant f_\alpha(y)$. Thus, the principal (F,G)-derivation f_α is isotone. □

One can easily verify that if the principal (F,G)-derivations on L are isotone, then F is increasing.

Proposition 3.14 leads to the following corollary.

Corollary 3.4. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, T be a t-norm and let S be a t-conorm on L. Then the principal (T,G)-derivations (resp. principal (S,G)-derivations) on L are isotone, for any idempotent binary operation G on L.*

Proposition 3.15. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let $f_\alpha \in \mathcal{A}_F(L)$ be a principal (F,G)-derivation on L. If F is increasing, then f_α satisfies that*

$$\textit{if } x \leqslant y, \textit{ then } f_\alpha(F(x,x)) \leqslant f_\alpha(F(y,y)), \textit{ for any } x, y \in L.$$

Proof. Let $x, y \in L$ such that $x \leqslant y$. Since F is associative, it holds that $f_\alpha(F(x,x)) = F(\alpha, F(x,x)) = F(F(\alpha,x),x)$ and $f_\alpha(F(y,y)) = F(F(\alpha,y),y)$.

The fact that $x \leqslant y$ and F is increasing imply that $F(F(\alpha, x), x) \leqslant F(F(\alpha, y), y)$. Thus, $f_\alpha(F(x, x)) \leqslant f_\alpha(F(y, y))$. □

3.3. Representations of a lattice in terms of its principal (F, G)-derivations

In this section, we provide two representations of a given lattice in terms of its principal (F, G)-derivations. We start by the first one.

Theorem 3.5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L such that F is commutative and associative, and G is idempotent. If F is increasing and having a neutral element $e \in L$, then the lattice $(L, \leqslant, \wedge, \vee)$ is isomorphic to the lattice $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L.*

Proof. Proposition 3.9 guarantees that the poset $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L is a lattice. Next, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha) = f_\alpha$, for any $\alpha \in L$. One can easily verify that ψ is surjective. Furthermore, from Proposition 3.11 (i), it holds that

$$\alpha \leqslant \beta \text{ if and only if } \psi(\alpha) \preceq \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Now, ψ is an order isomorphism between L and $\mathcal{A}_F(L)$. Thus, Proposition 1.1 guarantees that ψ is a lattice isomorphism. Therefore, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ are isomorphic. □

In the following, we present an illustrative example of Theorem 3.5.

Example 3.5. *Let $L = \mathbb{R}_+^*$ be the lattice of the positive real numbers ordered by the usual order, and let F, G be two binary operations on $\mathbb{R}_+^*$ defined for any $x, y \in \mathbb{R}_+^*$ as $F(x, y) = x \cdot y$ and* $G(x, y) = \begin{cases} x & \textit{if } x = y, \\ x + y & \textit{otherwise}. \end{cases}$
One can easily verify that F is commutative and associative, and G is idempotent. Furthermore, F is increasing and having the neutral element $1 \in \mathbb{R}_+^*$. Theorem 3.5 guarantees that the lattice $(\mathbb{R}_+^*, \leqslant, min, max)$ is isomorphic to the lattice $(\mathcal{A}_F(\mathbb{R}_+^*), \preceq, \frown, \smile)$ of principal (F, G)-derivations on $\mathbb{R}_+^*$.

Theorem 3.5 leads to the following corollary.

Corollary 3.5. *Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice, G be an idempotent binary operation, T be a t-norm and let S be a t-conorm on L. Then it holds that*
(i) The bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is isomorphic to the bounded lattice $(\mathcal{A}_T(L), \preceq, \frown, \smile, t_0, t_1)$ of principal (T, G)-derivations on L, where $t_0(x) = T(0, x) = 0$ and $t_1(x) = T(1, x) = x$, for any $x \in L$;
(ii) The bounded lattice $(L, \leqslant, \wedge, \vee, 0, 1)$ is isomorphic to the bounded lattice $(\mathcal{A}_S(L), \preceq, \frown, \smile, s_0, s_1)$ of principal (S, G)-derivations on L, where $s_0(x) = S(0, x) = x$ and $s_1(x) = S(1, x) = 1$, for any $x \in L$.

The following theorem provides the second representation of a lattice in terms of its principal (F, G)-derivations.
Theorem 3.6. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and let F, G be two binary operations on L such that G is idempotent. If F is the meet (resp. the join) operation of L, then the lattice $(L, \leqslant, \wedge, \vee)$ is isomorphic to the lattice $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L.*

***Proof*.** Since F is the meet (resp. the join) operation of L, it follows from Proposition 3.10 that the poset $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ of principal (F, G)-derivations on L is a lattice. Now, let ψ be a mapping from L into $\mathcal{A}_F(L)$ defined by $\psi(\alpha) = f_\alpha$, for any $\alpha \in L$. Using the same steps as Theorem 3.6, we obtain that ψ is a lattice isomorphism. Thus, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F(L), \preceq, \frown, \smile)$ are isomorphic. □

Remark 3.2. *We note that*

(i) if $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice, F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee, 0, 1)$ can be represented by using both Theorems 3.5 and 3.6;

(ii) if $(L, \leqslant, \wedge, \vee)$ is a lattice has not the least element 0 (resp. greatest element 1), F is the meet (resp. the join) operation of L and G is an idempotent binary operation on L, then $(L, \leqslant, \wedge, \vee)$ can be represented only by Theorem 3.6;

(iii) if $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and F, G are respectively the meet and the join operations of L, then Theorem 3.6 coincides with representation theorem given by Xin et al. (Theorem 3.29 in [35]). Thus, Theorem 3.6 is a generalization of Theorem 3.29 to an arbitrary lattice.

4 f-fixed points of isotone f-derivations on a lattice

In this chapter, we investigate the most important properties of isotone f-derivations on a lattice. Paying attention to the lattice (resp. ideal) structures of isotone f-derivations and the sets of their f-fixed points. As applications, we provide characterizations of distributive lattices and principal ideals of a lattice in terms of principal f-derivations.

4.1. Principal f-derivations on a lattice

In this section, we show a necessary and sufficient condition that the functions $d_{(\alpha,f)}$ on a lattice (given in Proposition 1.7) are f-derivations. Also, we show that their set has a lattice structure. Further, we provide a representation (resp. characterization) of any lattice (resp. distributive lattice) in terms of its principal f-derivations.

4.1.1. Poset structure for the set of principal f-derivations on a lattice

The following result shows a necessary and sufficient condition under which the functions $d_{(\alpha,f)}$ on a lattice L being f-derivations on L.

Theorem 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a function. Then f is a $\wedge$-homomorphism if and only if $d_{(\alpha,f)}$ is an f-derivation on L, for any $\alpha \in L$.*

Proof. The direct implication follows from Proposition 1.7. For the converse implication, we assume that $d_{(\alpha,f)}$ is an f-derivation on L, for any $\alpha \in L$. Then

$$\begin{aligned} d_{(\alpha,f)}(x \wedge y) &= \alpha \wedge f(x \wedge y) \\ &= (d_{(\alpha,f)}(x) \wedge fy) \vee (fx \wedge d_{(\alpha,f)}(y)) \\ &= (\alpha \wedge fx \wedge fy) \vee (fx \wedge \alpha \wedge fy) \\ &= \alpha \wedge fx \wedge fy, \text{ for any } \alpha, x, y \in L. \end{aligned}$$

Hence, $\alpha \wedge f(x \wedge y) = \alpha \wedge fx \wedge fy$, for any $\alpha, x, y \in L$. On the one hand, setting $\alpha = f(x \wedge y)$. Then it holds that $f(x \wedge y) = f(x \wedge y) \wedge (fx \wedge fy)$, for any $x, y \in L$. Hence, $f(x \wedge y) \leqslant fx \wedge fy$, for any $x, y \in L$. On the other hand, setting $\alpha = fx \wedge fy$. Then it follows that $(fx \wedge fy) \wedge f(x \wedge y) = (fx \wedge fy) \wedge (fx \wedge fy) = fx \wedge fy$, for any $x, y \in L$. Hence, $fx \wedge fy \leqslant f(x \wedge y)$, for any $x, y \in L$. Thus, $f(x \wedge y) = fx \wedge fy$, for any $x, y \in L$. Therefore, f is a $\wedge$-homomorphism. □

The following corollary expresses the relationship between the two sets $\mathcal{P}_f(L)$ and $\mathfrak{I}_f(L)$. Where, $\mathcal{P}_f(L) = \{d_{(\alpha,f)} \mid \alpha \in L\}$ and $\mathfrak{I}_f(L)$ is the set of isotone f-derivations on L.

Corollary 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a function. Then f is a $\wedge$-homomorphism if and only if $\mathcal{P}_f(L)$ is a subset of $\mathfrak{I}_f(L)$.*

Proof. The proof is directly from Theorem 4.1. □

In what follows, for a given lattice L, the f-derivations $d_{(\alpha,f)}$ will be called *principal f-derivations* on L, and $\mathcal{P}_f(L)$ denotes their set. One can easily verify that $\mathcal{P}_f(L)$ equipped with the usual order of functions $\leqslant'$ (i.e., $d_{(\alpha,f)} \leqslant' d_{(\beta,f)}$ if and only if $d_{(\alpha,f)}(x) \leqslant d_{(\beta,f)}(x)$, for any $x \in L$.) is a poset.

Remark 4.1. *If $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice, then the poset $(\mathcal{P}_f(L), \leqslant'$ $)$ is also bounded, where $0_{\mathcal{P}_f(L)} = d_{(0,f)}$ and $1_{\mathcal{P}_f(L)} = d_{(1,f)}$ such that $d_{(0,f)}(x) = 0$ and $d_{(1,f)}(x) = fx$, for any $x \in L$.*

4.1.2. Lattice structure for the poset of principal f-derivations on a lattice

In this subsection, we show some cases in which the poset $(\mathcal{P}_f(L), \leqslant')$ of principal f-derivations on a lattice L has a lattice structure. Also, we provide a representation of a lattice in terms of its principal f-derivations. First, we show that $(\mathcal{P}_f(L), \leqslant')$ is a *meet*-semilattice.

Proposition 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the poset $(P, \leqslant')$ is a meet-semilattice.*

Proof. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. It is easy to verify that $d_{(\alpha\wedge\beta,f)}$ is the greatest lower bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a *meet*-semilattice. □

The following theorem shows that the set of principal f-derivations on a complete lattice is also a complete lattice.

Theorem 4.2. *Let $(L, \leqslant, \wedge, \vee)$ be a complete lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the poset $(\mathcal{P}_f(L), \leqslant')$ is a complete lattice.*

Proof. Proposition 4.1 guarantees that $(\mathcal{P}_f(L), \leqslant')$ is a *meet*-semilattice. Let A be a non-empty subset of $\mathcal{P}_f(L)$ and P^u be the set of upper bounds of A. The fact that $(L, \leqslant, \wedge, \vee)$ is a complete lattice implies that $d_{(\beta,f)} = d_{(\bigwedge \alpha_i, f)}$ with $d_{(\alpha_i,f)} \in P^u$ is the least upper bound of A. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a complete lattice. □

The following corollary follows from the above theorem.

Corollary 4.2. *Let $(L, \leqslant, \wedge, \vee)$ be a finite lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L), \leqslant')$ is a finite lattice.*

In the following theorem, we show that the set of principal f-derivations on a distributive lattice is also a distributive lattice.

Theorem 4.3. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the poset $(\mathcal{P}_f(L), \leqslant')$ is a distributive lattice.*

Proof. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in L$. The fact that $(L, \leqslant, \wedge, \vee)$ is distributive implies that $d_{(\alpha\vee\beta,f)}$ is the least upper bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a lattice. Moreover, its distributivity follows from that of $(L, \leqslant, \wedge, \vee)$. □

Next, we provide a representation of a lattice L based on its principal f-derivations. This representation gives also another case where the poset $(\mathcal{P}_f(L), \leqslant')$ is a lattice. First, we need to show the following lemma.

Lemma 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a function. If f is surjective, then for any $\alpha, \beta \in L$, the following equivalence holds:*

$$\alpha \leqslant \beta \textit{ if and only if } d_{(\alpha,f)} \leqslant' d_{(\beta,f)}.$$

Proof. The direct implication is immediate. For the converse implication, assume that f is surjective. Let $\alpha, \beta \in L$ such that $d_{(\alpha,f)} \leqslant' d_{(\beta,f)}$. Then $\alpha \wedge fx \leqslant \beta \wedge fx$, for any $x \in L$. Since f is surjective, it holds that there exists $m \in L$ such that $fm = \alpha$. Hence, $\alpha \wedge fm \leqslant \beta \wedge fm$. Thus, $\alpha \leqslant \beta$. □

Now, we are able to provide a representation of a lattice in terms of its principal f-derivations.

Theorem 4.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and f be a $\wedge$-homomorphism. If f is a $\wedge$-epimorphism, then the poset $(\mathcal{P}_f(L), \leqslant')$ is a lattice, where $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ and $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$, for any $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. Moreover, $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ are isomorphic.*

Proof. Assume that $f : L \to L$ is a $\wedge$-epimorphism. Proposition 4.1 guarantees that $(\mathcal{P}_f(L), \leqslant')$ is a *meet*-semilattice, where $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ is the greatest lower bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$, for any $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. Now, we show that $(\mathcal{P}_f(L), \leqslant')$ is a *join*-semilattice. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$, since f is surjective, it follows from Lemma 4.1 that $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$ is the least upper bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Hence, $(\mathcal{P}_f(L), \leqslant')$ is a *join*-semilattice. Thus, the structure $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ is a lattice.
Next, let $\psi : L \to \mathcal{P}_f(L)$ be a mapping defined as $\psi(\alpha) = d_{(\alpha,f)}$, for any $\alpha \in L$. It is obvious to verify that ψ is surjective. Furthermore, Lemma 4.1 guarantees that $\alpha \leqslant \beta$ if and only if $\psi(\alpha) \leqslant' \psi(\beta)$, for any $\alpha, \beta \in L$. Thus, ψ is an order isomorphism between L and $\mathcal{P}_f(L)$. Proposition 1.1 guarantees that ψ is a lattice isomorphism. Therefore, $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ are isomorphic. □

In the following, we present an illustrative example of Theorem 4.4.
Example 4.1. *Let $(D(30), |, gcd, lcm)$ be the lattice of the positive divisors of 30 ordered by the divisibility relation and given by the Hasse diagram in Figure 4.1. Let $f : D(30) \to D(30)$ be a function defined by the following table:*

x	1	2	3	5	6	10	15	30
fx	1	2	5	3	10	6	15	30

The following table presents the elements of $\mathcal{P}_f(D(30))$.

x	1	2	3	5	6	10	15	30
$d_{(1,f)}(x)$	1	1	1	1	1	1	1	1
$d_{(2,f)}(x)$	1	2	1	1	2	2	1	2
$d_{(3,f)}(x)$	1	1	1	3	1	3	3	3
$d_{(5,f)}(x)$	1	1	5	1	5	1	5	5
$d_{(6,f)}(x)$	1	2	1	3	2	6	3	6
$d_{(10,f)}(x)$	1	2	5	1	10	2	5	10
$d_{(15,f)}(x)$	1	1	5	3	5	3	15	15
$d_{(30,f)}(x)$	1	2	5	3	10	6	15	30

One can easily verify that f is a lattice automorphism. Hence, Theorem 4.4 guarantees that the lattices $(\mathcal{P}_f(D(30)), \leqslant', \wedge', \vee')$ and $(D(30), |, gcd, lcm)$ are isomorphic.

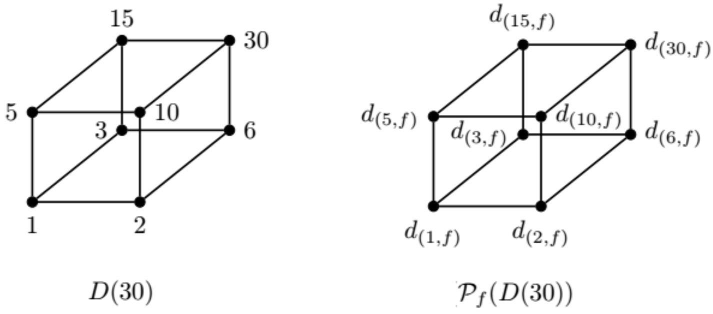

Figure 4.1: The Hasse diagrams of the lattices $(D(30), |, gcd, lcm)$ and $(\mathcal{P}_f(D(30)), \leqslant', \wedge', \vee')$.

Note that the converse of Theorem 4.4 does not necessarily hold, as can be seen in the following example.

Example 4.2. *Let $(D(6), |, gcd, lcm)$ be the lattice of the positive divisors of 6 given by the Hasse diagram in Figure 4.2. Let $f : D(6) \to D(6)$ be a function defined by the following table:*

x	1	2	3	6
fx	1	1	6	6

The following table presents the elements of $\mathcal{P}_f(D(6))$.

$d_{(1,f)}(x)$	1	1	1	1
$d_{(2,f)}(x)$	1	1	2	2
$d_{(3,f)}(x)$	1	1	3	3
$d_{(6,f)}(x)$	1	1	6	6

One easily verifies that f is a $\wedge$-homomorphism. Moreover, since there not exists $x \in D(6)$ such that $fx = 2$, it holds that f is not surjective. But, as can be seen in Figure 4.2 that the poset $(\mathcal{P}_f(D(6)), \leqslant')$ is a lattice and isomorphic to $(D(6), |, gcd, lcm)$.

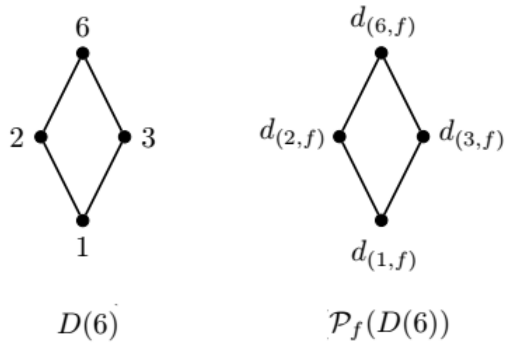

Figure 4.2: The Hasse diagrams of the lattices $(D(6), |, gcd, lcm)$ and $(\mathcal{P}_f(D(6)), \leqslant', \wedge', \vee')$.

4.2. A relationship between a distributive lattice and its lattice of isotone f-derivations

In this section, we give a relationship between a distributive lattice L and its lattice of isotone f-derivations $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$. Also, we show a characterization theorem of a distributive lattice in terms of its principal f-derivations. First, we need to recall the following result.

Proposition 4.2. *[38] Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ is a sublattice of the distributive lattice $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$.*

In the case of $(L, \leqslant, \wedge, \vee)$ is a distributive lattice, the following proposition shows that the lattice structures $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ coincide.

Proposition 4.3. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ coincides with $(\mathcal{P}_f(L), \leqslant'$ $, \wedge', \vee')$.*

Proof. On the one hand, $(d_{(\alpha,f)} \sqcap d_{(\beta,f)})(x) = (\alpha \wedge fx) \wedge (\beta \wedge fx) = (\alpha \wedge \beta) \wedge fx = d_{(\alpha\wedge\beta,f)}(x) = (d_{(\alpha,f)} \wedge' d_{(\beta,f)})(x)$, for any $\alpha, \beta, x \in L$. Then $\sqcap$ coincides with $\wedge'$ on $\mathcal{P}_f(L)$, for any lattice $(L, \leqslant, \wedge, \vee)$. On the other hand, we assume that $(L, \leqslant, \wedge, \vee)$ is a distributive lattice. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$, then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx = d_{(\alpha\vee\beta,f)}(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x)$, for any $x \in L$. Thus, $\sqcup$ coincides with $\vee'$ on $\mathcal{P}_f(L)$. Therefore, the lattice structures $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ and $(\mathcal{P}_f(L), \leqslant'$

$,\wedge',\vee')$ coincide. □

Combining Propositions 4.2 and 4.3 leads to the following corollary.

Corollary 4.3. *Let $(L,\leqslant,\wedge,\vee)$ be a distributive lattice and $f:L\to L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L),\leqslant',\wedge',\vee')$ is a sublattice of $(\mathfrak{I}_f(L),\preceq,\sqcap,\sqcup)$.*

The following theorem shows a relationship between a distributive lattice and its lattice of isotone f-derivations.

Theorem 4.5. *Let $(L,\leqslant,\wedge,\vee)$ be a distributive lattice and $f:L\to L$ be a $\wedge$-epimorphism. Then $(L,\leqslant,\wedge,\vee)$ is isomorphic to a sublattice of $(\mathfrak{I}_f(L),\preceq,\sqcap,\sqcup)$.*

Proof. Assume that $(L,\leqslant,\wedge,\vee)$ is a distributive lattice and $f:L\to L$ is a $\wedge$-epimorphism. On the one hand, Theorem 4.4 guarantees that $(L,\leqslant,\wedge,\vee)$ is isomorphic to $(\mathcal{P}_f(L),\leqslant',\wedge',\vee')$. On the other hand, Corollary 4.3 shows that $(\mathcal{P}_f(L),\leqslant',\wedge',\vee')$ is a sublattice of $(\mathfrak{I}_f(L),\preceq,\sqcap,\sqcup)$. Consequently, $(L,\leqslant,\wedge,\vee)$ is isomorphic to a sublattice of $(\mathfrak{I}_f(L),\preceq,\sqcap,\sqcup)$. □

We conclude this subsection by a characterization theorem of a distributive lattice in terms of its principal f-derivations.

Theorem 4.6. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and $f:L\to L$ be a $\wedge$-epimorphism. The following statements are equivalent:*

(i) $(L,\leqslant,\wedge,\vee)$ *is distributive;*

(ii) $(\mathcal{P}_f(L),\preceq,\sqcap,\sqcup)$ *is a distributive lattice;*

(iii) $\sqcup$ *is a binary operation on* $\mathcal{P}_f(L)$*;*

(iv) $\sqcup$ *coincides with* $\vee'$ *on* $\mathcal{P}_f(L)$*.*

Proof. $(i)\Rightarrow(ii)$: A straightforward application of Proposition 4.2.
$(ii)\Rightarrow(iii)$: The proof is immediate.
$(iii)\Rightarrow(iv)$: Let $d_{(\alpha,f)},d_{(\beta,f)}\in\mathcal{P}_f(L)$. The fact that $\sqcup$ is a binary operation on $\mathcal{P}_f(L)$ implies that there exists $d_{(\gamma,f)}\in\mathcal{P}_f(L)$ such that $d_{(\alpha,f)}\sqcup d_{(\beta,f)}=d_{(\gamma,f)}$, this equivalent to $(\alpha\wedge fx)\vee(\beta\wedge fx)=\gamma\wedge fx$, for any $x\in L$. Since f is surjective, it follows that there exist $a,b,c\in L$ such that $fa=\alpha$, $fb=\beta$ and $fc=\gamma$. Setting $x=a$ (resp. $x=b$), it holds that $\alpha=\gamma\wedge\alpha$ (resp. $\beta=\gamma\wedge\beta$). Moreover, setting $x=c$, we obtain that $\gamma=\alpha\vee\beta$. Hence, $d_{(\alpha,f)}\sqcup d_{(\beta,f)}=d_{(\alpha\vee\beta,f)}=d_{(\alpha,f)}\vee' d_{(\beta,f)}$. Thus, $\sqcup$ coincides with the binary

operation $\vee'$ on $\mathcal{P}_f(L)$.
$(iv) \Rightarrow (i)$: Let $\alpha, \beta, \gamma \in L$. Since $\sqcup$ coincides with $\vee'$ on $\mathcal{P}_f(L)$, it holds that $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x) = d_{(\alpha\vee\beta,f)}(x)$ for any $x \in L$, this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx, \text{ for any } x \in L.$$

The fact that f is surjective implies that there exists $c \in L$ satisfying $fc = \gamma$. Setting $x = c$, we obtain that $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge \gamma$. Thus, $(L, \leqslant, \wedge, \vee)$ is distributive. □

Remark 4.2. *From Theorem 4.6, we conclude that if $(L, \leqslant, \wedge, \vee)$ is not distributive and $f : L \to L$ is a $\wedge$-epimorphism, then $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ has not a lattice structure. It is only a meet-semilattice, indeed, in this case $\sqcup$ can not be a binary operation on $\mathcal{P}_f(L)$.*

4.3. Ideal structures of the sets of f-fixed points of an isotone and a principal f-derivation on a lattice

In this section, we present some cases that the set of f-fixed points of an isotone (resp. a principal) f-derivation on a lattice L is an ideal of L. First, we recall the following definition.
Definition 4.1. *[6] Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an f-derivation on L. The set of f-fixed points of d is given by:*

$$Fix_{(d,f)}(L) = \{x \in L \mid dx = fx\}.$$

Theorem 4.7. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and d be an isotone f-derivation on L. If f is a $\vee$-homomorphism and $Fix_{(d,f)}(L)$ is a non-empty set, then $Fix_{(d,f)}(L)$ is an ideal of L.*

Proof. Assume that $f : L \to L$ is a $\vee$-homomorphism. Let d be an isotone f-derivation on L such that $Fix_{(d,f)}(L)$ is a non-empty set. On the one hand, let $x, y \in L$ such that $x \in Fix_{(d,f)}(L)$ and $y \leqslant x$. The fact that d is an f-derivation on L implies from Proposition 1.6 that $dy \leqslant fy$. Since f is increasing, $y \leqslant x$ and $x \in Fix_{(d,f)}(L)$, it follows that $fy \leqslant fx = dx$. Hence, $dy = d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) = fy \vee dy$, and this implies that

$fy \leqslant dy$. Thus, $dy = fy$, i.e., $y \in Fix_{(d,f)}(L)$. On the other hand, let $x, y \in Fix_{(d,f)}(L)$. This implies that $dx = fx$ and $dy = fy$. Since $(L, \leqslant, \wedge, \vee)$ is distributive, f is a $\vee$-homomorphism and d is an isotone f-derivation on L, it follows from Proposition 1.6 that $d(x \vee y) = dx \vee dy = fx \vee fy = f(x \vee y)$. Hence, $x \vee y \in Fix_{(d,f)}(L)$. Finally, we conclude that $Fix_{(d,f)}(L)$ is an ideal of L. □

Remark 4.3. *In general, the set of f-fixed points of an f-derivation on a lattice L is a non-empty set. Indeed, if $(L, \leqslant, \wedge, \vee)$ is a lattice has a least element $0 \in L$ and $f0 = 0$, then 0 is an f-fixed point of any f-derivation on L.*

Theorem 4.8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $d_{(\alpha,f)}$ be a principal f-derivation on L such that $Fix_{(d_{(\alpha,f)},f)}(L)$ is a non-empty set. If f is a lattice homomorphism, then $Fix_{(d_{(\alpha,f)},f)}(L)$ is an ideal of L.*

Proof. Assume that $f : L \to L$ is a lattice homomorphism. Let $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $Fix_{(d_{(\alpha,f)},f)}(L)$ is a non-empty set. On the one hand, let $x, y \in L$ such that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ and $y \leqslant x$. The fact that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ implies that $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$. Hence, $fx \leqslant \alpha$. Now, since f is increasing and $y \leqslant x$, it holds that $fy \leqslant fx$. Hence, $fy \leqslant \alpha$. Thus, $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$. Therefore, $y \in Fix_{(d_{(\alpha,f)},f)}(L)$. On the other hand, let $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$. Then $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$ and $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$. This implies that $fx \vee fy \leqslant \alpha$. The fact that f is a $\vee$-homomorphism and $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$ imply that $f(x \vee y) = fx \vee fy \leqslant \alpha$. Hence, $d_{(\alpha,f)}(x \vee y) = \alpha \wedge f(x \vee y) = f(x \vee y)$. Thus, $x \vee y \in Fix_{(d_{(\alpha,f)},f)}(L)$. Therefore, $Fix_{(d_{(\alpha,f)},f)}(L)$ is an ideal of L. □

4.4. Characterization of principal ideals in terms of principal f-derivations on a lattice

In this section, we provide a characterization theorem of principal ideals of a lattice in terms of its principal f-derivations. First, we show the following key results.

Proposition 4.4. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $\downarrow x$ be a principal ideal of L and $f : L \to L$ be a $\wedge$-monomorphism. Then there exists a principal f-derivation $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$, where $\alpha = f(x)$.*

Proof. Let $\downarrow x$ be a principal ideal of L. Since f is a $\wedge$-monomorphism, it follows that

$$\begin{aligned}\downarrow x &= \{y \in L \mid y \leqslant x\}\\ &= \{y \in L \mid x \wedge y = y\}\\ &= \{y \in L \mid f(x \wedge y) = f(y)\}\\ &= \{y \in L \mid f(x) \wedge f(y) = f(y)\}\\ &= \{y \in L \mid d_{(f(x),f)}(y) = f(y)\}\\ &= Fix_{(d_{(f(x),f)},f)}(L).\end{aligned}$$

Thus, there exists $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$, where $\alpha = f(x)$. $\square$

Proposition 4.5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \to L$ be a lattice automorphism and $d_{(\alpha,f)}$ be a principal f-derivation on L. Then $Fix_{(d_{(\alpha,f)},f)}(L)$ is a principal ideal of L generated by $f^{-1}(\alpha)$.*

Proof. Let $d_{(\alpha,f)} \in \mathcal{P}_f(L)$. Since f is a lattice automorphism, it follows that

$$\begin{aligned}Fix_{(d_{(\alpha,f)},f)}(L) &= \{y \in L \mid d_{(\alpha,f)}(y) = fy\}\\ &= \{y \in L \mid \alpha \wedge fy = fy\}\\ &= \{y \in L \mid fy \leqslant \alpha\}\\ &= \{y \in L \mid y \leqslant f^{-1}(\alpha)\}\\ &= \downarrow f^{-1}(\alpha).\end{aligned}$$

$\square$

Notation 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a function. We denote by:*

(i) $\mathfrak{F}_f(L) := \{Fix_{(d,f)}(L) \mid d \in \mathfrak{I}_f(L)\}$;

(ii) $\mathcal{F}_f(L) := \{Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L)\}$.

Combining Propositions 4.4 and 4.5 leads to the following characterization theorem of principal ideals of a lattice L in terms of its principal f-derivations.

Theorem 4.9. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \to L$ be a lattice automorphism. Then $\mathcal{F}_f(L)$ is exactly the set of principal ideals of L.*

In the following, we present an illustrative example of the above Theorem 4.9.

Example 4.3. *Let $L = D(30)$ be the lattice of the positive divisors of 30 given by the Hasse diagram in Figure 4.1, and f be the lattice automorphism given in Example 4.1. Then the following holds:*

$$\begin{cases} Fix_{(d_{(1,f)},f)}(D(30)) = \{1\} = \downarrow 1 = \downarrow f^{-1}(1); \\ Fix_{(d_{(2,f)},f)}(D(30)) = \{1,2\} = \downarrow 2 = \downarrow f^{-1}(2); \\ Fix_{(d_{(3,f)},f)}(D(30)) = \{1,5\} = \downarrow 5 = \downarrow f^{-1}(3); \\ Fix_{(d_{(5,f)},f)}(D(30)) = \{1,3\} = \downarrow 3 = \downarrow f^{-1}(5); \\ Fix_{(d_{(6,f)},f)}(D(30)) = \{1,2,5,10\} = \downarrow 10 = \downarrow f^{-1}(6); \\ Fix_{(d_{(10,f)},f)}(D(30)) = \{1,2,3,6\} = \downarrow 6 = \downarrow f^{-1}(10); \\ Fix_{(d_{(15,f)},f)}(D(30)) = \{1,3,5,15\} = \downarrow 15 = \downarrow f^{-1}(15); \\ Fix_{(d_{(30,f)},f)}(D(30)) = D(30) = \downarrow 30 = \downarrow f^{-1}(30). \end{cases}$$

Thus, $\mathcal{F}_f(D(30))$ is the set of principal ideals of $D(30)$.

4.5. A link between prime ideals and f-derivations on a lattice

In this section, we show a link between prime ideals of a lattice L and f-derivations on L. This relationship is a generalization of the result of Theorem 4.13 given by Xin in [36].

Theorem 4.10. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \to L$ be a function and I be a prime ideal of L. The following implications hold:*

(i) if f is a $\wedge$-homomorphism, then there exists an f-derivation d on L such that $I \subseteq Fix_{(d,f)}(L)$;

(ii) if f is a $\wedge$-monomorphism, then there exists an f-derivation d on L such that $I = Fix_{(d,f)}(L)$.

Proof. Let $\alpha \in I$ and $d : L \to L$ be a function defined as

$$dx = \begin{cases} fx & , \ if x \in I; \\ f(\alpha \wedge x) & , \ otherwise. \end{cases}$$

(i) The fact that f is a $\wedge$-homomorphism and I is a prime ideal of L imply that $d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy)$, for any $x, y \in L$. Thus, d is an f-derivation on L. The proof of $I \subseteq Fix_{(d,f)}(L)$ is straightforward.

(ii) Assume that f is a $\wedge$-monomorphism. On the one hand, (i) guarantees that d is an f-derivation on L and $I \subseteq Fix_{(d,f)}(L)$. On the other hand, let $x \in Fix_{(d,f)}(L)$. Here, we distinguish two possible cases, which are $x \in I$ or $x \notin I$. Now, we prove that the case of $x \notin I$ is an impossible case. Suppose that $x \notin I$, then $dx = f(\alpha \wedge x)$. The fact that $x \in Fix_{(d,f)}(L)$ implies that $dx = fx$. Hence, $f(\alpha \wedge x) = fx$. Since f is injective, it holds that $\alpha \wedge x = x$, i.e., $x \leqslant \alpha$. Since $\alpha \in I$ and I is an *ideal*, it holds that $x \in I$, which contradicts the hypothesis that $x \notin I$. Hence, necessarily $x \in I$. Thus, $Fix_{(d,f)}(L) \subseteq I$. Finally, we conclude that $I = Fix_{(d,f)}(L)$.

□

4.6. Structure of the set of f-fixed points sets of isotone f-derivations on a distributive lattice

In this section, for a given distributive lattice L, we show that the set of f-fixed points sets $\mathfrak{F}_f(L)$ of its isotone f-derivations has also a structure of a distributive lattice. Moreover, we prove that the set of f-fixed points sets $\mathcal{F}_f(L)$ of principal f-derivations on L is a sublattice of $\mathfrak{F}_f(L)$. Finally, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations. First, we prove the following key result.

Proposition 4.6. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice. For any $Fix_{(d_1,f)}(L)$, $Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$, we define:*

$$Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcap d_2,f)}(L)\,,$$

and

$$Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcup d_2,f)}(L).$$

Then $\sqcap'$ and $\sqcup'$ are idempotent, commutative and associative binary operations on $\mathfrak{F}_f(L)$, and they satisfy the absorption laws.

Proof. Let $Fix_{(d_1,f)}(L), Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$. Then $d_1, d_2 \in \mathfrak{I}_f(L)$, i.e., d_1 and d_2 are two isotone f-derivations on L. Since $(L, \leqslant, \wedge, \vee)$ is distributive, it follows from Theorem 1.3 that $d_1 \sqcap d_2$ and $d_1 \sqcup d_2$ are also isotone f-derivations on L, i.e., $d_1 \sqcap d_2, d_1 \sqcup d_2 \in \mathfrak{I}_f(L)$. Hence, $Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L), Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$. Thus, $\sqcap'$ and $\sqcup'$ are binary operations on $\mathfrak{F}_f(L)$. Furthermore, the fact that $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ is a lattice implies that $\sqcap$ and $\sqcup$ are idempotent, commutative and associative binary operations on $\mathfrak{I}_f(L)$, and they satisfy the absorption laws. These imply that $\sqcap'$ and $\sqcup'$ are also idempotent, commutative and associative binary operations on $\mathfrak{F}_f(L)$, and they satisfy the absorption laws. $\square$

Theorem 4.11. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a function. Then the structure $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice, where the order relation $\preceq'$ is defined as $Fix_{(d_1,f)}(L) \preceq' Fix_{(d_2,f)}(L)$ if and only if $Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_2,f)}(L)$, for any $Fix_{(d_1,f)}(L), Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$.*

Proof. Proposition 4.6 guarantees that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a lattice. Moreover, from the distributivity of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$, we easily verify that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is also distributive. $\square$

The following Proposition lists some properties of the sets of f-fixed points of principal f-derivations on a lattice.

Proposition 4.7. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \to L$ be a $\wedge$-homomorphism and $d_{(\alpha,f)}, d_{(\beta,f)}$ be two principal f-derivations on L. Then it holds that*

(i) $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$;

(ii) $Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$;

(iii) If $(L, \leqslant, \wedge, \vee)$ is distributive, then $Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$.

Proof. (i) Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. We only prove that $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$, as the fact that $d_{(\alpha,f)} \sqcap d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ implies that $Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$. Then

$$\begin{aligned} Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) &= \{x \in L \mid d_{(\alpha,f)}(x) = d_{(\beta,f)}(x) = fx\} \\ &= \{x \in L \mid \alpha \wedge fx = \beta \wedge fx = fx\} \\ &= \{x \in L \mid fx \leqslant \alpha \wedge \beta\} \\ &= \{x \in L \mid (\alpha \wedge \beta) \wedge fx = fx\} \\ &= \{x \in L \mid d_{(\alpha\wedge\beta,f)}(x) = fx\} \\ &= Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L). \end{aligned}$$

Thus,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L).$$

(ii) On the one hand, let $x \in Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L)$. Then $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ or $x \in Fix_{(d_{(\beta,f)},f)}(L)$. Assume that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$, it holds that $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$. Then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = fx \vee (\beta \wedge fx) = fx$. Thus, $x \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$. The case of $x \in Fix_{(d_{(\beta,f)},f)}(L)$ can be proved similarly. Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)\,.$$

On the other hand, let $y \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$. Then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(y) = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$. This implies that $d_{(\alpha\vee\beta,f)}(y) = (\alpha \vee \beta) \wedge fy = (\alpha \vee \beta) \wedge [(\alpha \wedge fy) \vee (\beta \wedge fy)] = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$. Hence, $y \in Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$. Thus, $Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$. Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)\,.$$

(iii) Since $(L, \leqslant, \wedge, \vee)$ is distributive, it follows from Proposition 4.3 that $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$. Thus, $Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$. □

The following result shows that the set of f-fixed points sets $\mathcal{F}_f(L)$ of principal f-derivations on L is a sublattice of $\mathfrak{F}_f(L)$.

Theorem 4.12. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a sublattice of $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$.*

Proof. Since $(L, \leqslant, \wedge, \vee)$ is a distributive lattice, it holds form Theorem 4.11 that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice. The fact that f is a $\wedge$-homomorphism implies that $\mathcal{F}_f(L)$ is a subset of $\mathfrak{F}_f(L)$. Furthermore, Proposition 4.7 guarantees that

$$Fix_{(d_{(\alpha,f)},f)}(L) \sqcap' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$$

and

$$Fix_{(d_{(\alpha,f)},f)} \sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)\,,$$

for any $Fix_{(d_{(\alpha,f)},f)}(L), Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_f(L)$. Thus, $\mathcal{F}_f(L)$ is closed under $\sqcap'$ and $\sqcup'$. Therefore, $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a sublattice of $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$. □

Combining Theorems 4.11 and 4.12 leads to the following corollary.

Corollary 4.4. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice.*

Next, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations.

Theorem 4.13. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a lattice automorphism. Then $(L, \leqslant, \wedge, \vee)$ is isomorphic to $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$.*

Proof. Assume that $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and $f : L \to L$ is a lattice automorphism. Corollary 4.4 guarantees that $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice. Moreover, let $\psi : L \to \mathcal{F}_f(L)$ be a mapping defined as:

$$\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L), \text{ for any } \alpha \in L\,.$$

Now, we show that ψ is surjective. Let $Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_f(L)$. Since f is a lattice automorphism, it holds that there exists $\alpha \in L$ such that $f(\alpha) = \beta$. Then $\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L) = Fix_{(d_{(\beta,f)},f)}(L)$. Hence, ψ is surjective. Next, we prove that

$$\alpha \leqslant \beta \text{ if and only if } \psi(\alpha) \preceq' \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Since $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and f is a lattice automorphism, it follows from Propositions 4.4 and 4.7 that

$$\begin{aligned}
\alpha \leqslant \beta \quad &\Longleftrightarrow \alpha \vee \beta = \beta \\
&\Longleftrightarrow \downarrow (\alpha \vee \beta) = \downarrow \beta \\
&\Longleftrightarrow Fix_{(d_{(f(\alpha\vee\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\Longleftrightarrow Fix_{(d_{(f(\alpha)\vee f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\Longleftrightarrow Fix_{(d_{(f(\alpha),f)}\sqcup d_{(f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\Longleftrightarrow Fix_{(d_{(\alpha,f)},f)} \sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\Longleftrightarrow \psi(\alpha) \sqcup' \psi(\beta) = \psi(\beta) \\
&\Longleftrightarrow \psi(\alpha) \preceq' \psi(\beta),
\end{aligned}$$

for any $\alpha, \beta \in L$. Hence, ψ is an order isomorphism between L and $\mathcal{F}_f(L)$. Proposition 1.1 guarantees that ψ is a lattice isomorphism. Thus, the distributive lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ are isomorphic. □

Combining Theorems 4.4 and 4.13 leads to the following corollary.

Corollary 4.5. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \to L$ be a lattice automorphism. Then the three distributive lattices $(L, \leqslant, \wedge, \vee)$, $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ and $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ are isomorphic.*

Conclusion

In this thesis we have generalized the notion of $(\wedge, \vee)$-derivation to (F, G)-derivation on a lattice, and investigated its properties in detail. This generalization is based on two arbitrary binary operations F and G instead of the lattice meet $(\wedge)$ and join $(\vee)$ operations. To that end, a lot of preparatory work was required. In particular, several properties and characterizations of binary operations on an arbitrary lattice were investigated, and two representation theorems of a lattice based on a binary operation were provided.

Furthermore, we have studied the isotone and principal f-derivations on a lattice and investigated their properties. We have studied the lattice structure of isotone f-derivations on a lattice, and the ideal structures of the sets of their f-fixed points.

Finally, future work is anticipated in multiple directions. We intend to extend the different notions of derivation to other useful algebraic structures and investigate their fundamental properties. Moreover, we believe that the notion of (F, G)-derivations on a lattice is worthy of further investigations.

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