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Fuzzy ordered sets and aggregations

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Thesis submitted in fulfillment of the requirements for the degree of Master of Mathematics
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To the spirit of my mother who passed away, God have mercy on her.

Msila, December 12th 2018
The author

Table of contents

Table of contents

Introduction

Fuzzy sets and fuzzy relations have been introduced by Zadeh [69, 70]. Several different approaches of this concept such as L-fuzzy sets, L-fuzzy ordering relations, fuzzy lattices, fuzzy filters have been investigated by many authors [14, 29, 34, 37, 52]. Goguen introduced the concept of L-fuzzy set in [34] and studied its fundamental properties. In [37], Ulrich Höhle introduced the notion of L-fuzzy equivalence and L-fuzzy ordering relation based on L-equivalence relation (i.e., an ordering relation with respect to a binary operation $\otimes$ of the commutative monoïd L and an $\otimes$-equivalence relation E) briefly ($\otimes$-E-order). After, S. V. Ovchinnikov, in [52] has extended some set's operations. U. Bodenhofer in [14] restrict ordering relation with respect to $\otimes$ and an $\otimes$-equivalence relation E briefly ($\otimes$-E-order) replacing L by the unit interval $L = [0, 1]$ and the operation $\otimes$ by a t-norm T. Later, in [29], Mustafa Demirci studied the $\otimes$-E-order in an iccqm-lattice L.
One of the most important concept of fuzzy sets is the concepts of α-cuts and strong α-cuts. They play a principal role in the relationship between fuzzy sets and crisp sets and can be viewed as a bridge that connecting between fuzzy sets and crisp sets. The need to bridge the gap between a fuzzy structures and classical structures involves the concept cutworthy property which expresses that a special property (P) will be called cutworthy when it is true that a fuzzy structure possesses (P) (in the fuzzy sense) if and only if every α-cut of this structure possesses (P) (in the crisp sense) [6].

In [65], B. Seselja shows that an L-fuzzy relation is an L-fuzzy ordering relation if and only if all its α-cuts except 0-cut are crisp ordering relations.

In this thesis, we show that many fuzzy structures possess the cutworthy properties such as $\otimes$-fuzzy equivalence relations, and the $\otimes$-E-partial ordering relations. In the same direction, we focus on the aggregation of diverse families like fuzzy binary relations, fuzzy lattices, and fuzzy filters.

In fact, aggregating several informations in one information is an interesting operation in fields dealing with quantitative information. Therefore, many notions of fuzzy set theory can be expressed by aggregation functions [60]. Union and intersection are built by means of special aggregation functions.

Some links between these families and their images via an aggregation function and several characterizations for these were provided.

Given an aggregation function $A : U^n \to U$ and a family of fuzzy binary relations $\mathcal{L} = \big\{L_i : X^2 \to U, i \in \{1, ..., n\}\big\}$ on a domain X. A $(\mathcal{L}, A)$ fuzzy binary relation on X denote by $\mathcal{L}_A$ is obtained as the composition given by
$\mathcal{L}_A(x, y) = A(L_1(x, y), ..., L_n(x, y))$.

The special case L_i is a T_i-E_i order, where E_i is a T_i-equivalence relation will be studied. It was shown that the family of fuzzy preordering relations (L_i), $i = 1, 2, ..., n$ can be aggregated, and its aggregated image is also a preorder. The notion of generalized associativity law to prove that the image of a family of complete lattices via an aggregation function is also a complete lattice was introduced.

Moreover, the aggregation of a family of the left (respectively right) traces of a family of fuzzy relations was investigated and the conditions under which these images are left (respectively right) traces were given.

This thesis is organized as follows. The first chapter is devoted to some basic notions and definitions a complete residuated lattice, some cutworthy properties for fuzzy sets. Also, we recall some basic notion of triangular norms, aggregation functions, fuzzy lattices and fuzzy filters.

The second chapter states in a precise way, either a fuzzy structure possess the cutworthy properties or not, especially the L-fuzzy $\otimes$-equivalence relations (L-fuzzy $\otimes$-E-partial orderings) cases.

In the third chapter, we present the aggregation of some families of fuzzy structures, such as fuzzy relations, fuzzy lattices and fuzzy filters.

1 Preliminaries

In this chapter, we recall some definitions and well-known results on crisp ordered sets, crisp lattices, fuzzy sets, fuzzy sets operations, fuzzy relations, fuzzy orders, fuzzy lattice, fuzzy filters and ideals in fuzzy lattices, triangular norms, fuzzy implications, fuzzy intersection with respect to a triangular norm, fuzzy union with respect to a triangular conorm, complete residuated lattice and aggregation functions[12].

1.1. Generalities on ordered sets and complete lattices

This section deals with the basic notions of classical partial ordered sets and lattices needed in the sequel. For more detail, see [11, 24, 61, 63].

1.2. Crisp sets

There are three basic methods by which sets can be defined within a given universal set X,

1 by naming all its members;

2 by a property satisfied by its members;

3 by a function, usually called a characteristic function. We interest by the third method [17, 19, 31].

When A is a crisp set and x is an element of X, the proposition "x is an element of A" is necessarily either true or false, as required by two-valued logic. Let X be a universe of discourse. A subset A of X is defined by its characteristic function, that declares either an element x of X is a member of the set A or not. The set A is defined by the function $\mu_A : X \to \{0, 1\}$ as follows

$$\mu_A(x) = \begin{cases} 1 \text{ if and only if } x \in A, \\ 0 \text{ if and only if } x \notin A. \end{cases}$$

That is, the characteristic function maps elements of X to elements of the set $\{0, 1\}$. For each $x \in X$, when $\mu_A(x) = 1$, x is declared to be a member of A, when $\mu_A(x) = 0$, x is declared as a nonmember of A.

1.2.1. Sets operations

The relative complement of a set A with respect to a set B is the set containing all the members of B that are not also members of A. This can be written $B - A$. Thus, $B - A = \{x \in X / x \in B \text{ and } x \notin A\}$. If the set B is the universal set X, then this kind of complement is an absolute complement set A. That is, $\mathcal{C}A = X - A$ In general, a complement set means the absolute complement set. The complement set is always involutive

$$\mathcal{C}(\mathcal{C}A) = A.$$

1.2.2. Relation between sets

If every member of a set A is a member of the set B ($x \in A$ implies $x \in B$), then A is called a subset of B, and this is written as $A \subseteq B$. If the following relation is satisfied, $A \subseteq B$ and $B \subseteq A$ then A and B have the same elements and thus they are the same sets. This relation is denoted by

$$A = B.$$

If the following relations are satisfied between two sets A and B, $A \subseteq B$ and $A \neq B$ then B has elements which is not involved in A. In this case, A is called a proper subset of B and this relation is denoted by

$$A \subset B.$$

A set that has no element is called an empty set (denoted $\emptyset$). An empty set is a subset of any set.
The union of sets A and B is the set containing all the elements that belong either to set A alone, to set B alone, or to both set A and set B. This is denoted by $A \cup B$.
The intersection of sets A and B is the set containing all the elements belonging to both set A and set B. It is denoted by $A \cap B$. The most important properties of sets' operations is given in the **Table** 1.1.

Table 1.1:

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A, A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$
Absorption	$A \cup (A \cap B) = A, A \cap (A \cup B) = A$
Idempotence	$A \cup A = A, A \cap A = A$
Absorption by X and $\emptyset$	$A \times X = X, A \cap \emptyset = \emptyset$
Identity	$A \cup \emptyset = A$
Law of contradiction	$A \cap \overline{A} = \emptyset$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

The power set of a nonempty set X (denoted by $P(X)$) can be ordered by the set inclusion $\subseteq$.

1.2.3. Binary relations

Let X be a nonempty set. A subset of $X \times X$ is called a binary relation on X, and denoted by R.

Definition 1.1. *Let X be a nonempty set and let R be a binary relation on X. R is called:*

1. *Reflexive, if $(x, x) \in R$, for all $x \in X$.*
2. *Antisymmetric, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$, for all $x, y \in X$.*
3. *Symmetric, if $(x, y) \in R$ then $(y, x) \in R$, for all $x, y \in X$.*
4. *Transitive, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, for all $x, y, z \in X$.*

Definition 1.2. *A reflexive and transitive binary relation R on a set X is called a preordering relation. A preordering binary relation R which is symmetric is called an equivalence relation. A preordering binary relation R which is antisymmetric is called a partial ordering relation. A partial ordering relation R is said to be linearly ordered set or totally ordered relation, if and only if either $(x, y) \in R$ or $(y, x) \in R$, for all $x, y \in X$. A set X equipped with a partial order relation R denoted in general by (X, R) and called a partially ordered set (poset for short).*

Let (X, R) be a poset and S be a subset of X. Many important properties of the ordered set X are expressed in terms of the existence of upper bound or lower bound of the subset S. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices For more detail, see [10, 24, 50, 61].

Definition 1.3. *Let (X, R) be an ordered set and let $S \subseteq X$. An element $x \in X$ is an upper bound of S if $s \leq x$ for all $s \in S$. A lower bound is defined dually. The set of all upper bounds of S is denoted by S^u (read as 'S upper') and the set of all lower bounds by S^l (read as 'S lower'): $S^u := \{x \in X / (\forall s \in S) s \leq x\}$ and $S^l := \{x \in X / (\forall s \in S) s \geq x\}$. Since R is transitive, S^u is always an up-set and S^l a down-set. If S^u has a least element x, then x is called the least upper bound of S. Equivalently, x is the least upper bound of S if*

(i) *x is an upper bound of S, and*

(ii) *$x \leq y$ for all upper bounds y of S.*

The least upper bound of S exists if and only if there exists $x \in X$ such that $(\forall y \in X)[((\forall s \in S) s \leq y) \Leftrightarrow x \leq y]$, and this characterizes the least upper bound of S. Dually, if S^l has a greatest element, x, then x is called the greatest lower bound of S [24].

Notation. Let (X, R) be an ordered set. We adopt the notation, $x \vee y$ (read as 'x join y') in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as 'x meet y') in place of $\inf\{x, y\}$ when it exists. Similarly, we write $\vee S$ (the 'join of S') and $\wedge S$ (the 'meet of S') instead of $\sup S$ and $\inf S$ when these exist. When $\sup X$ and $\inf X$ they are called respectively top and bottom and denoted by 1 and 0.

Definition 1.4. *A poset (X, R) is called a lattice if for all $x, y \in X$, $x \wedge y$ and $x \vee y$ exist. If a lattice (X, R) has a top and a bottom, we say that (X, R) is a bounded lattice. A lattice (X, R) is called complete lattice if for all subset S of X, $\vee S$ and $\wedge S$ exist.*

1.3. Basic concepts of fuzzy sets

The characteristic function of a crisp set assigns a value of either 0 or 1 to each individual in the universal set, that is for to distinguish between members and non-members of this crisp set. In the fuzzy case, this function can be generalized such that any element of the universal set can take all values inserted between 0 and 1, this function is called a membership function, and the set correspondant is called a fuzzy set [17, 19, 31, 35, 36, 44, 55].

The most commonly used range of values of membership functions is the unit interval $[0, 1]$ [70] or a residuated lattice [34]. In this case, each membership function maps elements of a given universal set X, which is always a crisp set, into real numbers in $[0, 1]$ or this residuated lattice.

To denote membership functions of a fuzzy set A, we use either the function is denoted by μ_A,

$$\mu_A : X \longrightarrow [0,1],$$

or the function is denoted by A,

$$A : X \longrightarrow [0,1].$$

When we accept that the membership grades can be represented by any arbitrary partially ordered set L instead of the numbers in the unit interval $[0,1]$, we obtain fuzzy sets of another generalized type. They are called L-fuzzy sets, and their membership functions have the form

$$A : X \longrightarrow L.$$

Finally, we denote by $\mathcal{F}(X)$ the fuzzy power set of X (the set of all ordinary fuzzy sets of X).

1.3.1. Fuzzy sets operations

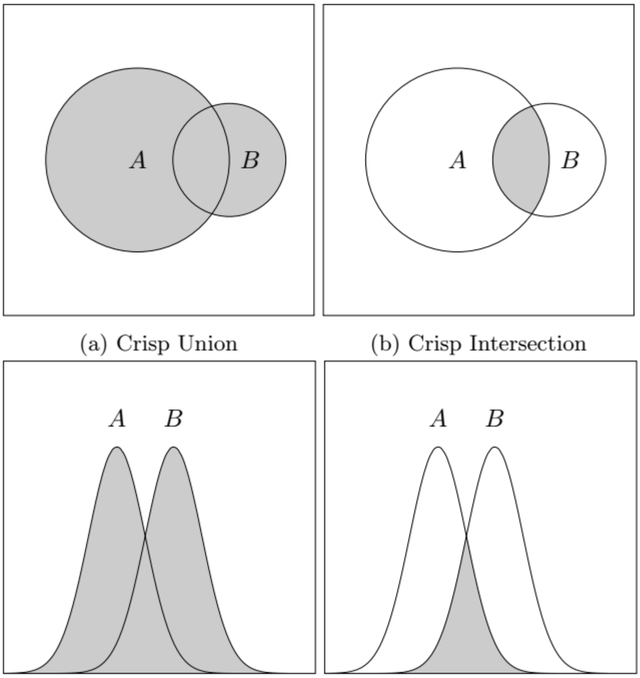
The most important fuzzy sets operations can be founed in [43, 57].

Fuzzy inclusion

Given two fuzzy sets $A, B \in \mathcal{F}(X)$, we say that A is a subset of B and write $A \subseteq B$ if and only if $A\,(x) \leq B\,(x)$ for all $x \in X$.

Standard union and intersection

Given two fuzzy sets, A and B, their standard intersection,
$A \cap B$, and standard union, $A \cup B$, are defined $forall x \in X$ by the equations $(A \cap B)(x) = \min[A(x), B(x)]$, $(A \cup B)(x) = \max[A(x), B(x)]$. In fuzzy logic, two Principle of classical logic are lost, that is to say the law of non contradiction and law of excluded middle. To verify, for example, that the law of non contradiction is unpreserved for fuzzy sets, we need only showing that the equation $\min[A(x), 1 - A(x)] = 0$, does not hold for at least for one $x \in X$. Or this equation is obviously unsatisfied for any value $A(x) \in \,]0,1[$ and it is satisfied only for $A(x) \in \{0,1\}$. The following **Figure** 1.1 illustrates the crisp (fuzzy) union and intersection.



(a) Crisp Union (b) Crisp Intersection

(c) Fuzzy Union (d) Fuzzy Intersection

Figure 1.1: Fuzzy and Crisp Union and Intersection

Fuzzy completion

Let $\mathcal{C}A$ denotes the fuzzy complement of A of type $\mathcal{C}$. Then, $\mathcal{C}A(x)$ means both the membership degree in $\mathcal{C}A$ and the non-membership in A.
The complement $\mathcal{C}A$ can be defined by a function $\mathcal{C} : [0,1] \longrightarrow [0,1]$.

should be satisfied the two following axioms.

(Axiom $\mathcal{C}1$) $\mathcal{C}(0) = 1$, $\mathcal{C}(1) = 0$ (boundary condition),

(Axiom $\mathcal{C}2$) If $x < y$, then $\mathcal{C}(x) \geq \mathcal{C}(y)$, for all x, y in $[0,1]$(monotonic nonincreasing).

$\mathcal{C}1$ and $\mathcal{C}2$ are fundamental requisites to be a complement function.

For particular purposes, we insert some additional axioms.

(Axiom $\mathcal{C}3$) $\mathcal{C}$ is a continuous function.

(Axiom $\mathcal{C}4$) $\mathcal{C}$ is involutive. $\mathcal{C}(\mathcal{C}(x)) = x$ for all x in $[0,1]$.

The set of elements of X verifying $\mathcal{C}A(x) = A(x)$ is called the equilibrium points of A. For the standard complement, clearly, membership grades of equilibrium points are 0.5.

Theorem 1.1. *(First Characterization Theorem of Fuzzy Complements) [44] Let $\mathcal{C}$ be a function from $[0,1]$ to $[0,1]$. Then, $\mathcal{C}$ is a fuzzy complement iff there exists a continuous function g from $[0,1]$ to $\mathbb{R}$ such that $g(0) = 0$, g is strictly decreasing,*

$$\mathcal{C}(a) = g^{-1}(g(1) - g(a)), \text{for all } a \in [0,1].$$

Theorem 1.2. *(Second Characterization Theorem of Fuzzy Complements) [44] Let $\mathcal{C}$ be a function from $[0,1]$ to $[0,1]$. Then, $\mathcal{C}$ is a fuzzy complement iff there exists a continuous function f from $[0,1]$ to $\mathbb{R}$ such that $f(1) = 0$, f is strictly decreasing, and*

$$\mathcal{C}(a) = f^{-1}(f(0) - f(a))$$

for all $a \in [0,1]$.

Example 1.1. *The function $\mathcal{C} : [0,1] \to [0,1]$ defined by*

$$\mathcal{C}(x) = \frac{1}{2}(1 + \cos \pi x),$$

verifies (Axiom $\mathcal{C}1$), (Axiom $\mathcal{C}2$) and (Axiom $\mathcal{C}3$), but not (Axiom $\mathcal{C}4$)

Example 1.2. *The function $\mathcal{C} : [0,1] \to [0,1]$ defined by*

$$\mathcal{C}(x) = \frac{1}{2}(1 + \cos \pi x),$$

verifies (Axiom $\mathcal{C}1$), (Axiom $\mathcal{C}2$) and (Axiom $\mathcal{C}3$), but not (Axiom $\mathcal{C}4$)

Example 1.3. *The function $\mathcal{C} : [0,1] \to [0,1]$ defined by*

$$\mathcal{C}_\omega(x) = (1 - x^\omega)^{\frac{1}{\omega}}, \ \omega \in]0, +\infty[$$

verifies (Axiom $\mathcal{C}1$), (Axiom $\mathcal{C}2$), (Axiom $\mathcal{C}3$) and (Axiom $\mathcal{C}4$).

*See **Figure** 1.2.*

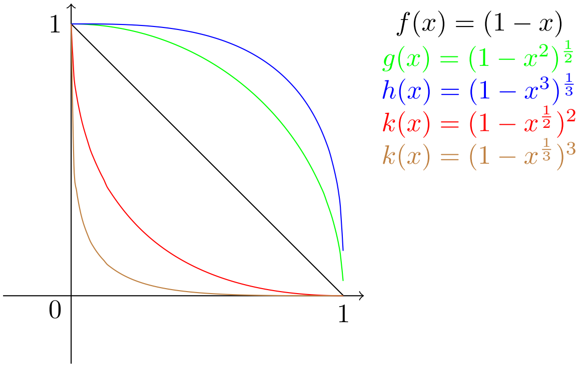


Figure 1.2: Yager completions for some values of ω

Example 1.4. *The function* $\mathcal{C} : [0,1] \to [0,1]$ *defined by*

$$\mathcal{C}_\lambda(x) = \frac{1-\lambda}{1+\lambda x},$$

where $\lambda \in]-1,\infty[$. for any $\lambda \in]-1,\infty[$, we obtain one particular involutive fuzzy complement.

Remark 1.1. *The fuzzy complement function* $\mathcal{C}$ *is not unique see* ***Example*** *1.4 and Yager completions* ***Figure*** *1.2.*

1.3.2. α-cuts

An L-fuzzy subset A (or simply fuzzy set A) on a nonempty set X is a mapping $A : X \to L$. The α-cut A_α and strong α-cut $A_\alpha^>$ of the fuzzy set A in the universe of discourse X are defined as follows. $A_\alpha = \{x \in X \diagup A(x) \geq \alpha\}$, where $\alpha \in L^0$, $A_\alpha^> = \{x \in X \diagup A(x) > \alpha\}$, where $\alpha \in L^1$.

The following definitions can be found in [14, 37].

1.3.3. Height and support of a fuzzy set

Let X be a nonempty set and let $A \in \mathcal{F}(X)$. The support of the fuzzy set A denoted by $Supp(A)$ is the crisp set that contains all the elements of X that have nonzero membership grades in A i.e., $Supp(A) = \{x \in X \diagup A(x) \neq 0\}$. It is easy to see that the support of A is exactly the same as the strong α-cut of A for $\alpha = 0$, i.e., $Supp(A) = A_0^>$. The 1-cut, A^1, is often called the kern of A. The height, $h(A)$, of a fuzzy set A is the greatest membership grade obtained by any element in that set. Formally, $h(A) = \sup_{x \in X} A(x)$. A fuzzy set A is called normal when $h(A) = 1$, it is called subnormal when $h(A) < 1$.

1.4. Triangular norms and triangular conorms

Some times the letter U is used to denote the close unite real interval $[0,1]$ and I denotes the set of the n integers $\{1, 2, ..., n\}$.

1.4.1. Triangular norms

Definition 1.5. *A triangular norm (briefly t-norm) is a binary fuzzy operation T on the unit interval U which is commutative, associative, monotone and has 1 as neutral element, see [2, 3, 40, 41].*

The following t-norms are the four basic t-norms.
The minimum t-norm $T_M(x,y) = \min(x,y)$.
The product t-norm $T_p(x,y) = xy$.
The Łukasiewicz t-norm $T_L(x,y) = \max(x+y-1, 0)$.
And the drastic product t-norm

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Definition 1.6. *[14] Let T_1 and T_2 be two t-norms. We say that T_2 is stronger than T_1, if $T_1(x,y) \leq T_2(x,y)$ for all $(x,y) \in U^2$, and we write $T_1 \leq T_2$.*

Definition 1.7. *[14] Let T_1 and T_2 be two t-norms. We say that T_1 dominates T_2 if and only if,*

$$T_1(T_2(x,y), T_2(z,t)) \geq T_2(T_1(x,z), T_1(y,t)), \textit{ for all } x,y,z,t \in U.$$

Remark 1.2. *The minimum t-norm is the only idempotent t-norm.*
Lemma 1.1. *[27, 41]*

(i) *Any t-norm T dominates itself.*

(ii) *The minimum t-norm T_M dominates any other t-norm.*

(iii) *If a t-norm T_1 dominates another t-norm T_2, then T_1 is stronger than T_2.*

Remark 1.3. *The dominance relation is reflexive, antisymmetric, but in general not transitive see [59].*
Definition 1.8. *[60] Let $(T_i)_{i\in I}$, be a family of t-norms. We say that $(T_i)_{i\in I}$ verify the generalized associativity law if and only if for all $1 \leq i,j \leq n$ and $x,y,z \in U$, $T_i(T_j(x,y),z) = T_i(x, T_j(y,z))$.*

Remark 1.4. *Between the four basic t-norms, we have these strict inequalities: $T_D < T_L < T_p < T_M$.*

1.4.2. Triangular conorms

Definition 1.9. *A triangular conorm (t-conorm for short) is a binary operation S on the unit interval $[0,1]$, i.e., it is a function $S:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]:$ the following four axioms are satisfied:*

(S1) $S(x,y) = S(y,x)$. *(commutativity)*

(S2) $S(x,S(y,z)) = S(S(x,y),z)$. *(associativity)*

(S3) $S(x,y) \leq S(x,z)$ *whenever* $y \leq z$. *(monotonicity)*

(S4) $S(x,0) = x$. *(boundary condition)*

The following are the four basic t-conorms S_M, S_P, S_L, and S_D given by, respectively:

$S_M(x,y) = \max(x,y)$	(maximum)
$S_P(x,y) = x + y - x \cdot y$	(probabilistic sum)
$S_L(x,y) = min(x+y,1)$	(Łukasiewicz t-conorm, bounded sum)
$S_D(x,y) = \begin{cases} 1, \text{ if } (x,y) \in]0,1]^2, \\ \max(x,y) \text{ otherwise.} \end{cases}$	(drastic sum)

Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.

Proposition 1.1. *A function $S:[0,1]^2 \to [0,1]$ is a t-conorm if and only if there exists a t-norm T such that for all $(x,y) \in [0,1]^2$*

$$S(x,y) = 1 - T(1-x, 1-y). \tag{1.1}$$

1.5. Fuzzy orders and fuzzy equivalences

We begin by recalling some basic properties of fuzzy binary relations see [9, 13, 14, 15, 16, 20, 22, 28, 29, 37, 42, 49, 53, 54, 62].

Definition 1.10. *Let X be a non-empty set and T a triangular norm on U. A mapping $R: X \times X \to [0,1]$ is called a fuzzy binary relation on X. A fuzzy binary relation R on X is said to be*

1. *Reflexive, if $R(x,x) = 1$, for all $x \in X$.*
2. *Antisymmetric, if $R(x,y) \wedge R(y,x) = 0$ whenever $x \neq y$, for all $x,y \in X$.*
3. *Symmetric, if $R(x,y) = R(y,x)$, for all $x,y \in X$.*
4. *Transitive, if $R(x,y) \wedge R(y,z) \leq R(x,z)$, for all $x,y,z \in X$.*

5. *T-Transitive, if $T(R(x,y),R(y,z)) \leq R(x,z)$, for all $x,y,z \in X$.*

Definition 1.11. *A reflexive, symmetric and transitive fuzzy relation is called a fuzzy equivalence relation. A reflexive, antisymmetric and transitive fuzzy relation is called a fuzzy partial ordering. A fuzzy partial ordering R is called a fuzzy total order if and only if either $R(x,y) > 0$ or $R(y,x) > 0$, for all $x,y \in X$. A set equipped with a fuzzy partial order relation is called fuzzy partially ordered set (fuzzy poset for short).*

Definition 1.12. *Let X be a non-empty set. A mapping $R : X^2 \to L$ is called an L-fuzzy relation on X. For $\alpha \in L^0$, an α-cut of R is a subset R_α of X^2, such that $R_\alpha = \{(x,y)|R(x,y) \geq \alpha\}$. A strong α-cut of R is a subset $R_\alpha^>$ of X^2 such that $\alpha \in L^1$ and $R_\alpha^> = \{(x,y)|R(x,y) > \alpha\}$.*

Definition 1.13. *Let E be a binary relation on X and let T be a t-norm, E is called fuzzy T-equivalence relation if and only if it is reflexive, symmetric and T-transitive.*

Definition 1.14. *[14] Let X be a nonempty set, and let E be a fuzzy T-equivalence relation on X. A fuzzy binary relation L on X is said to be a fuzzy ordering with respect to a t-norm T and a T-equivalence relation E (briefly T-E-ordering) if and only if it is T-transitive and fulfills the following two axioms*

1. *E-reflexivity, i.e., for all $x,y \in X, E(x,y) \leq L(x,y)$,*
2. *T-E-antisymmetry $T\left(L\left(x,y\right),L\left(y,x\right)\right) \leq E\left(x,y\right)$*

1.6. Fuzzy lattices

Next, we recall some definitions of lattice structures (as a relational structure) [1, 9, 21, 22, 25, 29, 38, 47, 51, 66, 67, 71].

1.6.1. Definitions and properties of fuzzy lattices

Definition 1.15. *Let (X,R) be a fuzzy poset and let A be a nonempty subset of X. An element $u \in X$ is said to be an upper bound of the subset A if and only if $R(x,u) > 0$ for all $x \in A$. An upper bound u_0 of A is said to be the least upper bound of A if and only if $R(u_0,u) > 0$, for every upper bound u of A. An element $l \in X$ is said to be the lower bound of a subset A if and only if $R(l,x) > 0$, for all $x \in A$. A lower bound l_0 of A is said to be the greatest lower bound of A if and only if $R(l,l_0) > 0$, for every lower bound l of A. The least upper bound and the greatest lower bound of the set $\{x,y\}$ are denoted by $x \vee y$ and $x \wedge y$ respectively.*

The least upper bound and the greatest lower bound of the set A are denoted by $\wedge A$ and $\vee A$ respectively.

Definition 1.16. *Let (X, R) be a fuzzy poset. (X, R) is a fuzzy lattice if and only if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.*

If $\sup_R(X)$ and $\inf_R(X)$ exist, they will be denoted respectively by 0 and 1. We say that this lattice is bounded and we denote it by $(X, R, \wedge, \vee, 0, 1)$.

Remark 1.5. *Since R is antisymmetric, it follows that the least upper (greatest lower) bound if it exists, is unique.*

Definition 1.17. *A fuzzy ordered set (X, R) is called a fuzzy complete lattice if $sup_R(A)$ and $inf_R(A)$ exist for every nonempty subset $A \subseteq X$.*

1.6.2. Fuzzy filters and fuzzy ideals in fuzzy lattices

In the following, we give some elementary definitions and properties of fuzzy filters, for more detail see [5, 46, 47, 51]

Definition 1.18. *Let (X, R) be a fuzzy lattice. A non-constant fuzzy subset F is said to be a fuzzy filter if the following hold for all x, y in X*

1. *$F(x) > 0$ and $R(x, y) > 0$, imply $F(y) > 0$,*

2. *$F(x) > 0$ and $F(y) > 0$, imply $F(x \wedge y) > 0$.*

A fuzzy filter F is said to be a fuzzy prime filter if $F(x \vee y) \leq F(x) \vee F(y)$ for all $x, y \in X$. A fuzzy filter F is said to be maximal, if for any filter G of X, $F(x) \leq G(x)$ for all $x \in X$, implies $F = G$.

Definition 1.19. *Let (X, R) be a fuzzy lattice. A non-constant fuzzy subset I is said to be a fuzzy ideal if the following hold for all x, y in X*

1. *$I(x) > 0$ and $R(y, x) > 0$, imply $I(y) > 0$,*

2. *$I(x) > 0$ and $I(y) > 0$, imply $I(x \vee y) > 0$.*

A fuzzy ideal I is said to be a fuzzy prime ideal if $I(x \wedge y) \leq I(x) \wedge I(y)$ for all $x, y \in X$. A fuzzy ideal I is said to be maximal, if for any ideal G of X, $I(x) \leq G(x)$ for all $x \in X$, implies $I = G$.

1.6.3. Fuzzy implications and co-implication

Fuzzy implications extend the classical implications as seen in the following definitions [60].

Definition 1.20. *A binary operation $\mathcal{I} : U^2 \to U$ is an implication operator if it satisfies the boundary conditions*

(I_1) $\mathcal{I}(1,1) = \mathcal{I}(0,1) = \mathcal{I}(0,0) = 1$ *and* $\mathcal{I}(1,0) = 0$.

A fuzzy implication $\mathcal{I}$ fulfills the following properties for all $x, y, z \in U$

(I_2) $x \le z$ implies $\mathcal{I}(x,y) \ge \mathcal{I}(z,y)$;

(I_3) $y \le z$ implies $\mathcal{I}(x,y) \le \mathcal{I}(x,z)$;

(I_4) $\mathcal{I}(0,y) = 1$ (see [14]).

The most used properties of fuzzy implication operators are listed in the **Table** 1.2, for more detail see [33, 48, 60].

I_5	$\mathcal{I}(x,1) = 1$;
I_6	$\mathcal{I}(1,y) = y$;
I_7	$\mathcal{I}(x,\mathcal{I}(y,z)) = \mathcal{I}(y,\mathcal{I}(x,z))$;
I_8	$x \le y \to \mathcal{I}(x,y) = 1$;
I_9	$N_{\mathcal{I}}(x) = \mathcal{I}(x,0)$ is FN;
I_{10}	$N_{\mathcal{I}}(x) = \mathcal{I}(x,0)$ is a continuous FN;
I_{11}	$N_{\mathcal{I}}(x) = \mathcal{I}(x,0)$ is strong FN;
I_{12}	$\mathcal{I}(x,y) \le y$;
I_{13}	$\mathcal{I}(x,x) = 1$;
I_{14}	$\mathcal{I}(x,y) = \mathcal{I}(N(y),N(x))$;
I_{15}	$x > 0 \to \mathcal{I}(x,0) < 1$;
I_{16}	$y < 0 \to \mathcal{I}(1,y) < 1$;
I_{17}	$\mathcal{I}$ is continuous;
I_{18}	$\mathcal{I}(x,y) = \mathcal{I}(x,\mathcal{I}(x,y))$;
I_{19}	$\mathcal{I}(x,\mathcal{I}(y,x)) = 1$;
I_{20}	$\mathcal{I}(x,N(x)) = N(x)$.

Table 1.2: Properties of implications

Definition 1.21. *[14] For a left-continuous t-norm T, the residual implication (residuum) $\mathcal{I}$ is defined as* $\mathcal{I}(x,y) = \sup\{u \in [0,1]/T(u,x) \le y\}$.

Fuzzy co-implications extend the classical co-implications as seen in the following definitions [60].

Definition 1.22. *A binary operation $\mathcal{J} : U^2 \to U$ is a co-implication operator if it satisfies the boundary conditions*

J1: $\mathcal{J}(1,1) = \mathcal{J}(1,0) = \mathcal{J}(0,0) = 0$ *and* $\mathcal{J}(0,1) = 1$.

A fuzzy co-implication $\mathcal{J}$ fulfills the following properties for all $x, y, z \in U$

J2: $x \leq z \to \mathcal{J}(x,y) \geq \mathcal{J}(z,y)$;

J3: $y \leq z \to \mathcal{J}(x,y) \leq \mathcal{J}(x,z)$;

J4: $\mathcal{J}(1,y) = 0$.

The most used properties of fuzzy co-implication operators are listed in the **Table 1.3**, for more detail see [33, 48, 60].

J_5	$\mathcal{J}(x,0) = 0$;
J_6	$\mathcal{J}(0,y) = y$
J_7	$\mathcal{J}(x,\mathcal{J}(y,z)) = \mathcal{J}(y,\mathcal{J}(x,z))$;
J_8	$x \geq y \to \mathcal{J}(x,y) = 0$;
J_9	$N_{\mathcal{J}}(x) = \mathcal{J}(x,1)$ is FN;
J_{9b}	$N_{\mathcal{J}}(x) = \mathcal{J}(x,1)$ is a continuous FN;
J_{9c}	$N_{\mathcal{J}}(x) = \mathcal{J}(x,1)$ is strong FN;
J_{10}	$\mathcal{J}(x,y) \leq y$;
J_{11}	$\mathcal{J}(x,x) = 0$;
J_{12}	$\mathcal{J}(x,y) = \mathcal{J}(N(y),N(x))$;
J_{13}	$x < 1 \to \mathcal{J}(x,1) > 0$;
J_{14}	$y > 0 \to \mathcal{J}(0,y) > 0$;
J_{15}	$\mathcal{J}$ is continuous;
J_{16}	$\mathcal{J}(x,y) = \mathcal{J}(x,\mathcal{J}(x,y))$;
J_{17}	$\mathcal{J}(x,\mathcal{J}(y,x)) = 0$;
J_{18}	$\mathcal{J}(N(x),x) = x$.

Table 1.3: Properties of co-implications

1.6.4. Fuzzy intersection with respect to a triangular norm

Let X a non empty set and let A, $B \in \mathcal{F}(X)$. The intersection of the two fuzzy sets is specified in general by a triangular norm T. For each element x of the universal set X, the intersection of A and B, $A \cap B$ is defined by:

$$(A \cap_T B) = T(A(x), B(x)) \text{ for all } x \in X.$$

The following are examples of some t-norms that are frequently used to express a fuzzy intersections.

And their graphs illustrate the intersections with respect to some basic t-norms.

(1) **Standard Intersection**
The standard intersection is defined by

$$(A \cap_{T_M} B)(x) = \min(A(x), B(x)).$$

whose graph is shown in **Figure** 1.3.

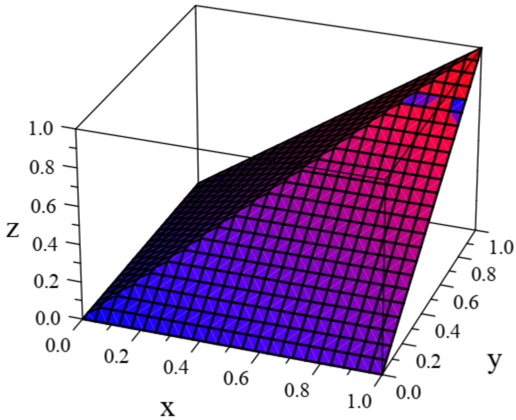


Figure 1.3: Min Intersection

(2) **Algebraic product intersection**

$$(A \cap_{T_p} B)(x) = T_p(A(x), B(x)) = (A(x) \cdot B(x)).$$

The following graph illustrates the intersection with respect to the product t-norm, whose graph is shown in **Figure** 1.4.

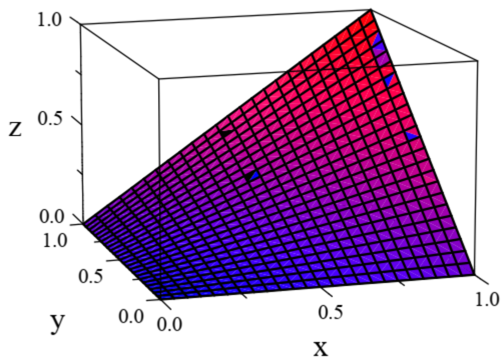


Figure 1.4: Algebraic Product Intersection

(3) **Łukasiewicz Intersection**.
The Łukasiewicz intersection (Bounded difference intersection), $(A \cap_{T_L} B)(x) = T_L(A(x), B(x)) = \max(0, A(x) + B(x) - 1)$.

(4) **Drastic intersection**
The Drastic intersection is given by

$$(A \cap_{T_D} B)(x) = T_D(A(x), B(x)) = \begin{cases} B(x) & \text{when } A(x) = 1, \\ A(x) & \text{when } B(x) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

These graphs verify the following inequalities

$(A \cap_{T_D} B)(x) \leq (A \cap_{T_L} B)(x) \leq (A \cap_{T_P} B)(x) \leq (A \cap_{T_M} B)(x)$,
for all $x \in X$.

1.6.5. Fuzzy union with respect to a triangular conorm

As the case of fuzzy intersection, fuzzy union can be defined by means of a triangular Conorm S. Hence, to the fuzzy union of two fuzzy sets A and B we associate function a function $S : [0,1] \times [0,1] \longrightarrow [0,1]$.

$(A \cup_S B)(x) = S(A(x), B(x))$
Remark 1.6. *The minimum t-norm (Zedah's t-norm) generates the smallest t-conorm among those produced by all possible t-norm.*
Theorem 1.3. *The standard fuzzy union is the only idempotent t-conorm.*

The following are the graphics of some frequently used t-conorms. They are associated to the four basic t-norms T_M, T_P, T_L and T_D, are the basic t-conorms, denoted respectively S_M, S_p, S_L, and S_D and given by:

(1) **The Standard Union**
The standard union is given by,

$$(A \cup_{S_M} B)(x) = S_M(A(x), B(x)) = \max(A(x), B(x)) \text{ (maximum)}.$$

As seen in **Figure** 1.5.

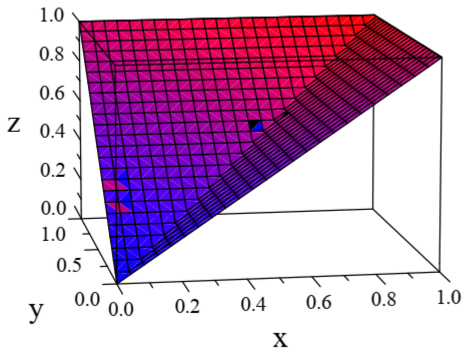


Figure 1.5: Standard Union

(2) **Probabilistic sum union**
The Probabilistic sum union is given by,

$$(A \cup_{S_p} B)(x) = S_p(A(x), B(x)) = A(x) + B(x) - A(x) \cdot B(x)$$

$S_L(x, y) = \min(x + y, 1)$,

(3) **Łukasiewicz t-conorm, bounded sum union**
The union associated to the Łukasiewicz t-conorm or (bounded sum) is given by
$(A \cup_{S_L} B)(x) = S_L(A(x), B(x)) = \min(A(x) + B(x), 1)$

(4) **Drastic Union**
The union associated to the Drastic t-conorm is given by
$$(A \cup_{S_D} B)(x) = S_D(A(x), B(x)) = \begin{cases} 1, \text{ if } (A(x), B(x)) \in]0,1]^2, \\ \max(A(x), B(x)) \text{ otherwise.} \end{cases}$$

2 Fuzzy set operations with respect to a binary operation and cutworthy property

A special property (P) will be called cutworthy when it is true that a fuzzy relation possesses (P) (in the fuzzy sense) if and only if every α -cut of the relation possesses (P) (in the crisp sense) [6, 35, 39, 64]. The concepts of α-cuts and strong α-cuts are the most important concepts of fuzzy sets.

They play a principal role in the relationship between fuzzy sets and crisp sets and can be viewed as a bridge that connecting between fuzzy sets and crisp sets. In [65] B. Šešelja shows that an L-fuzzy relation is an L-fuzzy ordering relation if and only if all its α-cuts except 0-cut are crisp ordering relations. In this chapter, we continue in the same direction by considering richer relational structures like similarity-based fuzzy orderings with respect to the binary operation. In addition, we study cut-set properties of an intersection with respect to the binary operation $\otimes$. and Cartesian product with respect to the binary operation $\otimes$. We begin this chapter by an example a fuzzy structure which possesses the cutworthy property.

2.1. Complete residuated lattice

In this subsection, we recall some basic definitions and properties of complete residuated lattices see [14, 56].

In the sequel, we suppose that the structure of truth values is a complete residuated lattice, i.e., an algebra $(L, \wedge, \vee, \rightarrow, \otimes, 0, 1)$ with four binary operations and two constants.

Definition 2.1. *[58] A complete residuated lattice is an algebraic structure* $(L, \otimes, \rightarrow)$*, or simply,* L*, where*

(i) $(L, \leq, \wedge, \vee, 0, 1)$ *forms a complete lattice with the smallest element* 0 *and the greatest element* 1*;*

(ii) $(\otimes, \rightarrow)$ *forms an adjoint couple on* L*, i.e., for any* $a, b, c \in L$*,*

(R1) if $a \leq b$*,* $c \leq d$*, then* $a \otimes c \leq b \otimes d$*;*

(R2) if $b \leq c$*, then* $a \rightarrow b \leq a \rightarrow c$*;*

(R3) *if* $a \leq b$*, then* $b \to c \leq a \to c$*;*

(R4) $a \otimes b \leq c \Leftrightarrow a \leq b \to c$*. (Adjointness condition)*

(iii) $(L, \otimes, 1)$ *forms a commutative monoïd, i.e., for any* $a, b, c \in L$*,*

(R5) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$*;*

(R6) $a \otimes b = b \otimes a$*;*

(R7) $1 \otimes a = a$*.*

Proposition 2.1. *In a residuated lattice* $(L, \wedge, \vee, \to, \otimes, 0, 1)$ *the following inequalities hold for every* x, y *in* L*.*

1. $x \otimes y \leq x \wedge y$*,*

2. $(\wedge_{i \in I} x_i) \otimes (\wedge_{i \in I} y_i) \leq (\wedge_{i \in I}(x_i \otimes y_i))$ *with* I *is a finite index set.*

Proof. Straightforward from **Definition** 2.1. □

As mentioned above, throughout this thesis, the structure of truth values $(L, \wedge, \vee, \to, \otimes, 0, 1)$ will be a complete residuated lattice,
and we denote $L^0 = L - \{0\}$ and $L^1 = L - \{1\}$.

2.1.1. Similarity based fuzzy ordering

Definition 2.2. *A fuzzy relation* R *on a non-empty set* X *is called:*

1. $\otimes$*-antisymmetric if and only if* $R(x, y) \otimes R(y, x) = 0$*, for all* $x, y \in X$*, whenever* $x \neq y$*.*

2. *Transitive if and only if* $R(x, y) \wedge R(y, z) \leq R(x, z)$*, for all* $x, y, z \in X$*.*

3. $\otimes$*-transitive if and only if*
$R(x, y) \otimes R(y, z) \leq R(x, z)$*, for all* $x, y, z \in X$*.*

Definition 2.3. *A reflexive and* $\otimes$*-transitive fuzzy relation is called fuzzy preordering with respect to* $\otimes$ *(*$\otimes$*-preordering for short). An* $\otimes$*-preordering which is in addition, symmetric is called fuzzy equivalence relation with respect to* $\otimes$ *(*$\otimes$*-equivalence for short).*

Definition 2.4. *An* $\otimes$*-preordering is called fuzzy ordering with respect to* $\otimes$ *(*$\otimes$*-ordering for short), if and only if the following axiom called* $\otimes$*-antisymmetric is fulfilled for all* $x, y \in X$*,*

$$x \neq y \Rightarrow R(x, y) \otimes R(y, x) = 0.$$

Definition 2.5. *Let $R : X^2 \to L$ be a binary fuzzy relation. R is called fuzzy ordering with respect to $\otimes$ and a fuzzy $\otimes$-equivalence relation E on the same domain X ($\otimes$-E-ordering for short) if and only if it fulfills the following three axioms:*
$(P.1)$ E-reflexivity, i.e., for all $x, y \in X$, $E(x,y) \leq R(x,y)$;
$(P.2)$ $\otimes$-E-antisymmetry, i.e.,
for all $x, y \in X$, $R(x,y) \otimes R(y,x) \leq E(x,y)$;
$(P.3)$ $\otimes$-transitivity, i.e., for all $x, y, z \in X$, $R(x,y) \otimes R(y,z)) \leq R(x,z)$.

In the boolean case, the fuzzy equivalence relation E and the fuzzy relation R on X correspond to an equivalence relation $\approx$ and a crisp binary relation $\preceq$ on X, and axioms (P1–P3) turn to the following conditions, respectively:

(1) $x \approx y \Rightarrow x \preceq y$, for all $x, y \in X$;

(2) $(x \preceq y$ and $y \preceq x) \Rightarrow x \approx y$, for all $x, y \in X$;

(3) $(x \preceq y$ and $y \preceq z) \Rightarrow x \preceq z$, for all $x, y, z \in X$.

2.2. An introductory example

Let X be a fuzzy set such that $X = \{0, a, b, c, d, 1\}$ and Let $R : X \times X \to [0,1]$ be a fuzzy relation defined on the set X by the **Table** 2.1.

R	0	a	b	c	d	1
0	1.0	0.1	0.3	0.3	0.5	0.7
a	0.0	1.0	0.2	0.2	0.4	0.6
b	0.0	0.0	1.0	0.0	0.2	0.3
c	0.0	0.0	0.0	1.0	0.2	0.3
d	0.0	0.0	0.0	0.0	1.0	0.2
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.1:

It is not difficult to verify that R is an $\wedge$-fuzzy ordering.

In the following, we give all α-*cuts* of R, for any $\alpha \in]0,1]$.

For this, we begin by the subinterval $]0.0, 0.1[$ in which all α-*cuts* of R are equal. In a similar way, we continue the calculation for intervals $[0.1, 0.2[$, $[0.2, 0.3[$,..., to $[0.7, 0.8[$, finally we finish by the interval $[0.8, 1]$. It should be noted that in all previous intervals, all α-*cuts* of R are equal.

Firstly, we calculate R_α for $\alpha \in]0.0, 0.1[$.

R_α	0	a	b	c	d	1
0	1.0	1.0	1.0	1.0	1.0	1.0
a	0.0	1.0	1.0	1.0	1.0	1.0
b	0.0	0.0	1.0	0.0	1.0	1.0
c	0.0	0.0	0.0	1.0	1.0	1.0
d	0.0	0.0	0.0	0.0	1.0	1.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.2: R_α for $\alpha \in]0.0, 0.1[$.

Whose Hasse diagram is shown in **Figure** 2.1.

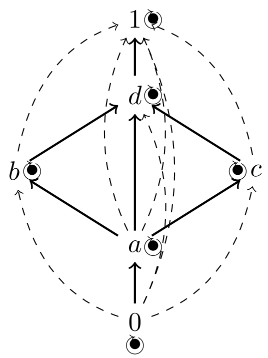


Figure 2.1: Hasse Diagram of R_α

We continue by calculating the cut $R_{0.1}$ given by **Table** 2.3.

$R_{0.1}$	0	a	b	c	d	1
0	1.0	1.0	1.0	1.0	1.0	1.0
a	0.0	1.0	1.0	1.0	1.0	1.0
b	0.0	0.0	1.0	0.0	1.0	1.0
c	0.0	0.0	0.0	1.0	1.0	1.0
d	0.0	0.0	0.0	0.0	1.0	1.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.3: $\alpha - cuts$ of R, for $\alpha \in [0.1, 0.2[$

Whose Hasse diagram is the same as that of R_α for $\alpha \in]0.0, 0.1[$.

And the cut $R_{0.2}$ given by **Table** 2.4.

$R_{0.2}$	0	a	b	c	d	1
0	1.0	0.0	1.0	1.0	1.0	1.0
a	0.0	1.0	1.0	1.0	1.0	1.0
b	0.0	0.0	1.0	0.0	1.0	1.0
c	0.0	0.0	0.0	1.0	0.0	1.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.4: $\alpha - cuts$ of R, for $\alpha \in [0.2, 0.3[$

And its Hasse diagram is given by **Figure** 2.2.

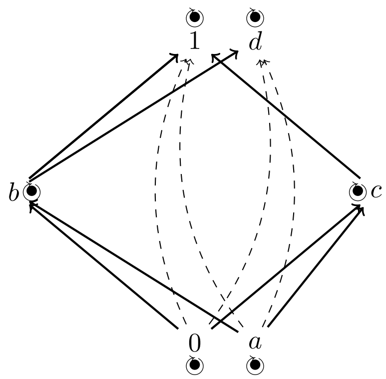


Figure 2.2: Hasse Diagram of the cut $R_{0.2}$

In the same way, we calculate $R_{0.3}$ given by **Table** 2.5.

$R_{0.3}$	0	a	b	c	d	1
0	1.0	0.0	1.0	1.0	1.0	1.0
a	0.0	1.0	0.0	0.0	1.0	1.0
b	0.0	0.0	1.0	0.0	0.0	1.0
c	0.0	0.0	0.0	1.0	0.0	1.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.5: $\alpha - cuts$ of R, for $\alpha \in [0.3, 0.4[$

Whose its Hasse digram is given by **Figure** 2.3.

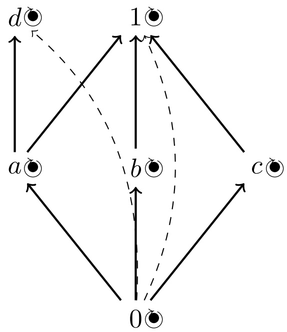


Figure 2.3: Hasse Diagram of the cut $R_{0.3}$

As above, we calculate $R_{0.4}$ given by **Table** 2.6.

$R_{0.4}$	0	a	b	c	d	1
0	1.0	0.0	0.0	0.0	1.0	1.0
a	0.0	1.0	0.0	0.0	1.0	1.0
b	0.0	0.0	1.0	0.0	0.0	0.0
c	0.0	0.0	0.0	1.0	0.0	0.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.6: $\alpha - cuts$ of R, for $\alpha \in [0.4, 0.5[$

Whose its Hasse digram is given by **Figure** 2.4.

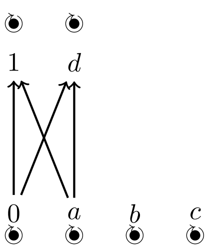


Figure 2.4: Hasse Diagram of the cut $R_{0.4}$

Similarly, we calculate $R_{0.5}$ given by **Table** 2.7.

$R_{0.5}$	0	a	b	c	d	1
0	1.0	0.0	0.0	0.0	1.0	1.0
a	0.0	1.0	0.0	0.0	0.0	1.0
b	0.0	0.0	1.0	0.0	0.0	0.0
c	0.0	0.0	0.0	1.0	0.0	0.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.7: $\alpha - cuts$ of R, for $\alpha \in [0.5, 0.6[$

Whose its Hasse digram is given by **Figure** 2.5.

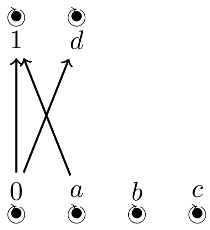


Figure 2.5: Hasse Diagram of the cut $R_{0.5}$

likewise, we calculate $R_{0.6}$ given by **Table** 2.8.

$R_{0.6}$	0	a	b	c	d	1
0	1.0	0.0	0.0	0.0	0.0	1.0
a	0.0	1.0	0.0	0.0	0.0	1.0
b	0.0	0.0	1.0	0.0	0.0	0.0
c	0.0	0.0	0.0	1.0	0.0	0.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.8: $\alpha - cuts$ of R, for $\alpha \in [0.6, 0.7[$

Whose its Hasse digram is given by **Figure** 2.6.

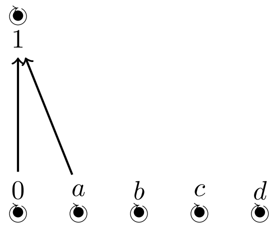


Figure 2.6: Hasse Diagram of $R_{0.6}$

And we calculate $R_{0.7}$ given by **Table** 2.9.

$R_{0.7}$	0	a	b	c	d	1
0	1.0	0.0	0.0	0.0	0.0	1.0
a	0.0	1.0	0.0	0.0	0.0	0.0
b	0.0	0.0	1.0	0.0	0.0	0.0
c	0.0	0.0	0.0	1.0	0.0	0.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.9: $\alpha - cuts$ of R, for $\alpha \in [0.7, 0.8[$

Whose its Hasse digram is given by **Figure** 2.7.

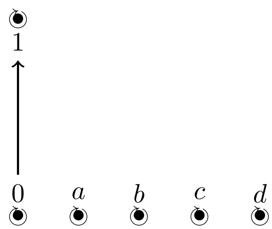


Figure 2.7: Hasse Diagram of R

Finally, we calculate $R_{0.8}$ given by **Table** 2.10.

$R_{0.8}$	0	a	b	c	d	1
0	1.0	0.0	0.0	0.0	0.0	0.0
a	0.0	1.0	0.0	0.0	0.0	0.0
b	0.0	0.0	1.0	0.0	0.0	0.0
c	0.0	0.0	0.0	1.0	0.0	0.0
d	0.0	0.0	0.0	0.0	1.0	0.0
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 2.10: The cut $R_{0.8}$

Whose Hasse diagram is shown in **Figure** 2.8.

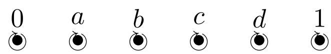


Figure 2.8: Hasse Diagram of the cut $R_{0.8}$

Remark 2.1. *Concerning the cuts of the interval* $[0.8, 1.0]$, *they are same likewise the cut* $R_{0.8}$. *As* R *is a fuzzy ordering whose its* α*-cuts are partially ordered sets (as shown in their Hasse diagrams), then* R *possesses the cut-worthy property.*

The next section continues the study of α*-cuts giving more properties and examples. After, we give and proof some properties related to some generalized sets operations, equivalence relations with respect to an extended t-norm, similarity-based fuzzy orderings with respect to a binary operation.*

2.3. α-cuts: An overview

An algebraic property (P) is cut-worthy if it can be transferred from the fuzzy set to its cuts and vice versa see [6, 35, 39, 64, 65].

2.3.1. Definitions and properties

Let X *be a nonempty set and let* A *be a fuzzy set defined on* X. *For any* $\alpha \in]0, 1]$, *the* α*-cut,* A_α, *and the strong* α*-cut,* $A_\alpha^>$, *are the crisp sets defined by*

$$\begin{aligned} A_\alpha &= \{x \in X \diagup A(x) \geq a\}; \\ A_\alpha^> &= \{x \in X \diagup A(x) > a\}. \end{aligned}$$

This means that, the α-cut (or the strong α-cut) of a fuzzy set A is the crisp set A_α (or the crisp set $A_\alpha^>$) that contains all the elements of the universal set X whose membership grades in A are greater than or equal to (or only greater than) the specified value of α. An important common property of both α-cuts and strong α-cuts, which comes immediately from their definitions, is that the total ordering of values of an α in $[0,1]$ is inversely preserved by set inclusion of the corresponding α-cuts as well as strong α-cuts. That is, for any fuzzy set A and pair $\alpha_1, \alpha_2 \in [0,1]$ of distinct values such that $\alpha_1 < \alpha_2$ we have $A_{\alpha_2} \subseteq A_{\alpha_1}$ and $A_{\alpha_2}^> \subseteq A_{\alpha_1}^>$. This property can also be expressed by the equations

$$A_{\alpha_2} \cap A_{\alpha_1} = A_{\alpha_2}, A_{\alpha_2}^> \cap A_{\alpha_1}^> = A_{\alpha_2}^>,$$

and

$$A_{\alpha_2} \cup A_{\alpha_1} = A_{\alpha_1}, A_{\alpha_2}^> \cup A_{\alpha_1}^> = A_{\alpha_1}^>.$$

In addition, from the definitions of α-cuts and strong α-cuts, it is easy to see that $A_\alpha^> \subseteq A_\alpha$.

Let $A, B \in \mathcal{F}(X)$. Then, for all $\alpha \in [0,1]$,

$A \subset B$ iff $A_\alpha \subset B_\alpha$ and $A \subset B$ iff $A_\alpha^> \subset B_\alpha^>$,
$A = B$ iff $A_\alpha = B_\alpha$ and $A = B$ iff $A_\alpha^> = B_\alpha^>$.

2.3.2. Example of a fuzzy set and its cuts

The following example illustrates the most basic concepts of fuzzy sets and its cuts introduced in this thesis.

Example 2.1. *Consider a universal set X which is defined on the age domain. $X = \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$, and $\mu : X \to [0,1]$ the membership function given by **Table** 2.11.*

Age	Infant	Young	Adult	Senior
5	0.00	0.00	0.00	0.00
15	0.00	0.20	0.10	0.00
25	0.00	1.00	0.90	0.00
35	0.00	0.80	1.00	0.00
45	0.00	0.40	1.00	0.10
55	0.00	0.10	1.00	0.20
65	0.00	0.00	1.00	0.60
75	0.00	0.00	1.00	1.00
85	0.00	0.00	1.00	1.00

Table 2.11:

*We can define fuzzy sets such as "Infant", "Young", "Adult" and "Senior" in X. The possibilities of each element of X to be in those four fuzzy sets are given by **Table** 2.11.*

We can think of a set that is made up of elements contained in A. This set is called "Support" of A, where Supp(A) $= \{x \in X | \mu_A(x) > 0\}$.

For example the support of the set "Young" is the below crisp set:

Supp(Young) $= \{15, 25, 35, 45, 55\}$.

The α-cut set is derived from fuzzy set "Young" by giving $\alpha = 0.2$.

$(Young)_{\ 0.2} = \{12, 25, 35, 45\}$. *This means the age that we can say youth with possibility not less than* 0.2.

If $\alpha = 0.4$, *(Young)*$_{0.4} = \{25, 35, 45\}$.

If $\alpha = 0.8$, *(Young)*$_{0.8} = \{25, 35\}$.

For any $\alpha, \beta \in [0, 1]$, *if* $\alpha \leq \beta$, *then* $A_\beta \subseteq A_\alpha$.

For example the set (Young)$_{0.8} \subseteq$ *(Young)*$_{0.2}$ *holds.*

If we denote the fuzzy set (Adult) by A, we have the standard complement
$\mathcal{C}A = \{(5, 1), (15, 0.9), (25, 0.1)\}$.

Now, we calculate the Yager fuzzy complement $(\mathcal{C}A_\omega)$, *for* $\omega = 2$,
$\mathcal{C}A_2 = \{(5, 1), (15, \sqrt[2]{0.99}), (25, \sqrt[2]{0.081})\}$.

Standard union of $(Young)$ *and (Adult) is*

$(Young) \cup_\vee$ *(Adult)*
$= \{(15, 0.2), (25, 1.0), (35, 1.0), (45, 1.0), (55, 1.0), (65, 1.0), (75, 1.0), (85, 1.0)\}$.

We calculate now the algebraic sum union of $(Young)$ *and (Adult) as B i.e.,* $(B(x) = \min(1, (Young)\,(x)) + (Adult)\,(x)))$.

$B = \{(15, 0.3), (25, 1), (35, 1), (45, 1), (55, 1), (65, 1), (75, 1), (85, 1)\}$. *Now, let's calculate the fuzzy set C given as* $(Young) \cap_\wedge$ *(Adult),*
$C(x) = ((Young) \cap_\wedge (Adult))(x) = (Young)(x) \wedge (Adult)(x)$. *A simple calculation gives,*
$C = \{(15, 0.1), (25, 0.9), (35, 0.8), (45, 0.4), (55, 0.1)\}$.
Analogously, we calculate the fuzzy set D as $(Young) \cap_P$ *(Adult), we obtain,*
$D = \{(15, 0.02), (25, 0.9), (35, 0.8), (45, 0.4), (55, 0.1)\}$.

2.4. Cut-set properties of an intersection with respect to the binary operation $\otimes$

In [65], B. Seselja shows that an L-fuzzy relation is an L-fuzzy ordering relation if and only if all its α-cuts except 0-cut are crisp ordering relations.

2.4.1. Some results related to the intersection with respect to the binary operation

In the following, we give and proof some properties related to some generalized sets operations.

Definition 2.6. *[54] Let A and B be two fuzzy sets on a non-empty set X. The fuzzy set $A \cap_{\otimes} B$ on X defined by*

$$(A \cap_{\otimes} B)(x) = A(x) \otimes B(x) \;\; \forall x \in X,$$

is called the fuzzy intersection of A and B with respect to $\otimes$.

Proposition 2.2. *Let A and B be two fuzzy sets on a non-empty set X. Then $(A \cap_{\otimes} B)_{\alpha} \subseteq A_{\alpha} \cap B_{\alpha}$ for any $\alpha \in L^0$.*

Proof. (Straightforward). □

Remark 2.2. *In general, this inclusion is strict, as shown in the following example.*

Example 2.2. *Let $A = \{(a, 0.5), (b, 0.8), (c, 0.6), (d, 0.4)\}$, $B = \{(a, 0.1), (b, 0.4), (c, 0.7), (d, 0.2)\}$, and take $\otimes = T_p, L = [0, 1]$ and $\alpha = 0.5$. Then $A_{0.5} = \{a, b, c\}$, $B_{0.5} = \{c\}$, hence $A_{0.5} \cap B_{0.5} = \{c\}$, furthermore we have $A \cap_{T_p} B = \{(a, 0.05), (b, 0.32), (c, 0.42), (d, 0.08)\}$, so $\left(A \cap_{T_p} B\right)_{0.5} = \phi$, hence $\left(A \cap_{T_p} B\right)_{0.5} \subset (A_{0.5} \cap B_{0.5})$.*

2.4.2. Cartesian product of fuzzy sets with respect to the binary operation $\otimes$

Definition 2.7. *[54] Consider a finite family of non-empty crisp sets $(X_1, ..., X_n)$, and a family of fuzzy sets $(A_1, ..., A_n)$, where A_i is a fuzzy set on X_i. The fuzzy set $\breve{A}$ defined by*

$$\breve{A} : \prod_{i=1}^{n} X_i \to L$$

$$(x_1, ..., x_n) \to \breve{A}(x_1, ..., x_n)$$

$$\breve{A}(x_1, ..., x_n) = A_1(x_1) \otimes ... \otimes A_n(x_n) = \prod_{i=1}^{n}{}_{\otimes} A_i(x_i),$$

is called the fuzzy product set of A_i with respect to the $\otimes$.

Proposition 2.3. *Consider a finite family of fuzzy sets $A_1, ..., A_n$ on a finite family of non-empty crisp sets $X_1, ..., X_n$ respectively. For any $\alpha \in L$, we have $(\breve{A})_\alpha \subseteq (A_1)_\alpha \times ... \times (A_n)_\alpha$.*

Proof. Let $\alpha \in L$, $(x_1, ..., x_n) \in (X_1, ..., X_n)$. $(x_1, ..., x_n) \in (\breve{A})_\alpha$ means that $\breve{A}(x_1, ..., x_n) \geq \alpha$, this is equivalent to $[A_1(x_1)] \otimes ... \otimes [A_n(x_n)] \geq \alpha$. Hence $A_1(x_1) \wedge ... \wedge A_n(x_n) \geq \alpha$, this implies that $A_i(x_i) \geq \alpha$ for any $i \in \{1, ..., n\}$, which gives $x_i \in (A_i)_\alpha$ for all $i \in \{1, ..., n\}$.
Thus $(x_1, ..., x_n) \in (A_1)_\alpha \times ... \times (A_n)_\alpha$.
Consequently, $(\breve{A})_\alpha \subseteq (A_1)_\alpha \times ... \times (A_n)_\alpha$. □

Remark 2.3. *In general, this inclusion is strict, as seen in the following example.*

Example 2.3. *Consider $X_1 = \{a, b\}$, $X_2 = \{c, d, e\}$ and $L = [0, 1]$. The fuzzy sets $A_1 = \{(a, 1), (b, 0.4)\}$, $A_2 = \{(c, 0.3), (d, 0.2), (e, 0.9)\}$, $\otimes = T_L$ and $\alpha = 0.3$,*
$(\prod_{i=1}^{2}{}_{T_L} A_i) =$
$\{((a, c), 0.3), ((a, d), 0.2), ((a, e), 0.9), ((b, c), 0), ((b, d), 0), ((b, e), 0.3)\}.$
Then $(\prod_{i=1}^{2}{}_{T_L} A_i)_{0.3} = \{(a, c), (a, e), (b, e)\}$.
Furthermore, we have $(A_1)_{0.3} = \{a, b\}, (A_2)_{0.3} = \{c, e\}$, *which implies*
$(A_1)_{0.3} \times (A_2)_{0.3} = \{(a, c), (a, e), (b, c), (b, e)\}$.
Hence, $(\prod_{i=1}^{2}{}_{T_L} A_i)_\alpha \subseteq (A_1)_{0.3} \times (A_2)_{0.3}$.

Remark 2.4. *If $\otimes = \wedge$, then the equality in **Theorem** 2.3 holds i.e.,*
$(\prod_{i=1}^{n}{}_{\wedge} A_i)_\alpha = (A_1)_\alpha \times ... \times (A_n)_\alpha$.
*Indeed, by **Theorem** 2.3, we have* $(\prod_{i=1}^{n}{}_{\wedge} A_i)_\alpha \subseteq (A_1)_\alpha \times ... \times (A_n)_\alpha$. *It remains to prove the converse inclusion.*
Let $(x_1, ..., x_n) \in (A_1)_\alpha \times ... \times (A_n)_\alpha$ *this is equivalent to* $x_i \in (A_i)_\alpha$ *for any* $i \in \{1, ..., n\}$ *which means that:*
$A_i(x_i) \geq \alpha$ *for any* $i \in \{1, ..., n\}$. *Hence,* $A_1(x_1) \wedge ... \wedge A_n(x_n) \geq \alpha$. *So,*
$\prod_{i=1}^{n}{}_{\wedge} A_i(x_i) \geq \alpha$, *i.e.,* $(x_1, ..., x_n) \in (\prod_{i=1}^{n}{}_{\wedge} A_i)_\alpha$. *This complete the prove.*

Example 2.4. *In **Example** 2.3, if we replace T_L by T_M the equality holds. Indeed,*
$(\prod_{i=1}^{2}{}_{T_M} A_i) =$
$\{((a, c), 0.3), ((a, d), 0.2), ((a, e), 0.9), ((b, c), 0.3), ((b, d), 0.2), ((b, e), 0.4)\}$. *Then*
$(\prod_{i=1}^{2}{}_{T_M} A_i)_{0.3} = \{(a, c), (a, e), (b, c), (b, e)\}$.
Furthermore, $(A_1)_{0.3} = \{a, b\}$, $(A_2)_{0.3} = \{c, e\}$.

Hence, $(A_1)_{0.3} \times (A_2)_{0.3} = \{(a,c),(a,e),(b,c),(b,e)\}$.

Consequently, $(A_1)_{0.3} \times (A_2)_{0.3} = (\prod_{T_M}{}_{i=1}^{2} A_i(x_i))_{0.3}$.

2.5. Similarity-based fuzzy orderings and cutworthy properties

In this section, we study ⊗-fuzzy orderings and similarity-based fuzzy orderings and their relationship with the Cutworthy properties.

2.5.1. Fuzzy orders and cutworthy properties

Theorem 2.1. *Let X be a nonempty set and let $R : X^2 \to L$ be a fuzzy relation on X. If all α-cuts of R are crisp ordering relations, then R is an $\otimes$-fuzzy ordering relation.*

Proof. Let $R : X^2 \to L$ be a fuzzy relation and let $\alpha \in L$. Since all α-cuts are reflexive relations, then $R(x,x) \geq \alpha$ for all $\alpha \in L^0$, hence $R(x,x) = 1$. Further, let $x, y \in X$ such that
$R(x,y) \otimes R(y,x) > 0$ this means that $R(x,y) \otimes R(y,x) = \alpha$ for some $\alpha \in L^0$.

Since $R(x,y) \wedge R(y,x) \geq R(x,y) \otimes R(y,x) = \alpha$, this gives $R(x,y) \geq \alpha$ and $R(y,x) \geq \alpha$. It follows that $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$, hence $x = y$ and the $\otimes$-antisymmetry of R is obtained.

Finally, to verify the $\otimes$-transitivity, take $x, y, z \in X$ and suppose that
$R(x,y) \otimes R(y,z) = \alpha$. Since, $\alpha = R(x,y) \otimes R(y,z) \leq R(x,y) \wedge R(y,z)$, then
$R(x,y) \geq \alpha$ and $R(y,z) \geq \alpha$. So, $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$, then $(x,z) \in R_\alpha$, and so $R(x,z) \geq R(x,y) \otimes R(y,z)$. □

Remark 2.5. *The reciprocal of **Theorem** 2.1 is not valid. It suffices to consider the following example*
Let $X = \{a,b,c\}$, and let $\otimes$ be the Łukasiewicz triangular norm T_L.

R	a	b	c
a	1	0.4	0.5
b	0.4	1	0.4
c	0.3	0.3	1

,

$R_{0.4}$	a	b	c
a	1	1	1
b	1	1	1
c	0	0	1

.

Obviously, the fuzzy relation R is an $\otimes$-partial ordering. But $R_{0.4}$ is not a crisp partial ordering relation. Indeed, $R_{0.4}(a,b) \wedge R_{0.4}(b,a) = 1$ but $a \neq b$, hence $R_{0.4}$ is not antisymmetric.

2.5.2. Fuzzy equivalence and cutworthy properties

Theorem 2.2. *Let X be a nonempty set and let $E : X^2 \to L$ be a fuzzy relation. If all cuts of E are equivalence relations, then E is an $\otimes$-fuzzy equivalence relation.*

Proof. As in **Theorem 2.1**, we can easily verify that E is reflexive and $\otimes$-transitive. To verify the symmetry of E, let $x, y \in X$, and suppose that $E(y,x) = \alpha$.

If $\alpha = 0, E(x,y) \geq E(y,x)$.

If $\alpha \neq 0$, then

$$\begin{aligned} E(y,x) = \alpha \quad &\Rightarrow \quad E(y,x) \geq \alpha \\ &\Rightarrow \quad (y,x) \in E_\alpha \\ &\Rightarrow \quad (x,y) \in E_\alpha \\ &\Rightarrow \quad E(x,y) \geq \alpha \\ &\Rightarrow \quad E(x,y) \geq E(y,x). \end{aligned}$$

In the same way, we can obtain that $E(y,x) \geq E(x,y)$.

Hence, $E(x,y) = E(y,x)$. □

Remark 2.6. *The converse of the above theorem is not true in general, as seen in the following example.*

Example 2.5. *Take $X = \{a,b,c\}$, $L = [0,1]$, $T = T_p$ and consider the relations E, $E_{0.5}$ given by*

E	a	b	c
a	1	0.5	0.5
b	0.5	1	0.25
c	0.5	0.25	1

,

$E_{0.5}$	a	b	c
a	1	1	1
b	1	1	0
c	1	0	1

.

It is easy to verify that E is T_P-equivalence, but $E_{0.5}$ is not a crisp equivalence relation ($E_{0.5}$ is not transitive).

Theorem 2.3. *A relation $E : X^2 \to L$ is a T_M-fuzzy equivalence relation if and only if all cuts of E are crisp equivalence relations.*

Proof. Let $E : X^2 \to L$ be a fuzzy relation. Suppose that all cuts are equivalence relations. As seen in **Theorem 2.2**, it is not difficult to verify that E is T_M-fuzzy equivalence relation.

On the other hand, suppose that E is a T_M-fuzzy equivalence relation and let $\alpha \in L^0$, $E(x,x) = 1 \geq \alpha$, hence $(x,x) \in E_\alpha$.

If $(x,y) \in E_\alpha$, then $E(x,y) \geq \alpha$, this implies that $E(y,x) \geq \alpha$, hence $(y,x) \in E_\alpha$ and then E_α is symmetric. For the transitivity, let $(x,y,z) \in X$ such that $(x,y) \in E_\alpha$ and $(y,z) \in E_\alpha$, then $E(x,y) \geq \alpha$ and $E(y,z) \geq \alpha$ so $E(x,y) \wedge E(y,z) \geq \alpha$. Consequently, $E(x,z) \geq E(x,y) \wedge E(y,z) \geq \alpha$, hence $(x,z) \in E_\alpha$. Which complete the proof. □

2.5.3. Similarity-based fuzzy orderings with respect to a binary operation and cutworthy properties

Theorem 2.4. *Let $E : X^2 \to L$ and $R : X^2 \to L$ be two fuzzy relations. If for any $\alpha \in L^0$ the cuts E_α are crisp equivalence relations and R_α are E_α-crisp partial orderings, then R is an $\otimes$-E-fuzzy ordering relation.*

Proof. Let R be a fuzzy relation, and suppose that for all $\alpha \in L^0$, R_α is E_α-crisp ordering relation. As seen in **Theorem** 2.2, E is an $\otimes$-equivalence relation and R is an $\otimes$-transitive relation. It stills to prove that R is E-reflexive and E-$\otimes$-antisymmetric. To prove the E-reflexivity of R. Take $E(x,y) = \alpha$,

$$\begin{aligned} E(x,y) = \alpha &\Rightarrow E(x,y) \geq \alpha \\ &\Rightarrow (x,y) \in E_\alpha \\ &\Rightarrow (x,y) \in R_\alpha \\ &\Rightarrow R(x,y) \geq \alpha. \end{aligned}$$

Hence, $R(x,y) \geq E(x,y)$. To prove the $\otimes$-E-antisymmetry property of R suppose that $R(x,y) \otimes R(y,x) = \alpha$,

$$\begin{aligned} R(x,y) \otimes R(y,x) = \alpha &\Rightarrow R(x,y) \wedge R(y,x) \geq \alpha \\ &\Rightarrow R(x,y) \geq \alpha \text{ and } R(y,x) \geq \alpha \\ &\Rightarrow (x,y) \in R_\alpha \text{ and } (y,x) \in R_\alpha \\ &\Rightarrow (x,y) \in E_\alpha \\ &\Rightarrow E(x,y) \geq \alpha. \end{aligned}$$

Hence, $E(x,y) \geq R(x,y) \otimes R(y,x)$.

The $\otimes$-transitivity is direct from **Theorem** 2.1. □

Remark 2.7. *The converse of the above theorem is not true in general as illustrated in the below example.*

Example 2.6. *Let R and E be the fuzzy relations given by the following tables*

R	a	b	c
a	1	0.6	0.7
b	0.8	1	0.8
c	0.8	0.8	1

,

E	a	b	c
a	1	0.5	0.6
b	0.5	1	0.7
c	0.6	0.7	1

and take $\otimes = T_P$ and $L = [0,1]$. It is easy to verify that E is an $\otimes$-equivalence relation and R is an $\otimes$-E-fuzzy ordering relation, but $R_{0.6}$ is not an $E_{0.6}$-crisp ordering because $E_{0.6}$ is not transitive. Indeed, according to the tables

$R_{0.6}$	a	b	c
a	1	1	1
b	1	1	1
c	1	1	1

,

$E_{0.6}$	a	b	c
a	1	0	1
b	0	1	1
c	1	1	1

.

$E_{0.6}(a,c) \wedge E_{0.6}(c,b) = 1 \nleq E_{0.6}(a,b) = 0.$

Theorem 2.5. *Let $E : X^2 \to L$ be a T_M-equivalence relation and let $R : X^2 \to L$ be a fuzzy relation. R is a T_M-E-fuzzy ordering relation if and only if for all $\alpha \in L^0$, the cuts R_α are E_α-partial orderings.*

Proof. Suppose the cuts R_α are E_α-partial orderings. The fact that R is a T_M-E-fuzzy ordering relation comes from the **Theorem** 2.4.

It thus remains to prove the converse: take $\alpha \in L^0$

$$\begin{aligned}(x,y) \in E_\alpha &\Rightarrow E(x,y) \geq \alpha \\ &\Rightarrow R(x,y) \geq E(x,y) \geq \alpha \\ &\Rightarrow (x,y) \in R_\alpha.\end{aligned}$$

Hence $R_\alpha(x,y) \geq E_\alpha(x,y)$, and R_α is E_α-reflexive.

To prove that R_α is E_α-antisymmetric, verify that for $(x,y) \in X^2$

$$R_\alpha(x,y) \wedge R_\alpha(y,x) = \begin{cases} 1 \text{ if } (x,y) \in R_\alpha \text{ and } (y,x) \in R_\alpha; \\ 0 \text{ otherwise.} \end{cases}$$

$$\begin{aligned}\text{If } (x,y) \in R_\alpha \text{ and } (y,x) \in R_\alpha &\Rightarrow R(x,y) \geq \alpha \text{ and } R(y,x) \geq \alpha \\ &\Rightarrow R(x,y) \wedge R(y,x) \geq \alpha \\ &\Rightarrow E(x,y) \geq \alpha \\ &\Rightarrow (x,y) \in E_\alpha.\end{aligned}$$

The case $R_\alpha(x,y) \wedge R_\alpha(y,x) = 0$ is trivial.

Hence R_α is T_M-E_α-antisymmetric.

Finally, the transitivity of R_α is obtained from [65]. □

Remark 2.8. *In fact, R is the crisp equality.*

Indeed, according to [26, 45], Theorem 4.4., we have

$$E(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

i.e., E is the crisp equality.

2.6. Intersection and Cartesian product of fuzzy relations with respect to a binary operation

2.6.1. Intersection of fuzzy relations with respect to a binary operation

This section studies the relationship between the families $(R_i)_{i\in\mathbb{N}}$ and $(E_i)_{i\in\mathbb{N}}$ and their intersection, where each E_i is an $\otimes$-fuzzy equivalence relation on X and R_i is an $\otimes$-E_i-fuzzy partial ordering on X.

In other words, we study the relation between $R(x,y) = \bigwedge_{i\in\mathbb{N}} R_i(x,y)$ and $E(x,y) = \bigwedge_{i\in\mathbb{N}} E_i(x,y)$.

The following proposition shows that to each fuzzy $\otimes$-preordering there corresponds a fuzzy $\otimes$-equivalence relation.

Proposition 2.4. *Let X be a nonempty set and let R be an $\otimes$-fuzzy preordering on X. Then the fuzzy relation defined by*

$$E(x,y) = R(x,y) \wedge R(y,x), \text{ for all } x,y \in X.$$

is an $\otimes$-equivalence relation.

Proof. Using the fact that $\min(x,y)$ is less than both x and y, it is easy to see that E is reflexive and symmetric. It remains to prove that E is $\otimes$-transitive.
Let $x,y,z \in X$,

$$\begin{aligned} E(x,y)\otimes E(y,z) &= (R(x,y)\wedge R(y,x))\otimes(R(y,z)\wedge R(z,y)) \\ &\leq (R(x,y)\otimes R(y,z))\wedge(R(y,x)\otimes R(z,y)),\ (\textbf{ Proposition } 2.1) \\ &= (R(x,y)\otimes R(y,z))\wedge(R(z,y)\otimes R(y,x)) \\ &\leq R(x,z)\wedge R(z,x)=E(x,z). \end{aligned}$$

Hence E is $\otimes$-transitive, thus E is an $\otimes$-equivalence relation. □

Theorem 2.6. *Let X be a nonempty set and let R be a fuzzy $\otimes$-preordering on X. Then R is an $\otimes$-E-fuzzy ordering. Where E is the $\otimes$-equivalence relation defined in **Proposition** 2.4.*

Proof. It is easy to see that $E(x,y)=R(x,y)\wedge R(y,x)\leq R(x,y)$ for any $x,y\in X$, hence R is E-reflexive. To prove the $\otimes$-E-antisymmetry of R, take $x,y\in X$, we have $R(x,y)\otimes R(y,x)\leq R(x,y)\wedge R(y,x)=E(x,y)$, hence R is an $\otimes$-E-antisymmetric. The $\otimes$-transitivity comes from the fact that R is $\otimes$-preordering. Hence, R is an $\otimes$-E-ordering. □

Theorem 2.7. *Let X be a nonempty set and $(R_i)_{i\in\mathbb{N}}$, $(E_i)_{i\in\mathbb{N}}$ two families of fuzzy relations on X, where each E_i is an $\otimes$-equivalence relation and R_i is an $\otimes$-E_i-fuzzy partial ordering. Then $R(x,y)=\underset{i\in\mathbb{N}}{\wedge}R_i(x,y)$ is an $\otimes$-E-ordering, where $E(x,y)=\underset{i\in\mathbb{N}}{\wedge}E_i(x,y)$.*

Proof. Let $x\in X$, we have for any $i\in\mathbb{N}$, E_i is an $\otimes$-equivalence relation, so $E_i(x,x)=1,\forall i\in\mathbb{N}$, hence $\underset{i\in\mathbb{N}}{\wedge}(E_i(x,x))=1$, this means $E(x,x)=1$, and E is reflexive.
Let $x,y\in X$, $E(x,y)=\underset{i\in\mathbb{N}}{\wedge}E_i(x,y)=\underset{i\in\mathbb{N}}{\wedge}E_i(y,x)=E(y,x)$, then E is symmetric.
Take $x,y,z\in X$.

$$\begin{aligned} E(x,y)\otimes E(y,z) &= (\underset{i\in\mathbb{N}}{\wedge}E_i(x,y))\otimes(\underset{i\in\mathbb{N}}{\wedge}E_i(y,z)) \\ &\leq \underset{i\in\mathbb{N}}{\wedge}(E_i(x,y)\otimes E_i(y,z))\ (\text{by } \textbf{Proposition } 2.1) \\ &\leq \underset{i\in\mathbb{N}}{\wedge}E_i(x,z) \\ &= E(x,z). \end{aligned}$$

Hence, E is $\otimes$-transitive. So, E is an $\otimes$-equivalence relation.
To prove the E-reflexivity of R, take $x,y\in X$. We have for all $i\in\mathbb{N}$, $R_i(x,y)\geq E_i(x,y)$, then $\underset{i\in\mathbb{N}}{\wedge}R_i(x,y)\geq\underset{i\in\mathbb{N}}{\wedge}E_i(x,y)$.
Hence, $R(x,y)\geq E(x,y)$. So, R is E-reflexive. To prove the $\otimes$-E-antisymmetry, let $x,y\in X$,

$$\begin{aligned} R(x,y)\otimes R(y,x) &= (\underset{i\in\mathbb{N}}{\wedge} R_i(x,y))\otimes(\underset{i\in\mathbb{N}}{\wedge} R_i(y,x)) \\ &\leq \underset{i\in\mathbb{N}}{\wedge}(R_i(x,y)\otimes R_i(y,x)) \text{ (by } \textbf{Proposition } 2.1\text{)} \\ &\leq \underset{i\in\mathbb{N}}{\wedge} E_i(x,y) \\ &= E(x,y). \end{aligned}$$

Hence R is $\otimes$-E-antisymmetric. To verify the $\otimes$-transitivity of R, take $x,y,z\in X$,

$$\begin{aligned} R(x,y)\otimes R(y,z) &= (\underset{i\in\mathbb{N}}{\wedge} R_i(x,y))\otimes(\underset{i\in\mathbb{N}}{\wedge} R_i(y,z)) \\ &\leq \underset{i\in\mathbb{N}}{\wedge}(R_i(x,y)\otimes R_i(y,z)) \text{ (by } \textbf{Proposition } 2.1\text{)} \\ &\leq \underset{i\in\mathbb{N}}{\wedge}(R_i(x,z)) \\ &= R(x,z). \end{aligned}$$

Hence R is $\otimes$-transitive. Consequently, R is $\otimes$-E-order. □

Proposition 2.5. *Let X be a nonempty set and E_1, E_2 two fuzzy $\otimes$-equivalence relations on X. Then the relation E defined on X by*

$$E(x,y) = E_1(x,y)\otimes E_2(x,y)$$

is an $\otimes$-equivalence.

Proof. The reflexivity and symmetry are trivial.
The $\otimes$-transitivity of E comes directly from the commutativity and associativity of $\otimes$. □

Proposition 2.6. *Let E_1, E_2 be two fuzzy $\otimes$-equivalence relations on X, and R_1, R_2 fuzzy relations, where R_1 is a fuzzy $\otimes$-E_1-ordering on X and R_2 is a fuzzy $\otimes$-E_2-order on X. The relation R defined by $R(x,y) = R_1(x,y)\otimes R_2(x,y)$ is an $\otimes$-E-partial order, where E is the $\otimes$-equivalence relation defined in **Proposition** 2.5.*

Proof. For $x,y\in X$, we have $E_1(x,y)\leq R_1(x,y)$ and $E_2(x,y)\leq R_2(x,y)$,
hence $E_1(x,y)\otimes E_2(x,y)\leq R_1(x,y)\otimes R_2(x,y)$.
So, $E(x,y)\leq R(x,y)$, thus R is E-reflexive.
The $\otimes$-E-antisymmetry and $\otimes$-transitivity of R come directly from the commutativity and associativity of $\otimes$.
Hence, R is an $\otimes$-E-ordering relation. □

2.6.2. Cartesian product of fuzzy relations with respect to a binary operation

Theorem 2.8. *Consider a family of non-empty crisp sets $(X_i)_{i\in\mathbb{N}}$, and two families of fuzzy relations $(R_i)_{i\in\mathbb{N}}$ and $(E_i)_{i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$, E_i is an $\otimes$-equivalence relation on X_i and R_i is an $\otimes$-E_i-partial ordering on X_i. The binary relation*

$$\begin{array}{llll} R: & (\prod\limits_{i\in\mathbb{N}} X_i)^2 & \to & L \\ & ((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) & \to & \bigwedge\limits_{i\in\mathbb{N}} R_i(x_i,y_i), \end{array}$$

is an $\otimes$-E-fuzzy partial ordering, where $E((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) = \bigwedge\limits_{i\in\mathbb{N}} E_i(x_i,y_i)$.

Proof. Let $((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) \in (\prod\limits_{i\in\mathbb{N}} X_i)^2$, we have $R_i(x_i,y_i) \geq E_i(x_i,y_i)$ for all $i \in \mathbb{N}$, then $\bigwedge\limits_{i\in\mathbb{N}} R_i(x_i,y_i) \geq \bigwedge\limits_{i\in\mathbb{N}} E_i(x_i,y_i)$.
So, $R((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) \geq E((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}})$, hence R is E-reflexive.
For $((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) \in (\prod\limits_{i\in\mathbb{N}} X_i)^2$,

$$\begin{array}{ll} & R((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}) \otimes R((y_i)_{i\in\mathbb{N}},(x_i)_{i\in\mathbb{N}}) \\ = & (\bigwedge\limits_{i\in\mathbb{N}} R_i(x_i,y_i)) \otimes (\bigwedge\limits_{i\in\mathbb{N}} R_i(y_i,x_i)) \\ \leq & \bigwedge\limits_{i\in\mathbb{N}} (R_i(x_i,y_i) \otimes R_i(y_i,x_i)) \text{ (by } \textbf{Proposition 2.1}) \\ \leq & \bigwedge\limits_{i\in\mathbb{N}} E_i(x_i,y_i) \\ = & E\left((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}\right). \end{array}$$

Hence R is $\otimes$-E-antisymmetric.

Let $((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}},(z_i)_{i\in\mathbb{N}}) \in (\prod\limits_{i\in\mathbb{N}} X_i)^3$,

$$\begin{array}{l} R\left((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}}\right) \otimes R\left((y_i)_{i\in\mathbb{N}},(z_i)_{i\in\mathbb{N}}\right) \\ = (\bigwedge\limits_{i\in\mathbb{N}} R_i(x_i,y_i)) \otimes (\bigwedge\limits_{i\in\mathbb{N}} R_i(y_i,z_i)) \\ \leq \bigwedge\limits_{i\in\mathbb{N}} (R_i(x_i,y_i) \otimes R_i(y_i,z_i)) \text{ (by } \textbf{Proposition 2.1}) \\ \leq \bigwedge\limits_{i\in\mathbb{N}} (R_i(x_i,z_i)) \\ = R\left((x_i)_{i\in\mathbb{N}},(z_i)_{i\in\mathbb{N}}\right). \end{array}$$

Then R is $\otimes$-transitive. Consequently, R is $\otimes$-E-ordering. □

3 Aggregating some finite families of fuzzy sets

The main goal of this chapter is to investigate the aggregation of diverse families of binary fuzzy relations, fuzzy filters, and fuzzy lattices. Some links between these families and their images via an aggregation.

3.1. Aggregation functions

Definition 3.1. *[60] An n-ary aggregation is an application $A : U^n \to U$ fulfills, the following conditions for all $\overrightarrow{x} = (x_1, ..., x_n), \overrightarrow{y} = (y_1, ..., y_n) \in U^n$,*

(a) $A(\overrightarrow{1}) = 1$; $A(\overrightarrow{0}) = 0$;

(b) *if for all $i \in I$, $x_i \leq y_i$, then $A(\overrightarrow{x}) \leq A(\overrightarrow{y})$.*

Furthermore,

1. *An aggregation A is said to be strictly monotone if for all $\overrightarrow{x}, \overrightarrow{y} \in U^n$, with $x_1 \leq y_1, ..., x_n \leq y_n$ and $\overrightarrow{x} \neq \overrightarrow{y}$, then $A(\overrightarrow{x}) < A(\overrightarrow{y})$.*

2. *An aggregation A is said to be jointly strictly monotone if for all $\overrightarrow{x}, \overrightarrow{y} \in U^n$ with $x_1 < y_1, ..., x_n < y_n$, then $A(\overrightarrow{x}) < A(\overrightarrow{y})$.*

3. *An aggregation A is said to be idempotent, if for all $x \in U$, $A(x, x, ..., x) = x$ (idempotency property).*

4. *An aggregation A is said to be without zero divisor other than 0, if $A(x_1, x_2, ..., x_n) = 0 \Leftrightarrow x_1 = 0$ or $x_2 = 0$ or...or $x_n = 0$.*

5. *An aggregation A is said to be (left-) rigth-continuous for the first component, if, for any (non-decreasing) non-increasing sequence $(x_n)_{n\in\mathbb{N}}$, it holds that $\lim\limits_n A(x_n, y) = A(\lim\limits_n x_n, y)$.*

Remark 3.1. **(a)** *Let $\vee, \wedge : U^2 \to U$ be two binary idempotent aggregation functions defined as $\vee(x, y) = \max(x, y)$ and $\wedge(x, y) = \min(x, y)$. So, when A is an idempotent aggregation function, then $\wedge(x, y) \leq A(x, y) \leq \vee(x, y)$ for all $x, y \in U$.*

(b) *For all $\overrightarrow{x}, \overrightarrow{y} \in U^n$, we have*
(1) $A(\overrightarrow{x} \vee \overrightarrow{y}) \geq A(\overrightarrow{x}) \vee A(\overrightarrow{y})$, where $\overrightarrow{x} \vee \overrightarrow{y} = (x_1 \vee y_1, ..., x_n \vee y_n)$,
(2) $A(\overrightarrow{x} \wedge \overrightarrow{y}) \leq A(\overrightarrow{x}) \wedge A(\overrightarrow{y})$, where $\overrightarrow{x} \wedge \overrightarrow{y} = (x_1 \wedge y_1, ..., x_n \wedge y_n)$,

(3) $(A(\overrightarrow{x}.\overrightarrow{y}))^2 \leq A(\overrightarrow{x}).A(\overrightarrow{y})$, *where* $\overrightarrow{x}.\overrightarrow{y} = (x_1.y_1, ..., x_n.y_n)$.

Definition 3.2. *An aggregation A_1 dominates another aggregation A_2 if and only if the following inequality holds $A_1(A_2(x,y), A_2(u,v)) \geq A_2(A_1(x,u), A_1(y,v))$, for all $x, y, u, v \in U$.*

Definition 3.3. *An aggregation A_1 bidominates another aggregation A_2 if and only if the following equality holds $A_1(A_2(x,y), A_2(u,v)) = A_2(A_1(x,u), A_1(y,v))$, for all $x, y, u, v \in U$.*

Definition 3.4. *[60] Let A be an aggregation, we said that the t-norms T satisfies the distributive property if and only if for all $x, y_1, ..., y_n \in X$, $A(T(x,y_1), ..., T(x,y_n)) = T(x, A(y_1, ..., y_n))$.*

3.2. Aggregating of a finite family of fuzzy binary relations and their traces

3.2.1. Aggregating of a finite family of fuzzy binary relations

In this subsection, we aggregate some finite families of fuzzy relations and fuzzy complete lattices, for more detail see [4].

Definition 3.5. *[60] Let $A : U^n \to U$ be an aggregation function and $\mathcal{L} = \{L_i/X^k \to U, i \in I\}$ a family of fuzzy k-ary relations on a domain X. A $(\mathcal{L}, A)$ k-ary relation on X denoted by $\mathcal{L}_A$ is obtained as the composition given by*

$$\mathcal{L}_A(x_1, ..., x_k) = A(L_1(x_1, ..., x_k), ..., L_n(x_1, ..., x_k)).$$

Definition 3.6. *[60] Let $A : U^n \to U$ be an aggregation function and $\mathcal{F} = \{L_i : X^2 \to U, i \in I\}$ a family of fuzzy binary relations on a domain X. A $(\mathcal{L}, A)$ fuzzy binary relation on X denoted by $\mathcal{L}_A$ is obtained as the composition given by*

$$\mathcal{L}_A(x,y) = A(L_1(x,y), ..., L_n(x,y)).$$

Definition 3.7. *Let $(X_i, R_i)_{i \in I}$ be a family of fuzzy lattices, $(F_i)_{i \in I}$ a family of fuzzy subsets of X_i, $A : U^n \to U$ an aggregation and $\Re_A$, $\mathcal{F}_A$ two operators defined on $(\prod_{i=1}^{n} X_i)^2$, $(\prod_{i=1}^{n} X_i)$ respectively by*
$\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) = A(R_1(x_1, y_1), ..., R_n(x_n, y_n))$ and
$\mathcal{F}_A(x_1, ..., x_n) = A(F_1(x_1), ..., F_n(x_n))$.

In what follow, we define a special aggregation that we need in sequel.
Let $T_{M(\alpha,\beta)}$ be the function defined from U^2 to U by

$$T_{M(\alpha,\beta)}(x,y) = x^\alpha \wedge y^\beta, \text{ where } \alpha,\beta \in \mathbb{R}_+^*.$$

Lemma 3.1. *The function $T_{M(\alpha,\beta)}$ is an aggregation.*

Proof. $T_{M(\alpha,\beta)}(0,0) = 0$ and $T_{M(\alpha,\beta)}(1,1) = 1$.

As $\alpha,\beta \in \mathbb{R}_+^*$, the inceasing of $T_{M(\alpha,\beta)}$ is direct, hence $T_{M(\alpha,\beta)}$ is an aggregation function. □

Lemma 3.2. *Let T_M be the minimum t-norm and $T_{M(\alpha,\beta)}$ the aggregation defined by $T_{M(\alpha,\beta)}(x,y) = x^\alpha \wedge y^\beta$, where $\alpha,\beta \in \mathbb{R}_+^*$ then each one of them dominates the other.*

Proof. Let $x,y,u,v \in U$,

$$\begin{aligned} T_M(T_{M(\alpha,\beta)}(x,y), T_{M(\alpha,\beta)}(u,v)) &= (x^\alpha \wedge y^\beta) \wedge (u^\alpha \wedge v^\beta) \\ &= (x^\alpha \wedge u^\alpha) \wedge (y^\beta \wedge v^\beta) \\ &= (x \wedge u)^\alpha \wedge (y \wedge v)^\beta \\ &= T_{M(\alpha,\beta)}(T_M(x,u), T_M(y,v)). \end{aligned}$$

□

Remark 3.2.
1. *Contrary to the t-norms, there are aggregations which do not dominate themselves.*
2. *If an aggregation A_1 dominates another aggregation A_2, it is not necessary that A_1 be stronger than A_2.*
3. *The minimum aggregation dominates all other aggregations.*

Indeed,

1. *take $A(x,y) = \frac{x^2+y^2}{2}$*

$$\begin{aligned} A(A(x,y), A(u,v)) &= A\left(\frac{x^2+y^2}{2}, \frac{u^2+v^2}{2}\right) \\ &= \frac{(\frac{x^2+y^2}{2})^2 + (\frac{u^2+v^2}{2})^2}{2} \\ &= \frac{x^4+2x^2y^2+y^4+u^4+2u^2v^2+v^4}{8}. \text{ And} \\ A(A(x,u), A(y,v)) &= A\left(\frac{x^2+u^2}{2}, \frac{y^2+v^2}{2}\right) \\ &= \frac{(\frac{x^2+u^2}{2})^2 + (\frac{y^2+v^2}{2})^2}{2} \\ &= \frac{x^4+2x^2u^2+u^4+y^4+2y^2v^2+v^4}{8}. \end{aligned}$$

It is easy to see that neither $A(A(x,y), A(u,v)) \leq A(A(x,u), A(y,v))$ nor $A(A(x,u), A(y,v))$ $A(A(x,y), A(u,v))$. Hence, A does not dominate itself.

2. *We prove that $T_{M(2,2)}$ dominates T_p this means that for all $x,y,u,v \in U$ the inequality $T_{M(2,2)}(T_p(x,y), T_p(u,v)) \geq T_p(T_{M(2,2)}(x,u), T_{M(2,2)}(y,v))$ holds. This is equivalent to $(xy)^2 \wedge (uv)^2 \geq (x^2 \wedge u^2)(y^2 \wedge v^2)$. To show this, we have four possible cases as in* ***Table** 3.1.*

Cases	$(x \leq u)$	$(y \leq v)$	$\Leftrightarrow$
a	1	1	$(x \leq u) \wedge (y \leq v)$
b	1	0	$(x \leq u) \wedge (v < y)$
c	0	1	$(u < x) \wedge (y \leq v)$
d	0	0	$(u < x) \wedge (v < y)$

Table 3.1:

Case (a)- If $x \leq u$ and $y \leq v$, then $x^2 \leq u^2$ and $y^2 \leq v^2$, which gives $x^2y^2 \leq u^2v^2$, hence
$(xy)^2 \wedge (uv)^2 = (x^2y^2) \wedge (u^2v^2) = x^2y^2$.
Obviously that $(x^2 \wedge u^2)(y^2 \wedge v^2) = x^2y^2$.
Hence $(xy)^2 \wedge (uv)^2 = (x^2 \wedge u^2)(y^2 \wedge v^2)$.
Case (b)- If $x \leq u$ and $v < y$, then $x^2 \leq u^2$ and $v^2 < y^2$, this implies that
$(x^2 \wedge u^2)(y^2 \wedge v^2) = x^2v^2$.
And $x^2v^2 \leq u^2v^2$ and $v^2x^2 < y^2x^2$, which implies that $x^2v^2 \leq (u^2v^2) \wedge (y^2x^2)$.
Consequently, $(xy)^2 \wedge (uv)^2 \geq (x^2 \wedge u^2)(y^2 \wedge v^2)$.

Case (c)- If $u < x$ and $y \leq v$, then $u^2 < x^2$ and $y^2 \leq v^2$, this implies that
$(x^2 \wedge u^2)(y^2 \wedge v^2) = u^2y^2$. *And $u^2y^2 < x^2y^2$ and $u^2y^2 \leq v^2u^2$, so*
$u^2y^2 \leq (x^2y^2) \wedge (v^2u^2)$.
Hence $(x^2 \wedge u^2)(y^2 \wedge v^2) \leq (xy)^2 \wedge (uv)^2$.
Case (d)- If $u < x$ and $v < y$ give $u^2 < x^2$ and $v^2 < y^2$.
Hence $(x^2 \wedge u^2)(y^2 \wedge v^2) = u^2v^2$.
And $u^2v^2 < x^2y^2 \Leftrightarrow (x^2y^2) \wedge (v^2u^2) = u^2v^2$, then
$(xy)^2 \wedge (uv)^2 = (x^2 \wedge u^2)(y^2 \wedge v^2)$.
It can be seen that assertions a, b, c and d give that for all $x, y, u, v \in U$, $T_{M(2,2)}(T_p(x,y), T_p(u,v)) \geq T_p(T_{M(2,2)}(x,u), T_{M(2,2)}(y,v))$. Hence $T_{M(2,2)}$ dominates T_p.
But $T_{M(2,2)}(x,y) \leq T_p(x,y)$ for all $x, y \in U$. Indeed, if $x \leq y \Rightarrow x^2 \leq y^2$, then $T_{M(2,2)}(x,y) = x^2 \wedge y^2 = x^2 \leq xy = T_p(x,y)$.
For $y < x$, $T_{M(2,2)}(x,y) = x^2 \wedge y^2 = y^2 < T_p(x,y)$ hence $T_{M(2,2)}$ is not stronger than T_p.

Proposition 3.1. *Let $(X_i)_{i \in I}$ be a family of non empty sets, $(R_i)_{i \in I}$ a family of fuzzy binary relations on $(X_i)_{i \in I}$, $A : U^n \to U$ an aggregation and $\Re_A$ a fuzzy set defined on $(\prod_{i=1}^{n} X_i)^2$ by $\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) = A(R_1(x_1, y_1), ..., R_n(x_n, y_n))$. It holds that:*

1. *If R_i is reflexive for all $i \in I$, then $\Re_A$ is reflexive.*
2. *If R_i is symmetric for all $i \in I$, then $\Re_A$ is symmetric.*

Proof. (1) Suppose that R_i is reflexive for all $i \in I$. Let $(x_1, ..., x_n) \in (\prod_{i=1}^{n} X_i)$

$\Re_A((x_1, ..., x_n), (x_1, ..., x_n)) = A(R_1(x_1, x_1), ..., R_n(x_n, x_n)) = A(1, ..., 1) = 1$. Then, $\Re_A$ is reflexive.

(2) Let $(x_1, ..., x_n), (y_1, ..., y_n) \in (\prod_{i=1}^{n} X_i)$. Suppose that for all $i \in I$, R_i is symmetric, then

$$\begin{aligned}\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) &= A(R_1(x_1, y_1), ..., R_n(x_n, y_n))\\ &= A(R_1(y_1, x_1), ..., R_n(y_n, x_n))\\ &= \Re_A((y_1, ..., y_n), (x_1, ..., x_n)).\end{aligned}$$

Hence, $\Re_A$ is symmetric. □

Proposition 3.2. *Let $(R_i)_{i \in I}$ be a finite family of antisymmetric (respectively transitive) relations. If the aggregation A is defined by $A(x_1, ..., x_n) = x_1^{\alpha_1} \wedge ... \wedge x_n^{\alpha_n}$ where $\alpha_1, ..., \alpha_n \in \mathbb{R}_+^*$, then $\Re_A$ is an antisymmetric (respectively transitive) relation.*

Proof. (1) Suppose that R_i is antisymmetric for all $i \in I$, i.e., for all $x_i, y_i \in X_i$, $R_i(x_i, y_i) \wedge R_i(y_i, x_i) > 0$ implies $x_i = y_i$. Let $(x_1, ..., x_n), (y_1, ..., y_n) \in (\prod_{i=1}^{n} X_i)$, using **Lemma** 3.2,

$$\begin{aligned}&\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) \wedge \Re_A((y_1, ..., y_n), (x_1, ..., x_n))\\ &= A(R_1(x_1, y_1), ..., R_n(x_n, y_n)) \wedge A(R_1(y_1, x_1), ..., R_n(y_n, x_n))\\ &= A(R_1(x_1, y_1) \wedge R_1(y_1, x_1), ..., R_n(x_n, y_n) \wedge R_n(y_n, x_n)) > 0,\end{aligned}$$

this means $R_i(x_i, y_i) \wedge R_i(y_i, x_i) > 0$ for all $i \in I$, thus $x_i = y_i$ for all $i \in I$, consequently $(x_1, ..., x_n) = (y_1, ..., y_n)$. Therefore, $\Re_A$ is antisymmetric.

(2) Suppose that R_i is transitive for all $i \in I$ i.e., for all $x_i, y_i, z_i \in X_i$ we have $R_i(x_i, y_i) \wedge R_i(y_i, z_i) \leq R_i(x_i, z_i)$.

Let $(x_1, ..., x_n), (y_1, ..., y_n), (z_1, ..., z_n) \in (\prod_{i=1}^{n} X_i)$,

$$\begin{aligned}&\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) \wedge \Re_A((y_1, ..., y_n), (z_1, ..., z_n))\\ &= A(R_1(x_1, y_1), ..., R_n(x_n, y_n)) \wedge A(R_1(y_1, z_1), ..., R_n(y_n, z_n))\\ &= A(R_1(x_1, y_1) \wedge R_1(y_1, z_1), ..., R_n(x_n, y_n) \wedge R_n(y_n, z_n))\\ &\leq A(R_1(x_1, z_1), ..., R_n(x_n, z_n))\\ &= \Re_A((x_1, ..., x_n), (z_1, ..., z_n)),\end{aligned}$$

hence, $\Re_A$ is transitive. □

Corollary 3.1. *Let $(X_i, R_i)_{i \in I}$ be a family of fuzzy posets, $A : U^n \to U$ an aggregation defined by $A(x_1, ..., x_n) = x_1^{\alpha_1} \wedge ... \wedge x_n^{\alpha_n}$, where $\alpha_1, ..., \alpha_n \in \mathbb{R}_+^*$, then $\Re_A$ is a fuzzy order on $(\prod_{i=1}^{n} X_i)$.*

3.2.2. Aggregating of a finite family of fuzzy lattices

Proposition 3.3. *Let $(X_i, R_i)_{i\in I}$ be a family of fuzzy complete lattices, A an aggregation function defined by $A(x_1, ..., x_n) = x_1^{\alpha_1} \wedge ... \wedge x_n^{\alpha_n}$ and B_i a subset of X_i. If l_i respectively u_i) is the greatest lower (respectively the least upper) bound of B_i for all $i \in I$, then $(l_1, ..., l_n)$ (respectively $(u_1, ..., u_n)$) is the greatest lower (the least upper) bound of $\prod_{i=1}^{n} B_i$.*

Moreover, $(\prod_{i=1}^{n} X_i, \Re_A)$ is a fuzzy complete lattice.

Proof. According to **Corollary** 3.1, $(\prod_{i=1}^{n} X_i, \Re_A)$ is a poset. Suppose that l_i is the greatest lower bound of B_i for all $i \in I$. Prove that $(l_1, ..., l_n)$ is a lower bound of $\prod_{i=1}^{n} B_i$ indeed,

for all $(x_1, ..., x_n) \in \prod_{i=1}^{n} B_i$, since $R_i(x_i, l_i) > 0$ for all $i \in I$ this gives $\Re_A((l_1, ..., l_n), (x_1, ..., x_n)) = A(R_1(x_1, l_1), ..., R_n(x_n, l_n)) > 0$. Hence, $(l_1, ..., l_n)$ is a lower bound of $\prod_{i=1}^{n} B_i$. Suppose by way of contradiction that there exists an other lower bound $(l_1^{'}, ..., l_n^{'})$ of $\prod_{i=1}^{n} B_i$ greater than $(l_1, ..., l_n)$. Then $\Re_A((l_1, ..., l_n), (l_1^{'}, ..., l_n^{'})) > 0$, which implies $A(R_1(l_1, l_1^{'}), ..., R_n(l_n, l_n^{'})) > 0$. According to **Definition** 3.1 (4.), this equivalent to $R_i(l_i, l_i^{'}) > 0$ for all $i \in I$. But this contradicts the fact that l_i is the greatest lower bound of B_i. Thus, $(l_1, ..., l_n)$ is the greatest lower bound of $\prod_{i=1}^{n} B_i$. In a similar way, we can prove that $(u_1, ..., u_n)$ is the least upper bound of $\prod_{i=1}^{n} B_i$. Let B be an arbitrary subset of $\prod_{i=1}^{n} X_i$. Then there exists a family of subsets $(B_i)_{i\in I}$ of $(X_i)_{i\in I}$ such that $B = \prod_{i=1}^{n} B_i$. Since (X_i, R_i) is a complete lattice for all $i \in I$, then for all subset B_i of X_i, there exists a greatest lower (respectively least upper) bound l_i (respectively u_i) of B_i. So the subset B has a greatest lower bound $(l_1, ..., l_n)$ and a least upper bound $(u_1, ..., u_n)$. Consequently, $(\prod_{i=1}^{n} X_i, \Re_A)$ is a complete fuzzy lattice. □

3.3. Aggregating of a finite family of fuzzy binary relations and their traces

3.3.1. Traces of fuzzy binary relations

Definition 3.8. *[14] For a binary fuzzy relation R on a domain X, its right trace R^r and its left trace R^l are defined as the following binary fuzzy relations:*

$$R^l(x,y) = \inf_{z\in X} \mathcal{I}(R(z,x), R(z,y));$$
$$R^r(x,y) = \inf_{z\in X} \mathcal{I}(R(y,z), R(x,z)).$$

Proposition 3.4. *[14]*

1. *For a binary fuzzy relation R on a domain X and some left-continuous t-norm T, the following three statements are equivalent:*

   (i) *R is reflexive,*

   (ii) *$R^l \subseteq R$,*

   (iii) *$R^r \subseteq R$.*

2. *For a binary fuzzy relation R on a domain X and some left-continuous t-norm T, the following three statements are equivalent:*

   (i) *R is T-transitive,*

   (ii) *$R \subseteq R^l$,*

   (iii) *$R \subseteq R^r$.*

3.3.2. Aggregating of a finite family of traces of a binary relation

In what follows, we will define the aggregation of finite family of traces of a binary relation, see [4, 7, 8, 18, 23, 30].

Definition 3.9. *Let $A : U^n \to U$ be an idempotent aggregation and $\{T_1, ..., T_n\}$ a family of t-norms. Define the aggregation T_A of the family $\{T_1, ..., T_n\}$ by $T_A(x,y) = A(T_1(x,y), ..., T_n((x,y))$.*

Remark 3.3. *[60] For a family $\{T_1, ..., T_n\}$ of left continuous t-norms which satisfies the distributivity and generalized associativity, the aggregation T_A is a left continuous t-norm.*

Definition 3.10. *Let T_A be a left continuous t-norm given in **Definition** 3.9 and $\mathcal{I}$ the residual implication associated to T_A, and R a fuzzy binary relation on a* *domain X. The left (respectively right) trace of R denoted by R^l_A respectively (R^r_A) are defined as follow:*

$$R^l_A(x,y) = \inf_{z\in X} \mathcal{I}(R(z,x), R(z,y)),$$

$$R^r_A(x,y) = \inf_{z\in X} \mathcal{I}(R(y,z), R(x,z)).$$

Now, we characterize the aggregation of left and right trace relations of a fuzzy binary relation R in term of an aggregation fuzzy implication.

Definition 3.11. *Let A be an aggregation on U, R a fuzzy binary relation on a domain X, $\{\mathcal{I}_i/i \in I\}$ a family of residual implications and $R^l_{\mathcal{I}_i}, R^r_{\mathcal{I}_i}$ the corresponding left (respectively right) traces of R. We define the relations L^l_A, L^r_A as follows*

$$\begin{aligned} L^l_A(x,y) &= A(R^l_{\mathcal{I}_1}(x,y), ..., R^l_{\mathcal{I}_n}(x,y)) \\ &= A(\inf_{z_1\in X} \mathcal{I}_1(R(z_1,x), R(z_1,y)), ..., \inf_{z_n\in X} \mathcal{I}_n(R(z_n,x), R(z_n,y))). \end{aligned}$$

$$\begin{aligned} L^r_A(x,y) &= A(R^r_{\mathcal{I}_1}(x,y), ..., R^r_{\mathcal{I}_n}(x,y)) \\ &= A(\inf_{z_1\in X} \mathcal{I}_1(R(y,z_1), R(x,z_1)), ..., \inf_{z_n\in X} \mathcal{I}_n(R(y,z_n), R(x,z_n))). \end{aligned}$$

The following proposition establishes the relationship between R, L^l_A, L^r_A for a given relation R and an aggregation A.

Remark 3.4. *[32] For any relation R, R^l and R^r are always reflexive.*

Proposition 3.5. *Let R be a fuzzy binary relation on a domain X, $\{\mathcal{I}_i/i \in I\}$ a family of fuzzy implications and A an idempotent aggregation on U. The following statements are equivalents:*

1. *R is reflexive;*
2. *$L^l_A \subset R$;*
3. *$L^r_A \subset R$.*

Proof. (1) implies (2) Suppose that R is reflexive. By **Proposition** 3.4, we get for all $i \in I$, $R^l_i \subset R$. Then, for all $x, y \in X$, $A(R^l_1(x,y), ..., R^l_n(x,y)) \leq A(R(x,y), ..., R(x,y)) = R(x,y)$. Hence $L^l_A \subset R$.
(1) implies (3) is obtained in the same manner.
(2) implies (1), for all $x, y \in X$, we have $L^l_A(x,y) \leq R(x,y)$, take $x = y$. Thus $R(x,x) \geq L^l_A(x,x) = A(R^l_1(x,x), ..., R^l_n(x,x)) = A(1, ..., 1) = 1$. Hence, R is reflexive.
For (3) implies (1) is obtained in the same manner as mentioned before. □

Proposition 3.6. *Let R be a fuzzy binary relation on a domain X, $\{T_i/i \in I\}$ a family of left continuous t-norms, $\{\mathcal{I}_i/i \in I\}$ a family of corresponding fuzzy residual implications and A an idempotent aggregation on U, the following statements holds:*

1. *If for all $i \in I$ R is T_i-transitive, then $R \subset L^l_A$;*
2. *If for all $i \in I$ R is T_i-transitive, then $R \subset L^r_A$.*

Proof. For the first assertion, suppose that R is T_i-transitive then for all $i \in I$, by **Proposition** 3.4, we get $R \subset R^l_i$. Then, $R(x,y) \leq R^l_i(x,y)$, hence $R(x,y) = A(R(x,y),...,R(x,y)) \leq A(R^l_1(x,y),...,R^l_n(x,y)) = L^l_A(x,y)$ for all $x,y \in X$. The first assertion is proved.
The second assertion can be proved in a similar way. □

3.4. Aggregating of a finite family of t-orders

Now, we introduce an aggregation function to aggregate T-preordering relations.
Proposition 3.7. *Let $(L_i)_{i\in I}$ be a family of T-preordering relations on a domain X and let $A : U^n \to U$ be an aggregation without zero divisors other than zero which dominate T. Then the relation $\mathcal{L}_A$ defined by*

$$\mathcal{L}_A(x,y) = A(L_1(x,y),...,L_n(x,y)),$$

is a T-$\mathcal{E}_A$-ordering, where

$$\mathcal{E}_A(x,y) = A(T(L_1(x,y),L_1(y,x)),...,T(L_n(x,y),L_n(y,x))).$$

Proof. Firstly, we prove that $\mathcal{E}_A$ is a T-equivalence relation. Obviously, for all $x \in X$, $\mathcal{E}_A(x,x) = 1$. Hence, $\mathcal{E}_A$ is reflexive.
Clearly, $\mathcal{E}_A(x,y) = \mathcal{E}_A(y,x)$ for all $x,y \in X$. Then $\mathcal{E}_A$ is symmetric. To prove the T-transitivity, let $x,y,z \in X$.

$$\begin{aligned}
&T(\mathcal{E}_A(x,y),\mathcal{E}_A(y,z))\\
&= T(A(T(L_1(x,y),L_1(y,x)),...,T(L_n(x,y),L_n(y,x))),\\
&\qquad\qquad A(T(L_1(y,z),L_1(z,y)),...,T(L_n(y,z),L_n(z,y))))\\
&\leq A(T(T(L_1(x,y),L_1(y,x)),T(L_1(y,z),L_1(z,y))),...,\\
&\qquad\qquad T(T(L_n(x,y),L_n(y,x)),T(L_n(y,z),L_n(z,y))))\\
&\leq A(T(T(L_1(x,y),L_1(y,z)),T(L_1(y,x),L_1(z,y))),...,\\
&\qquad\qquad T(T(L_n(x,y),L_n(y,z)),T(L_n(y,x),L_n(z,y))))\\
&= A(T(T(L_1(x,y),L_1(y,z)),T(L_1(z,y),L_1(y,x))),...,\\
&\qquad\qquad T(T(L_n(x,y),L_n(y,z)),T(L_n(z,y),L_n(y,x))))\\
&\leq A(T(L_1(x,z),L_1(z,x)),...,T(L_n(x,z),L_n(z,x)))
\end{aligned}$$

$= \mathcal{E}_A(x,z)$. According to the definition $\mathcal{E}_A$.

Hence $\mathcal{E}_A$ is T-transitive. Consequently $\mathcal{E}_A$ is T-equivalence relation.

Secondly. To prove that $\mathcal{L}_A$ is T-$\mathcal{E}_A$-order, let $x, y \in X$,

$$\begin{array}{rcl} \mathcal{E}_A(x,y) & = & A(T(L_1(x,y), L_1(y,x)), ..., T(L_n(x,y), L_n(y,x))) \\ & \leq & A(L_1(x,y), ..., L_n(x,y)), \text{ using } T(x,y) \leq x \\ & = & \mathcal{L}_A(x,y), \text{ by the definition of } \mathcal{L}_A. \end{array}$$

Hence, $\mathcal{L}_A$ is $\mathcal{E}_A$-reflexive.

Let $x, y \in X$.

$$\begin{array}{rcl} T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) & = & T(A(L_1(x,y), ..., L_n(x,y)), A(L_1(y,x), ..., L_n(y,x))) \\ & \leq & A(T(L_1(x,y), L_1(y,x)), ..., T(L_n(x,y), L_n(y,x))) \\ & = & \mathcal{E}_A(x,y). \end{array}$$

So, $\mathcal{L}_A$ is T-$\mathcal{E}_A$-antisymmetric.

Finally, to prove that $\mathcal{L}_A$ is T-transitive. Let $x, y, z \in X$,

$$\begin{array}{rcl} T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)) & = & T(A(L_1(x,y), ..., L_n(x,y)), A(L_1(y,z), ..., L_n(y,z))) \\ & \leq & A(T(L_1(x,y), L_1(y,z)), ..., T(L_n(x,y), L_n(y,z))) \\ & \leq & A(L_1(x,z), ..., L_n(x,z)) \\ & = & \mathcal{L}_A(x,z). \end{array}$$

Which complete the proof of $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order. □

Proposition 3.8. *Let $(L_i)_{i \in I}$ be a family of T-preordering relations, $A : U^n \to U$ an aggregation dominating T and $\tilde{A} : U^2 \to U$ a binary aggregation dominating T and satisfying $T \leq \tilde{A} \leq T_M$. Then the relation $\mathcal{L}_A$ defined by $\mathcal{L}_A(x,y) = A(L_1(x,y), ..., L_n(x,y))$, is a T-$\widetilde{E}$-ordering, where $\widetilde{E}(x,y) = \tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$. Besides,*
$T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \leq \widetilde{E}(x,y) \leq \min(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$ for all $x, y \in X$, are T-equivalence relations.

Proof. To prove that $\widetilde{E}$ is a T-equivalence relation, let $x \in X$,

$$\begin{array}{rcl} \widetilde{E}(x,x) & = & \tilde{A}(\mathcal{L}_A(x,x), \mathcal{L}_A(x,x)) \\ & = & \tilde{A}(A(L_1(x,x), ..., L_n(x,x)), A(L_1(x,x), ..., L_n(x,x))) \\ & = & \tilde{A}(1,1) \\ & = & 1. \end{array}$$

Hence, $\widetilde{E}$ is reflexive. Clearly, $\widetilde{E}$ is symmetric. To show that $\widetilde{E}$ is transitive, let $x, y, z \in X$.

$$\begin{array}{rcl} T(\widetilde{E}(x,y), \widetilde{E}(y,z)) & = & T(\tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)), \tilde{A}(\mathcal{L}_A(y,z), \mathcal{L}_A(z,y))) \\ & \leq & \tilde{A}(T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)), T(\mathcal{L}_A(y,x), \mathcal{L}_A(z,y))) \\ & = & \tilde{A}(T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)), T(\mathcal{L}_A(z,y), \mathcal{L}_A(y,x))) \\ & \leq & \tilde{A}(\mathcal{L}_A(x,z), \mathcal{L}_A(z,x)) \\ & = & \widetilde{E}(x,z). \end{array}$$

This means that $\widetilde{E}$ is T-transitive, hence $\widetilde{E}$ is T-equivalence relation. To prove that $\mathcal{L}_A$ is $\widetilde{E}$-reflexive, let $x, y \in X$

$$\begin{aligned}\widetilde{E}(x,y) &= \tilde{A}(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x))\\ &\leq T_M(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x))\\ &\leq \mathcal{L}_A(x,y).\end{aligned}$$

Hence $\mathcal{L}_A$ is $\widetilde{E}$-reflexive.
Let $x,y \in X$,

$$\begin{aligned}T(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x)) &\leq \tilde{A}(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x))\\ &= \widetilde{E}(x,y).\end{aligned}$$

Therefore, $\mathcal{L}_A$ is T-$\widetilde{E}$-antisymmetric.
Let $x,y,z \in X$,

$$\begin{aligned}T(\mathcal{L}_A(x,y),\mathcal{L}_A(y,z)) &= T(A(L_1(x,y),...,L_n(x,y)),A(L_1(y,z),...,L_n(y,z)))\\ &\leq A(T(L_1(x,y),L_1(y,z)),...,T(L_n(x,y),L_n(y,z)))\\ &\leq A(L_1(x,z),...,L_n(x,z))\\ &= \mathcal{L}_A(x,z) \text{ (by } \textbf{Proposition } 3.7\text{)}.\end{aligned}$$

Hence, $\mathcal{L}_A$ is T-transitive which complet the proof of $\mathcal{L}_A$ is T-$\widetilde{E}$-order.
Finally, since $T(x,y) \leq \tilde{A}(x,y) \leq T_M(x,y)$, for all $x,y \in X$, then $T(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x)) \leq \widetilde{E}(x,y) \leq T_M(\mathcal{L}_A(x,y),\mathcal{L}_A(y,x))$, and it is not difficult to show that the two bounds are T-equivalence relations. □

The next lemma is used to demonstrate ***Proposition*** *3.9.*

Lemma 3.3. *Let $(T_i)_{i\in I}$ be a family of t-norms and A a continuous aggregation which dominates all T_i, then A dominates $g = \wedge_{i\in I}T_i$.*

Proof. Let $x,y,u,v \in U$, $p \in I$, and put $g_p = T_1 \wedge ... \wedge T_p$. Hence, $g = \lim\limits_{p\to n} T_p$

$$\begin{aligned}g(A(x,y),A(u,v)) &= \lim_{p\to n} g_p(A(x,y),A(u,v))\\ &\leq \lim_{p\to n}(A(g_p(x,u),g_p(y,v)))\\ &= A(\lim_{p\to n} g_p(x,u), \lim_{p\to n} g_p(y,v))\\ &= A(g(x,u),g(y,v)).\end{aligned}$$

Hence, A dominate g. □

Remark 3.5. *Let $(T_i)_{i\in I}$ be a family of t-norms and $T = \wedge_{i\in I}T_i$, T is not necessary a t-norm. Indeed, let*

$$T_1(x,y) = \begin{cases} 0 & if (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ \min(x,y) & otherwise. \end{cases}$$

and T_p be the product t-norm. Put $T = T_1 \wedge T_p$, then the new t-norm T is given by

$$T(x,y) = \begin{cases} 0 & if (x,y) \in \left[0,\frac{1}{2}\right]^2, \\ x.y & otherwise. \end{cases}$$

It is easy to see that T is not a t-norm. Indeed, take $(x,y,z) = (0.5,0.7,0.7)$, $T(T(0.7,0.7),0.5) = T(0.49,0.5) = 0 \neq T(0.7,T(0.7,0.5)) = T(0.7,0.35) = 0.7 \times 0.35$, hence T is not associative.

Proposition 3.9. *Let $(T_i)_{i\in I}$ be a family of t-norms, (L_i) a family of T_i-preordering relations on X_i and $A : U^n \to U$ an aggregation such that for all $i \in I$, A dominates T_i, then the fuzzy relation $\mathcal{L}_A$ defined on $\prod_{i=1}^{n} X_i$ by $\mathcal{L}_A((x_1, ..x_n), (y_1, ..., y_n)) = A(L_1(x_1, y_1), ..., L_n(x_n, y_n))$. If $g_p = T_1 \wedge ... \wedge T_p$. is a t-norm, then $\mathcal{L}_A$ is a g-$\mathcal{E}_A$-ordering relation, where $\mathcal{E}_A$ is a fuzzy binary relation on $\prod_{i=1}^{n} X_i$ defined by*
$\mathcal{E}_A((x_1, ..x_n), (y_1, ..., y_n)) = A(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), ..., T_n(L_n(x_n, y_n), L_n(y_n, x_n)))$.

Proof. It is not difficult to prove that $\mathcal{E}_A$ is reflexive and symmetric. It remains to prove that $\mathcal{E}_A$ is g-transitive. Let $(x_1, ..x_n), (y_1, ..., y_n), (z_1, ..., z_n) \in \prod_{i=1}^{n} X_i$,

$$g(\mathcal{E}_A((x_1, ..., x_n), (y_1, ..., y_n)), \mathcal{E}_A((y_1, ..., y_n), (z_1, ..., z_n)))$$

$$= g[A(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), ..., T_n(L_n(x_n, y_n), L_n(y_n, x_n))), A(T_1(L_1(y_1, z_1), L_1(z_1, y_1)), ..., T_n(L_n(y_n, z_n), L_n(z_n, y_n)))]$$

$$\leq A[g(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), T_1(L_1(y_1, z_1), L_1(z_1, y_1))), ..., g(T_n(L_n(x_n, y_n), L_n(y_n, x_n)), T_n(L_n(y_n, z_n), L_n(z_n, y_n)))]$$

$$\leq A[T_1(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), T_1(L_1(y_1, z_1), L_1(z_1, y_1))), ..., T_n(T_n(L_n(x_n, y_n), L_n(y_n, x_n)), T_n(L_n(y_n, z_n), L_n(z_n, y_n)))]$$

$$\leq A[T_1(T_1(L_1(x_1, y_1), L_1(y_1, z_1)), T_1(L_1(y_1, x_1), L_1(z_1, y_1))), ..., T_n(T_n(L_n(x_n, y_n), L_n(y_n, z_n)), T_n(L_n(y_n, x_n), L_n(z_n, y_n)))]$$

$$= A[T_1(T_1(L_1(x_1, y_1), L_1(y_1, z_1)), T_1(L_1(z_1, y_1), L_1(y_1, x_1))), ..., T_n(T_n(L_n(x_n, y_n), L_n(y_n, z_n)), T_n(L_n(z_n, y_n), L_n(y_n, x_n)))]$$

$$\leq A(T_1(L_1(x_1, z_1), L_1(z_1, x_1)), ..., T_n(L_n(x_n, z_n), L_n(z_n, x_n)))$$
$$= \mathcal{E}_A((x_1, ..., x_n), (z_1, ..., z_n)).$$

Hence $\mathcal{E}_A$ is a g-equivalence relation. To prove that $\mathcal{L}_A$ is a g-$\mathcal{E}_A$-order. Let $(x_1, ..., x_n), (y_1, ..., y_n) \in \prod_{i=1}^{n} X_i$,

$$\mathcal{E}_A((x_1, ..., x_n), (y_1, ..., y_n))$$
$$= A(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), ..., T_n(L_n(x_n, y_n), L_n(y_n, x_n)))$$
$$\leq A(L_1(x_1, y_1), ..., L_n(x_n, y_n))$$
$$= \mathcal{L}_A((x_1, ..., x_n), (y_1, ..., y_n)).$$

Hence, $\mathcal{L}_A$ is a $\mathcal{E}_A$-reflexive.

Let $(x_1, ..., x_n), (y_1, ..., y_n) \in \prod_{i=1}^{n} X_i$.

$$g(\mathcal{L}_A((x_1, ..., x_n), (y_1, ..., y_n)), \mathcal{L}_A((y_1, ..., y_n), (x_1, ..., x_n))) = g(A(L_1(x_1, y_1), ..., L_n(x_n, y_n)), A(L$$

Hence, $\mathcal{L}_A$ is g-transitive. Consequently, $\mathcal{L}_A$ is g-$\mathcal{E}_A$-order. □

Proposition 3.10. *Let T be a t-norm on U, A an aggregation on U which dominates T, and let $(L_i)_{i\in I}$, $(E_i)_{i\in I}$ be two families of fuzzy binary relations on a domain X such that for each $i \in I$, E_i is a T-equivalence relation, where L_i is T-E_i-order, then the relation $\mathcal{E}_A$ defined by $\mathcal{E}_A(x,y) = A(E_1(x,y),...,E_n(x,y))$, is a T-equivalence relation. And the relation $\mathcal{L}_A$ defined by*

$$\mathcal{L}_A(x,y) = A(L_1(x,y),...,L_n(x,y)),$$

is T-$\mathcal{E}_A$-order.

Proof. It is not difficult to show that $\mathcal{E}_A$ is a T-equivalence relation.
Let us prove now that $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order relation. For $x, y \in X$,
$\mathcal{L}_A(x,y) = A(L_1(x,y),...,L_n(x,y))$. Since for each $i \in I$, $L_i(x,y) \geq E_i(x,y)$. Hence $\mathcal{L}_A(x,y) \geq A(E_1(x,y),...,E_n(x,y)) = \mathcal{E}_A(x,y)$. Consequently, $\mathcal{L}_A$ is $\mathcal{E}_A$-reflexive.
For all $x, y \in X$, we have $T(L_i(x,y), L_i(y,x)) \leq E_i(x,y)$ and prove that $\mathcal{L}_A$ is T-$\mathcal{E}_A$-antisymmetric.
Let $x, y \in X$

$$T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$$

$$= T(A(L_1(x,y),...,L_n(x,y)), A(L_1(y,x),...,L_n(y,x)))$$
$$\leq A(T(L_1(x,y), L_1(y,x)),...,T(L_n(x,y), L_n(y,x))).$$

Or for all $i \in I$, $T(L_i(x,y), L_i(y,x)) \leq E_i(x,y)$, then
$T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \leq A(E_1(x,y),...,E_n(x,y))$

$$= \mathcal{E}_A(x,y).$$

Hence $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-antisymmetric relation on X.
Finally, it is easy to show that $\mathcal{L}_A$ is T-transitive, hence $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order. □

Proposition 3.11. *Let A, $\tilde{A}$ be two aggregations on U and T a left-continuous t-norm dominated by both A and $\tilde{A}$. And let $(E_i^j)_{i,j\in I}$ be n families of T-equivalence relations on a domain X, where (R_i^j) be n-families of fuzzy binary relations such that each (R_i^j) is T-E_i^j-order, then the relation $\check{R}$ defined by*

$$\check{R}(x,y) = \tilde{A}(A(R_1^1(x,y),...,R_1^n(x,y)),...,A(R_n^1(x,y),...R_n^n(x,y)))$$

for all $x, y \in X$ is a T-$\widetilde{E}$-order where

$$\widetilde{E}(x,y) = \tilde{A}(A(E_1^1(x,y),...,E_1^n(x,y)),...,A(E_n^1(x,y),...,E_n^n(x,y))).$$

Proof. Firstly, we prove that $\widetilde{E}$ is a T-equivalence relation. Let $x \in X$,

$\widetilde{E}(x,x) = \tilde{A}(A(E_1^1(x,x),...,E_1^n(x,x))...,A(E_n^1(x,x),...,E_n^n(x,x)))$

$= \tilde{A}(A(1,...,1),...,A(1,...,1))$

$= \tilde{A}(1,...,1)$

$= 1.$
Hence, $\widetilde{E}$ is reflexive.
It is not difficult to show that $\widetilde{E}$ is symmetric.
Let us prove now that $\widetilde{E}$ is T-transitive.
For $x,y,z \in X$,

$T(\widetilde{E}(x,y),\widetilde{E}(y,z))$

$= T[\tilde{A}(A(E_1^1(x,y),...,E_1^n(x,y)),...,A(E_n^1(x,y),...,E_n^n(x,y))),$
$\tilde{A}(A(E_1^1(y,z),...,E_1^n(y,z)),...,A(E_n^1(y,z),...,E_n^n(y,z)))]$

$\leq \tilde{A}[T(A\left[E_1^1(x,y),...,E_1^n(x,y)\right],A\left[E_1^1(y,z),...,E_1^n(y,z)\right]),...,$
$T(A\left[E_n^1(x,y),...,E_n^n(x,y)\right],A\left[E_n^1(y,z),...,E_n^n(y,z)\right])]$

$\leq \tilde{A}[A(T(E_1^1(x,y),E_1^1(y,z)),...,T(E_1^n(x,y)),E_1^n(y,z)),...,$
$A(T(E_n^1(x,y),E_n^1(y,z)),...,T(E_n^n(x,y),E_n^n(y,z))]$

$\leq \tilde{A}(A(E_1^1(x,z),...,E_1^n(x,z)),...,A(E_n^1(x,z),...,E_n^n(x,z)))$

$= \widetilde{E}(x,z)$. Hence $\widetilde{E}$ is T-equivalence relation.

Secondly, we prove that $\check{R}$ is T-$\widetilde{E}$-order. Let $x,y \in X$,

$\check{R}(x,y)$

$= \tilde{A}(A(R_1^1(x,y),...,R_1^n(x,y)),...,A(R_n^1(x,y),...,R_n^n(x,y)))$

$\geq \tilde{A}\left[A(E_1^1(x,y),...,E_1^n(x,y)),...,A(E_n^1(x,y),...,E_n^n(x,y))\right]$

$= \widetilde{E}(x,y).$

Hence, $\check{R}$ is $\widetilde{E}$-reflexive. To prove the T-$\widetilde{E}$-antisymmetry of $\check{R}$, let $x,y \in X$,

$T(\check{R}(x,y),\check{R}(y,x))$

$= T[\tilde{A}(A(R_1^1(x,y),...,R_1^n(x,y)),...,A(R_n^1(x,y),...,R_n^n(x,y))),$
$\tilde{A}(A(R_1^1(y,x),...,R_1^n(y,x)),...,A(R_n^1(y,x),...,R_n^n(y,x))]$

$\leq \tilde{A}[T(A(R_1^1(x,y),...,R_1^n(x,y)),A(R_1^1(y,x),...,R_1^n(y,x))),...,$
$T(A(R_n^1(x,y),...,R_n^n(x,y)),A(R_n^1(y,x),...,R_n^n(y,x)))]$ ($\tilde{A}$ dominates T).

$\leq \tilde{A}[A(T(R_1^1(x,y),R_1^1(y,x)),...,T(R_1^n(x,y),R_1^n(y,x))),...,$
$A(T(R_n^1(x,y),R_n^1(y,x)),...,T(R_n^n(x,y),R_n^n(y,x)))]$

$\leq \tilde{A}(A(E_1^1(x,y),...,E_1^n(x,y)),...,A(E_n^1(x,y),...,E_n^n(x,y)))$ (A dominates T).

$= \widetilde{E}(x,y)$. Thus, the T-$\widetilde{E}$-antisymmetry of $\check{R}$ is got.

Now, let us verify that $\check{R}$ is T-transitive. Let $x,y,z \in X$,

$T(\check{R}(x,y),\check{R}(y,z))$

$= T[\tilde{A}(A(R_1^1(x,y),...,R_1^n(x,y)),...,A(R_n^1(x,y),...,R_n^n(x,y))),$
$\tilde{A}(A(R_1^1(y,z),...,R_1^n(y,z)),...,A(R_n^1(y,z),...,R_n^n(y,z)))]$

$\leq \tilde{A}[T(A(R_1^1(x,y),...,R_1^n(x,y)),A(R_1^1(y,z),...,R_1^n(y,z))),...,$
$T(A(R_n^1(x,y),...,R_n^n(x,y))),A(R_n^1(y,z),...,R_n^n(y,z)))]$

$\leq \tilde{A}[A(T(R_1^1(x,y),R_1^1(y,z)),...,T(R_1^n(x,y)),R_1^n(y,z))),...,$
$A(T(R_n^1(x,y),R_n^1(y,z)),...,T(R_n^n(x,y),R_n^n(y,z)))]$

$\leq \tilde{A}(A(R_1^1(x,z),...,R_1^n(x,z)),...,A(R_n^1(x,z),...,R_n^n(x,z)))$

$= \check{R}(x,z)$.

Thus, $\check{R}$ is T-equivalence. Which complete the proof of the proposition. □

3.5. Aggregating of a finite family of fuzzy filters and fuzzy ideals

3.5.1. Aggregating of a finite family of fuzzy filters

In this section, we introduce and study some proprieties of the operator $(A,\mathcal{F})$ defined on a nonempty set X, where $A: U^n \to U$ is an aggregation on U and $\mathcal{F}$ is a finite family of fuzzy subsets of X.

Definition 3.12. *[60] Let (X,R) be a fuzzy lattice, $\mathcal{F} = \{F_i : X \longrightarrow U, i \in I\}$ a family of fuzzy subsets of X, and $A: U^n \to U$ an aggregation on U. The $(A,\mathcal{F})$ operator defined on X by $\mathcal{F}_A : X \to U$, is obtained as the composition given by $\mathcal{F}_A(x) = A(F_1(x),...,F_n(x))$.*

Remark 3.6. *If $(F_i)_{i\in I}$ is a family of fuzzy filters, $\mathcal{F}_A$ is not necessary a fuzzy filter and the converse as well.*

Example 3.1. *Let (X, R) be a fuzzy lattice with $X = \{0, a, b, c, 1\}$ and R given by the **Table** 3.2.*

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.0	0.5	0.0	0.7
b	0.0	0.0	1.0	0.0	0.9	0.9
c	0.0	0.0	0.0	1.0	0.0	0.3
d	0.0	0.0	0.0	0.0	1.0	0.4
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 3.2:

*whose Hasse diagram is shown in **Figure** 3.1.*

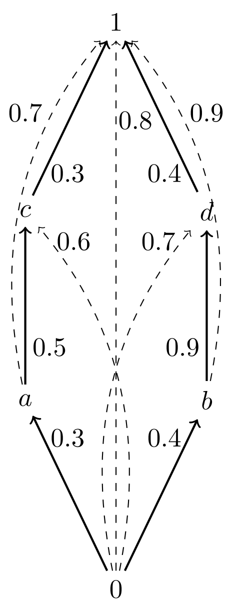


Figure 3.1: Hasse Diagram of R

*Define three fuzzy filters F_1, F_2, and F_3 as in **Table** 3.3.*

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.2	0.0	0.0
b	0.0	0.0	0.3
c	0.4	0.7	0.0
d	0.0	0.0	0.6
1	0.5	0.9	0.8

Table 3.3:

And $A(x, y, z) = \frac{x+y+z}{3}$. *Then* $\mathcal{F}_A$ *as in* ***Table*** *3.4.*

x	$\mathcal{F}_A(x)$
0	0.0
a	$\frac{0.2}{3}$
b	0.1
c	$\frac{1.1}{3}$
d	0.2
1	$\frac{2.2}{3}$

.

Table 3.4:

It is easy to verify that $\mathcal{F}_A$ *is not a fuzzy filter. Indeed,* $\mathcal{F}_A(a) = \frac{0.2}{3} > 0$ *and* $\mathcal{F}_A(b) = 0.1 > 0$, *but* $\mathcal{F}_A(a \wedge b) = \mathcal{F}_A(0) = 0$ *(the second condition is not satisfies.) Conversely, we can define* $\mathcal{F}_A$ *to be a fuzzy filter on* X *as in* ***Table*** *3.5.*

x	$\mathcal{F}_A(x)$
0	0.0
a	$\frac{0.4}{3}$
b	0.0
c	0.2
d	0.0
1	$\frac{0.7}{3}$

Table 3.5:

And choose F_1, F_2, *and* F_3, *for example as in* ***Table*** *3.6.*

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.2	0.1	0.1
b	0.0	0.0	0.0
c	0.4	0.2	0.0
d	0.0	0.0	0.0
1	0.5	0.0	0.2

.

Table 3.6:

Clearly, F_2 and F_3 are not fuzzy filters.

Now, we give a sufficient condition under which an aggregation of a family of fuzzy filters is a fuzzy filter.

Proposition 3.12. *Let (X, R) be a fuzzy lattice, $A : U^n \to U$ an aggregation such that A has no zero divisors other than 0 and let $\mathcal{F} = \{F_i : X \to U, i \in I\}$ be a family of fuzzy subsets of X. If $\mathcal{F}$ is a family of fuzzy filters of (X, R), then $\mathcal{F}_A$ is a fuzzy filter of (X, R).*

Proof. (a) Suppose that $\mathcal{F} = \{F_i : X \to U, i \in I\}$ is a family of fuzzy filters of (X, R) and A be an aggregation on U such that A has no zero divisors other than 0, i.e., $A(x_1, ..., x_n) = 0 \Leftrightarrow x_1 = 0$ or...or $x_n = 0$. Let $x, y \in X$ such that $\mathcal{F}_A(x) > 0$ and $R(x, y) > 0$. This is equivalent to $A(F_1(x), ..., F_n(x)) > 0$ and $R(x, y) > 0$, which implies $F_i(x) > 0$ and $R(x, y) > 0$ for all $i \in I$. Hence $F_i(y) > 0$ for all $i \in I$, this implies that $A(F_1(y), ..., F_n(y)) > 0$. Consequently $\mathcal{F}_A(y) > 0$.
(b) Let $x, y \in X$ such that $\mathcal{F}_A(x) > 0$ and $\mathcal{F}_A(y) > 0$, this means $A(F_1(x), ..., F_n(x)) > 0$ and $A(F_1(y), ..., F_n(y)) > 0$, hence $F_i(x) > 0$ and $F_i(y) > 0$ for any $i \in I$, then $F_i(x \wedge y) > 0$ for all $i \in I$ which implies $A(F_1(x \wedge y), ..., F_n(x \wedge y)) > 0$. Thus, $\mathcal{F}_A(x \wedge y) > 0$. Consequently, $\mathcal{F}_A$ is a fuzzy filter of (X,R). □

Remark 3.7. *The converse of* ***Proposition*** *3.14 is not true. Indeed, take R as in* ***Table*** *3.7.*

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.4	0.3	0.5	0.7
b	0.0	0.0	1.0	0.0	0.4	0.9
c	0.0	0.0	0.0	1.0	0.2	0.3
d	0.0	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 3.7:

And $A(x, y, z) = x \wedge y \wedge z$. Take $\mathcal{F}_A$ as in ***Table*** *3.8.*

x	$\mathcal{F}_A(x)$
0	0.0
a	0.0
b	0.0
c	0.0
d	0.3
1	0.5

Table 3.8:

And choose F_1, F_2, and F_3, for example as in ***Table*** *3.9.*

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.0	0.0	0.0
b	0.2	0.0	0.3
c	0.2	0.3	0.0
d	0.3	0.4	0.3
1	0.5	0.6	0.7

Table 3.9:

It is easy to see that $\mathcal{F}_A$ is a filter, but F_1, F_2, F_3 are not all filters (F_1 is not a filter).
Remark 3.8. *Let R be a fuzzy relation defined on the set $X = \{0, b, c, d, e, 1\}$ by the* ***Table*** *3.10.*

R	0	b	c	d	1
0	1.0	0.4	0.6	0.7	0.8
b	0.0	1.0	0.0	0.4	0.7
c	0.0	0.0	1.0	0.2	0.7
d	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	1.0

.

Table 3.10:

Also, let F_1 and F_2 two filters, and $\mathcal{F}_A$ their aggregation as in ***Table*** *3.11.*

x	$F_1(x)$	$F_2(x)$	$\mathcal{F}_A(x)$
0	0.0	0.0	0.0
b	0.3	0.0	0.0
c	0.0	0.2	0.0
d	0.6	0.4	0.4
1	0.8	0.7	0.7

Table 3.11:

Put $\mathcal{F}_A(x) = \inf(F_1(x), F_2(x))$.
The ***Table*** *3.11 shows that the aggregation of a finite family of prime (respectively maximal) filters is not prime (respectively maximal) filter. Even* $\mathcal{F}_A(x) = \inf(F_1(x), F_2(x))$.
Proposition 3.13. *Let* $(X_i, R_i)_{i\in I}$ *be a family of fuzzy lattices,* $(F_i)_{i\in I}$ *a family of fuzzy subsets such that* $F_i : X_i \to U$ *and* $A : U^n \to U$ *is an aggregation defined by* $A(x_1, ..., x_n) = x_1^{\alpha_1} \wedge ... \wedge x_n^{\alpha_n}$*, where* $\alpha_1, ..., \alpha_n \in \mathbb{R}_+^*$*. Let* $\Re_A$ *and* $\mathcal{F}_A$ *be two fuzzy sets defined on* $(\prod_{i=1}^{n} X_i)^2$ *and* $\prod_{i=1}^{n} X_i$ *by*

$$\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) = A(R_1(x_1, y_1), ..., R_n(x_n, y_n)) \text{ and}$$

$$\mathcal{F}_A(x_1, ..., x_n) = A(F_1(x_1), ..., F_n(x_n))$$

respectively. If F_i *is a fuzzy filters of* (X_i, R_i) *for all* $i \in I$*, then* $\mathcal{F}_A$ *is a fuzzy filter of* $(\prod_{i=1}^{n} X_i, \Re_A)$.

Proof. (a) Let $(x_1, ..., x_n), (y_1, ..., y_n) \in \prod_{i=1}^{n} X_i$ such that $\mathcal{F}_A(x_1, ..., x_n) > 0$ and $\Re_A((x_1, ..., x_n), (y_1, ..., y_n)) > 0$. This is equivalent to $A(F_1(x_1), ..., F_n(x_n)) > 0$ and $A(R_1(x_1, y_1), ..., R_n(x_n, y_n)) > 0$. Hence, $F_i(x_i) > 0$ and $R_i(x_i, y_i) > 0$, for all $i \in I$, thus $F_i(y_i) > 0$ for all $i \in I$. Consequently $A(F_1(y_1), ..., F_n(y_n)) > 0$, i.e., $\mathcal{F}_A(y_1, ..., y_n) > 0$.
(b) Let $(x_1, ..., x_n), (y_1, ..., y_n) \in \prod_{i=1}^{n} X_i$ such that

$$\mathcal{F}_A(x_1, ..., x_n) > 0 \text{ and } \mathcal{F}_A(y_1, ..., y_n) > 0.$$

This is equivalent to $A(F_1(x_1), ..., F_n(x_n)) > 0$ and $A(F_1(y_1), ..., F_n(y_n)) > 0$. Hence, $F_i(x_i) > 0$ and $F_i(y_i) > 0$ for all $i \in I$, this imply that $F_i(x_i \wedge y_i) > 0$ for all $i \in I$ and that $A(F_1(x_1 \wedge y_1), ..., F_n(x_n \wedge y_n)) > 0$, this means $\mathcal{F}r_A(x_1 \wedge y_1, ..., x_n \wedge y_n) > 0$. Hence, $\mathcal{F}_A((x_1, ..., x_n) \wedge (y_1, ..., y_n)) > 0$, which complete the proof of this proposition. □

3.5.2. Aggregating of a finite family of fuzzy ideals

In this subsection, we introduce and study some proprieties of the operator $(A, \mathfrak{I})$ *defined on a nonempty set* X*, where* $A : U^n \to U$ *is an aggregation on* U *and* $\mathfrak{I}$ *is a finite family of fuzzy subsets of* X*.*

Definition 3.13. *[60] Let* (X, R) *be a fuzzy lattice,* $\mathfrak{I} = \{\mathcal{I}_i : X \longrightarrow U, i \in I\}$ *a family of fuzzy subsets of* X*, and* $A : U^n \to U$ *an aggregation on* U*. The* $(A, \mathfrak{I})$ *operator defined on* X *by* $\mathfrak{I}_A : X \to U$*, is obtained as the composition given by* $\mathfrak{I}_A(x) = A(\mathcal{I}_1(x), ..., \mathcal{I}_n(x))$*.*

Remark 3.9. *If* $(\mathcal{I}_i)_{i\in I}$ *is a family of fuzzy ideals,* $\mathfrak{I}_A$ *is not necessary a fuzzy ideal and the converse as well.*

Example 3.2. *Let* (X, R) *be a fuzzy lattice with* $X = \{0, a, b, c, 1\}$ *and* R *given by the* ***Table*** *3.12.*

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.0	0.5	0.0	0.7
b	0.0	0.0	1.0	0.0	0.9	0.9
c	0.0	0.0	0.0	1.0	0.0	0.3
d	0.0	0.0	0.0	0.0	1.0	0.4
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 3.12:

Define three fuzzy ideals $\mathcal{I}_1$*,* $\mathcal{I}_2$*, and* $\mathcal{I}_3$ *as in* ***Table*** *3.13.*

x	$\mathcal{I}_1(x)$	$\mathcal{I}_2(x)$	$\mathcal{I}_3(x)$
0	0.5	0.5	0.4
a	0.2	0.0	0.0
b	0.0	0.1	0.3
c	0.4	0.0	0.0
d	0.0	0.0	0.6
1	0.0	0.0	0.0

Table 3.13:

And $A(x, y, z) = \frac{x+y+z}{3}$*. Then* $\mathfrak{I}_A$ *is as in* ***Table*** *3.14.*

x	$\mathfrak{I}_A(x)$
0	$\frac{1.4}{3}$
a	$\frac{0.2}{3}$
b	$\frac{0.4}{3}$
c	$\frac{0.4}{3}$
d	0.2
1	0.0

Table 3.14:

*It is easy to verify that $\mathfrak{I}_A$ is not a fuzzy ideal. Indeed, $\mathfrak{I}_A(c) = \frac{1.1}{3} > 0$ and $\mathfrak{I}_A(d) = 0.2 > 0$, but $\mathfrak{I}_A(c \vee d) = \mathfrak{I}_A(1) = 0$. Conversely, we can define $\mathfrak{I}_A$ to be a fuzzy ideal on X as in **Table** 3.15.*

x	$\mathfrak{I}_A(x)$
0	0.3
a	$\frac{0.4}{3}$
b	0.0
c	0.2
d	0.0
1	0.0

Table 3.15:

*And choose $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ as in **Table** 3.16.*

x	$\mathcal{I}_1(x)$	$\mathcal{I}_2(x)$	$\mathcal{I}_3(x)$
0	0.6	0.3	0.0
a	0.2	0.0	0.2
b	0.0	0.0	0.0
c	0.2	0.2	0.2
d	0.0	0.0	0.0
1	0.0	0.0	0.0

Table 3.16:

Clearly, $\mathcal{I}_2$ and $\mathcal{I}_3$ are not fuzzy ideals. As in filters case, we give a sufficient condition under which an aggregation of a family of fuzzy ideals is a fuzzy ideal.

Proposition 3.14. *Let (X, R) be a fuzzy lattice, $A : U^n \to U$ an aggregation such that A has no zero divisors other than 0 and let $\mathfrak{I} = \{\mathcal{I}_i : X \to U, i \in I\}$ be a family of fuzzy subsets of X. If $\mathfrak{I}$ is a family of fuzzy ideals of (X, R), then $\mathfrak{I}_A$ is a fuzzy ideal of (X, R).*

Proof. a) Suppose that $\mathfrak{I} = \{\mathcal{I}_i : X \to U, i \in \mathcal{I}\}$ is a family of fuzzy ideals of (X, R) and A an aggregation on U such that A has no zero divisors other than 0, i.e., $A(x_1, ..., x_n) = 0 \Leftrightarrow x_1 = 0$ or,...,or $x_n = 0$. Let $x, y \in X$ such that $\mathfrak{I}_A(x) > 0$ and $R(y, x) > 0$. This is equivalent to $A(\mathcal{I}_1(x), ..., \mathcal{I}_n(x)) > 0$ and $R(y, x) > 0$, which implies $\mathcal{I}_i(x) > 0$ and $R(y, x) > 0$ for all $i \in I$. Hence $\mathcal{I}_i(y) > 0$ for all $i \in I$ and this implies that $A(\mathcal{I}_1(y), ..., \mathcal{I}_n(y)) > 0$. Consequently $\mathfrak{I}_A(y) > 0$.
b) Let $x, y \in X$ such that $\mathfrak{I}_A(x) > 0$ and $\mathfrak{I}_A(y) > 0$, this means $A(\mathcal{I}_1(x), ..., \mathcal{I}_n(x)) > 0$ and $A(\mathcal{I}_1(y), ..., \mathcal{I}_n(y)) > 0$, hence, $\mathcal{I}_i(x) > 0$ and $\mathcal{I}_i(y) > 0$ for any $i \in I$, then $\mathcal{I}_i(x \vee y) > 0$ for all $i \in I$ which implies $A(\mathcal{I}_1(x \vee y), ..., \mathcal{I}_n(x \vee y)) > 0$. Hence, $\mathfrak{I}_A(x \vee y) > 0$. Thus, $\mathfrak{I}_A$ is a fuzzy ideal of (X, R). □

Remark 3.10. *The converse of* ***Proposition*** *3.14 is not true. Indeed, take R as in* ***Table*** *3.17.*

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.4	0.3	0.5	0.7
b	0.0	0.0	1.0	0.0	0.4	0.9
c	0.0	0.0	0.0	1.0	0.2	0.3
d	0.0	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 3.17:

And $A(x, y, z) = x \wedge y \wedge z$. Take $\mathfrak{I}_A$ as in ***Table*** *3.18.*

x	$\mathfrak{I}_A(x)$
0	0.4
a	0.3
b	0.0
c	0.0
d	0.0
1	0.0

Table 3.18:

And choose $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, for example as in ***Table*** *3.19.*

x	$\mathcal{I}_1(x)$	$\mathcal{I}_2(x)$	$\mathcal{I}_3(x)$
0	0.4	0.5	0.5
a	0.3	0.5	0.3
b	0.2	0.3	0.0
c	0.2	0.3	0.0
d	0.3	0.0	0.3
1	0.0	0.0	0.0

Table 3.19:

It is easy to see that $\mathfrak{I}_A$ is a ideal, but $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ are not all ideals ($\mathcal{I}_2$, $\mathcal{I}_3$ are not ideals).

Remark 3.11. *Let R be a fuzzy relation defined on the set $X = \{0, b, c, d, e, 1\}$ by the* ***Table*** *3.20.*

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.0	0.0	0.5	0.7
b	0.0	0.0	1.0	0.0	0.4	0.7
c	0.0	0.0	0.0	1.0	0.2	0.7
d	0.0	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	0.0	1.0

Table 3.20:

Also, let $\mathcal{I}_1$ and $\mathcal{I}_2$ two ideals, and $\mathfrak{I}_A$ their aggregation as in ***Table 3.21***.

x	$\mathcal{I}_1(x)$	$\mathcal{I}_2(x)$	$\mathfrak{I}_A(x)$
0	0.7	0.5	0.5
a	0.2	0.0	0.0
b	0.0	0.3	0.0
c	0.0	0.0	0.0
d	0.7	0.6	0.6
1	0.0	0.0	0.0

Table 3.21:

Put $\mathfrak{I}_A(x) = \inf(\mathcal{I}_1(x), \mathcal{I}_2(x))$.

Table *3.21 shows that the aggregation of a finite family of prime (resp. maximal) ideals is not prime (resp. maximal) ideal. Even $\mathfrak{I}_A(x) = \inf(\mathcal{I}_1(x), \mathcal{I}_2(x))$.*

Proposition 3.15. *Let $(X_i, R_i)_{i\in I}$ be a family of fuzzy lattices, $(\mathcal{I}_i)_{i\in I}$ a family of fuzzy subsets such that $\mathcal{I}_i : X_i \to U$ and $A : U^n \to U$ is an aggregation defined* *by* $A(x_1,...,x_n) = x_1^{\alpha_1} \wedge ... \wedge x_n^{\alpha_n}$, *where* $\alpha_1,...,\alpha_n \in \mathbb{R}_+^*$. *Let* $\Re_A$ *and* $\Im_A$ *be two fuzzy sets defined on* $(\prod_{i=1}^{n} X_i)^2$ *and* $\prod_{i=1}^{n} X_i$ *by*

$$\Re_A((x_1,...,x_n),(y_1,...,y_n)) = A(R_1(x_1,y_1),...,R_n(x_n,y_n))$$

and

$$\Im_A(x_1,...,x_n) = A(\mathcal{I}_1(x_1),...,\mathcal{I}_n(x_n))$$

respectively. If $\mathcal{I}_i$ *is a fuzzy ideals of* (X_i,R_i) *for all* $i \in I$, *then* $\Im_A$ *is a fuzzy ideal of* $(\prod_{i=1}^{n} X_i, \Re_A)$.

Proof. (a) Let $(x_1,...,x_n),(y_1,...,y_n) \in \prod_{i=1}^{n} X_i$ such that $\Im_A(x_1,...,x_n) > 0$ and $\Re_A((y_1,...,y_n),(x_1,...,x_n)) > 0$. This is equivalent to $A(\mathcal{I}_1(x_1),...,\mathcal{I}_n(x_n)) > 0$ and $A(R_1(y_1,x_1),...,R_n(y_n),x_n) > 0$. Hence, $\mathcal{I}_i(x_i) > 0$ and $R_i(y_i,x_i) > 0$, for all $i \in I$, thus $\mathcal{I}_i(y_i) > 0$ for all $i \in I$. Consequently $A(\mathcal{I}_1(y_1),...,\mathcal{I}_n(y_n)) > 0$, i.e., $\Im_A(y_1,...,y_n) > 0$.

(b) Let $(x_1,...,x_n),(y_1,...,y_n) \in \prod_{i=1}^{n} X_i$ such that

$$\Im_A(x_1,...,x_n) > 0 \text{ and } \Im_A(y_1,...,y_n) > 0.$$

This is equivalent to $A(\mathcal{I}_1(x_1),...,\mathcal{I}_n(x_n)) > 0$ and $A(\mathcal{I}_1(y_1),...,\mathcal{I}_n(y_n)) > 0$. Hence, $\mathcal{I}_i(x_i) > 0$ and $\mathcal{I}_i(y_i) > 0$ for all $i \in I$, this implies that $\mathcal{I}_i(x_i \vee y_i) > 0$ for all $i \in I$ and that $A(\mathcal{I}_1(x_1 \vee y_1),...,\mathcal{I}_n(x_n \vee y_n)) > 0$, this means $\Im_A(x_1 \vee y_1,...,x_n \vee y_n) > 0$. Hence, $\Im_A((x_1,...,x_n) \vee (y_1,...,y_n)) > 0$, which complete the proof of this proposition. □

4 General conclusions and future research

In this thesis, we have established that $\otimes$-E-fuzzy partial ordering structures presented in this work support cutworthy properties.

The connection between the intersection of L-fuzzy sets and their α-cuts are given. In addition, we have studied the aggregation of some finite families of fuzzy structures, (Fuzzy binary relations and fuzzy filters).

We have also studied the relation between those families and their aggregations.

It has established that the aggregation of a family of fuzzy ordering relations is a fuzzy ordering relation. Furthermore, the aggregation of a family of complete lattices is a complete lattice. In addition, the aggregation of a family of right (left) traces is a right (left) trace. Finally, the aggregation of a family of fuzzy filters is a filter.

As open questions:

1. *Is it also possible to obtain such results for the intuitionistic fuzzy case?*
2. *Is it possible to extend this study to the aggregation of L-fuzzy structures, where L is any lattice?*

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