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Thesis

AM-GM AND CAUCHY-SCHWARZ INEQUALITIES

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Our thanks also go to the jury members for the honor they have done for us by accepting to judge this modest work.

Dedication

I dedicate this modest work

To my dear parents

To my dear sisters .

To my brothers .

To all the members of my familly.

To my friends.

Fatima

Dedication

The simplest words are the strongest,
I send all my love to my father and my mother ,
thank you for making me what I am today
I dedicate this work to them.
To my husband.
To my brothers and my sisters.
To the best person in my life.

Zehor

Abstract

الملخص: في هذه المذكرة، قمنا بدراسة و اثبات العلاقة بين الوسط الحسابي والهندسي ومتباينة كوشي شوارتز. تم تقديم العديد من التطبيقات لتوضيح أهمية هاتين المتباينتين.

الكلمات المفتاحية : الوسط الحسابي، الوسط الهندسي، كوشي شوارتز،أولومبياد الرياضيات العالمية.

Dans ce mémoire, Nous énoncons et prouvons les deux inégalités : moyenne aritmétique-moyenne géométrique et l'inégalité de Cauchy-Schwarz. Plusieurs exemples sont donnés pour illustrer l'importance de ces inégalités.

Mots-clés: *Moyenne arithmétique, moyenne géométrique, Cauchy-Schwarz, IMO.*

In this thesis, we state and prove the two inequalities : arithmetic mean-geometric mean inequality and Cauchy-Schwarz inequality. To illustrate the importance of these inequalities we give many examples.

Keywords : *Arithmetic mean, Geometric mean, Cauchy-Schwarz, Inequality, IMO.*

Contents

General Introduction

In Mathematics, it is not always easy to find exact formulas, therefore, sometimes we take approximate values for certain terms, that is why we need inequalities.

The inequality is a fundamental relation in mathematics. It is a useful tool for problem solving and building relationships with other mathematics. The Arithmetic-Geometric mean (AM-GM for short) and Cauchy-Schwartz inequalities are of particular importance and interest. These inequalities are easily stated and have many useful and important applications.

The aim of this thesis is to investigate these inequalities. In the first chapter we give the definition of arithmetic mean and geometric mean, some proofs of the AM-GM inequality as well as some applications.
In the second chapter we do the same with the other well known inequality, i.e; we state and prove Cauchy-Schwartz inequality. As for the AM-GM inequality we give some applications illustrating its importance.
Inequalities is one among the many topics encountered in the International Mathematical Olympiad. The last chapter, is devoted to some problems taken from some sessions of the IMO.

Chapter 1

Arithmetic Mean-Geometric Mean

In this chapter, we state and prove the AM-GM Inequality. Different proofs are supplied. Also, many examples are treated to illustrate the importance of the AM-GM inequality.

We start by giving some elementary inequalities.

1.1 Elementary inequalities

Let $a, b, c \in \mathbb{R}^+$

- if $a \geq b$ and $b \geq c$ then $a \geq c$.
- if $a \geq b$ then $a + c \geq b + c$.
- if $a \geq b$ and $x \geq y$ then $a + x \geq b + y$ for any $x, y \in \mathbb{R}^+$.
- if $a \geq b$ and $x \geq y$ then $ax \geq by$ for any $x, y \in \mathbb{R}^+$.

1.2 Arithmetic Mean and Geometric Mean

Let $a_1, ..., a_n$ be n positive real numbers.

Definition 1.1 (Arithmetic Mean) The arithmetic Mean of n real numbers $a_1, ..., a_n$ is defined as

$$AM = \frac{a_1 + \cdots + a_n}{n}.$$

Definition 1.2 (Geometric Mean) The geometric Mean of n real numbers $a_1, ..., a_n$ is defined as

$$GM = \sqrt[n]{a_1 \ldots a_n}.$$

1.3 Basic AM-GM Inequality

Theorem 1.3 *For all real numbers x, y the inequality*

$$x^2 + y^2 \geq 2\,x\,y, \qquad \left(\frac{x+y}{2}\right)^2 \geq xy, \qquad \frac{x+y}{2} \geq \sqrt{x\,y}.$$

With equality, if and only if $x = y$.

Proof 1.4

The proof of this theorem is direct, because

1-

$$\begin{aligned} x^2 + y^2 \geq 2\,x\,y \quad &\Longleftrightarrow \quad x^2 - 2xy + y^2 \geq 0; \\ &\Longleftrightarrow \quad (x-y)^2 \geq 0. \end{aligned}$$

2-

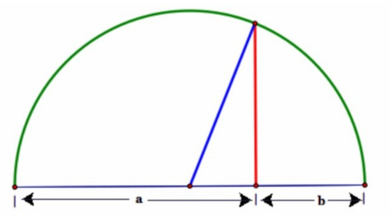
$$\begin{aligned} \left(\frac{x+y}{2}\right)^2 \geq xy \quad &\Longleftrightarrow \quad \frac{x+y}{2} \geq \sqrt{x\,y}; \\ &\Longleftrightarrow \quad x + y \geq 2\sqrt{xy}; \\ &\Longleftrightarrow \quad x - 2\sqrt{x\,y} + y \geq 0; \\ &\Longleftrightarrow \quad (\sqrt{x} - \sqrt{y})^2 \geq 0. \end{aligned}$$

3-

$$\begin{aligned} \frac{x+y}{2} \geq \sqrt{xy} \quad &\Longleftrightarrow \quad \left(\frac{x+y}{2}\right)^2 \geq xy; \\ &\Longleftrightarrow \quad (x+y)^2 \geq 4\,x\,y; \\ &\Longleftrightarrow \quad (x+y)^2 - 4\,x\,y \geq 0; \\ &\Longleftrightarrow \quad (x-y)^2 \geq 0. \end{aligned}$$

Geometric Proof 1.5

Construct a semicircle with diameter $a+b$. The radius will be the arithmetic Mean of a and b. Construct a perpendicular to the diameter from common endpoint of the segments of length a and b. This perpendicular will always have a length less than or equal to the radius of the circle and it will be equal only if $a = b$. This may be shown by the application of the Pythagorean Theorem:



The radius has length $\dfrac{a+b}{2}$.

The red segment has length $\sqrt{a\,b}$. So, $\dfrac{a+b}{2} \geq \sqrt{a\,b}$.

With equality, if and only if $a = b$.

Example 1.6

Here is our first significant example, showing the strength of the AM-GM inequality.

Prove the following inequality for every $a, b, c \in \mathbb{R}^+$

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Solution

Applying AM-GM inequality on the two couples $(\frac{a^2}{b^2}, \frac{b^2}{c^2})$, $(\frac{b^2}{c^2}, \frac{c^2}{a^2})$ gives us :

$$\frac{1}{2}\left(\frac{a^2}{b^2} + \frac{b^2}{c^2}\right) \geq \sqrt{\frac{a^2b^2}{b^2c^2}} = \frac{a}{c}$$
$$\frac{1}{2}\left(\frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq \sqrt{\frac{b^2c^2}{c^2a^2}} = \frac{b}{a}$$

And

$$\frac{1}{2}\left(\frac{c^2}{a^2} + \frac{a^2}{b^2}\right) \geq \sqrt{\frac{c^2a^2}{a^2b^2}} = \frac{c}{b}$$
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Note that equality occurs when $a = b = c$.

Example 1.7

For positive real numbers a, b and c prove

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution

The AM-GM gives us:

$$\begin{aligned} a+b &\geq 2\sqrt{ab}; \\ b+c &\geq 2\sqrt{bc}; \\ \textit{and } c+a &\geq 2\sqrt{ca} \\ \textit{so } (a+b)(b+c)(c+a) &\geq 8(\sqrt{ab})(\sqrt{bc})(\sqrt{ca}) \\ &\geq 8\sqrt{a^2b^2c^2} = 8abc. \end{aligned}$$

Example 1.8

For real positive numbers a, b, c, prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution

By the AM-GM inequality we have

$$a^2 + b^2 \geq 2\,a\,b$$

$$b^2 + c^2 \geq 2\,b\,c$$

$$c^2 + a^2 \geq 2\,c\,a.$$

Adding the three inequalities and then dividing by 2, we get the desired result. It is easily seen that equality holds if and only if $a = b = c$.

1.4 General AM-GM Inequality

Theorem 1.9 ($AM \geq GM$) *For non negative real numbers* $a_1, ..., a_n$, we have

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}.$$

Equality holds, if and only if $a_1 = a_2 = ... = a_n$.

There are many different proofs. Here, we give just 3 of them.

1.4.1 First proof : Cauchy Proof

Here we present the well known proof due to Cauchy. It is based on a strong variant of the proof by induction. This special kind of induction is done by performing the following three steps:

- Basic case
- $\Omega_n \Rightarrow \Omega_{2n}$
- $\Omega_n \Rightarrow \Omega_{n-1}$

Where Ω_n denote the proposition that if $a_1, a_2, \ldots, a_n$ are non negative numbers, not all equal, then

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Step 1 :

For $n = 2$, the inequality

$$\sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2} \ . \tag{1.1}$$

Is equivalent to

$$\begin{aligned} \sqrt{a_1 a_2} \leq \frac{a_1 + a_2}{2} &\iff 2\sqrt{a_1 a_2} \leq a_1 + a_2; \\ &\iff 0 \leq a_1 - 2\sqrt{a_1 a_2} + a_2; \\ &\iff 0 \leq (\sqrt{a_1} - \sqrt{a_2})^2. \end{aligned}$$

Let us prove it for $n = 4$. Put $a_1 = a, a_2 = b, a_3 = c, a_4 = d$, then

$$\sqrt[4]{a\,b\,c\,d} \leq \frac{a + b + c + d}{4}. \tag{1.2}$$

Now, use the AM-GM in case $n = 2$

$$\begin{aligned} \frac{a + b + c + d}{4} &= \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2}; \\ &\geq \frac{\sqrt{a\,b} + \sqrt{c\,d}}{2}, \text{ by the case } n = 2; \\ &\geq \sqrt{(\sqrt{a\,b})(\sqrt{c\,d})}, \text{ by the case } n = 2; \\ &= \sqrt[4]{a\,b\,c\,d}. \end{aligned} \tag{1.3}$$

Now let us prove it for $n = 3$

$$\begin{aligned} \frac{a_1 + a_2 + a_3}{3} &= \frac{a_1 + a_2 + a_3 + \dfrac{a_1 + a_2 + a_3}{3}}{4} \\ &\geq \sqrt[4]{\left(a_1\, a_2\, a_3 \frac{a_1 + a_2 + a_3}{3}\right)}. \end{aligned}$$

By (1.4), the case $n = 4$ is just

$$\sqrt[3]{a_1\, a_2\, a_3} \leq \frac{a_1 + a_2 + a_3}{3}.$$

Step 2 :

$\Omega_n \Rightarrow \Omega_{2n}$

Assuming that Ω_n is true , we have

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Now it's not difficult to notice that

$$a_1 + a_2 + \ldots + a_{2\,n} \geq n\,\sqrt[n]{a_1 a_2 \ldots a_n} + n\,\sqrt[n]{a_{n+1} a_{n+2} \ldots a_{2\,n}} \geq 2\,n\,\sqrt[2n]{a_1 a_2 \ldots a_{2\,n}}$$

Implying Ω_{2n} is true.

Step 3 :

$\Omega_n \Rightarrow \Omega_{n-1}$

First we assume that Ω_n is true i.e.

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

As this is true for all positive a_i , we let $a_n = \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}$. So now we have

$$\begin{aligned} \frac{a_1 + a_2 + \ldots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_{n-1}\,\sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}} \\ &= \sqrt[n]{(a_1 a_2 \ldots a_{n-1})^{\frac{n}{n-1}}} \\ &= \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}} \\ &= a_n. \end{aligned}$$

Which in turn is equivalent to

$$\frac{a_1 + a_2 + \ldots + a_{n-1}}{n-1} \geq a_n = \sqrt[n-1]{a_1 a_2 \ldots a_{n-1}}.$$

The proof is complete.

1.4.2 Second proof : Induction Proof

Lemma *Let* $a_1, a_2, ..., a_n$ *be positive real numbers.*
If $a_1 a_2 \dots a_n = 1$, *then* $a_1 + a_2 + \dots + a_n \geq n$.

Proof

By induction on n. The case $n = 1$ is trivial. So, suppose that $n \geq 2$. Let $a_1, a_2, ..., a_n$ be positive numbers with $a_1 a_2 \dots a_n = 1$. Without loss of generality, we can assume that $a_1 = max\{a_1, a_2, ..., a_n\} \geq 1$ and $a_2 = min\{a_1, a_2, ..., a_n\} \leq 1$. Thus we have

$$a_1 + a_2 - a_1 a_2 - 1 = (a_1 - 1)(1 - a_2) \geq 0,$$

i.e.,

$$a_1 + a_2 - a_1 a_2 \geq 1. \tag{1.4}$$

Since $a_1 a_2, ..., a_n$ are $(n-1)$ positive numbers with product equal to 1, by the induction hypothesis we have

$$a_1 a_2 + a_3 + \dots + a_n \geq n - 1. \tag{1.5}$$

Adding the inequalities (1.4) (1.5), we get

$$a_1 + a_2 + \dots + a_n \geq 1 + n - 1 = n. \tag{1.6}$$

If equality holds in (1.7) then we must have equality in (1.5), that is, $a_1 = 1$ or $a_2 = 1$, hence $a_1 = a_2 = \dots = a_n = 1$.

Proof 1.10

For positive real numbers $a_1, a_2, ..., a_n$, we set:
$x_1 = \dfrac{a_1}{\sqrt[n]{a_1\, a_2 \dots a_n}}, \dots, x_n = \dfrac{a_n}{\sqrt[n]{a_1\, a_2 \dots a_n}}.$
The numbers $x_1, x_2, ..., x_n$ satisfy the condition $x_1 x_2 \dots x_n = 1$. So we have

$$\frac{a_1}{\sqrt[n]{a_1\, a_2 \dots a_n}} + \dots + \frac{a_n}{\sqrt[n]{a_1\, a_2 \dots a_n}} \geq n,$$

hence we obtain

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \dots a_n},$$

with the equality holding only for $a_1 = \dots = a_n$.

1.4.3 Third proof : Convex Function

Definition 1.11(Convex and concave functions) Let X be a subset of R

Let $\Phi : X \to \mathbb{R}$, be a function

1-We say that Φ is convex on X if

$$\Phi((1-\lambda)x_1 + \lambda\, x_2) \leq (1-\lambda)\Phi(x_1) + \lambda\Phi(x_2),$$

for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$.

2-We say that Φ is concave on X if

$$\Phi((1-\lambda)x_1 + \lambda x_2) \geq (1-\lambda)\Phi(x_1) + \lambda\Phi(x_2),$$

for all x_1, x_2 in X and $\lambda \in [0,1]$.

Proposition 1.12

Let $\Phi : X \to \mathbb{R}$, twice differentiable function. Then Φ is convex (or concave up)if $\Phi''(x) \geq 0$ for all $x \in X$.

Proof (by convex function)

We take $f(x) = e^x$. First, we claim that f(x) is a convex function that is strictly increasing on its domain of definition, which is $[0, \infty[$. We have that

$(e^x)' = e^x > 0$ for all $x \in [0, \infty[$. Hence, e^x is strictly increasing. Differentiate another time to get

$$(e^x)'' = (e^x)' = e^x > 0 \ \ for\ all\ x \in [0, \infty[$$

By the previous proposition, we see that e^x is a convex function.

Let $(x_i)_{1\leq i\leq n}$ and $(\lambda_i)_{1\leq i\leq n}$be real number satisfying

$$x_i \geq 0, \lambda_i \in [0,1]\,, \sum_{i=1}^{n} \lambda_i = 1.$$

Taking $\lambda_1 = \cdots = \lambda_n = \frac{1}{n}$ and $y_i = \ln(x_i)$, and using the convexity of e^x, we get

$$\begin{aligned} e^{\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n} &\leq \lambda_1 e^{y_1} + \lambda_2 e^{y_2} + \cdots + \lambda_n e^{y_n} \\ e^{\frac{1}{n}(\ln(x_1) + \cdots + \ln(x_n))} &\leq \frac{1}{n} e^{\ln(x_1)} + \cdots + \frac{1}{n} e^{\ln(x_n)} \\ e^{\frac{1}{n}\ln(x_1 x_2 \ldots x_n)} &\leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \\ (x_1 x_2 \ldots x_n)^{\frac{1}{n}} &\leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n). \end{aligned}$$

Which is the desired result.

1.5 Applications

1.5.1 Applications of AM-GM

Example 1.14

Let $a_1, a_2, ..., a_n$ be a positive real numbers. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} \geq n.$$

Solution

By the AM-GM inequality we have

$$\frac{\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}}{n} \geq \sqrt[n]{\frac{a_1}{a_2}\frac{a_2}{a_3} \ldots \frac{a_n}{a_1}} = 1.$$

Example 1.15

Show that for positive real numbers $a, b \; and \; c$

$$(a^2\,b + b^2\,c + c^2\,a)(a\,b^2 + b\,c^2 + c\,a^2) \geq 9\,a^2\,b^2\,c^2.$$

Solution

Just expand the right hand side to get

$(a^2\,b + b^2\,c + c^2\,a)(a\,b^2 + b\,c^2 + c\,a^2) = a^3\,b^3 + a^2\,b^2\,c^2 + a^4\,b\,c + a\,b^4\,c + b^3\,c^3 + a^2\,b^2\,c^2 + a^2\,b^2\,c^2 + a\,b\,c^3 + a^3\,c^3$

Then apply the AM-GM inequality , we obtain

$$\frac{(a^2\,b + b^2\,c + c^2\,a)(a\,b^2 + b\,c^2 + c\,a^2)}{9} \geq \sqrt[9]{a^{18}\,b^{18}\,c^{18}} = a^2\,b^2\,c^2$$

and from then, $\quad (a^2\,b + b^2\,c + c^2\,a)(a\,b^2 + b\,c^2 + c\,a^2) \geq 9\,a^2\,b^2\,c^2.$

Example 1.16

Let $a_1, ..., a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 2^n.$$

Solution

By AM-GM

$$1 + a_1 \geq 2\sqrt{a_1},$$
$$1 + a_2 \geq 2\sqrt{a_2},$$
$$\vdots$$
$$1 + a_n \geq 2\sqrt{a_n}.$$

Multiplying the above inequalities and using the fact $a_1 a_2 \ldots a_n = 1$ we get the desired result. Equality holds for $a_i = 1, i = 1, 2, ..., n$.

Example 1.17

Let $a, b, c \geq 0$. Prove that

$$\frac{a^3}{b\,c} + \frac{b^3}{c\,a} + \frac{c^3}{a\,b} \geq a + b + c.$$

Solution

By AM-GM we easily deduce that

$$\frac{a^3}{b\,c} + b + c \geq 3\sqrt[3]{\frac{a^3}{b\,c}\,b\,c} = 3\,a;$$
$$\frac{b^3}{c\,a} + c + a \geq 3\sqrt[3]{\frac{b^3}{c\,a}\,a\,a} = 3\,b;$$
$$\frac{c^3}{a\,b} + a + b \geq 3\sqrt[3]{\frac{c^3}{a\,b}\,a\,b} = 3\,c.$$

Adding the three inequalities to get

$$\frac{a^3}{b\,c} + \frac{b^3}{c\,a} + \frac{c^3}{a\,b} + 2(a + b + c) \geq 3(a + b + c).$$

Which is the desired result.

Example 1.18

Let a, b, c, d be positive real numbers. Prove that

$$16(a\,b\,c + b\,c\,d + c\,d\,a + d\,a\,b) \leq (a + b + c + d)^3.$$

Solution

Applying AM-GM for two numbers, we obtain

$$\begin{aligned} 16\,(a\,b\,c + b\,c\,d + c\,d\,a + d\,a\,b) &= 16\,a\,b\,(c + d) + 16\,c\,d\,(a + b) \\ &\leq 4\,(a + b)^2(c + d) + 4\,(c + d)^2(a + b) \\ &= 4\,(a + b + c + d)(a + b)(c + d) \\ &\leq (a + b + c + d)^3. \end{aligned}$$

The equality holds for $a = b = c = d$.

Example 1.19

Let a, b, c be the side-lengths of a triangle. Prove that

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a\, b^b\, c^c.$$

Solution

Applying the weighted AM-GM inequality, we conclude that

$$\begin{aligned}\sqrt[a+b+c]{\left(\frac{a+b-c}{a}\right)\left(\frac{b+c-a}{b}\right)\left(\frac{c+a-b}{c}\right)} &\leq \frac{1}{a+b+c}\left(a\cdot\frac{a+b-c}{a}+b\cdot\frac{b+c-a}{b}+c\cdot\frac{c+a-b}{c}\right)\\ &= 1.\end{aligned}$$

In other words, we have

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a\, b^b\, c^c.$$

Equality holds if and only if $a = b = c$.

Example 1.20

Let a, b, c be positive real numbers. Prove that

$$\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right) \geq 2+\frac{2(x+y+z)}{\sqrt[3]{x\,y\,z}}.$$

Solution

The problem may be transformed to be equivalent to

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{2(x+y+z)}{\sqrt[3]{x\,y\,z}}.$$

This is true by AM-GM, indeed

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)=\left(\frac{2\,x}{y}+\frac{y}{z}\right)+\left(\frac{2\,y}{z}+\frac{z}{x}\right)+\left(\frac{2\,z}{x}+\frac{x}{y}\right) \geq \frac{3\,x}{\sqrt[3]{x\,y\,z}}+\frac{3\,y}{\sqrt[3]{x\,y\,z}}+\frac{3\,z}{\sqrt[3]{x\,y\,z}}.$$

Example 1.21

Suppose that a, b, c, d are four positive real numbers and $a+b+c+d = 4$. Show that

$$\frac{a}{1+b^2\,c}+\frac{b}{1+c^2\,d}+\frac{c}{1+d^2\,a}+\frac{d}{1+a^2\,b} \geq 2.$$

Solution

By the AM-GM, we deduce that

$$\begin{aligned}\frac{a}{1+b^2\,c} &= a-\frac{a\,b^2\,c}{1+b^2\,c}\geq a-\frac{a\,b^2\,c}{2\,b\sqrt{c}}=a-\frac{a\,b\,\sqrt{c}}{2}\\ &= a-\frac{b\sqrt{a\cdot a\,c}}{2}\geq a-\frac{b(a+a\,c)}{4}.\end{aligned}$$

And so,

$$\sum_{cyc}\frac{a}{1+b^2\,c}\geq\sum_{cyc}a-\frac{1}{4}\sum_{cyc}a\,b-\frac{1}{4}\sum_{cyc}a\,b\,c$$

By AM-GM once again, we have

$$\sum_{cyc}a\,b\leq\frac{1}{4}\left(\sum_{cyc}a\right)^2=4\quad;\sum_{cyc}a\,b\,c\leq\frac{1}{16}\left(\sum_{cyc}a\right)^3=4$$

Therefore

$$\frac{a}{1+b^2\,c}+\frac{b}{1+c^2\,d}+\frac{c}{1+d^2\,a}+\frac{d}{1+a^2\,b}\geq a+b+c+d-2=2.$$

Example 1.22

Let $a,b,c>0$, such that $a\,b+b\,c+c\,a+2\,a\,b\,c=1$, show that

$$\sqrt{a\,b}+\sqrt{b\,c}+\sqrt{c\,a}\leq\frac{3}{2}.$$

Solution

Let $x=\sqrt{a\,b}, y=\sqrt{b\,c}, z=\sqrt{c\,a}, s=x+y+z$. The given relation becomes

$$x^2+y^2+z^2+2\,x\,y\,z=2$$

and, by the AM-GM inequality, we have

$$\begin{aligned}s^2-2\,s+1 &= (x+y+z)^2-2(x+y+z)+1\\ &= 1-2\,x\,y\,z+2(x\,y+x\,z+y\,z)-2(x+y+z)+1\\ &= 2(x\,y+x\,z+y\,z-x\,y\,z-x-y-z+1)\\ &= 2(1-x)(1-y)(1-z)\\ &\leq 2\left(\frac{1-x+1-y+1-z}{3}\right)^3\quad\text{(AM-GM)}\\ &= 2\left(\frac{3-s}{3}\right)^3.\end{aligned}$$

Therefore

$2\,s^3+9\,s^2-27\le 0 \Longleftrightarrow (2\,s-3)(s+3)^2\le 0 \Longleftrightarrow s\le \dfrac{3}{2}$, the wanted inequality.

Example 1.23

Let $a, b, c > 0$, prove that

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\ge 2\left(1+\frac{a+b+c}{\sqrt[3]{a\,b\,c}}\right).$$

Solution

We have

$$\begin{aligned}
\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) &= 2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\\
&= \left(\frac{a}{b}+\frac{a}{c}+\frac{a}{a}\right)+\left(\frac{b}{c}+\frac{b}{a}+\frac{b}{b}\right)+\left(\frac{c}{a}+\frac{c}{b}+\frac{c}{c}\right)-1\\
&\ge 3\left(\frac{a}{\sqrt[3]{a\,b\,c}}+\frac{b}{\sqrt[3]{a\,b\,c}}+\frac{c}{\sqrt[3]{a\,b\,c}}\right)-1\\
&= 2\left(\frac{a+b+c}{\sqrt[3]{a\,b\,c}}\right)+\left(\frac{a+b+c}{\sqrt[3]{a\,b\,c}}\right)-1\\
&\ge 2\left(\frac{a+b+c}{\sqrt[3]{a\,b\,c}}\right)+3-1\\
&= 2\left(1+\frac{a+b+c}{\sqrt[3]{a\,b\,c}}\right).
\end{aligned}$$

As wanted.

1.5.2 Applications for extrema

Example 1.24

Find the Maximum of

$$f(x)=(1-x)(1+x)(1+x) \quad , x\in[0,1].$$

Solution

We have three factors in the function and we want to know when the function reaches a maximum in the interval $[0,1]$.

$$f(x)=(1-x)(1+x)(1+x).$$

In order to apply the AM-GM inequality, we need the sum of these factors to be equal a constant. For this, we need $-2x$ instead of $-x$. We can get that by the following

$$\begin{aligned}
f(x) &= (1-x)(1+x)(1+x); \\
&= \left(\frac{1}{2}\right) 2(1-x)(1+x)(1+x); \\
&= \left(\frac{1}{2}(2-2x)(1+x)(1+x)\right) \\
&\leq \left(\frac{1}{2}\right) \left(\frac{2-2x+1+x+1+x}{3}\right)^3 \text{(by AM-GM)} \\
&= \left(\frac{1}{2}\right) \left(\frac{4}{3}\right)^3 = \frac{32}{27} = 1.185185185...
\end{aligned}$$

The function attains this maximum value for x in $[0,1]$ if and only if

$$\begin{aligned}
2(1-x) &= 1+x, \\
2-2x &= 1+x, \\
1 &= 3x.
\end{aligned}$$

That is

$$f(x) \leq \frac{32}{27} = 1.185185185...$$

With equality if and only if

$$x = \frac{1}{3}.$$

Example 1.25

Find the minimum of

$$h(x) = \frac{x}{1+x^2}.$$

Solution

Let us write

$$h(x) = \frac{x}{1+x^2} = \frac{1}{x+\dfrac{1}{x}}.$$

By the AM-GM inequality,

$$\frac{1}{x} + x \geq 2\sqrt{\frac{1}{x}x} = 2.$$

Equality holds if and only if $x = \frac{1}{x}$ of, That is $x = 1$.

Therefore the value of the function is always ≤ 0.5 and it is equal to 0.5 only when $x = 1$.

Example 1.26

Find the minimum

$$f(x,y) = x + \frac{8}{y(x-y)},$$

such that $x > y > 0$. **Solution**

$$f(x,y) = x + \frac{8}{y(x-y)},$$

We write $\quad x = (x-y) + y$,

$$\begin{aligned} f(x,y) &= (x-y) + y + \frac{8}{y(x-y)} \\ &\geq 3\sqrt[3]{(x-y)y\frac{8}{y(x-y)}}, \quad \text{By AM-GM Inequality} \\ &= 3\sqrt[3]{8} = 6. \end{aligned}$$

The minimum value of $f(x,y)$ is 6. The minimum occurs when

$$x - y = y = \frac{8}{y(x-y)} = \frac{6}{3}.$$

So $\qquad x = 4 \; and \; y = 2.$

Example 1.27

Find the minimum

$$f(x,y) = \frac{12}{x} + \frac{18}{y} + xy,$$

such that $x > 0$ and $y > 0$. **Solution**

$$f(x,y) = \frac{12}{x} + \frac{18}{y} + xy;$$

By AM-GM inequality

$$\begin{aligned} \frac{12}{x} + \frac{18}{y} + x\,y &\geq 3\sqrt[3]{\left(\frac{12}{x}\right)\left(\frac{18}{y}\right)(x\,y)} \\ &\geq 3\sqrt[3]{216} = 18. \end{aligned}$$

The minimum value of $f(x,y)$ is 18 and occurs when
$\frac{12}{x} = \frac{18}{y} = x\,y = \frac{18}{3}$,
So $x = 2$, $y = 3$.

Example 1.28

Find the maximum g

$$g(x,y) = x\,y(72 - 3\,x - 4\,y),$$

where $x > 0$ and $y > 0$. **Solution**

$$\begin{aligned} g(x,y) &= x\,y(72 - 3\,x - 4\,y) \\ &= \frac{1}{12}(3\,x)(4\,y)(72 - 3\,x - 4\,y) \\ &\leq \frac{1}{12}\left(\frac{3\,x + 4\,y + 72 - 3\,x - 4\,y}{3}\right)^3, \quad \text{(By AM-GM Inequality).} \\ &\leq \frac{1}{12}\left(\frac{72}{3}\right)^3 = 1152. \end{aligned}$$

Hence the maximum limit value of the function is 1152 with equality, if and only if

$3\,x = 4\,y = 72 - 3\,x - 4\,y = \frac{1152}{3}$,

so, $x = 128$ and $y = 96$.

Chapter 2

Cauchy-Schwarz inequality

In this chapter, we state and prove Cauchy-Shwartz inequality , some proofs for C-S, are given. We also mention Cauchy-Schwarz inequality for integrals.

2.1 Cauchy-Schwarz inequality

Theorem 2.1.1

For any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ the following inequality holds.

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2.$$

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i \right)^2$$

Equality holds the two sequences are proportional, that is, if $a_i = \lambda b_i$.

2.2 Some different proofs of the Cauchy-Schwarz inequality

Proof 2.2.1

This is the classical proof of Cauchy-Schwartz inequality. Consider the quadratic

$$f(x) = \sum_{i=1}^{n} (a_i x - b_i)^2 = x^2 \sum_{i=1}^{n} a_i^2 - x \sum_{i=1}^{n} 2a_i b_i + \sum_{i=1}^{n} b_i^2 = A\,x^2 + B\,x + C.$$

Clearly $f(x) \geq 0$ for all real x. Hence if D is the discriminant of f, we must have $D \leq 0$. This implies

$$B^2 \leq 4\,A\,C \Rightarrow \left(\sum_{i=1}^{n} 2\,a_i b_i\right)^2 \leq 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

With is equivalent to

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Equality holds when $f(x) = 0$ for some x, which happens if $x = \dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \cdots = \dfrac{b_n}{a_n}$.

Proof 2.2.2

Consider the arithmetic-geometric means inequality

$$\sum_{i=1}^{n} \sqrt{x_i\,y_i} \leq \sum_{i=1}^{n} \frac{x_i + y_i}{2}. \tag{2.1}$$

Let $A = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$, $B = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$.

By the AM-GM inequality, we have

$$\sum_{i=1}^{n} \frac{a_i b_i}{A\,B} \leq \sum_{i=1}^{n} \frac{1}{2}\left(\frac{a_i^2}{A^2} + \frac{b_i^2}{B^2}\right) = 1,$$

so that

$$\sum_{i=1}^{n} a_i\,b_i \leq A\,B = \sqrt{a_1^2 + a^2{}_2 + \cdots + a_n^2}\,\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}.$$

Thus

$$\left(\sum_{i=1}^{n} a_i\,b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Proof 2.2.3

Consider the AM-GM inequality (2.1).

Let $A = \sqrt{\sum_{i=1}^{n} a_i^2}$ and $B = \sqrt{\sum_{i=1}^{n} b_i^2}$.

By choosing $x_i = \dfrac{B\,a_i^2}{A}$and $y_i = \dfrac{A\,b_i^2}{B}$ in inequality (2.1) we get

$$\begin{aligned}
\sum_{i=1}^{n} |a_i b_i| &\leq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{B\,a_i^2}{A} + \frac{A\,b_i^2}{B}\right) \\
&\leq \frac{1}{2}\left(\frac{B}{A}\sum_{i=1}^{n} a_i^2 + \frac{A}{B}\sum_{i=1}^{n} b_i^2\right) \\
&\leq \frac{1}{2}\left(\frac{B}{A}A^2 + \frac{A}{B}B^2\right) \\
&= A\,B.
\end{aligned}$$

From which we get the C-S inequality.

Proof 2.2.4

Expanding out the brackets and collecting together identical terms we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i\, b_j - a_j\, b_i)^2 &= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_j^2 - 2\sum_{i=1}^{n} a_i\, b_i \sum_{j=1}^{n} b_j\, a_j \\ &= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - 2\left(\sum_{i=1}^{n} a_i\, b_i\right)^2.\end{aligned}$$

Because the left hand side of the equation is a sum of the squares of real numbers it is greater than or equal to zero, thus

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i\, b_i\right)^2.$$

2.3 Cauchy-Schwarz inequality for integrals

There is a generalisation of this inequality to integrals

Theorem 2.3.1

Let $(f, g) \in (C([a, b], \mathbb{R}))^2$, *then*

$$\left|\int_a^b f(t)\, g(t)\, d(t)\right| \leq \sqrt{\int_a^b f(t)^2\, dt}\sqrt{\int_a^b g(t)^2\, dt}.$$

2.4 Examples

Example 2.4.1

Let $a, b, c > 0$, prove that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution

Using the Cauchy-Schwartz inequality we have

$$\begin{aligned}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 &= \left(\sum_{cyc}\frac{\sqrt{b+c}}{a}\frac{1}{\sqrt{b+c}}\right)^2\\ &\leq \left(\sum_{cyc}\frac{b+c}{a^2}\right)\left(\frac{1}{b+c}+\frac{1}{a+c}+\frac{1}{a+b}\right) \quad \text{(Cauchy-Schwarz)}\\ &\leq \left(\sum_{cyc}\frac{b+c}{a^2}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\end{aligned}$$

Therefore

$$\left(\sum_{cyc}\frac{b+c}{a^2}\right) \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Example 2.4.2

Suppose that $a, b, c \geq 1$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2$. Prove that

$$\sqrt{a+b+c} \geq \sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1}.$$

Solution

By Cauchy-Schwartz, we have

$$\begin{aligned}\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} &= \left(\sum_{cyc}\frac{\sqrt{a-1}}{\sqrt{a}}\sqrt{a}\right)\\ &\leq \left(\sum_{cyc}\frac{a-1}{a}\right)^{1/2}(a+b+c)^{1/2} \quad \text{(by Cauchy-Schwarz)}\\ &= \left(3-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)^{1/2}\sqrt{a+b+c}\\ &= \sqrt{a+b+c}.\end{aligned}$$

Which implies

$$\sqrt{a+b+c} \geq \sqrt{a-1}+\sqrt{b-1}+\sqrt{z-1}.$$

Example 2.4.3 (Nesbitt's inequality)

For positive real numbers a,b,c prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}.$$

First Solution

Our equality is equivalent to

$$\frac{a}{b+c}+1+\frac{b}{c+a}+1+\frac{c}{a+b}+1 \quad \geq \quad \frac{9}{2},$$

$$\text{or} \quad (a+b+c)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}) \quad \geq \quad \frac{9}{2}.$$

This can be written as

$$(x^2+y^2+z^2)(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}) \geq (1+1+1)^2.$$

Where $x=\sqrt{b+c}, y=\sqrt{c+a}, z=\sqrt{a+b}$. Then this is true by Cauchy-Schwartz.

Second Solution

After the substitution $x=b+c, y=c+a, z=a+b$, the inequality becomes

$$\sum_{cyc}\frac{y+z-x}{2\,x} \geq \frac{3}{2} \quad \text{or} \quad \sum_{cyc}\frac{y+z}{x} \geq 6,$$

and this is deduced from the AM-GM inequality as follows:

$$\sum_{cyc}\frac{y+z}{x}=\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z} \geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6.$$

Example 2.4.4

Prove that for all non negative real numbers $a, b\ and\ c$

$$6(a+b-c)(a^2+b^2+c^2)+27a\,b\,c \leq 10(a^2+b^2+c^2)^{\frac{3}{2}}$$

Solution

By Cauchy-Schwartz inequality, we deduce:

$10(a^2+b^2+c^2)^{\frac{3}{2}}-6(a+b-c)(a^2+b^2+c^2)$

$=(a^2+b^2+c^2)\left(10\sqrt{a^2+b^2+c^2}-6(a+b-c)\right)$

$=(a^2+b^2+c^2)\left(\frac{10}{3}\sqrt{(a^2+b^2+c^2)(2^2+2^2+1^2)}-6(a+b-c)\right)$

$\geq (a^2+b^2+c^2)\left(\frac{10(2a+2b+c)}{3}-6(a+b-c)\right)$

$=\frac{10(a^2+b^2+c^2)(2a+2b+28c)}{3}.$

Then, by the the AM-GM inequality, we have

$$a^2+b^2+c^2=4\,\frac{a^2}{4}+4\,\frac{b^2}{4}+c^2 \geq 9\sqrt[9]{\frac{a^8b^8c^8}{4^8}}$$

$$2a+2b+28c = 2a+2b+7.4c \geq 9\sqrt[9]{(2a)(2b)(2c)^7} \geq 9\sqrt[9]{4^8}\,a\,b\,c^7.$$

Thus,

$$10(a^2+b^2+c^2)^{\frac{3}{2}} - 6(a+b-c)(a^2+b^2+c^2) \geq 27a\,b\,c.$$

$$6(a+b-c)(a^2+b^2+c^2) + 27a\,b\,c \leq 10(a^2+b^2+c^2)^{\frac{3}{2}}$$

Example 2.4.5

Let a, b and c be positive real numbers such that, $a\,b\,c = 1$.

Prove that

$$a+b+c \leq a^2+b^2+c^2$$

using the C-S inequality.

Solution

The C-S inequlity can be written as

$$a\cdot 1 + b\cdot 1 + c\cdot 1 \leq \sqrt{a^2+b^2+c^2}\sqrt{1^2+1^2+1^2}$$

Or

$$a+b+c \leq \sqrt{a^2+b^2+c^2}\sqrt{3}. \tag{2.2}$$

Since the AM-GM inequality can be written as

$$\sqrt[3]{a^2b^2c^2} \leq \frac{a^2+b^2+c^2}{3}.$$

Using the given fact that $abc = 1$ yields

$$\sqrt{3} \leq \sqrt{a^2+b^2+c^2}. \tag{2.3}$$

From inequalities (2.2) and (2.3) we deduce that

$a+b+c \leq \sqrt{a^2+b^2+c^2}\sqrt{3} \leq \sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2} = a^2+b^2+c^2.$

Example 2.4.6

Let $a_i, b_i \in \mathbb{R}, b_i > 0$ for $i = 1, \cdots, n.$ Prove that

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}.$$

Solution

By the C-S inequality we have

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \left(\sum_{i=1}^{n} \frac{a_i}{\sqrt{b_i}}\sqrt{b_i}\right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i^2}{b_i}\right)\left(\sum_{i=1}^{n} b_i\right).$$

$$\frac{\left(\sum_{i=1}^n a_i\right)^2}{\sum_{i=1}^n b_i} \le \sum_{i=1}^n \frac{a_i^2}{b_i}.$$

Example 2.4.7 (Problem 3, IMC 2016).

Let n be a positive integer and $a_1, \dots, a_n, b_1, \dots, b_n$ be real number such that $a_i + b_i > 0$ for $i = 1, \dots, n$. Prove that

$$\sum_{i=1}^n \frac{a_i b_i - b_i^2}{a_i + b_i} \le \frac{\sum_{i=1}^n a_i \sum_{i=1}^n b_i - \left(\sum_{i=1}^n b_i\right)^2}{\sum_{i=1}^n (a_i + b_i)}. \tag{2.4}$$

Proof

The side of (2.4) are similar. For $A, B \in \mathbb{R}$,we have

$$\frac{A\,B - B^2}{A + B} \quad = \quad B - \frac{2\,B^2}{A + B} \tag{2.5}$$

Applying (2.5) for $A = a_i, B = b_i$, we get

$$\frac{A\,B - B^2}{A + B} = \sum_{i=1}^n \frac{a_i b_i - b_i^2}{a_i + b_i} = \sum_{i=1}^n \left(b_i - \frac{2\,b_i^2}{a_i + b_i}\right) = \sum_{i=1}^n b_i - \sum_{i=1}^n \frac{2\,b_i^2}{a_i + b_i}.$$

Similarly, apply (2.5) for $A = \sum_{i=1}^n a_i, B = \sum_{i=1}^n b_i$, we get

$$B - \frac{2\,B^2}{A + B} = \sum_{i=1}^n b_i - \frac{2\,\left(\sum_{i=1}^n b_i\right)^2}{\sum_{i=1}^n a_i + b_i}$$

then (2.4) is equivalent to

$$\frac{\left(\sum_{i=1}^n b_i\right)^2}{\sum_{i=1}^n a_i + b_i} \le \sum_{i=1}^n \frac{b_i^2}{a_i + b_i}. \tag{2.6}$$

By Cauchy-Schwarz inequality we deduce

$$\left(\sum_{i=1}^n b_i\right)^2 = \sum_{i=1}^n b_i \frac{b_i}{\sqrt{a_i + b_i}} \sqrt{a_i + b_i}^2 \le \left(\sum_{i=1}^n \frac{b_i^2}{a_i + b_i}\right)\left(\sum_{i=1}^n a_i + b_i\right),$$

which implies (2.6).

Example 2.4.8

Let $a, b, c > 0$, such that $a + b + c = 1$. Prove that

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \le \frac{1}{\sqrt{3}}$$

Solution

By Cauchy-Schwarz inequqlity

$$\begin{aligned}\left(a\sqrt{b}+b\sqrt{c}+c\sqrt{a}\right)^2 &= \left(\sum_{cyc}\sqrt{a}\sqrt{a\,b}\right)^2\\ &\leq \left(\sum_{cyc}a\right)\left(\sum_{cyc}a\,b\right)\\ &= (a\,b+b\,c+c\,a)\\ &\leq \frac{1}{3}(a+b+c)^2=\frac{1}{3}\end{aligned}$$

Just extract the square root to obtain the desired inequality.

Chapter 3

Some IMO problems

We give some problems from the International Mathematical olympiad.

The solutions use AM-GM and Cauchy-Schwarz inequalities.

Example 3.1

Let a, b, c be positive real numbers such that $a\,b\,c = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution

Let $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$. Then x, y, z are positive real numbers and $x\,y\,z = 1$. We have

$$\frac{1}{a^3(b+c)} = \frac{1}{\frac{1}{x^3}\left(\frac{1}{y} + \frac{1}{z}\right)} = \frac{x^2}{y+z}.$$

Similarly

$$\frac{1}{b^3(c+a)} = \frac{1}{\frac{1}{y^3}\left(\frac{1}{z} + \frac{1}{x}\right)} = \frac{y^2}{z+x}, \frac{1}{c^3(a+b)} = \frac{1}{\frac{1}{z^3}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{z^2}{x+y}.$$

By Cauchy-Shwarz inequality we have

$$((y+z) + (z+x) + (x+y))\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \geq (x+y+z)^2$$

So that, by AM-GM inequality we have

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{(x+y+z)^2}{2(x+y+z)} = \frac{x+y+z}{2} \geq \frac{3\sqrt[3]{x\,y\,z}}{2} = \frac{3}{2}$$

.

Example 3.2

Let a, b and c positive real numbers such that $a^2+b^2+c^2=3$. Prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\geq\frac{3}{2}.$$

Solution

By choosing the vectors

$x=(\sqrt{1+a\,b},\sqrt{1+b\,c},\sqrt{1+c\,a})$ and $y=\left(\frac{1}{\sqrt{1+a\,b}},\frac{1}{\sqrt{1+b\,c}},\frac{1}{\sqrt{1+c\,a}}\right)$ and by using the C-S inequality we get

$$((1+a\,b)+(1+b\,c)+(1+c\,a))(\frac{1}{1+a\,b}+\frac{1}{1+b\,c}+\frac{1}{1+c\,a})\geq 9,$$

Whence

$$\frac{1}{1+a\,b}+\frac{1}{1+b\,c}+\frac{1}{1+a\,c}\geq\frac{3}{2}\geq\frac{9}{3+a\,b+b\,c+c\,a}.$$

We know that

$$\frac{9}{3+a\,b+b\,c+c\,a}\geq\frac{9}{3+a^2+b^2+c^2},$$

And since $a^2+b^2+c^2=3$, we obtain

$$\frac{1}{1+a\,b}+\frac{1}{1+b\,c}+\frac{1}{1+a\,c}\geq\frac{3}{2}.$$

Example 3.3

Let a, b, c and d be positive real numbers such that $a+b+c+d=1$. Prove that

$$\frac{1}{2}\leq\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}.$$

Solution

By choosing the vectors

$x=\left(\sqrt{a+b},\sqrt{b+c},\sqrt{c+d},\sqrt{d+a}\right)$ and $y=\left(\frac{a}{\sqrt{a+b}},\frac{b}{\sqrt{b+c}},\frac{c}{\sqrt{c+d}},\frac{d}{\sqrt{d+a}}\right)$.

And by using the C-S inequality we get

$$\begin{aligned}(a+b+c+d)^2 &\leq ((a+b)+(b+c)+(c+d)+(d+a))\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)\\ \frac{(a+b+c+d)^2}{2\,(a+b+c+d)} &\leq \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\end{aligned} \tag{3.2}$$

Since $a+b+c+d=1$ we get the desired result.

Example 3.4

Prove that

$$\frac{a}{\sqrt{a^2+8\,b\,c}}+\frac{b}{\sqrt{b^2+8\,c\,a}}+\frac{c}{\sqrt{c^2+8\,a\,b}}\geq 1,$$

for all positive real numbers a, b and c.

Solution

First we shall prove that

$$\frac{a}{\sqrt{a^2+8\,b\,c}}\geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$

or equivalently, that

$$(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2\geq a^{\frac{2}{3}}(a^2+8\,b\,c).$$

The AM-GM inequality yields

$$\begin{aligned}(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2-(a^{\frac{4}{3}})^2 &= (b^{\frac{4}{3}}+c^{\frac{4}{3}})(a^{\frac{4}{3}}+a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})\\ &\geq 2\,b^{\frac{2}{3}}\,c^{\frac{2}{3}}\cdot 4\,a^{\frac{2}{3}}b^{\frac{1}{3}}\,c^{\frac{1}{3}}\\ &= 8\,a^{\frac{2}{3}}\,b\,c.\end{aligned}$$

Thus

$$\begin{aligned}(a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}})^2 &\geq (a^{\frac{4}{3}})^2+8\,a^{\frac{2}{3}}\,b\,c\\ &= a^{\frac{2}{3}}(a^2+8\,b\,c),\end{aligned}$$

so

$$\frac{a}{\sqrt{a^2+8\,b\,c}}\geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}.$$

Similarly, we have

$$\frac{b}{\sqrt{b^2+8\,c\,a}}\geq \frac{b^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}}, \text{and} \quad \frac{c}{\sqrt{c^2+8\,a\,b}}\geq \frac{c^{\frac{4}{3}}}{a^{\frac{4}{3}}+b^{\frac{4}{3}}+c^{\frac{4}{3}}},$$

Adding these three inequalities yields

$$\frac{a}{\sqrt{a^2+8\,b\,c}}+\frac{b}{\sqrt{b^2+8\,c\,a}}+\frac{c}{\sqrt{c^2+8\,a\,b}}\geq 1.$$

Example 3.5

Prove that

$$\frac{a}{b+2\,c+3\,d}+\frac{b}{c+2\,d+3\,a}+\frac{c}{d+2\,a+3\,b}+\frac{d}{a+2\,b+3\,c}\geq \frac{2}{3}$$

For all positive real numbers a, b, c, d.

Solution

By the Cauchy-Schwarz inequality, if $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ are positive numbers, then

$$\left(\sum_{i=1}^{n} \frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n} x_i\, y_i\right) \geq \left(\sum_{i=1}^{n} x_i\right)^2$$

Applying this to the numbers a, b, c, d and $b + 2\,c + 3\,d, c + 2\,d + 3a, d + 2\,a + 3\,b, a + 2\,b + 3\,c$ (here $n = 4$), we obtain

$$\begin{aligned} \frac{a}{b+2\,c+3\,d} + \frac{b}{c+2\,d+3\,a} + \frac{c}{d+2\,a+3\,b} + \frac{d}{a+2\,b+3\,c} &\geq \frac{(a+b+c+d)^2}{4\,(a\,b+a\,c+a\,d+b\,c+b\,d+c\,d)} \\ &\geq \frac{2}{3}. \end{aligned}$$

The last inequality follows, for example, from $(a-b)^2 + (a-c)^2 + \ldots + (c-d)^2 \geq 0$. Equality holds if and only if $a = b = c = d$.

Example 3.6

Denote by a, b, c the lengths of the sides of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3\,a\,b\,c.$$

Solution

By substituting $a = x + y, b = y + z$, and $c = z + x(x, y, z > 0)$ the given inequality becomes

$$6\,x\,y\,z \leq x^2\,y + x\,y^2 + y^2\,z + y\,z^2 + z^2\,x + z\,x^2.$$

Which follows immediately by the AM-GM inequality applied to $x^2\,y, x\,y^2, x^2\,z, x\,z^2, y^2\,z, y\,z^2$.

Example 3.7

Prove that for arbitrary positive numbers the following inequality holds:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3\,b^3\,c^3}.$$

Solution

Using the A-G mean inequality we obtain

$$\begin{aligned} 8\,a^2\,b^3\,c^3 &\leq 2\,a^8 + 3\,b^8 + 3\,c^8, \\ 8\,a^3\,b^2\,c^3 &\leq 3\,a^8 + 2\,b^8 + 3\,c^8, \\ 8\,a^3\,b^3\,c^2 &\leq 3\,a^8 + 3\,b^8 + 2\,c^8. \end{aligned}$$

By adding these inequalities and dividing by $3\,a^3b^3c^3$ we obtain the desired inequality.

Example 3.8

Let a, b, c be the sides of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}+\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\le 3.$$

Solution

Due to the symmetry of variables, it can be assumed that $a\ge b\ge c$. We claim that

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\le 1 \quad\text{and}\quad \frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}\le 2$$

The frst inequality follows from

$$\sqrt{a+b-c}-\sqrt{a}=\frac{(a+b-c)-a}{\sqrt{a+b-c}+\sqrt{a}}\le\frac{b-c}{\sqrt{b}+\sqrt{c}}=\sqrt{b}-\sqrt{c}.$$

For proving the second inequality, let $p=\sqrt{a}+\sqrt{b}$ and $q=\sqrt{a}-\sqrt{b}$. Then $a-b=p\,q$ and the inequality becomes

$$\frac{\sqrt{c-p\,q}}{\sqrt{c}-q}+\frac{\sqrt{c+p\,q}}{\sqrt{c}+q}\le 2.$$

From $a\ge b\ge c$ we have $p\ge 2\sqrt{c}$. By Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{\sqrt{c-p\,q}}{\sqrt{c}-q}+\frac{\sqrt{c+p\,q}}{\sqrt{c}+q}^2 &\le \left(\frac{c-p\,q}{\sqrt{c}-q}+\frac{c+p\,q}{\sqrt{c}+q}\right)\left(\frac{1}{\sqrt{c}-q}+\frac{1}{\sqrt{c}+q}\right)\\ &= \frac{2(c\sqrt{c}-p\,q^2)}{c-q^2}\cdot\frac{2\sqrt{c}}{c-q^2}\\ &= 4\cdot\frac{c^2-\sqrt{c}\,p\,q^2}{(c-q^2)^2}\le 4\cdot\frac{c^2-2\,c\,q^2}{(c-q^2)^2}\le 4.\end{aligned}$$

Conclusion

In this thesis, we have stated and proved two important inequalities. The Arithmetic-Geometric inequality (AM-GM) and Cauchy-Schwarz inequality.

Their importance and use to prove other inequalities, as well as to find extrema of functions of several variables is illustrated by many examples.

Last not least, we can not talk about AM-GM and Cauchy-Schwarz without mentioning the International Mathematical Olympiad, where these inequalities are among the basic tools to be mastered by any good contestant. For this, we gave some problems with their solutions as examples from some IMO sessions.

Finally, we must say that our investigation of these inequalities is not exhaustive. For instance, the AM-GM could be applied to deduce many interesting geometric inequalities, and the Cauchy-Schwartz inequality has a continuous analogous leading to many interesting inequalities. We did not see almost any application of this this kind.

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